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MANUAL of the THEORY of ELASTICITY

by V. G. Rekach

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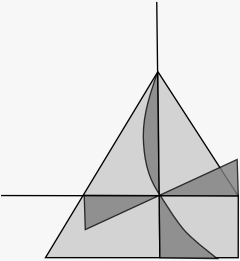

MANUAL OF THE THEORY OF ELASTICITY

В. Г. Рекач

проф. докт. техн. наук

РУКОВОДСТВО К РЕШЕНИЮ ЗАДАЧ ПО ТЕОРИИ УПРУГОСТИ

МОСКВА «ВЫСШАЯ ШКОЛА»

V. G. Rekach

MANUAL *of the* THEORY *of* ELASTICITY

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CONTENTS

NOTATION

$\alpha_1, \alpha_2, \alpha_3$	orthogonal curvilinear co-ordinates
x, y, z	rectangular Cartesian co-ordinates
r, β, z	cylindrical co-ordinates
r, β, α	spherical co-ordinates
$\left.\begin{matrix} u_x, u_y, u_z \\ u_r, u_\beta, u_z \\ u_r, u_\beta, u_\alpha \end{matrix}\right\}$	projections of the displacement u of a point on fixed co-ordinate axes (x, y, z; r, β, z; r, β, α), i.e., the components of the displacement vector
$\left.\begin{matrix} X_x, Y_x, Z_x \\ X_y, Y_y, Z_y \\ X_z, Y_z, Z_z \end{matrix}\right\}$	components of the stress tensor in rectangular co-ordinates
$\left.\begin{matrix} R_r, B_r, Z_r \\ R_\beta, B_\beta, Z_\beta \\ R_z, B_z, Z_z \end{matrix}\right\}$	components of the stress tensor in cylindrical co-ordinates
$\left.\begin{matrix} R_r, B_r, A_r \\ R_\beta, B_\beta, A_\beta \\ R_\alpha, B_\alpha, A_\alpha \end{matrix}\right\}$	components of the stress tensor in spherical co-ordinates
ρ	density of material
$E = G\frac{3\lambda+2G}{\lambda+G}$	modulus of longitudinal elasticity

$\sigma = \frac{\lambda}{2(\lambda+G)}$	Poisson's ratio
$\mu = G = \frac{E}{2(1+\sigma)}$, $\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$	Lamé's coefficients
e_{xx}, e_{xy}, e_{xz}; e_{yx}, e_{yy}, e_{yz}; e_{zx}, e_{zy}, e_{zz}	components of the strain tensor in rectangular co-ordinates
τ	time
t	temperature
$\sigma_\nu, \tau_\nu, p_\nu$	normal, tangential, and total stress on a plane with normal ν
$\sigma_1, \sigma_2, \sigma_3$	principal normal stresses at a point
$\tau_{12}, \tau_{23}, \tau_{31}$	principal shearing stresses at a point
J_1, J_2, J_3	invariants of the state of stress at a point
I_1, I_2, I_3	invariants of the state of strain at a point

Chapter 1

THEORY OF STRESS

I. STATIC AND DYNAMIC EQUILIBRIUM EQUATIONS

1. Orthogonal curvilinear co-ordinates

$$\frac{\partial (H_h \Delta)}{\partial \alpha_h} - \frac{1}{2} \sum_{\nu} \frac{V_\nu \Delta}{g_\nu} \frac{\partial g_\nu}{\partial \alpha_h} + \sum \frac{\partial}{\partial \alpha_\nu} \left(\frac{V_h \Delta g_h}{\sqrt{g_\nu g_h}} \right) +$$

$$+ H \Delta \sqrt{g_h} = \rho \Delta \sqrt{g_h} \frac{\partial^2 u_h}{\partial \tau^2}, \tag{1.1}$$

where α_ν, α_h = orthogonal curvilinear co-ordinates (mutually perpendicular surfaces),

H_h, V_ν = normal stresses,

$V_h = H_\nu$ = shearing stresses,

g_h, g_ν = coefficients in the first quadratic form,

$$ds^2 = \sum_1^3 g_h d^2\alpha_h,$$

$$g_h = \left(\frac{\partial x}{\partial \alpha_h} \right)^2 + \left(\frac{\partial y}{\partial \alpha_h} \right)^2 + \left(\frac{\partial z}{\partial \alpha_h} \right)^2,$$

$$\frac{\partial x}{\partial \alpha_h} = \sqrt{g_h} \frac{\partial \alpha_h}{\partial x}, \quad \text{etc.}$$

$\sqrt{g_h}\, d\alpha_h = ds_h$ = length of an element of a co-ordinate line,

$$\left.\begin{aligned} \cos(\alpha_1, x) &= \sqrt{g_1} \frac{\partial \alpha_1}{\partial x}, \\ \cos(\alpha_1, y) &= \sqrt{g_1} \frac{\partial \alpha_1}{\partial y}, \\ \cos(\alpha_1, z) &= \sqrt{g_1} \frac{\partial \alpha_1}{\partial z} \end{aligned}\right\} = \text{direction cosines of a normal to surface } \alpha_1 \text{ at point } (\alpha_1, \alpha_1, \alpha_3),$$

$\Delta = \sqrt{g_1 g_2 g_3}$ = unit volume factor,

$\Delta V = \Delta d\alpha_1\, d\alpha_2\, d\alpha_3$ = element of volume,

H = body force per unit volume in the direction of the α_h co-ordinate,

$u_h,\ u_v$ = displacements in the direction of curvilinear co-ordinates.

2. Rectangular co-ordinates

$\alpha_1 = x,\quad \alpha_2 = y,\quad \alpha_3 = z$ = planes,
$g_1 = g_2 = g_3 = 1,\quad \Delta = 1,\quad dV = dx\, dy\, dz,$
$ds^2 = dx^2 + dy^2 + dz^2.$
The equilibrium equations are

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X = 0 \left(= \rho \frac{\partial^2 u_x}{\partial \tau^2} \right),$$
$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y = 0 \left(= \rho \frac{\partial^2 u_y}{\partial \tau^2} \right), \qquad (1.1a)$$
$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z = 0 \left(= \rho \frac{\partial^2 u_z}{\partial \tau^2} \right),$$

where X, Y, Z are the projections of body forces on the x, y, z axes.

3. Cylindrical co-ordinates

$\alpha_1 = r$ = circular cylinders,
$\alpha_2 = \beta$ = planes through the Oz axis,
$\alpha_3 = z$ = planes parallel to the xOy plane (Fig. 1).
Referring to Fig. 1,

$$x = r\cos\beta,\quad y = r\sin\beta,\quad z,$$

from which $g_1 = 1,\ g_2 = r^2,\ g_3 = 1,\ \Delta = r,$
$dV = r\, dr\, d\beta\, dz,\quad ds^2 = dr^2 + r^2\, d\beta^2 + dz^2.$
The equilibrium equations are

$$\frac{1}{r}\frac{\partial}{\partial r}(rR_r) + \frac{1}{r}\frac{\partial R_\beta}{\partial \beta} - \frac{B_\beta}{r} + \frac{\partial R_z}{\partial z} + R = 0 \left(= \rho \frac{\partial^2 u_r}{\partial \tau^2} \right),$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r}\frac{\partial B_\beta}{\partial \beta} + \frac{\partial B_z}{\partial z} + B = 0 \left(= \rho \frac{\partial^2 u_\beta}{\partial \tau^2} \right),$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rZ_r)+\frac{1}{r}\frac{\partial Z_\beta}{\partial \beta}+\frac{\partial Z_z}{\partial z}+Z=0\left(=\rho\frac{\partial^2 u_z}{\partial \tau^2}\right), \qquad (1.1b)$$

where R, B, Z are the projections of body forces on the r, β, z axes.

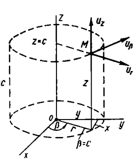

Fig. 1

4. Spherical co-ordinates

$\alpha_1 = r$ = spheres,
$\alpha_2 = \beta$ = planes through the Oz axis,
$\alpha_3 = \alpha$ = cones with vertex at the point O (Fig. 2).
Referring to Fig. 2,

$$x = r\sin\alpha\cos\beta, \quad y = r\sin\alpha\sin\beta, \quad z = r\cos\alpha,$$

from which

$$g_1 = 1, \quad g_2 = r^2\sin^2\alpha, \quad g_3 = r^2, \quad \Delta = r^3\sin\alpha,$$

$$dV = r^2\sin\alpha\, dr\, d\beta\, d\alpha,$$
$$ds^2 = dr^2 + r^2\sin^2\alpha\, d\beta^2 + r^2\, d\alpha^2.$$

The equilibrium equations are

$$\frac{\partial R_r}{\partial r}+\frac{1}{r\sin\alpha}\frac{\partial R_\beta}{\partial \beta}+\frac{1}{r}\frac{\partial R_\alpha}{\partial \alpha}+$$
$$+\frac{2R_r-B_\beta-A_\alpha+R_\alpha\cot\alpha}{r}+R=0\left(=\rho\frac{\partial^2 u_r}{\partial \tau^2}\right),$$

$$\frac{\partial B_r}{\partial r}+\frac{1}{r\sin\alpha}\frac{\partial B_\beta}{\partial \beta}+\frac{1}{r}\frac{\partial B_\alpha}{\partial \alpha}+\frac{3B_r+2B_\alpha\cot\alpha}{r}+$$
$$+B=0\left(=\rho\frac{\partial^2 u_\beta}{\partial t^2}\right), \tag{1.1c}$$
$$\frac{\partial A_r}{\partial r}+\frac{1}{r\sin\alpha}\frac{\partial A_\beta}{\partial \beta}+\frac{1}{r}\frac{\partial A_\alpha}{\partial \alpha}+$$
$$+\frac{(A_\alpha-B_\beta)\cot\alpha+3A_r}{r}+A=0\left(=\rho\frac{\partial^2 u_\alpha}{\partial t^2}\right),$$

where R, B, A are the projections of body forces on the r, β, α axes.

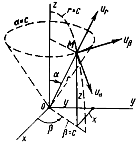

Fig. 2

II. SURFACE CONDITIONS

The local boundary conditions, which are valid for every point of the surface of a body with normal ν, are of the form

$$X_\nu = X_x l + X_y m + X_z n,$$
$$Y_\nu = Y_x l + Y_y m + Y_z n, \tag{1.2}$$
$$Z_\nu = Z_x l + Z_y m + Z_z n,$$

where $l=\cos(x,\nu)$, $m=\cos(y,\nu)$, $n=\cos(z,\nu)$, $l^2+m^2+n^2=1$.

The integral boundary conditions, which are valid for a part of the surface of a body (usually a plane), specify that the sum of stresses acting on the surface is equal to the external forces (Problem 5.3).

The normal and tangential stresses on a surface element with normal ν are, respectively,

$$\sigma_\nu = X_x l^2 + Y_y m^2 + Z_z n^2 + 2X_y lm + 2Y_z mn + 2Z_x nl, \tag{1.3}$$

$$\tau_\nu = \sqrt{X_\nu^2 + Y_\nu^2 + Z_\nu^2 - \sigma_\nu^2}.$$

The total stress is

$$p_\nu = \sqrt{\sigma_\nu^2 + \tau_\nu^2}.$$

III. STATE OF STRESS AT A POINT

The principal normal stresses σ_1, σ_2, σ_3 at a point are determined as the roots of the cubic equation

$$\sigma^3 - J_1\sigma^2 + J_2\sigma - J_3 = 0, \tag{1.4}$$

where J_i are the invariants (quantities independent of the choice of co-ordinate axes) of the state of stress, respectively, equal to

$$J_1 = X_x + Y_y + Z_z\ (R_r + B_\beta + Z_z,\ \text{etc.}),$$
$$J_2 = X_xY_y + X_xZ_z + Y_yZ_z - X_y^2 - Y_z^2 - Z_x^2,$$
$$J_3 = \begin{vmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{vmatrix}. \tag{1.5}$$

The invariants of the state of stress are expressed in terms of the principal stresses as

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3,\quad J_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3,$$
$$J_3 = \sigma_1\sigma_2\sigma_3. \tag{1.6}$$

The principal shearing stresses are determined by the formulas

$$\tau_{12} = \pm\frac{1}{2}(\sigma_1 - \sigma_2),\quad \tau_{23} = \pm\frac{1}{2}(\sigma_2 - \sigma_3),$$
$$\tau_{31} = \pm\frac{1}{2}(\sigma_3 - \sigma_1). \tag{1.7}$$

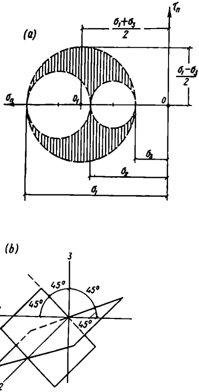

Fig. 3

The values of the normal and shearing stresses on any planes passing through a given point lie on the shaded part of the $\sigma_n\tau_n$ plane (Fig. 3*a*).

If $\sigma_1 > \sigma_2 > \sigma_3$, the extreme values of the shearing stress are determined by the formulas

$$\tau_{\substack{max \\ min}} = \pm \frac{\sigma_1 - \sigma_3}{2}. \qquad (1.8)$$

The planes on which the extreme shearing stresses are acting are shown in Fig. 3*b*.

PROBLEMS

1.1. Write the equilibrium equations for an infinitesimal parallelepiped isolated from a body acted on by the force of attraction of a mass M located at a point ξ, η, ζ (Fig. 4).

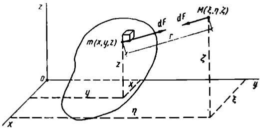

Fig. 4

The mass of the parallelepiped is $dm = \rho\, dV$, where $dV = dx\, dy\, dz$.

The distance between the masses dm and M is

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}.$$

According to Newton's law, the force of attraction acting between the masses dm and M is

$$dF = k^2 \frac{M\, dm}{r^2} = k^2 \frac{\rho M}{r^2}\, dV,$$

where k^2 is the gravitational constant.

The projections of the force dF on the co-ordinate axes are

$$dF_x = k^2 \frac{\rho M}{r^2} dV \cos(r, x) = k^2 \frac{\rho M}{r^3} (\xi - x)\, dV,$$

$$dF_y = k^2 \frac{\rho M}{r^2} dV \cos(r, y) = k^2 \frac{\rho M}{r^3} (\eta - y)\, dV,$$

$$dF_z = k^2 \frac{\rho M}{r^2} dV \cos(r, z) = k^2 \frac{\rho M}{r^3} (\zeta - z)\, dV.$$

Substituting the values of $dF_{x,\,y,\,z}$ in Eqs. (1.1a), and cancelling out the element of volume dV, we obtain

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho \frac{k^2 M}{r^3} (\xi - x) = 0,$$

$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + \rho \frac{k^2 M}{r^3} (\eta - y) = 0,$$

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + \rho \frac{k^2 M}{r^3} (\zeta - z) = 0.$$

1.2. Write the equilibrium equations for an infinitesimal parallelepiped isolated from a body that is located on the surface of the earth and is subjected to the gravitational attraction (Fig. 5).

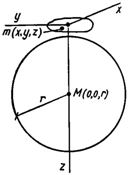

Fig. 5

Assuming $(\xi - x) = (\eta - y) = 0$ and $(\zeta - z) \cong r$ (Problem 1.1), we obtain

$$dF_x = dF_y = 0, \quad dF_z = \rho \frac{k^2 M}{r^2} \, dV = \rho g \, dV,$$

where $g = k^2M/r^2 = 980.616$ cm/s² = acceleration of gravity,
$k^2 = 6.67 \times 10$ cm³/g·s² = gravitational constant,
$M = 5.98 \times 10^{27}$ g = mass of the earth,
$r = 6.3783 \times 10^8$ cm = radius of the earth.

Equations (1.1a) become

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0,$$

$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = 0,$$

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + \rho g = 0.$$

1.3. Write the boundary conditions, calculate the stresses and strains for a body *ABCD* of small thickness that is

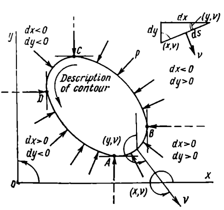

Fig. 6

acted on by a compressive load of intensity p normal to the contour (Fig. 6). Assume no body forces.

According to Eqs. (1.2),

$$X_\nu = X_x \cos(x, \nu) + X_y \cos(y, \nu),$$

$$Y_\nu = Y_x \cos(x, \nu) + Y_y \cos(y, \nu).$$

The values of the cosine must be taken for positive values of dx and dy, i.e., for the section AB:

$$\cos(x, \nu) = \frac{dy}{ds} \quad \text{(fourth quadrant)},$$

$$\cos(y, \nu) = -\frac{dx}{ds} \quad \text{(third quadrant)}.$$

The boundary conditions become

$$-p \cos(x, \nu) = X_x \cos(x, \nu) + X_y \cos(y, \nu),$$

$$-p \cos(y, \nu) = Y_x \cos(x, \nu) + Y_y \cos(y, \nu)$$

or

$$p = -X_x + X_y \frac{dx}{dy}, \quad p = Y_x \frac{dy}{dx} - Y_y.$$

The state of stress in the body is characterized by a stress system satisfying the equilibrium equations and the boundary conditions:

$$X_x = Y_y = -p, \quad X_y = Y_x = 0.$$

The strains are

$$e_{xx} = e_{yy} = -\frac{p(1-\sigma)}{E}, \quad e_{xy} = 0.$$

For a body of arbitrary shape we obtain

$$X_x = Y_y = Z_z = -p, \quad X_y = Y_z = Z_x = 0,$$

$$e_{xx} = e_{yy} = e_{zz} = -p/K, \quad e_{xy} = e_{yz} = e_{zx} = 0,$$

$$0 = -3p/K,$$

where $K = E/(1 - 2\sigma)$ is thrice the bulk modulus.

1.4. Write the boundary conditions for a triangular section of small thickness to which a load $q = \gamma y$ is applied along the line OB (Fig. 7).

Along the line OA ($x = y \tan \beta$):

$\cos(x, \nu) = \cos(360 - \beta) = \cos\beta$,

$\cos(y, \nu) = \cos(270 - \beta) = -\sin\beta$,

$X_\nu = Y_\nu = 0$, and the boundary conditions become $X_x \cos\beta - X_y \sin\beta = 0$, $\quad Y_x \cos\beta - Y_y \sin\beta = 0$.

Along the line OB ($x = 0$):

$\cos(x, \nu) = \cos 180° = -1$, $\quad \cos(y, \nu) = \cos 90° = 0$,

$X_\nu = \gamma y$ (the load coincides with the positive Ox axis), $Y_\nu = 0$, and the boundary conditions are

$X_x = -\gamma y$, $\quad Y_x = 0$.

1.5. Examine what conditions are satisfied by Maxwell's

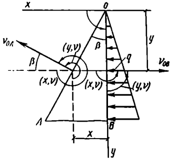

Fig 7

electrostatic stress system

$$X_x = \frac{1}{8\pi}\left[\left(\frac{\partial U}{\partial x}\right)^2 - \left(\frac{\partial U}{\partial y}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2\right], \quad X_y = \frac{1}{4\pi}\frac{\partial U}{\partial x}\frac{\partial U}{\partial y},$$

$$Y_y = \frac{1}{8\pi}\left[\left(\frac{\partial U}{\partial y}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2 - \left(\frac{\partial U}{\partial x}\right)^2\right], \quad Y_z = \frac{1}{4\pi}\frac{\partial U}{\partial y}\frac{\partial U}{\partial z},$$

$$Z_z = \frac{1}{8\pi}\left[\left(\frac{\partial U}{\partial z}\right)^2 - \left(\frac{\partial U}{\partial x}\right)^2 - \left(\frac{\partial U}{\partial y}\right)^2\right], \quad Z_x = \frac{1}{4\pi}\frac{\partial U}{\partial z}\frac{\partial U}{\partial x}.$$

Answer. (1) The system is balanced when

$$\nabla^2 U = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U = 0.$$

(2) The system is in equilibrium with the body forces

$$X,\ Y,\ Z = -\frac{1}{4\pi} \nabla^2 U \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right).$$

1.6. In the absence of body forces the stresses can be expressed in terms of three stress functions.

Check the following stress systems by substitution in the homogeneous equations (1.1a):

(1) Maxwell's system (1870)

$$X_x = \frac{\partial^2 \chi_1}{\partial y^2} + \frac{\partial^2 \chi_3}{\partial z^2}, \quad X_y = -\frac{\partial^2 \chi_1}{\partial x\,\partial y},$$

$$Y_y = \frac{\partial^2 \chi_2}{\partial z^2} + \frac{\partial^2 \chi_1}{\partial x^2}, \quad Y_z = -\frac{\partial^2 \chi_2}{\partial y\,\partial z},$$

$$Z_z = \frac{\partial^2 \chi_3}{\partial x^2} + \frac{\partial^2 \chi_2}{\partial y^2}, \quad Z_x = -\frac{\partial^2 \chi_3}{\partial z\,\partial x}.$$

(2) Morera's system (1892)*

$$X_x = \frac{\partial^3 \varphi_1}{\partial x\,\partial y\,\partial z}, \quad Y_y = \frac{\partial^3 \varphi_2}{\partial y\,\partial z\,\partial x}, \quad Z_z = \frac{\partial^3 \varphi_3}{\partial z\,\partial x\,\partial y},$$

$$X_y = -\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial y^2} - \frac{\partial^2 \varphi_3}{\partial z^2}\right),$$

$$Y_z = -\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial^2 \varphi_2}{\partial y^2} + \frac{\partial^2 \varphi_3}{\partial z^2} - \frac{\partial^2 \varphi_1}{\partial x^2}\right),$$

$$Z_x = -\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial^2 \varphi_3}{\partial z^2} + \frac{\partial^2 \varphi_1}{\partial x^2} - \frac{\partial^2 \varphi_2}{\partial y^2}\right).$$

The indicated functions are determined from the equations [1], [2]:

$$\left[\frac{\partial^2}{\partial z^2} + (\sigma - 2)\nabla^2\right]\chi_1 + \frac{\partial^2}{\partial z^2}\chi_2 + \frac{\partial^2}{\partial y^2}\chi_3 = 0,$$

$$\frac{\partial^2}{\partial z^2}\chi_1 + \left[\frac{\partial^2}{\partial x^2} + (\sigma - 2)\nabla^2\right]\chi_2 + \frac{\partial^2}{\partial y^2}\chi_3 = 0,$$

$$\frac{\partial^2}{\partial z^2}\chi_1 + \frac{\partial^2}{\partial x^2}\chi_2 + \left[\frac{\partial^2}{\partial y^2} + (\sigma - 2)\nabla^2\right]\chi_3 = 0$$

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* For other forms of representation, see Blokh V. I., "Theory of Elasticity", Izd. Khar'kov. Univ., Khar'kov, 1964, p. 314 (in Russian).

and

$$\left(\frac{\partial^2}{\partial x^2}+\sigma\nabla^2\right)\varphi_1-\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\varphi_2+\varphi_3)=0,$$

$$\left(\frac{\partial^2}{\partial y^2}+\sigma\nabla^2\right)\varphi_2-\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)(\varphi_3+\varphi_1)=0,$$

$$\left(\frac{\partial^2}{\partial z^2}+\sigma\nabla^2\right)\varphi_3-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\varphi_1+\varphi_2)=0.$$

1.7. Lamé's problem (1859) [1].

For plane stress, derive the equilibrium equations in Lamé's curvilinear isostatic co-ordinates (curves coinciding at each point with the directions of the principal stresses σ_1 and σ_2).

Let s_1 be an isostath coinciding with the direction of the principal stress σ_1, and let s_2 be an isostath coinciding

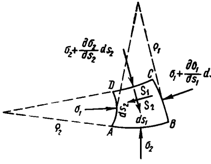

Fig. 8

with σ_2; ρ_1 and ρ_2 are the radii of curvature of these isostaths (Fig. 8).

By isolating a plane element *ABCD* by two pairs of infinitesimally close isostaths, and setting up the equilibrium conditions for it, we obtain

$$\frac{\partial\sigma_1}{\partial s_1}+\frac{\sigma_1-\sigma_2}{\rho_2}+S_1=0,$$

$$\frac{\partial\sigma_2}{\partial s_2}+\frac{\sigma_1-\sigma_2}{\rho_1}+S_2=0,$$

where S_l is the projection of the body force on the direction ds_l.*

1.8. The principal stresses at a point M of an elastic body are: $\sigma_1 = 50\ \mathrm{N/cm^2}$, $\sigma_2 = -50\ \mathrm{N/cm^2}$, $\sigma_3 = 75\ \mathrm{N/cm^2}$.

Find the total stress p_ν, the normal stress σ_ν, and the tangential stress τ_ν on a plane equally inclined to the principal axes (Fig. 9).

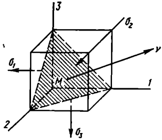

Fig. 9

The problem is solved by using Eqs. (1.2) and (1.3).

Answer. $p_\nu = 59.5\ \mathrm{N/cm^2}$, $\sigma_\nu = 25\ \mathrm{N/cm^2}$, $\tau_\nu = 54.1\ \mathrm{N/cm^2}$.

1.9. The stresses X_x, Y_y, Z_z, Y_x, and X_y are acting at a point of an elastic body (Fig. 10). The stresses $Y_z = Z_x = 0$.

Find the principal normal stresses σ_i and the stresses on a plane parallel to the z axis whose normal makes an angle α with the x axis.

Use Eqs. (1.4) and (1.3) to solve the problem.

Answer.

$$\sigma_\nu = X_x \cos^2\alpha + Y_y \sin^2\alpha + X_y \sin 2\alpha,$$

$$\tau_\nu = \frac{1}{2}(Y_y - X_x)\sin 2\alpha + X_y \cos 2\alpha,$$

$$Z_\nu = 0, \quad \sigma_{1,2} = \frac{(X_x + Y_y)}{2} \pm \sqrt{\left(\frac{X_x - Y_y}{2}\right)^2 + X_y^2}.$$

* For Lamé's equilibrium equations in isostatic co-ordinates for a three-dimensional problem, see the monograph 1, p. 42.

1.10. The stresses at a point of an elastic body are: $X_x = 50$ N/cm², $Y_y = 0$, $Z_z = -30$ N/cm², $X_y = 50$ N/cm², $Y_z = -75$ N/cm², $Z_x = 80$ N/cm².

Find the principal normal and shearing stresses.

Use Eqs. (1.4), (1.5), and (1.8) to solve the problem.

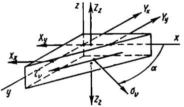

Fig. 10

Answer. $\sigma_1 = 99.3$ N/cm², $\sigma_2 = 58.8$ N/cm², $\sigma_3 = -138$ N/cm², $\tau_{max} = 118.6$ N/cm².

Chapter 2

THEORY OF STRAIN

I. STRAIN EQUATIONS IN ORTHOGONAL CO-ORDINATES

1. Curvilinear co-ordinates

$$
\begin{aligned}
e_{hh} &= \frac{1}{\sqrt{g_h}}\frac{\partial u_h}{\partial \alpha_h} + \sum_{\nu} \frac{1}{\sqrt{g_h g_\nu}}\frac{\partial \sqrt{g_h}}{\partial \alpha_\nu} u_\nu, \\
e_{h\nu} &= \sqrt{\frac{g_h}{g_\nu}}\frac{\partial}{\partial \alpha_h}\left(\frac{u_h}{\sqrt{g_h}}\right) + \sqrt{\frac{g_\nu}{g_h}}\frac{\partial}{\partial \alpha_h}\left(\frac{u_\nu}{\sqrt{g_\nu}}\right),
\end{aligned} \tag{2.1}
$$

where e_{hh} = linear strains,
$e_{h\nu}$ = shearing strains.

The dilatation is

$$
\theta = \sum_{1}^{3} e_{hh} = \frac{1}{\Delta}\left[\frac{\partial}{\partial \alpha_1}\left(\sqrt{g_2 g_3}\,u_1\right) + \frac{\partial}{\partial \alpha_2}\left(\sqrt{g_3 g_1}\,u_2\right) + \frac{\partial}{\partial \alpha_3}\left(\sqrt{g_1 g_2}\,u_3\right)\right]. \tag{2.2}
$$

The components of elementary rotation are

$$
\begin{aligned}
\omega_1 &= \frac{1}{2\sqrt{g_2 g_3}}\left[\frac{\partial}{\partial \alpha_2}\left(\sqrt{g_3}\,u_3\right) - \frac{\partial}{\partial \alpha_3}\left(\sqrt{g_2}\,u_2\right)\right], \\
\omega_2 &= \frac{1}{2\sqrt{g_3 g_1}}\left[\frac{\partial}{\partial \alpha_3}\left(\sqrt{g_1}\,u_1\right) - \frac{\partial}{\partial \alpha_1}\left(\sqrt{g_3}\,u_3\right)\right], \\
\omega_3 &= \frac{1}{2\sqrt{g_1 g_2}}\left[\frac{\partial}{\partial \alpha_1}\left(\sqrt{g_2}\,u_2\right) - \frac{\partial}{\partial \alpha_2}\left(\sqrt{g_1}\,u_1\right)\right].
\end{aligned} \tag{2.3}
$$

Note that on the basis of formulas of the calculus of vectors (div rot $u = 0$) the components of rotation identically satisfy the equality

$$\frac{\partial}{\partial \alpha_1}\left(\sqrt{g_2 g_3}\,\omega_1\right)+\frac{\partial}{\partial \alpha_2}\left(\sqrt{g_3 g_1}\,\omega_2\right)+\frac{\partial}{\partial \alpha_3}\left(\sqrt{g_1 g_2}\,\omega_3\right)=0.$$

For the strain compatibility equations in curvilinear co-ordinates, see [3].

2. Rectangular co-ordinates

Co-ordinates measured in terms of displacements:

$$\delta x = u_x, \quad \delta y = u_y, \quad \delta z = u_z.$$

The strain equations are

$$e_{xx}=\frac{\partial u_x}{\partial x}, \quad e_{yy}=\frac{\partial u_y}{\partial y}, \quad e_{zz}=\frac{\partial u_z}{\partial z},$$
$$e_{xy}=\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}, \quad e_{yz}=\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}, \tag{2.1a}$$
$$e_{zx}=\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}.$$

The dilatation is

$$\theta=\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}. \tag{2.2a}$$

The components of rotation are

$$\omega_x=\frac{1}{2}\left(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}\right), \quad \omega_y=\frac{1}{2}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right),$$
$$\omega_z=\frac{1}{2}\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right). \tag{2.3a}$$

The strain compatibility equations are

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2 e_{xy}}{\partial x\,\partial y},$$
$$\frac{\partial^2 e_{yy}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial y^2}=\frac{\partial^2 e_{yz}}{\partial y\,\partial z},$$
$$\frac{\partial^2 e_{zz}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial z^2}=\frac{\partial^2 e_{zx}}{\partial z\,\partial x},$$

$$\frac{\partial}{\partial x}\left[\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}-\frac{\partial e_{yz}}{\partial x}\right]=2\,\frac{\partial^2 e_{xx}}{\partial y\,\partial z},$$
$$\frac{\partial}{\partial y}\left[\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}-\frac{\partial e_{zx}}{\partial y}\right]=2\,\frac{\partial^2 e_{yy}}{\partial z\,\partial x},$$
$$\frac{\partial}{\partial z}\left[\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}-\frac{\partial e_{xy}}{\partial z}\right]=2\,\frac{\partial^2 e_{zz}}{\partial x\,\partial y}. \tag{2.4a}$$

The first group of Saint-Venant's identities expresses the continuity of curvatures of deformed fibres of a body, the second group expresses the continuity of relative angles of twist [4].

For a plane problem in rectangular co-ordinates, only the first equation of (2.4a) will remain.

3. Cylindrical co-ordinates

Co-ordinates measured in terms of displacements:

$\delta r = u_r, \quad \delta\beta = u_\beta/r, \quad \delta z = u_z.$

The strain equations are

$$e_{rr}=\frac{\partial u_r}{\partial r},\quad e_{\beta\beta}=\frac{1}{r}\,\frac{\partial u_\beta}{\partial \beta}+\frac{u_r}{r},\quad e_{zz}=\frac{\partial u_z}{\partial z},$$
$$e_{r\beta}=\frac{1}{r}\,\frac{\partial u_r}{\partial \beta}+\frac{\partial u_\beta}{\partial r}-\frac{u_\beta}{r}, \tag{2.1b}$$
$$e_{\beta z}=\frac{\partial u_\beta}{\partial z}+\frac{1}{r}\,\frac{\partial u_z}{\partial \beta},\quad e_{zr}=\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}.$$

The dilatation is

$$\theta=\frac{1}{r}\,\frac{\partial (ru_r)}{\partial r}+\frac{1}{r}\,\frac{\partial u_\beta}{\partial \beta}+\frac{\partial u_z}{\partial z}. \tag{2.2b}$$

The components of rotation are

$$\omega_r=\frac{1}{2r}\left(\frac{\partial u_z}{\partial \beta}-r\,\frac{\partial u_\beta}{\partial z}\right),$$
$$\omega_\beta=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right), \tag{2.3b}$$
$$\omega_z=\frac{1}{2r}\left[\frac{\partial}{\partial r}(ru_\beta)-\frac{\partial u_r}{\partial \beta}\right].$$

The strain compatibility equations are

$$
\begin{aligned}
&\frac{\partial^2 e_{rr}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial r^2}=\frac{\partial^2 e_{rz}}{\partial r\,\partial z},\\
&\frac{\partial^2 e_{\beta\beta}}{\partial z^2}+\frac{1}{r^2}\frac{\partial^2 e_{zz}}{\partial \beta^2}+\frac{1}{r}\frac{\partial e_{zz}}{\partial r}=\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial e_{\beta z}}{\partial \beta}+e_{rz}\right),\\
&\frac{1}{r}\frac{\partial^2 e_{rr}}{\partial \beta^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial e_{\beta\beta}}{\partial r}\right)-\frac{\partial e_{rr}}{\partial r}=\frac{1}{r}\frac{\partial^2 (re_{\beta r})}{\partial r\,\partial \beta},\\
&\frac{\partial^2 e_{r\beta}}{\partial z^2}-r\frac{\partial^2}{\partial r\,\partial z}\left(\frac{e_{\beta z}}{r}\right)-\frac{1}{r}\frac{\partial^2 e_{rz}}{\partial \beta\,\partial z}=-2\frac{\partial^2}{\partial \beta\,\partial r}\left(\frac{e_{zz}}{r}\right), \qquad (2.4b)\\
&\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (re_{\beta z})}{\partial r}\right]-\frac{1}{r^2}\frac{\partial^2 (r^2 e_{r\beta})}{\partial r\,\partial z}-\frac{\partial^2}{\partial r\,\partial \beta}\left(\frac{e_{rz}}{r}\right)=-\frac{2}{r}\frac{\partial^2 e_{rr}}{\partial \beta\,\partial z},\\
&\frac{\partial^2 e_{rz}}{\partial \beta^2}-\frac{\partial^2 (re_{\beta z})}{\partial r\,\partial \beta}-\frac{\partial^2 (re_{\beta r})}{\partial z\,\partial \beta}=r\frac{\partial}{\partial z}\left[e_{rr}-\frac{\partial (re_{\beta\beta})}{\partial r}\right].
\end{aligned}
$$

For a plane problem in polar co-ordinates r and β, only the third equation will remain since $e_{rz}=e_{\beta z}=e_{zz}=0$, and the other strains will be functions of r and β.

4. Spherical co-ordinates

Co-ordinates measured in terms of displacements:

$$\delta r=u_r,\quad \delta\beta=u_\beta/(r\sin\alpha),\quad \delta\alpha=u_\alpha/r.$$

The strain equations are

$$
\begin{aligned}
&e_{rr}=\frac{\partial u_r}{\partial r},\quad e_{\alpha\alpha}=\frac{1}{r}\left(\frac{\partial u_\alpha}{\partial \alpha}+u_r\right),\\
&e_{\beta\beta}=\frac{1}{r}\left(\frac{1}{\sin\alpha}\frac{\partial u_\beta}{\partial \beta}+u_r+u_\alpha\cot\alpha\right),\\
&e_{r\beta}=\frac{1}{r\sin\alpha}\frac{\partial u_r}{\partial \beta}+r\frac{\partial}{\partial r}\left(\frac{u_\beta}{r}\right), \qquad (2.1c)\\
&e_{r\alpha}=r\frac{\partial}{\partial r}\left(\frac{u_\alpha}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \alpha},\\
&e_{\beta\alpha}=\frac{1}{r\sin\alpha}\left[\frac{\partial}{\partial \alpha}(u_\beta\sin\alpha)+\frac{\partial u_\alpha}{\partial \beta}\right].
\end{aligned}
$$

The dilatation is

$$\theta=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)+\frac{1}{r\sin\alpha}\left[\frac{\partial}{\partial\alpha}(u_\alpha\sin\alpha)+\frac{\partial u_\beta}{\partial\beta}\right]. \qquad (2.2c)$$

The components of rotation are

$$\begin{aligned}\omega_r&=\frac{1}{2r\sin\alpha}\left[\frac{\partial u_\alpha}{\partial\beta}-\frac{\partial}{\partial\alpha}(u_\beta\sin\alpha)\right],\\ \omega_\beta&=\frac{1}{2r}\left[\frac{\partial u_r}{\partial\alpha}-\frac{\partial}{\partial r}(ru_\alpha)\right],\\ \omega_\alpha&=\frac{1}{2r\sin\alpha}\left[\frac{\partial}{\partial r}(ru_\beta\sin\alpha)-\frac{\partial u_r}{\partial\beta}\right].\end{aligned} \qquad (2.3c)$$

For the strain compatibility equations, see [3].

II. STATE OF STRAIN AT A POINT

Tho principal strains e_1, e_2, e_3* at a point are determined by three roots of the equation

$$\begin{vmatrix}2(e_{xx}-e) & e_{xy} & e_{xz}\\ e_{yx} & 2(e_{yy}-e) & e_{yz}\\ e_{zx} & e_{zy} & 2(e_{zz}-e)\end{vmatrix}=0. \qquad (2.5)$$

Three real roots of the cubic equation (2.5) give three principal extensions e_1, e_2, e_3.

Determinant (2.5) in expanded form is

$$e^3-I_1e^2+I_2e-I_3=0, \qquad (2.6)$$

where

$$\begin{aligned}I_1&=e_{xx}+e_{yy}+e_{zz}=e_1+e_2+e_3,\\ I_2&=e_{xx}e_{yy}+e_{yy}e_{zz}+e_{zz}e_{xx}-(e_{xy}^2+e_{yz}^2+e_{zx}^2)/4=\\ &=e_1e_2+e_2e_3+e_3e_1,\\ I_3&=e_{xx}e_{yy}e_{zz}+e_{xy}e_{yz}e_{zx}/4-\\ &-(e_{xx}e_{yz}^2+e_{yy}e_{zx}^2+e_{zz}e_{xy}^2)/4=e_1e_2e_3;\end{aligned} \qquad (2.7)$$

---

* For the principal directions (*1*, *2*, *3*) the shear components are zero.

I_i are the invariants of the state of strain at the given point.

The direction cosines of the principal strains e_i are determined from the equations

$$\begin{aligned} &2\,(e_{xx}-e_i)\,l_i+e_{xy}m_i+e_{xz}n_i=0,\\ &e_{yx}l_i+2\,(e_{yy}-e_i)\,m_i+e_{yz}n_i=0,\\ &e_{zx}l_i+e_{zy}m_i+2\,(e_{zz}-e_i)\,n_i=0 \end{aligned} \tag{2.8}$$

with the relation

$$l_i^2+m_i^2+n_i^2=1. \tag{2.9}$$

The extension of a line element whose direction is defined by the cosines l, m, and n is determined by the formula

$$e=e_{xx}l^2+e_{yy}m^2+e_{zz}n^2+e_{xy}lm+e_{yz}mn+e_{zx}nl. \tag{2.10}$$

III. CESARO'S FORMULAS [1]

The determination of the displacements u_h, u_v from Eqs. (2.1) will be given for the case of rectangular coordinates [formulas (2.1a)]

$$\begin{aligned} &u_{x1}=u_{x0}+\omega_{y0}\,(z_1-z_0)/2-\omega_{z0}\,(y_1-y_0)/2+\\ &+\int\limits_{M_1M_0}(U_x\,dx+U_y\,dy+U_z\,dz),\\ &u_{y1}=u_{y0}+\omega_{z0}\,(x_1-x_0)-\omega_{x0}\,(z_1-z_0)/2+\\ &+\int\limits_{M_1M_0}(V_x\,dx+V_y\,dy+V_z\,dz),\\ &u_{z1}=u_{z0}+\omega_{x0}\,(y_1-y_0)-\omega_{y0}\,(x_1-x_0)/2+\\ &+\int\limits_{M_1M_0}(W_x\,dx+W_y\,dy+W_z\,dz), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} &U_x=e_{xx}+(y_1-y)\left(2\,\frac{\partial e_{xx}}{\partial y}-\frac{\partial e_{xy}}{\partial x}\right)+\\ &+(z_1-z)\left(2\,\frac{\partial e_{xx}}{\partial z}-\frac{\partial e_{zx}}{\partial x}\right)\Big/2,\\ &U_y=e_{xy}/2+(y_1-y)\left(\frac{\partial e_{xy}}{\partial y}-2\,\frac{\partial e_{yy}}{\partial x}\right)\Big/2+\\ &+(z_1-z)\left(\frac{\partial e_{xy}}{\partial z}-\frac{\partial e_{yz}}{\partial x}\right)\Big/2, \end{aligned} \tag{2.12}$$

$$U_z = e_{zx}/2 + (y_1 - y)\left(\frac{\partial e_{zx}}{\partial y} - \frac{\partial e_{yz}}{\partial x}\right)\Big/2 +$$
$$+ (z_1 - z)\left(\frac{\partial e_{zx}}{\partial z} - 2\,\frac{\partial e_{zz}}{\partial x}\right)\Big/2.$$

The quantities V_x, V_y, V_z and W_x, W_y, W_z are obtained from (2.12) by cyclic permutation of the letters x, y, z. The subscript 0 refers to the point M_0, and the subscript 1 to the point M_1.

PROBLEMS

2.1. The following displacements are given:

(1) corresponding to Maxwell's stress system

$$2Gu_x = \frac{\partial}{\partial x}(\chi_2 - \chi_3 - \chi_1),$$
$$2Gu_y = \frac{\partial}{\partial y}(\chi_3 - \chi_1 - \chi_2),$$
$$2Gu_z = \frac{\partial}{\partial z}(\chi_1 - \chi_2 - \chi_3);$$

(2) corresponding to Morera's stress system

$$Eu_x = \frac{\partial^2}{\partial y\,\partial z}[\varphi_1 - \sigma(\varphi_2 + \varphi_3)],$$
$$Eu_y = \frac{\partial^2}{\partial z\,\partial x}[\varphi_2 - \sigma(\varphi_3 + \varphi_1)],$$
$$Eu_z = \frac{\partial^2}{\partial x\,\partial y}[\varphi_3 - \sigma(\varphi_1 + \varphi_2)]$$

(see Problem 1.6);

(3) $u_z = -xz/a$, $u_y = \sigma xy/a$, $u_x = [z^2 + \sigma(x^2 - y^2)]/(2a)$, where a is a constant.

Find the strains and show that they satisfy the strain compatibility equations (2.4a).

2.2. Saint-Venant's problem (1855).

A cylindrical or a prismatic bar with generators parallel to the z axis coinciding with the line of centroids of cross sections is bent by terminal couples $M = EI_y/a$, which lie in the xOz plane (Fig. 11a).

Find the state of stress and strain.

Assume

$$Z_z = -Ex/a,$$

where a is a constant, and the remaining components of the stress tensor are zero. The assumed stress system satisfies Eqs. (1.1a) in the absence of body forces and

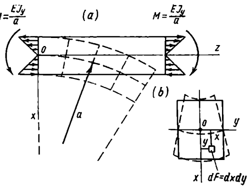

Fig. 11

the boundary conditions (1.2) on the lateral surface. At the end sections we have

$$M = \frac{EI_y}{a} = -\int Z_z x\, dF = \iint\limits_F \frac{E}{a} x^2\, dx\, dy,$$

from which $I_y = \iint\limits_F x^2\, dx\, dy$ (the moment of inertia of the cross-sectional area with respect to the y axis).

The moment of the stresses about the x axis is zero since the x and y axes are assumed to be principal axes.

The resultant vector of the stresses $\iint\limits_F Z_z\, dx\, dy$ is zero since the Oz axis coincides with the line of centroids.

The strain components are, by Eqs. (3.1),

$$\frac{\partial u_x}{\partial x}=\frac{\partial u_y}{\partial y}=\frac{\sigma x}{a}, \quad \frac{\partial u_z}{\partial z}=-\frac{x}{a},$$
$$\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}=\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}=\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}=0. \tag{a}$$

The strains obtained satisfy the compatibility conditions (2.4a). By integrating Eqs. (a), we find

$$u_x=\frac{1}{2a}\left[z^2+\sigma\left(x^2-y^2\right)\right], \quad u_y=\frac{\sigma}{a}xy, \quad u_z=-\frac{xz}{a}.$$

The line of centroids of cross sections is displaced according to the law $u_x = z^2/(2a)$, and for large values of a it may be considered a circumference of radius $a = EI_y/M$ centred at the point $x = a$, $z = 0$ called the centre of curvature.

The distortion of a cross section in the form of a rectangle is shown in Fig. 11*b*.

2.3. When a concentrated force P acts on the half-space $z \geqslant 0$, the displacements are obtained as (see Problem 4.4)

$$u_x=\frac{P}{4\pi G}\left(\frac{1-2\sigma}{R+z}-\frac{z}{R^2}\right)\frac{x}{R},$$

$$u_y=\frac{P}{4\pi G}\left(\frac{1-2\sigma}{R+z}-\frac{z}{R^2}\right)\frac{y}{R},$$

$$u_z=\frac{P}{4\pi G}\left[2\left(1-\sigma\right)+\frac{z^2}{R^2}\right]\frac{1}{R},$$

where

$$R=\sqrt{x^2+y^2+z^2}.$$

Find the strains and see whether they satisfy the strain compatibility equations (2.4a).

2.4. Calculate the dilatation for the following cases of plane orthogonal co-ordinates: parabolic (confocal α_1 and α_2 parabolas), elliptic (confocal α_1 hyperbolas and α_2 ellipses).

For a plane problem, the dilatation, according to formula (2.2), is calculated by the formula

$$\theta=\sum_{1}^{2} e_{hh}=\frac{1}{\Delta}\left[\frac{\partial}{\partial\alpha_1}\left(\sqrt{g_2}\,u_1\right)+\frac{\partial}{\partial\alpha_2}\left(\sqrt{g_1}\,u_2\right)\right],$$

where

$$g_h=\left(\frac{\partial x}{\partial\alpha_h}\right)^2+\left(\frac{\partial y}{\partial\alpha_h}\right)^2,\quad \Delta=\sqrt{g_1g_2},\quad dV=\Delta\,d\alpha_1\,d\alpha_2,$$

$$ds^2=ds_1^2+ds_2^2=g_1\,d\alpha_1^2+g_2\,d\alpha_2^2.$$

By using the complex expression, we assume

$\alpha_1+i\alpha_2=f(x+iy)$,

where $f(\ldots)$ is an analytic function.

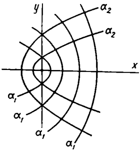

Fig. 12

Parabolic co-ordinates (Fig. 12):

$\alpha_1+i\alpha_2=\sqrt{2(x+iy)},\quad \alpha_1=\sqrt{r+x},\quad \alpha_2=\sqrt{r-x},$

where

$r=\sqrt{x^2+y^2}$

is the distance from the focus ($x=y=0$),

$$x=\frac{\alpha_1^2-\alpha_2^2}{2},\quad 0\leqslant\alpha_1<\infty,$$

$$y=\pm\alpha_1\alpha_2,\quad 0\leqslant\alpha_2<\infty,$$

$$g_1=g_2=\alpha_1^2+\alpha_2^2,$$

$$\Delta=\sqrt{(\alpha_1^2+\alpha_2^2)^2}=\alpha_1^2+\alpha_2^2,$$

$$\theta=\frac{1}{\alpha_1^2+\alpha_2^2}\left[\frac{\partial}{\partial\alpha_1}\sqrt{\alpha_1^2+\alpha_2^2}\,u_1+\frac{\partial}{\partial\alpha_2}\sqrt{\alpha_1^2+\alpha_2^2}\,u_2\right].$$

Elliptic co-ordinates (Fig. 13):

$\alpha_1 + i\alpha_2 = \arcsin (x + iy)/a$,

$\sin \alpha_1 = (s_1 - s_2)/2a, \quad \cosh \alpha_2 = (s_1 + s_2)/2a$,

where

$s_1 = \sqrt{(x + a)^2 + y^2}, \quad s_2 = \sqrt{(x - a)^2 - y^2}$

are the distances of the point $M\ (x,\ y)$ from the foci lying

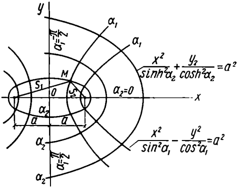

Fig. 13

on the x axis at the points $x_1 = -a$ and $x_2 = a$,

$x = a \sin \alpha_1 \cosh \alpha_2, \quad 0 \leqslant \alpha_1 < 2\pi$,

$y = \pm \cos \alpha_1 \sinh \alpha_2, \quad 0 \leqslant \alpha_2 < \infty$,

$g_1 = g_2 = a^2 (\cos^2 \alpha_1 + \sinh^2 \alpha_2) = a^2 (\cosh^2 \alpha_2 - \sin^2 \alpha_1)$,

$\Delta = a^2 (\cos^2 \alpha_1 + \sinh^2 \alpha_2)$,

$$\theta = \frac{1}{a^2 (\cos^2 \alpha_1 + \sinh^2 \alpha_2)} \Big[\frac{\partial}{\partial \alpha_1} \left(\sqrt{\cos^2 \alpha_1 + \sinh^2 \alpha_2}\, u_1\right) + \frac{\partial}{\partial \alpha_2} \left(\sqrt{\cos^2 \alpha_1 + \sinh^2 \alpha_2}\, u_2\right) \Big].$$

By adding to the co-ordinates α_1 and α_2 a third coordinate $\alpha_3 = z$ independent of them, we obtain, respectively, space parabolic and elliptic cylindrical co-ordinates.

2.5. Find the values of displacements u_1, u_2, and u_3 for which the components of rotation are zero.

By equating expressions (2.3) to zero, we find

$$u_1 = \frac{\partial \Phi}{\sqrt{g_1}\,\partial \alpha_1} = \frac{\partial \Phi}{\partial s_1}, \quad u_2 = \frac{\partial \Phi}{\sqrt{g_2}\,\partial \alpha_2} = \frac{\partial \Phi}{\partial s_2}, \quad u_3 = \frac{\partial \Phi}{\sqrt{g_3}\,\partial \alpha_3} = \frac{\partial \Phi}{\partial s_3}.$$

Thus, the rotation vanishes ($\omega_1 = \omega_2 = \omega_3 = 0$) when the projections of the displacement are partial derivatives

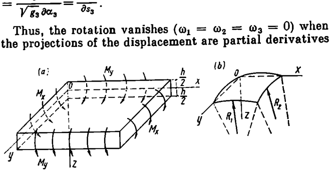

Fig. 14

with respect to the arc lengths of the co-ordinate lines of the single function Φ, the displacement potential.

In the case of rectangular co-ordinates ($g_1 = g_2 = g_3 = 1$)

$$u_x = \frac{\partial \Phi}{\partial x}, \quad u_y = \frac{\partial \Phi}{\partial y}, \quad u_z = \frac{\partial \Phi}{\partial z}.$$

2.6. A rectangular plate of thickness h is bounded by planes $z = \pm h/2$ and bent by moments M_x and M_y uniformly distributed along its edges (Fig. 14*a*).

Determine the values of the moments for which the curvatures in the xOz and yOz planes are positive, i.e., the centres of curvature lie in the positive direction of the z axis.

Assume

$$X_x = Eaz, \quad Y_y = Ebz,$$

where a and b are constants, and the remaining components of the stress tensor are zero.

The assumed stress system satisfies Eqs. (1.1a) in the absence of body forces and the boundary conditions (1.2) on the free planes $z = \pm h/2$.

By integrating the strain equations (2.1a), we obtain the following expressions for displacements:

$$u_x = (a - \sigma b)\, xz, \quad u_y = (b - \sigma a)\, yz,$$
$$u_z = -\frac{a - \sigma b}{2} x^2 - \frac{b - \sigma a}{2} y^2 - \frac{\sigma (a + b)}{2} z^2. \tag{a}$$

The displacements (a) satisfy the compatibility equations (2.4a).

According to the equations (a), each plane z = constant is bent with curvatures in the xOz and yOz planes equal, respectively, to $(\sigma b - a)$ and $(\sigma a - b)$.

Assuming R_1 and R_2 to be the radii of curvature (Fig. 14*b*), we obtain

$$\frac{1}{R_1} = \frac{\partial^2 u_z}{\partial x^2} = \sigma b - a, \quad \frac{1}{R_2} = \frac{\partial^2 u_z}{\partial y^2} = \sigma a - b,$$

from which

$$a = -\frac{1}{1 - \sigma^2}\left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right), \quad b = -\frac{1}{1 - \sigma^2}\left(\frac{1}{R_2} + \frac{\sigma}{R_1}\right).$$

The intensity of bending moments is

$$M_x = \int_{-h/2}^{h/2} X_x z\, dz = -D\,(1/R_1 + \sigma/R_2),$$

$$M_y = \int_{-h/2}^{h/2} Y_y z\, dz = -D\,(1/R_2 + \sigma/R_1),$$

where $D = \frac{Eh^3}{12(1 - \sigma^2)}$ is the flexural rigidity of the plate.

2.7. Find all components of strain for a deformation of a body symmetrical about the origin of co-ordinates O.

If the deformation is symmetrical about the origin of co-ordinates, the displacement u_R of any point is directed along the radius vector R and is a function of it. We thus have (Fig. 15)

$$u_x = \frac{u_R}{R}x = f(R)\,x, \quad u_y = \frac{u_R}{R}y = f(R)\,y, \quad u_z = $$
$$= \frac{u_R}{R}z = f(R)\,z,$$

where

$$R = \sqrt{x^2+y^2+z^2}, \quad f(R) = \frac{u_R}{R}.$$

According to Eqs. (2.1a) we obtain

$$e_{xx} = f(R) + \frac{x^2}{R}\frac{df(R)}{dR}, \quad e_{xy} = \frac{2xy}{R}\frac{df(R)}{dR},$$
$$e_{yy} = f(R) + \frac{y^2}{R}\frac{df(R)}{dR},$$
$$e_{yz} = \frac{2yz}{R}\frac{df(R)}{dR}, \quad e_{zz} = f(R) + \frac{z^2}{R}\frac{df(R)}{dR}, \quad e_{zx} = \frac{2zx}{R}\frac{df(R)}{dR}.$$

2.8. Find all components of strain corresponding to a rectangular co-ordinate system for a deformation of a body symmetrical about the Oz axis.

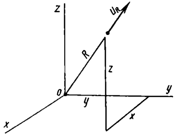

Fig. 15

Let the projection of the displacement on the xOy plane be denoted by u_r, and the projection on the Oz axis by u_z.

Because of the axial symmetry, u_r and u_z are functions of r and z, where

$r = \sqrt{x^2 + y^2}$ (Fig. 16).

The displacements are

$$u_x = u_r x/r = f(r, z)\, x, \quad u_y = u_r y/r = f(r, z)\, y,$$
$$u_z = u_z(r, z),$$

where $f(r, z) = u_r/r$.

According to Eqs. (2.1a) we obtain

$$e_{xx} = f(r, z) + \frac{x^2}{r}\frac{\partial f(r, z)}{\partial r}, \quad e_{xy} = \frac{2xy}{r}\frac{\partial f(r, z)}{\partial r},$$

$$e_{yy} = f(r, z) + \frac{y^2}{r}\frac{\partial f(r, z)}{\partial r}, \quad e_{yz} = y\frac{\partial f(r, z)}{\partial z} + \frac{\partial u_z}{\partial r}\frac{y}{r}.$$

$$e_{zz} = \frac{\partial u_z}{\partial z}, \quad e_{zx} = x\frac{\partial f(r, z)}{\partial z} + \frac{\partial u_z}{\partial r}\frac{x}{r}.$$

2.9. At some point of a body

$$e_{xx} = 0.001, \quad e_{yy} = -0.0005, \quad e_{zz} = 0.0005,$$
$$e_{xy} = 0.003, \quad e_{yz} = 0.001, \quad e_{zx} = 0.0008.$$

Determine the principal strains and their orientation with respect to the Ox, Oy, Oz axes.

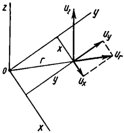

Fig. 16

The problem is solved by using Eqs. (2.5) to (2.9).

2.10. For the case of plane strain, when $u_z = 0$, $u_x = u_x(x, y)$, $u_y = u_y(x, y)$, determine the principal strains and their orientation.

Answer. One of the roots of Eq. (2.5) is zero, the other two are determined by the quadratic equation

$$e^2 - (e_{xx} + e_{yy})\, e + e_{xx}e_{yy} - e_{xy}^2/4 = 0.$$

One of the three principal strains coincides with the Oz axis, the other two lie in the xOy plane.

Chapter 3

BASIC EQUATIONS OF THE THEORY OF ELASTICITY AND THEIR SOLUTION FOR SPECIAL CASES

I. ORTHOGONAL CURVILINEAR CO-ORDINATES

1. Direct and inverse forms of Hooke's law

$$e_{hh}=\frac{1}{2G}\left(H_h-\frac{\sigma}{1+\sigma}\Theta\right),\quad e_{h\nu}=\frac{1}{G}H_\nu, \tag{3.1}$$

where

$$\Theta=\sum_1^3 H_h,\quad H_h=\lambda\theta+2Ge_{hh},\quad H_\nu=Ge_{h\nu}. \tag{3.2}$$

2. Equations in terms of displacements

$$(\lambda+2G)\sqrt{\frac{g_2g_3}{g_1}}\frac{\partial\theta}{\partial\alpha_1}-2G\left[\frac{\partial}{\partial\alpha_2}\left(\sqrt{g_3}\,\omega_3\right)-\frac{\partial}{\partial\alpha_3}\left(\sqrt{g_2}\,\omega_2\right)\right]+$$
$$+\sqrt{g_2g_3}\,H_1=0,$$
$$(\lambda+2G)\sqrt{\frac{g_3g_1}{g_2}}\frac{\partial\theta}{\partial\alpha_2}-2G\left[\frac{\partial}{\partial\alpha_3}\left(\sqrt{g_1}\,\omega_1\right)-\frac{\partial}{\partial\alpha_1}\left(\sqrt{g_3}\,\omega_3\right)\right]+$$
$$+\sqrt{g_3g_1}\,H_2=0,$$
$$(\lambda+2G)\sqrt{\frac{g_1g_2}{g_3}}\frac{\partial\theta}{\partial\alpha_3}-2G\left[\frac{\partial}{\partial\alpha_1}\left(\sqrt{g_2}\,\omega_2\right)-\frac{\partial}{\partial\alpha_2}\left(\sqrt{g_1}\,\omega_1\right)\right]+$$
$$+\sqrt{g_1g_2}\,H_3=0, \tag{3.3}$$

where ω_i are determined by formulas (2.3), and θ by formula (2.2).

3. Equations in terms of stresses

Three equilibrium equations (1.1) and six strain compatibility equations [3] expressed in terms of stresses according to formulas (3.1) constitute a complete system of equations in the solution of problems in terms of stresses.

II. RECTANGULAR CO-ORDINATES

1. Direct and inverse forms of Hooke's law

$$e_{xx}=\frac{1}{2G}\left(X_x-\frac{\sigma}{1+\sigma}\Theta\right)^{*},\quad e_{xy}=\frac{1}{G}X_y,$$
$$e_{yy}=\frac{1}{2G}\left(Y_y-\frac{\sigma}{1+\sigma}\Theta\right),\quad e_{yz}=\frac{1}{G}Y_z,\tag{3.1a}$$
$$e_{zz}=\frac{1}{2G}\left(Z_z-\frac{\sigma}{1+\sigma}\Theta\right),\quad e_{zx}=\frac{1}{G}Z_x.$$

$$X_x=\lambda\theta+2G\frac{\partial u_x}{\partial x}^{**},\quad X_y=G\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right),$$
$$Y_y=\lambda\theta+2G\frac{\partial u_y}{\partial y},\quad Y_z=G\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right),\tag{3.2a}$$
$$Z_z=\lambda\theta+2G\frac{\partial u_z}{\partial z},\quad Z_x=G\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right),$$

where θ is determined by formula (2.2a).

2. Equations in terms of displacements

$$(\lambda+2G)\frac{\partial\theta}{\partial x}-2G\left(\frac{\partial\omega_z}{\partial y}-\frac{\partial\omega_y}{\partial z}\right)+X=0,$$
$$(\lambda+2G)\frac{\partial\theta}{\partial y}-2G\left(\frac{\partial\omega_x}{\partial z}-\frac{\partial\omega_z}{\partial x}\right)+Y=0,\tag{3.3a}$$
$$(\lambda+2G)\frac{\partial\theta}{\partial z}-2G\left(\frac{\partial\omega_y}{\partial x}-\frac{\partial\omega_x}{\partial y}\right)+Z=0$$

* An alternative form is $e_{xx}=\frac{1}{E}[X_x-\sigma(Y_y+Z_z)]$, etc.

** An alternative form is

$$X_x=\frac{E}{1-\sigma^2}\left[\frac{\partial u_x}{\partial x}+G\left(\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)\right],\quad \text{etc.}$$

or

$$G\nabla^2 u_x + (\lambda + G)\frac{\partial\theta}{\partial x} + X = 0,$$

$$G\nabla^2 u_y + (\lambda + G)\frac{\partial\theta}{\partial y} + Y = 0, \qquad (3.3a)$$

$$G\nabla^2 u_z + (\lambda + G)\frac{\partial\theta}{\partial z} + Z = 0,$$

where

$$\nabla^2(\ldots) = \frac{\partial^2(\ldots)}{\partial x^2} + \frac{\partial^2(\ldots)}{\partial y^2} + \frac{\partial^2(\ldots)}{\partial z^2}.$$

Equations (3.3a′) may be represented as

$$(1-2\sigma)\nabla^2 u_x + \frac{\partial\theta}{\partial x} + \frac{1-2\sigma}{G}X = 0,$$

$$(1-2\sigma)\nabla^2 u_y + \frac{\partial\theta}{\partial y} + \frac{1-2\sigma}{G}Y = 0, \qquad (3.3a')$$

$$(1-2\sigma)\nabla^2 u_z + \frac{\partial\theta}{\partial z} + \frac{1-2\sigma}{G}Z = 0.$$

3. Equations in terms of stresses

Three equilibrium equations (1.1a) and six compatibility equations, viz. the Beltrami-Michell equations:

$$\nabla^2 X_x + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2} = -2\frac{\partial X}{\partial x} - \frac{\sigma}{1-\sigma}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)^*,$$

$$\nabla^2 Y_y + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y^2} = -2\frac{\partial Y}{\partial y} - \frac{\sigma}{1-\sigma}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right),$$

$$\nabla^2 Z_z + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} = -2\frac{\partial Z}{\partial z} - \frac{\sigma}{1-\sigma}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right),$$

$$\nabla^2 X_y + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x\,\partial y} = -\left(\frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x}\right), \qquad (3.4a)$$

---

* In dynamic problems, instead of X, Y, Z on the right-hand sides of the equations we must have, respectively,

$$\left(X - \rho\frac{\partial^2 u_x}{\partial \tau^2}\right), \quad \left(Y - \rho\frac{\partial^2 u_y}{\partial \tau^2}\right), \quad \left(Z - \rho\frac{\partial^2 u_z}{\partial \tau^2}\right).$$

$$\nabla^2 Y_z + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y\,\partial z} = -\left(\frac{\partial Y}{\partial z} + \frac{\partial Z}{\partial y}\right),$$

$$\nabla^2 Z_x + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z\,\partial x} = -\left(\frac{\partial Z}{\partial x} + \frac{\partial X}{\partial z}\right),$$

where $\Theta = X_x + Y_y + Z_z$.

When the body forces X, Y, Z are constant, the right-hand sides of Eqs. (3.4a) are zero (Beltrami's equations).

III. CYLINDRICAL CO-ORDINATES

1. Direct and inverse forms of Hooke's law

$$e_{rr} = \frac{1}{2G}\left(R_r - \frac{\sigma}{1+\sigma}\Theta\right), \quad e_{r\beta} = \frac{1}{G}R_\beta,$$

$$e_{\beta\beta} = \frac{1}{2G}\left(B_\beta - \frac{\sigma}{1+\sigma}\Theta\right), \quad e_{\beta z} = \frac{1}{G}B_z, \tag{3.1b}$$

$$e_{zz} = \frac{1}{2G}\left(Z_z - \frac{\sigma}{1+\sigma}\Theta\right), \quad e_{zr} = \frac{1}{G}Z_r.$$

$$R_r = \lambda\theta + 2G\frac{\partial u_r}{\partial r}, \quad R_\beta = G\left[\frac{1}{r}\frac{\partial u_r}{\partial\beta} + r\frac{\partial}{\partial r}\left(\frac{u_\beta}{r}\right)\right],$$

$$B_\beta = \lambda\theta + \frac{2G}{r}\left(\frac{\partial u_\beta}{\partial\beta} + u_r\right), \quad B_z = G\left(\frac{\partial u_\beta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\beta}\right), \tag{3.2b}$$

$$Z_z = \lambda\theta + 2G\frac{\partial u_z}{\partial z}, \quad Z_r = G\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right),$$

where θ is determined by formula (2.2b).

2. Equations in terms of displacements

$$(\lambda + 2G)\,r\frac{\partial\theta}{\partial r} - 2G\left[\frac{\partial\omega_z}{\partial\beta} - \frac{\partial}{\partial z}(r\omega_\beta)\right] + rR = 0,$$

$$(\lambda + 2G)\frac{1}{r}\frac{\partial\theta}{\partial\beta} - 2G\left(\frac{\partial\omega_r}{\partial z} - \frac{\partial\omega_z}{\partial r}\right) + B = 0, \tag{3.3b}$$

$$(\lambda + 2G)\,r\frac{\partial\theta}{\partial z} - 2G\left[\frac{\partial}{\partial r}(r\omega_\beta) - \frac{\partial\omega_r}{\partial\beta}\right] + rZ = 0,$$

where ω_r, ω_β, and ω_z are determined by formulas (2.3b), or

$$\nabla^2 u_r + \frac{1}{1-2\sigma}\frac{\partial\theta}{\partial r} - \frac{2}{r^2}\frac{\partial u_\beta}{\partial\beta} - \frac{u_r}{r^2} = -\frac{R}{G},$$

$$\nabla^2 u_\beta + \frac{1}{1-2\sigma}\frac{\partial\theta}{r\,\partial\beta} + \frac{2}{r^2}\frac{\partial u_r}{\partial\beta} - \frac{u_\beta}{r^2} = -\frac{B}{G}, \qquad (3.3b')$$

$$\nabla^2 u_z + \frac{1}{1-2\sigma}\frac{\partial\theta}{\partial z} = -\frac{Z}{G},$$

where $\nabla^2(\ldots)$ is given by formula (4.1b).

3. Equations in terms of stresses

Three equilibrium equations (1.1b) and six strain compatibility equations:

$$\nabla^2 R_r + \frac{2}{r^2}(R_r - B_\beta) - \frac{4}{r^2}\frac{\partial R_\beta}{\partial\beta} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial r^2} = 0,$$

$$\nabla^2 B_\beta + \frac{2}{r^2}(R_r - B_\beta) + \frac{4}{r^2}\frac{\partial B_r}{\partial\beta} + $$

$$+ \frac{1}{1+\sigma}\frac{1}{r}\left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial\beta^2}\right)\Theta = 0,$$

$$\nabla^2 Z_z + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} = 0, \qquad (3.4b)$$

$$\nabla^2 R_\beta + \frac{1}{1+\sigma}\frac{\partial}{\partial r}\left(\frac{\partial\Theta}{r\,\partial\beta}\right) + \frac{2}{r^2}\frac{\partial}{\partial\beta}(R_r - B_\beta) - \frac{4}{r^2}R_\beta = 0,$$

$$\nabla^2 B_z + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{r\,\partial\beta\,\partial z} + \frac{2}{r^2}\frac{\partial Z_r}{\partial\beta} - \frac{B_z}{r^2} = 0,$$

$$\nabla^2 Z_r + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial r\,\partial z} - \frac{2}{r^2}\frac{\partial Z_\beta}{\partial\beta} - \frac{Z_r}{r^2} = 0,$$

where $\Theta = R_r + B_\beta + Z_z$.

IV. SPHERICAL CO-ORDINATES

1. Direct and inverse forms of Hooke's law

$$e_{rr} = \frac{1}{2G}\left(R_r - \frac{\sigma}{1+\sigma}\Theta\right), \quad e_{r\beta} = \frac{1}{G}R_\beta,$$

$$e_{\beta\beta} = \frac{1}{2G}\left(B_\beta - \frac{\sigma}{1+\sigma}\Theta\right), \quad e_{\beta\alpha} = \frac{1}{G}B_\alpha,$$

$$e_{\alpha\alpha}=\frac{1}{2G}\left(A_\alpha-\frac{\sigma}{1+\sigma}\Theta\right),\quad e_{\alpha r}=\frac{1}{G}A_r. \tag{3.1c}$$

$$\begin{aligned}
R_r&=\lambda\theta+2G\frac{\partial u_r}{\partial r},\\
B_\beta&=\lambda\theta+\frac{2G}{r}\left(\frac{1}{\sin\alpha}\frac{\partial u_\beta}{\partial\beta}+u_r+u_\alpha\cot\alpha\right),\\
A_\alpha&=\lambda\theta+\frac{2G}{r}\left(\frac{\partial u_\alpha}{\partial\alpha}+u_r\right),\\
R_\beta&=\frac{G}{r}\left[\frac{1}{\sin\alpha}\frac{\partial u_r}{\partial\beta}+r^2\frac{\partial}{\partial r}\left(\frac{u_\beta}{r}\right)\right],\\
B_\alpha&=\frac{G}{r}\left(\frac{\partial u_\beta}{\partial\alpha}+u_\beta\cot\alpha+\frac{1}{\sin\alpha}\frac{\partial u_\alpha}{\partial\beta}\right),\\
A_r&=\frac{G}{r}\left[r^2\frac{\partial}{\partial r}\left(\frac{u_\alpha}{r}\right)+\frac{\partial u_r}{\partial\alpha}\right],
\end{aligned} \tag{3.2c}$$

where θ is determined by formula (2.2c).

2. Equations in terms of displacements

$$\begin{aligned}
&(\lambda+2G)\,r\sin\alpha\frac{\partial\theta}{\partial r}-2G\left[\frac{\partial\omega_\alpha}{\partial\beta}-\frac{\partial}{\partial\alpha}(\omega_\beta\sin\alpha)\right]+\\
&+r\sin\alpha R=0,\\
&(\lambda+2G)\frac{1}{\sin\alpha}\frac{\partial\theta}{\partial\beta}-2G\left[\frac{\partial\omega_r}{\partial\alpha}-\frac{\partial}{\partial r}(r\omega_\alpha)\right]+rB=0,\\
&(\lambda+2G)\sin\alpha\frac{\partial\theta}{\partial\alpha}-2G\left[\frac{\partial}{\partial r}(r\omega_\beta\sin\alpha)-\frac{\partial\omega_r}{\partial\beta}\right]+\\
&+r\sin\alpha A=0,
\end{aligned} \tag{3.3c}$$

where ω_r, ω_β, and ω_α are determined by formulas (2.3c).

For an axially symmetric problem, the strains are independent of the β co-ordinate, $u_\beta=0$, and, in addition,

$$\omega_r=\omega_\alpha=0,\quad \omega_\beta=\omega;$$

Eqs. (3.3c) become

$$\begin{aligned}
&(\lambda+2G)\,r\sin\alpha\frac{\partial\theta}{\partial r}+2G\frac{\partial}{\partial\alpha}(\omega\sin\alpha)+Rr\sin\alpha=0,\\
&(\lambda+2G)\frac{\partial\theta}{\partial\alpha}-2G\frac{\partial}{\partial z}(r\omega)+rA=0,
\end{aligned} \tag{3.3c'}$$

where

$$\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\alpha}\frac{\partial}{\partial \alpha}(u_\alpha \sin\alpha),$$

$$\omega = \frac{1}{2r}\left[\frac{\partial u_r}{\partial \alpha} - \frac{\partial}{\partial r}(r u_\alpha)\right].$$

For an alternative form of the equations in terms of displacements, see the monograph [5], p. 141.

3. Equations in terms of stresses

Three equilibrium equations (1.1c) and six strain compatibility equations [3] expressed in terms of stresses according to formulas (3.1).

PROBLEMS

3.1. Write the basic equations in terms of stresses for an axially symmetric (axis z) problem in cylindrical co-ordinates.

According to Eqs. (1.1b) and (3.4b) we have:

the equilibrium equations

$$\frac{\partial R_r}{\partial r} + \frac{\partial R_z}{\partial z} + \frac{R_r - B_\beta}{r} + R = 0, \quad \frac{\partial Z_r}{\partial r} + \frac{\partial Z_z}{\partial z} + \frac{Z_r}{r} + Z = 0;$$

the strain compatibility equations

$$\nabla^2 R_r - \frac{2}{r^2}(R_r - B_\beta) + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial r^2} = 0,$$

$$\nabla^2 B_\beta + \frac{2}{r^2}(R_r - B_\beta) + \frac{1}{1+\sigma}\frac{1}{r}\frac{\partial \Theta}{\partial r} = 0$$

$$\nabla^2 Z_z + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial z^2} = 0,$$

$$\nabla^2 Z_r + \frac{1}{1+\sigma}\frac{\partial^2 \Theta}{\partial r\,\partial z} - \frac{1}{r^2} Z_z = 0,$$

where

$$\nabla^2(\ldots) = \frac{\partial^2(\ldots)}{\partial r^2} + \frac{1}{r}\frac{\partial(\ldots)}{\partial r} + \frac{\partial^2(\ldots)}{\partial z^2}$$

and

$$\Theta = R_r + B_\beta + Z_z.$$

3.2. Write the basic equations in terms of displacements for an axially symmetric (axis z) problem in cylindrical co-ordinates, and find their solution.

Since the problem is axially symmetric, it follows that the displacements and strains are independent of the angle β and, in addition,

$$u_\beta = \omega_r = \omega_z = 0.$$

Equations (3.3b') reduce to

$$\left(\nabla^2 - \frac{1}{r^2}\right) u_r + \frac{1}{1-2\sigma}\left[\left(\nabla^2 - \frac{1}{r^2} - \frac{\partial^2}{\partial z^2}\right) u_r + \frac{\partial^2 u_z}{\partial r\, \partial z}\right] = -\frac{R}{G}, \quad \text{(a)}$$

$$\nabla^2 u_z + \frac{1}{1-2\sigma}\left(\frac{\partial^2 u_z}{\partial z^2} + \frac{\partial^2 u_r}{\partial z\, \partial r} + \frac{1}{r}\frac{\partial u_r}{\partial z}\right) = -\frac{Z}{G},$$

where

$$\nabla^2 \ldots = \frac{\partial^2 \ldots}{\partial r^2} + \frac{1}{r}\frac{\partial \ldots}{\partial r} + \frac{\partial^2 \ldots}{\partial z^2}.$$

If the body forces have a potential Π, i.e., $R = \frac{\partial \Pi}{\partial r}$ and $Z = \frac{\partial \Pi}{\partial z}$, the particular solution of the non-homogeneous equations (a) is [6]:

$$\bar{u}_r = \frac{1}{2G}\frac{\partial \Phi}{\partial r}, \quad \bar{u}_z = \frac{1}{2G}\frac{\partial \Phi}{\partial z}, \quad \text{(b)}$$

where $\Phi = \Phi(r, z)$ is determined by substituting the expressions (b) in the equations (a), giving the equation

$$\nabla^2 \Phi = \frac{1-2\sigma}{1-\sigma}\Pi.$$

The stresses are determined by formulas (3.2b) and are equal to

$$\bar{R}_r = \frac{\partial^2 \Phi}{\partial r^2} + \frac{\sigma}{1-\sigma}\Pi, \quad \bar{B}_\beta = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\sigma}{1-\sigma}\Pi,$$
$$\bar{Z}_z = \frac{\partial^2 \Phi}{\partial z^2} + \frac{\sigma}{1-\sigma}\Pi, \quad \bar{R}_z = \bar{Z}_r = \frac{\partial^2 \Phi}{\partial r\, \partial z}. \quad \text{(c)}$$

The general solution of the equations (a) in generalized Neuber's form is

$$u_r = \frac{1}{2G}\frac{\partial}{\partial r}[F + 4(1-\sigma)\varphi], \quad u_z = \frac{1}{2G}\frac{\partial}{\partial z}[F + 4(1-\sigma)\varphi],$$

where

$$\nabla^2\nabla^2 F = 0, \quad \nabla^2\varphi = 0, \quad \nabla^2 F = 4\frac{\partial^2\varphi}{\partial z^2}.$$

The stresses are

$$\begin{aligned} R_r &= \frac{\partial^2 F}{\partial z^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1-\sigma}{r}\frac{\partial}{\partial r}\int \nabla^2 F\,dz^2, \\ B_\beta &= \nabla^2 F - \frac{1}{r}\frac{\partial F}{\partial r} + (1-\sigma)\frac{\partial^2}{\partial r^2}\int \nabla^2 F\,dz^2, \\ Z_z &= \nabla^2 F - \frac{\partial^2 F}{\partial z^2}, \quad R_z = Z_r = -\frac{\partial^2 F}{\partial r\,\partial z}. \end{aligned} \qquad \text{(d)}$$

The answer is the sum of the solutions (c) and (d); without changing the notation for the stresses, we obtain

$$R_r = \frac{\partial^2(\Phi - F)}{\partial r^2} + \nabla^2 F + \frac{1-\sigma}{r}\frac{\partial}{\partial r}\int \nabla^2 F\,dz^2 + \frac{\sigma}{1-\sigma}\Pi,$$

$$B_\beta = \frac{1}{r}\frac{\partial(\Phi - F)}{\partial r} + \nabla^2 F + (1-\sigma)\frac{\partial^2}{\partial r^2}\int \nabla^2 F\,dz^2 + \frac{\sigma}{1-\sigma}\Pi,$$

$$Z_z = \frac{\partial^2(\Phi - F)}{\partial z^2} + \nabla^2 F + \frac{\sigma}{1-\sigma}\Pi,$$

$$R_z = Z_r = \frac{\partial^2(\Phi - F)}{\partial r\,\partial z}.$$

►For the same conditions $u_\beta = \omega_r = \omega_z = 0$, $R = \frac{\partial\Pi}{\partial r}$, $Z = \frac{\partial\Pi}{\partial z}$, by using Eqs. (3.3b), determine $\theta = \theta(r, z)$ and $\omega_\beta = \omega(r, z)$.

Equations (3.3b) become

$$\begin{aligned} \frac{\partial\theta}{\partial r} + \frac{K}{r}\frac{\partial}{\partial z}(r\omega) &= -\frac{1}{\lambda + 2G}R, \\ \frac{\partial\theta}{\partial z} - \frac{K}{r}\frac{\partial}{\partial r}(r\omega) &= -\frac{1}{\lambda + 2G}Z, \end{aligned} \qquad \text{(a)}$$

where

$$K = \frac{2G}{\lambda + 2G}.$$

From the first equation of (a)

$$\frac{\partial \theta}{\partial r} = -K \frac{\partial \omega}{\partial z} - \frac{1}{\lambda + 2G} R. \tag{b}$$

By differentiating the expression (b) with respect to z, and substituting the result in the second equation of (a) differentiated with respect to r, we find

$$\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial (r\omega)}{\partial r} \right] = 0.$$

Assuming in the last equation

$$\omega = R(r) Z(z), \tag{c}$$

and separating the variables, we obtain two equations

$$\frac{d^2 Z}{dz^2} - p^2 Z = 0, \quad \frac{d^2 R}{d\alpha^2} + \frac{1}{\alpha} \frac{dR}{d\alpha} + \left(1 - \frac{1}{\alpha^2}\right) R = 0,$$

where $\alpha = pr$, p is an arbitrary number.

By solving the resulting equations, we find, according to (c), that ω is equal to

$$\omega = Z_1(\alpha) (A_p \cosh pz + B_p \sinh pz)$$

and to the corresponding sum of the solutions over p, where $Z_1(\alpha) = E_p J_1(\alpha) + F_p N_1(\alpha)$ is the cylindrical function of the first order; $J_1(\alpha)$ is the Bessel function of the first kind of the first order; $N_1(\alpha)$ is the Neumann function of the first order; A_p, B_p, E_p, F_p are arbitrary constants. According to the equation (b),

$$\frac{\partial \theta}{\partial r} = -Kp (A_p \sinh pz + B_p \cosh pz) Z_1(\alpha) - \frac{1}{\lambda + 2G} R,$$

from which

$$\theta = -K (A_p \sinh pz + B_p \cosh pz) \times$$
$$\times \int Z_1(\alpha) \, d\alpha - \frac{1}{p(\lambda + 2G)} \int R \, d\alpha + f(z) =$$
$$= K (A_p \sinh pz + B_p \cosh pz) Z_0(\alpha) -$$
$$- \frac{1}{p(\lambda + 2G)} \int R \, d\alpha + f(z),$$

where $Z_0(\alpha) = E_p J_0(\alpha) + F_p N_0(\alpha)$ is the cylindrical function of zero order.

3.3. Write the basic equations in terms of displacements for a polarly symmetric problem in spherical co-ordinates, and give their general and particular solutions.

Since the problem is polarly symmetric, it follows that all quantities depend only on r, and, in addition, $u_\alpha = u_\beta = \omega_r = \omega_\beta = \omega_\alpha = 0$.

The dilatation is, by (2.2c),

$$\theta = \frac{du_r}{dr} + 2\frac{u_r}{r}.$$

Thus, there remains only the first equation of (3.3c), which assumes the form

$$\frac{d}{dr}\left(\frac{du_r}{dr} + 2\frac{u_r}{r}\right) + R_1 = 0, \tag{a}$$

where

$$R_1 = \frac{R}{\lambda + 2G}.$$

The general solution of the homogeneous equation

$$\frac{d}{dr}\left(\frac{du_r}{dr} + 2\frac{u_r}{r}\right) = 0$$

is found successively:

$$\frac{du_r}{dr} + \frac{2u_r}{r} = \frac{1}{r^2}\frac{d}{dr}(r^2 u_r) = 3C_1$$

and

$$u_r = C_1 r + \frac{C_2}{r^2}. \tag{b}$$

A particular solution of the non-homogeneous equation (a) is obtained in the form of the general solution (b) by the method of variation of arbitrary constants, assuming C_1 and C_2 to be functions of r:

$$\bar{u}_r = C_1(r)\, r + \frac{C_2(r)}{r^2}. \tag{c}$$

The first derivative of the particular solution is

$$\frac{d\bar{u}_r}{dr} = C_1(r) - \frac{2}{r^3}C_2(r) \tag{d}$$

on condition that

$$C_1'(r)\,r+\frac{1}{r^2}\,C_2'(r)=0. \tag{e}$$

The second derivative is

$$\frac{d^2\bar{u}_r}{dr^2}=C_1'(r)+\frac{6}{r^4}\,C_2(r)-\frac{2}{r^3}\,C_2'(r). \tag{f}$$

Substituting (c), (d), and (f) in the equation (a), we obtain

$$C_1'(r)-\frac{2}{r^3}\,C_2'(r)+R_1=0. \tag{g}$$

By solving the equations (e) and (g) simultaneously, we find

$$C_1(r)=-\int\limits_c^r\frac{R_1}{3}\,dr,\quad C_2(r)=\int\limits_c^r\frac{R_1r^3}{3}\,dr. \tag{h}$$

Substituting the expressions (h) in the equation (c), we obtain a particular solution.

3.4. Lamé's problem (1859) [5].
Determine the displacements and stresses in a closed spherical shell loaded internally ($r = a$) and externally ($r = b$) by uniformly distributed pressures (p_i and p_0).

According to the equation (a) of Problem 3.3,

$$u_r=C_1r+\frac{C_2}{r^2}.$$

The dilatation is

$$\theta=\frac{du_r}{dr}+2\,\frac{u_r}{r}=\frac{1}{r^2}\,\frac{d}{dr}\,(r^2u_r)=3C_1.$$

The stresses are, by (3.2c),

$$R_r=\lambda\theta+2G\,\frac{du_r}{dr}=(3\lambda+2G)\,C_1-\frac{4G}{r^3}\,C_2,$$

$$B_\beta=A_\alpha=\lambda\theta+\frac{2G}{r}\,u_r=(3\lambda+2G)\,C_1+\frac{2G}{r^3}\,C_2.$$

The boundary conditions of the problem are: when $r=a$, $R_r = -p_i$; when $r = b$, $R_r = -p_0$.

By satisfying the boundary conditions, we obtain

$$C_1 = \frac{1}{3\lambda+2G}\,\frac{p_i a^3 - p_0 b^3}{b^3-a^3}, \quad C_2 = \frac{1}{4G}\,\frac{a^3 b^3 (p_i - p_0)}{b^3 - a^3}.$$

During the deformation, the sphere retains the same shape.

3.5. Lamé's problem (1852) [5].

Determine the deformation of a sphere of radius a due to the mutual attraction of its particles.

Each unit volume of the sphere is acted on by a radial force $R = -\rho g r/a$, where g is the acceleration of gravity at the surface of the sphere (Problem 1.2).

The equilibrium equation in the presence of the radial force is of the form (Problem 3.3)

$$\frac{d}{dr}\left(\frac{du_r}{dr} + \frac{2u_r}{r}\right) - \frac{\rho g r}{a(\lambda+2G)} = 0. \tag{a}$$

By using the results of Problem 3.3, we obtain the general solution of the equation (a) in the form

$$u_r = C_1 r.$$

For a solid sphere $C_2 = 0$.

A particular solution is taken in the form

$$u_r = B r^3. \tag{b}$$

Substituting the solution (b) in the equation (a), we obtain

$$B = \frac{\rho g}{10a(\lambda+2G)},$$

and the displacement is equal to

$$u_r = C_1 r + \frac{\rho g r^3}{10a(\lambda+2G)}.$$

Since the surface of the sphere is free from stresses ($R_{r,a} = 0$), we find, finally,

$$u_r = -\frac{1}{10}\frac{\rho g a r}{\lambda+2G}\left(\frac{5\lambda+6G}{3\lambda+2G}-\frac{r^2}{a^2}\right).$$

It is interesting to note that the radial strain inside a sphere of radius $a\sqrt{\frac{3-\sigma}{3+3\sigma}}$ is a contraction, and outside the sphere it is an extension. Thus, significant initial stresses are induced in very large bodies due to the mutual attraction of particles.

3.6. Write the basic equations in terms of displacements for an axially symmetric problem in spherical coordinates, and find their general solution.

Since the problem is axially symmetric, it follows that all quantities depend on r and α; in addition, $u_\beta = \omega_r = \omega_\alpha = 0$, and Eqs. (3.3c) reduce to

$$(\lambda+2G)\frac{\partial\theta}{\partial r}+\frac{2G}{r^2\sin\alpha}\frac{\partial}{\partial\alpha}(r\omega_\beta\sin\alpha)+R=0,$$

$$(\lambda+2G)\frac{\partial\theta}{\partial\alpha}-2G\frac{\partial}{\partial r}(r\omega_\beta)+rA=0,$$

where, according to (2.2c) and (2.3c),

$$\theta=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)+\frac{1}{r\sin\alpha}\frac{\partial}{\partial\alpha}(u_\alpha\sin\alpha),$$

$$\omega_\beta=\frac{1}{2r}\left[\frac{\partial u_r}{\partial\alpha}-\frac{\partial}{\partial r}(ru_\alpha)\right].$$

The equations are homogeneous when $R=A=0$:

$$\frac{\partial\theta}{\partial r}+\frac{K}{r^2\sin\alpha}\frac{\partial}{\partial\alpha}(\omega\sin\alpha)=0,\quad \frac{\partial\theta}{\partial\alpha}-K\frac{\partial\omega}{\partial r}=0, \qquad (a$$

where

$$\omega=r\omega_\beta,\quad K=\frac{2G}{\lambda+2G}.$$

Assuming

$$\theta=K\frac{\partial\Phi}{\partial r},\quad \omega=\frac{\partial\Phi}{\partial\alpha}, \qquad (b$$

the second equation of (a) is satisfied identically, and the first equation is

$$\frac{\partial^2\Phi}{\partial\alpha^2}+\cot\alpha\,\frac{\partial\Phi}{\partial\alpha}+r^2\,\frac{\partial^2\Phi}{\partial r^2}=0.$$

Assuming in the last equation

$$\Phi = R\ (r)\ A\ (\alpha),$$

and separating the variables, we obtain two equations

$$\frac{d^2R}{dr^2}-\frac{k\,(k+1)}{r^2}\,R=0,\quad \frac{d^2A}{d\alpha^2}+\cot\alpha\,\frac{dA}{d\alpha}+k\,(k+1)\,A=0,$$

where k is an arbitrary number.

By solving the resulting equations, we find

$$\Phi = (A_k r^{k+1} + B_k r^{-k})\ [E_k P_k\ (\mu) + F_k Q_k\ (\mu)],$$

where $\mu = \cos\alpha$, $P_k\ (\mu)$ are Legendre's functions of the first kind, or Legendre's polynomials*; $Q_k\ (\mu)$ are Legendre's functions of the second kind (see Chap. 4 and [7]); A_k, B_k, E_k, F_k are arbitrary constants.

According to the formulas (b), θ and ω_β are equal to

$$\theta=K\,\frac{\partial\Phi}{\partial r}=K\,[A_k\,(k+1)\,r^k-B_k k r^{-(k+1)}]\times$$

$$\times\,[E_k P_k\,(\mu)+F_k Q_k\,(\mu)],$$

$$\omega_\beta=\frac{\omega}{r}=\frac{1}{r}\,\frac{\partial\Phi}{\partial\alpha}=$$

$$=-(A_k r^k+B_k r^{-k-1})\,[E_k P_k'\,(\mu)+F_k Q_k'\,(\mu)]$$

and to the corresponding sum of the solutions over k.

3.7. Michell's problem (1900) [5].

Find the stress distribution in an infinite cone $\alpha = \alpha_1$, with a force P acting along its axis at the vertex (Fig. 17).

By symmetry about the z axis,

$$u_\beta = A_\beta = R_\beta = 0,\quad \omega_r = \omega_\alpha = 0.$$

Equations (3.3c) in the absence of body forces take the form of the equations (a) of Problem 3.6.

* If k is an integer.

The solutions of the equations (a) of Problem 3.6, in which the displacements are inversely proportional to the

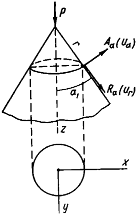

Fig. 17

radius r, are as follows:

$$
\begin{aligned}
(1)\quad & u_r = \frac{F}{4\pi G}\,\frac{\cos\alpha}{r}, \\
& u_\alpha = -\frac{\lambda+3G}{2(\lambda+2G)}\,\frac{F}{4\pi G}\,\frac{\sin\alpha}{r}, \\
& R_r = -\frac{3\lambda+4G}{\lambda+2G}\,\frac{F}{4\pi}\,\frac{\cos\alpha}{r^2}, \\
& A_\alpha = B_\beta = \frac{G}{\lambda+2G}\,\frac{F}{4\pi}\,\frac{\cos\alpha}{r^2}, \\
& R_\alpha = \frac{G}{\lambda+2G}\,\frac{F}{4\pi}\,\frac{\sin\alpha}{r^2}.
\end{aligned}
\qquad \text{(a)}
$$

$$
\begin{aligned}
(2)\quad & u_r = \frac{B}{r}, \quad u_\alpha = -\frac{B}{r}\,\frac{\sin\alpha}{1+\cos\alpha}, \\
& R_r = -2G\frac{B}{r^2}, \quad R_\alpha = 2G\,\frac{B}{r^3}\,\frac{\sin\alpha}{1+\cos\alpha}, \\
& A_\alpha = 2G\,\frac{B}{r^2}\,\frac{\cos\alpha}{1+\cos\alpha}, \\
& B_\beta = 2G\,\frac{B}{r^2(1+\cos\alpha)},
\end{aligned}
$$

where F and B are arbitrary constants.

The boundary conditions of the problem $A_\alpha = B_\alpha = 0$ when $\alpha = \alpha_1$ reduce, after cancelling proportional terms, to a single equation of the form

$$\frac{1}{\lambda+2G}\frac{F}{4\pi}+\frac{2B}{1+\cos\alpha_1}=0,$$

giving

$$B=-\frac{1+\cos\alpha_1}{8\pi(\lambda+2G)}F.$$

The second condition for the determination of F is obtained by forming the sum of the projections on the axis z of the cone of the force P and the stresses on a spherical surface centred at the vertex of the cone. It follows from the last condition that

$$P=\frac{F}{2(\lambda+2G)}[\lambda(1-\cos^3\alpha_1)+G(1-\cos\alpha_1)(1+\cos^2\alpha_1)].$$

When $\alpha_1 = \pi/2$, we obtain a pressure at some point on a body bounded by a plane.

3.8. By using the results of Problem 3.2 ▶, determine the displacements for an axially symmetric problem in cylindrical co-ordinates.

According to Eqs. (2.2b) and (2.3b),

$$\theta=\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\partial u_z}{\partial z},\quad 2\omega=\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}. \tag{a}$$

By eliminating the displacement u_z from the equations (a), we obtain

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right]+\frac{\partial^2 u_r}{\partial z^2}=\frac{\partial\theta}{\partial r}+2\frac{\partial\omega}{\partial z}. \tag{b}$$

The general solution of the equation (b) is taken in the form

$$u_r = R(r)\,Z(z). \tag{c}$$

Substituting the expression (c) in the homogeneous equation (b), and separating the variables, we obtain two equations

$$\frac{d^2Z}{dz^2}-m^2Z=0,\quad \frac{d^2R}{d\alpha^2}+\frac{1}{\alpha}\frac{dR}{d\alpha}+\left(1-\frac{1}{\alpha^2}\right)R=0, \tag{d}$$

where $\alpha = mr$, m is an arbitrary number.

The equations (d) have the solutions

$$Z = A_m \cosh mz + B_m \sinh mz,$$

$$R = Z_1(\alpha) = G_m J_1(\alpha) + H_m N_1(\alpha).$$

According to the equation (c), u_r is equal to

$$u_r = (A_m \cosh mz + B_m \sinh mz)\,[G_m J_1(\alpha) + H_m N_1(\alpha)]$$

and to the corresponding sum of the solutions over m.

According to the second homogeneous equation of (a),

$$\frac{\partial u_z}{\partial r} = \frac{\partial u_r}{\partial z} = m\,(A_m \sinh mz + B \cosh mz)\, Z_1(\alpha),$$

from which u_z is equal to

$$u_z = -(A_m \sinh mz + B_m \cosh mz)\,[G_m J_0(\alpha) + H_m N_0(\alpha)] + f(z)$$

(see [7]) and to the corresponding sum of the solutions over m.

To find a particular solution, we substitute the values of 0 and ω from Problem 3.2 ▶ in the equation (b). After some rearrangement, we obtain

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\,\frac{\partial (r\overline{u}_r)}{\partial r}\right] + \frac{\partial^2 \overline{u}_r}{\partial r^2} = (\overline{A}_p \sinh pz + \overline{B}_p \cosh pz)\, Z_1(\alpha)^*, \qquad \text{(e)}$$

where

$$Z_1(\alpha) = [E_p J_1(\alpha) + F_p N_1(\alpha)],$$

$$\overline{A}_p = (2 - K)\, pA_p,$$

$$\overline{B}_p = (2 - K)\, pB_p, \quad \alpha = pr.$$

Assuming

$$\overline{u}_r = \overline{R}(r)\,(\overline{A}_p \sinh pz + \overline{B}_p \cosh pz),$$

where $\overline{R}(r)$ is an unknown function of r, and substituting in the equation (e), we obtain

$$\frac{d^2\overline{R}}{d\alpha^2} + \frac{1}{\alpha}\,\frac{d\overline{R}}{d\alpha} + \left(1 - \frac{1}{\alpha^2}\right)\overline{R} = \frac{1}{p^2}\, Z_1(\alpha). \qquad \text{(f)}$$

* The body forces R and Z are taken to be zero.

A particular solution of the equation (f) is taken in the form

$$\overline{R}(r) = -\frac{\alpha}{2p^2} Z_0(\alpha)^*.$$

The displacements are finally

$$u_r = \sum_m [G_m J_1(\alpha) + H_m N_1(\alpha)] (A_m \cosh mz + B_m \sinh mz) -$$

$$- \sum_p \frac{\alpha}{2p^2} [E_p J_0(\alpha) + F_p N_0(\alpha)] (\overline{A}_p \sinh pz + \overline{B}_p \cosh pz),$$

where $\overline{A}_p$ and $\overline{B}_p$ are determined by the formulas (e).

According to the second equation of (a),

$$u_z = \int \left(2\omega - \frac{\partial u_r}{\partial z}\right) dr + f(z) =$$

$$= \sum_m [G_m J_0(\alpha) + H_m N_0(\alpha)] (A_m \sinh mz + B_m \cosh mz) +$$

$$+ \sum_p \frac{1}{p} \left\{\frac{\alpha}{2p} (\overline{A}_p \cosh pz + \overline{B}_p \sinh pz) \times\right.$$

$$\times [E_p J_1(\alpha) + F_p N_1(\alpha)] -$$

$$\left. -2(A_p \cosh pz + B_p \sinh pz) [E_p J_0(\alpha) + F_p N_0(\alpha)]\right\}^{**} + f(z).$$

3.9. By using the results of Problem 3.6, determine the displacements for an axially symmetric problem in spherical co-ordinates.

According to Eqs. (2.2c) and (2.3c),

$$0 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r^2 \sin\alpha} \frac{\partial}{\partial \alpha} (r u_\alpha \sin\alpha),$$

$$\omega = \omega_\beta = \frac{1}{2r} \left[\frac{\partial u_r}{\partial \alpha} - \frac{\partial}{\partial r} (r u_\alpha)\right]$$

---

* In seeking a particular solution use is made of the formulas

$$Z_0' = -Z_1, \quad Z_1' = Z_0 - \frac{1}{\alpha} Z_1.$$

** In deriving these expressions use is made of the formulas

$$\int Z_1(\alpha)\, d\alpha = -Z_0(\alpha), \quad \int \alpha Z_0(\alpha)\, d\alpha = \alpha Z_1(\alpha).$$

or

$$\frac{\partial U_r}{\partial r}+\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}(U_\alpha\sin\alpha)=r^2\theta,\quad \frac{\partial U_r}{\partial\alpha}-r^2\frac{\partial U_\alpha}{\partial r}=2r^3\omega, \tag{a}$$

where

$$U_r=r^2u_r,\quad U_\alpha=ru_\alpha. \tag{b}$$

For the homogeneous system (a), by taking

$$U_r=r^2\frac{\partial\Phi}{\partial r},\quad U_\alpha=\frac{\partial\Phi}{\partial\alpha}, \tag{c}$$

we identically satisfy the second equation of (a), and the first equation is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\partial\Phi}{\partial\alpha}\sin\alpha\right)=0.$$

Assuming, in the last equation,

$$\Phi(\alpha,\ r)=R(r)\,A(\alpha),$$

and separating the variables, we obtain two equations

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-n(n+1)R=0,$$

$$\frac{d^2A}{d\alpha^2}+\cot\alpha\frac{dA}{d\alpha}+n(n+1)A=0.$$

By solving these equations, we find

$$\Phi=(C_nr^n+D_nr^{-n-1})[E_nP_n(\mu)+F_nQ_n(\mu)].$$

According to the equations (b) and (c),

$$u_r=\frac{U_r}{r^2}=\frac{\partial\Phi}{\partial r}=[C_nnr^{n-1}-D_n(n+1)r^{-n-2}\times$$

$$\times[E_nP_n(\mu)+F_nQ_n(\mu)], \tag{d}$$

$$u_\alpha=\frac{U_\alpha}{r}=\frac{1}{r}\frac{\partial\Phi}{\partial\alpha}=$$

$$=(C_nr^{n-1}+D_nr^{-n-2})[E_nP_n'(\mu)+F_nQ_n'(\mu)]. \tag{e}$$

For the non-homogeneous system (a), according to (b) and the results of Problem 3.6, we have

$$2r^3\omega=-2(A_kr^{k+3}+B_kr^{-k+2})[E_kP_k'(\mu)+F_kQ_k'(\mu)],$$

$$r^2\theta=K[A_k(k+1)r^{k+2}-B_kkr^{-k+1}][E_kP_k(\mu)+F_kQ_k(\mu)].$$

By taking a particular solution in the form

$$U_r = r^2 \frac{\partial \overline{\Phi}}{\partial r} + \int 2r^3 \omega \, d\alpha, \quad U_\alpha = \frac{\partial \overline{\Phi}}{\partial \alpha}, \tag{f}$$

where

$$\int 2r^3 \omega \, d\alpha = -2 (A_k r^{k+3} + B_k r^{-k+2}) [E_k P_k (\mu) + F_k Q_k (\mu)],$$

we identically satisfy the second equation of (a), and the first equation is

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial \overline{\Phi}}{\partial r} \right) + \frac{1}{\sin \alpha} \frac{\partial}{\partial \alpha} \left(\frac{\partial \overline{\Phi}}{\partial \alpha} \sin \alpha \right) =$$

$$= (A_k r^{k+2} + \overline{B} r^{-k+1}) [E_k P_k (\mu) + F_k Q_k (\mu)], \tag{g}$$

where

$$\overline{A}_k = [K (k + 1) + 2 (k + 3)] A_k,$$

$$\overline{B}_k = - [Kk + 2 (k - 2)] B_k.$$

Assuming

$$\overline{\Phi} (\alpha, r) = \overline{R} (r) [E_k P_k (\mu) + F_k Q_k (\mu)],$$

and substituting in the equation (g), we obtain, after some necessary cancellations,

$$\frac{d}{dr} \left(r^2 \frac{d\overline{R}}{dr} \right) - k (k+1) \overline{R} = A_k r^{k+2} + \overline{B}_k r^{-k+1},$$

from which

$$\overline{R} (r) = \frac{\overline{A}_k}{2 (2k+3)} r^{k+2} - \frac{\overline{B}_k}{2 (2k-1)} r^{-k+1}$$

and

$$\overline{\Phi} = (\overline{\overline{A}}_k r^{k+2} + \overline{\overline{B}}_k r^{-k+1}) [E_k P_k (\mu) + F_k Q_k (\mu)].$$

where

$$\overline{\overline{A}}_k = \frac{1}{2 (2k+3)} [K (k+1) + 2 (k+3)] A_k,$$

$$\overline{\overline{B}}_k = \frac{1}{2 (2k-1)} [Kk + 2 (k-2)] B_k.$$

According to the formulas (b) and (f),

$$\bar{u}_r = \frac{U_r}{r^2} = \frac{\partial \bar{\Phi}}{\partial r} + \frac{1}{r^2} \int 2r^3 \omega \, d\alpha =$$

$$= \{[(k+2)\, \bar{\bar{A}}_k + 2A_k]\, r^{k+1} - [(k-1)\, \bar{\bar{B}}_k + 2B_k]\, r^{-k}\} \times$$

$$\times [E_k P_k(\mu) + F_k Q_k(\mu)], \qquad \text{(h)}$$

$$\bar{u}_\alpha = \frac{U_\alpha}{r} = \frac{1}{r} \frac{\partial \bar{\Phi}}{\partial \alpha} = (\bar{\bar{A}}_k r^{k+1} + \bar{\bar{B}}_k r^{-k})\, [E_k P'_k(\mu) + F_k Q'_k(\mu)]. \qquad \text{(i)}$$

The final values of displacements are obtained by adding together the expressions (d), (e) and (h), (i):

$$u_r = \sum_n [C_n n r^{n-1} - D_n (n+1)\, r^{-n-2}]\, [E_n P_n(\mu) + F_n Q_n(\mu)] +$$

$$+ \sum_k \{[(k+2)\, \bar{\bar{A}}_k + 2A_k]\, r^{k+1} - [(k-1)\, \bar{\bar{B}}_k + 2B_k]\, r^{-k}\} \times$$

$$\times [E_k P_k(\mu) + F_k Q_k(\mu)],$$

$$u_\alpha = \sum_n (C_n r^{n-1} + D_n r^{-n-2})\, [E_n P'_n(\mu) + F_n Q'_n(\mu)] +$$

$$+ \sum_k (\bar{\bar{A}}_k r^{k+1} + \bar{\bar{B}}_k r^{-k})\, [E_k P'_k(\mu) + F_k Q'_k(\mu)]. \qquad \text{(j)}$$

If the numbers n and k are of the same nature, the results of the solution (j) may be represented by a single sum; in this case we obtain

$$u_r = \sum_n [a_n r^{n+1} + b_n r^{-n} + C_n n r^{n-1} - D_n (n+1)\, r^{-n-2}] \times$$

$$\times [E_n P_n(\mu) + F_n Q_n(\mu)],$$

$$u_\alpha = \sum_n (\bar{\bar{A}}_n r^{n+1} - \bar{\bar{B}}_n r^{-n} + C_n r^{n-1} + D_n r^{-n-2}) \times$$

$$\times [E_n P'_n(\mu) + F_n Q'_n(\mu)],$$

where

$$a_n = (n+2)\, \bar{\bar{A}}_n - 2A_n = \frac{n+1}{2(2n+3)}\, [K(n+2) + 2(n-1)]\, A_n,$$

$$b_n = -[(n-1)\, \bar{\bar{B}}_n + 2B_n] = \frac{n}{2(2n-1)}\, [K(n-1) +$$

$$+2(n+1)]B_n,$$

$$\overline{\overline{A}}_n=\frac{1}{2(2n+3)}[K(n+1)+2(n+3)]A_n,$$

$$\overline{\overline{B}}_n=\frac{1}{2(2n-1)}[Kn+2(n-2)]B_n.$$

Here A_n, B_n, C_n, D_n, E_n, F_n are arbitrary constants.
If the origin of co-ordinates ($r = 0$) belongs to the body under consideration (internal problem), the constants B_n and D_n are zero; if $r \to \infty$ (external problem), the constants A_n and C_n are zero; if the poles ($\alpha = 0$ and $\alpha = \pi$) belong to the body, the constant F_n is zero.

3.10. Neuber's problem (1931) [8].
Determine the state of stress in a sphere of radius a compressed by forces P applied at the poles (Fig. 18a).

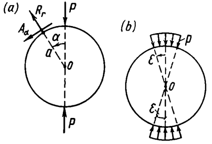

Fig. 18

When external loads are given on the surface of the sphere, where $r = a$ (internal problem), we have

$$R_r = \sigma(\alpha), \quad A_r = \tau(\alpha). \tag{a}$$

These functions must be represented by series

$$\sigma(\alpha)=\sum_{n=0}^{\infty}\sigma_n P_n(\mu),$$

$$\tau(\alpha)=\sum_{n=1}^{\infty}\tau_n\frac{dP_n}{d\mu}=-\sum_{n=1}^{\infty}\tau_n\frac{dP_n}{d\alpha}\sin\alpha,$$

where the coefficients are determined by the formulas

$$\sigma_n = \frac{2n+1}{2} \int_0^{\pi} \sigma(\alpha) P_n(\cos\alpha) \sin\alpha\, d\alpha,$$

$$\tau_n = \frac{2n+1}{2n(n+1)} \int_0^{\pi} \tau(\alpha) \frac{dP_n}{d\alpha} \sin\alpha\, d\alpha. \qquad \text{(b)}$$

In the present problem

$$\sigma(\alpha) = \begin{cases} -p & \text{when } 0 < \alpha < \varepsilon \text{ and } \pi - \varepsilon < \alpha < \pi, \\ 0 & \text{when } \varepsilon < \alpha < \pi - \varepsilon; \end{cases}$$

$\tau(\alpha) = 0.$

Let us break the normal load into two component loads

$\sigma(\alpha) = \sigma^{(1)}(\alpha) + \sigma^{(2)}(\alpha),$

so that

$$\sigma^{(1)}(\alpha) = \begin{cases} -p & \text{when } 0 < \alpha < \varepsilon, \\ 0 & \text{when } \varepsilon < \alpha < \pi; \end{cases}$$

$$\sigma^{(2)}(\alpha) = \begin{cases} -p & \text{when } \pi - \varepsilon < \alpha < \pi, \\ 0 & \text{when } 0 < \alpha < \pi - \varepsilon. \end{cases}$$

Consider the first case of loading

$$\sigma^{(1)}(\alpha) = \sum_{n=0}^{\infty} \sigma_n^{(1)} P_n(\cos\alpha),$$

where, according to the formulas (b),

$$\sigma_n^{(1)} = -\frac{2n+1}{2} p \int_0^{\varepsilon} P_n(\cos\alpha) \sin\alpha\, d\alpha =$$

$$= (2n+1) P_n(\cos\alpha) = \frac{dP_{n+1}}{d\mu} - \frac{dP_{n-1}}{d\mu} \Big|_0^{\varepsilon} =$$

$$= -\frac{p}{2} \{[P_{n+1}(1) - P_{n+1}(\cos\varepsilon)] -$$

$$- [P_{n-1}(1) - P_{n-1}(\cos\varepsilon)]\} =$$

$$= -\frac{p}{2} [P_{n-1}(\cos\varepsilon) - P_{n+1}(\cos\varepsilon)].$$

The area, over which the uniform load p is distributed when the angle ε is small, is approximately equal to $\pi a^2 \varepsilon^2$ (Fig. 18*b*).

By applying a limiting process, i.e., letting $\varepsilon \to 0$ and $p\pi a^2 \varepsilon^2 \to P$, we obtain

$$\sigma_n^{(1)} = -\frac{P}{2\pi a^2} \lim_{\varepsilon \to 0} \frac{P_{n-1}(\cos \varepsilon) - P_{n+1}(\cos \varepsilon)}{\varepsilon^2} = -\frac{P(2n+1)}{4\pi a^2}.$$

Similarly, for a concentrated force at the pole $\alpha = \pi$ we have

$$\sigma_n^{(2)} = -\frac{P}{2\pi a^2} \frac{2n+1}{2} P_n(-1) = -\frac{P}{4\pi a^2}(2n+1)(-1)^n.$$

Thus,

$$\sigma_n = \sigma_n^{(1)} + \sigma_n^{(2)} = \begin{cases} 0 & \text{when } n = 1, 3, 5, \ldots \\ -\frac{P}{2\pi a^2}(2n+1) & \text{when } n = 0, 2, 4, \ldots \end{cases}$$

and the load is expressed by the series

$$\sigma(\alpha) = \sum_{n=0}^{\infty} \sigma_n P_n(\mu) = -\frac{P}{\pi a^2} \sum_{n=0,\,2,\,4,\,\ldots}^{\infty} \frac{2n+1}{2} P_n(\mu),$$

which diverges at the poles $\alpha = 0$ and $\alpha = \pi$, and converges at the other points.

For the solution of the problem we must have an expression for the stresses in terms of Legendre's polynomials. For this, we use the results of Problem 3.9 and formulas (3.2c)

$$\begin{aligned} R_r &= \lambda\theta + 2G\frac{\partial u_r}{\partial r} = \sum_{n=0,\,2,\,4,\,\ldots}^{\infty} [(n+1)(2Ga_n + \\ &+ K\lambda A_n) r^n + 2Gn(n-1) C_n r^{n-2}] P_n(\mu), \\ A_r &= \frac{G}{r}\left(r\frac{\partial u_\alpha}{\partial r} - u_\alpha + \frac{\partial u_r}{\partial \alpha}\right) = \\ &= \sum_{n=0,\,2,\,4,\,\ldots}^{\infty} G[(a_n + n\bar{\bar{A}}_n) r^n + 2(n-1) C_n r^{n-2}] P_n'(\mu). \end{aligned} \quad \text{(c)}$$

To determine the arbitrary constants A_n and C_n, we use the equations (a):

$$(n+1)(2Ga_n + K\lambda A_n)a^n + 2Gn(n-1)a^{n-2}C_n = = -\frac{(2n+1)P}{2\pi a^2},$$
$$(a_n + n\bar{A}_n) + 2(n-1)a^{-2}C_n = 0. \tag{d}$$

By solving the equations (d), we obtain

$$A_n = = -\frac{(2n+3)(2n+1)P/\pi a^{n+2}G}{4\{K(n+1)^2 + 2(n+3)(1-n) - (2n+3)[(n+2) - 2(n+1)\sigma]\}},$$

$$C_n = -\frac{a_n + n\bar{A}_n}{2(n-1)} = \frac{(1+n)a^2 A_n}{2(1-n)(2n+3)}[K(n+1)^2 + + 2(n+3)(n+1) - 2(2n+3)].$$

In carrying out the calculations use has been made of the fact that

$$\sigma = \frac{\lambda}{2(\lambda + G)}.$$

For the technique of calculating the series (c), see the monograph [6], Chap. VI.

When $r = 0$, the only non-vanishing terms are those corresponding to the coefficients A_0 and C_2, and it is easy to calculate the stresses at the centre of the sphere.

Chapter 4

GENERAL SOLUTIONS OF THE BASIC EQUATIONS OF THE THEORY OF ELASTICITY. SOLUTION OF THREE-DIMENSIONAL PROBLEMS

Since various forms of the general solutions of the basic equations involve harmonic, ψ, and biharmonic, φ, functions, we shall consider the solutions of the harmonic and biharmonic equations.

I. HARMONIC EQUATION (LAPLACE'S)

1. Orthogonal curvilinear co-ordinates (Lamé's)

$$\nabla^2\psi = \frac{1}{\Delta}\left[\frac{\partial}{\partial\alpha_1}\left(\sqrt{\frac{g_2 g_3}{g_1}}\frac{\partial}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\sqrt{\frac{g_3 g_1}{g_2}}\frac{\partial}{\partial\alpha_2}\right) + \frac{\partial}{\partial\alpha_3}\left(\sqrt{\frac{g_1 g_2}{g_3}}\frac{\partial}{\partial\alpha_3}\right)\right]\psi = 0, \tag{4.1}$$

where $\Delta = \sqrt{g_1 g_2 g_3}$.

By applying the method of separation of variables, we obtain the standard forms of the solutions of Eq. (4.1):

$$\psi = \psi_1(\alpha_1)\,\psi_2(\alpha_2)\,\psi_3(\alpha_3),$$

$$\psi = \sum \psi_1(\alpha_1)\,\psi_2(\alpha_2)\,\psi_3(\alpha_3). \tag{4.2}$$

2. Rectangular co-ordinates

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0, \tag{4.1a}$$

$$\psi = X(x)\,Y(y)\,Z(z). \tag{4.2a}$$

Substituting (4.2a) in (4.1a), we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

or, after separating the variables,

$$\frac{d^2X}{dx^2}-a^2X=0,\quad \frac{d^2Y}{dy^2}-b^2Y=0,\quad \frac{d^2Z}{dz^2}-c^2Z=0$$

with $a^2+b^2+c^2=0$.

By using the last equations, we obtain the standard forms of the solution

$$\psi=e^{\pm ax}e^{\pm by}e^{\pm cz}=e^{\pm ax\pm by\pm cz} \qquad (4.2a')$$

and a number of special cases:

$$\psi=\sum_{m,\,n} f(m,\,n)\,e^{\pm\sqrt{m^2+n^2}\,z}(\sin mx+\cos mx)\times$$

$$\times(\sin ny+\cos ny) \qquad (4.2a'')$$

when $a=im,\ b=in,\ c=\pm\sqrt{m^2+n^2}$;

$$\psi=\sum_{a,\,b}\psi(a,\,b)\,e^{\pm ax\pm by}\,{}^{\cos}_{\sin}(\sqrt{a^2+b^2}\,z) \qquad (4.2a''')$$

when $c=i\sqrt{a^2+b^2}$.

The most general expression for a harmonic function is [9]

$$\psi=\int_{-\pi}^{\pi} f\,[(x\cos\alpha+y\sin\alpha+iz),\ \alpha]\,d\alpha,$$

from which various particular solutions can be obtained.

Of interest is a particular solution of Eq. (4.1a) in the form

$$\psi=[(x-a)^2+(y-b)^2+(z-c)^2]^{1/2},$$

where a, b, c are constants.

3. Cylindrical co-ordinates

$$\nabla^2\psi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\beta^2}+\frac{\partial^2\psi}{\partial z^2}=0, \qquad (4.1b)$$

$$\psi=R(r)\,B(\beta)\,Z(z). \qquad (4.2b)$$

Substituting the ψ value (4.2b) in Eq. (4.1b), we obtain

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{1}{r^2}\frac{1}{B}\frac{d^2B}{d\beta^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}\quad 0$$

or, after separating the variables,

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=m^2,\quad \frac{1}{B}\frac{d^2B}{d\beta^2}=-p^2,$$

$$\alpha^2\frac{d^2R}{d\alpha^2}+a\frac{dR}{d\alpha}+(\alpha^2-p^2)R=0,$$

where $\alpha = mr$, m, and p are positive integers.
By solving the last equations, we find

$$Z = A_m \cosh mz + B_m \sinh mz,$$

$$B = C_p \cos p\beta + D_p \sin p\beta,$$

$$R = E_pJ_p(\alpha) + F_pN_p(\alpha),$$

where

$$J_p(\alpha)=\left(\frac{\alpha}{2}\right)^p\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(p+k)!}\left(\frac{\alpha}{2}\right)^{2k}$$

is the Bessel function of the first kind of the pth order,

$$N_p(\alpha)=\frac{1}{\pi}\left[\frac{\partial J_p(\alpha)}{\partial p}-(-1)^p\frac{\partial J_{-p}(\alpha)}{\partial p}\right]$$

is the Bessel function of the second kind of the pth order, or the Neumann function.
Formula (4.2b) is finally

$$\psi(r,\beta,r)=\sum_{m,p}[A_{m,p}J_p(mr)+$$

$$+B_{m,p}N_p(mr)]^{\cos}_{\sin}(p\beta)^{\cosh}_{\sinh}(mz). \qquad (4.2b')$$

4. Spherical co-ordinates

$$\nabla^2\psi=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\partial\psi}{\partial\alpha}\sin\alpha\right)+\right.$$

$$\left.+\frac{1}{\sin^2\alpha}\frac{\partial^2\psi}{\partial\beta^2}\right]=0, \qquad (4.1c)$$

where

$$\psi = R(r)\,B(\beta)\,A(\alpha) \qquad (4.2c)$$

is a spatial spherical function (see below).

Substituting the ψ value (4.2c) in Eq. (4.1c), we obtain

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{B\sin^2\alpha}\frac{d^2B}{d\beta^2}\ \frac{1}{A\sin\alpha}\frac{d}{d\alpha}\left(\frac{dA}{d\alpha}\sin\alpha\right)=0$$

or, after separating the variables,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=n(n+1),$$

$$\frac{1}{B}\frac{d^2B}{d\beta^2}=-m^2,$$

$$\frac{1}{\sin\alpha}\frac{d}{d\alpha}\left(\frac{dA}{d\alpha}\sin\alpha\right)+\left[n(n+1)-\frac{m^2}{\sin^2\alpha}\right]A=0,$$

where $m = 0, 1, 2, \ldots$ is an integer, n is any number.

By solving the first two equations, we find

$$R(r)=A_nr^n+B_nr^{-n-1},\quad B(\beta)=C_m\cos m\beta+D_m\sin m\beta.$$

The third equation, on putting $\cos\alpha=\mu$, reduces to an equation for Legendre's associated functions

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dA}{d\mu}\right]+\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]A=0.$$

The standard forms of the solution of the last equation, when $m = 0$ (symmetrical problems), n is an integer, and $-1<\mu<1$, are as follows:

$$A(\mu)=E_nP_n(\mu)+F_nQ_n(\mu),$$

where

$$P_n(\mu)=\frac{1}{2^nn!}\frac{d^n(\mu^2-1)^n}{d\mu^n}$$

are Legendre's functions of the first kind, or Legendre's polynomials,

$$Q_n(\mu)=\frac{1}{2}P_n(\mu)\ln\frac{1+\mu}{1-\mu}-\sum_{k=1}^{n}\frac{1}{k}P_{k-1}(\mu)P_{n-k}(\mu)$$

are Legendre's functions of the second kind.

The consecutive values of the above quantities are

$$P_0(\mu)=1,\qquad Q_0(\mu)=\frac{1}{2}\ln\frac{1+\mu}{1-\mu},$$

$$P_1(\mu)=\mu,\qquad Q_1(\mu)=P_1(\mu)Q_0(\mu)-1,$$

$$P_2(\mu)=\frac{1}{2}(3\mu^2-1), \qquad Q_2(\mu)=P_2(\mu)Q_0(\mu)-\frac{3}{2}\mu,$$

$$P_3(\mu)=\frac{1}{2}(5\mu^3-3\mu), \quad Q_3(\mu)=P_3(\mu)Q_0(\mu)-\frac{5}{2}\mu^2-\frac{2}{3},$$

. . . .

Continued in [7].

When $n \geqslant m$ are integers and $-1 < \mu < 1$,

$$A(\mu)=E_{n,m}P_{n,m}(\mu)+F_{n,m}Q_{n,m}(\mu),$$

where

$$P_{n,m}(\mu)=(1-\mu^2)^{m/2}\frac{d^m P_n(\mu)}{d\mu^m}=\frac{(1-\mu^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n,$$

$$Q_{n,m}(\mu)=(1-\mu^2)^{m/2}\frac{d^m Q_n(\mu)}{d\mu^m}$$

are Legendre's associated functions of the first and second kind of degree n and order m.

In particular, $P_{n,0}(\mu) = P_n(\mu)$.

The consecutive values of the above quantities are

$$P_{1,1}(\mu)=\sqrt{1-\mu^2},$$

$$P_{2,1}(\mu)=3\mu\sqrt{1-\mu^2}, \quad P_{2,2}(\mu)=3(1-\mu^2),$$

$$P_{3,1}(\mu)=\frac{3}{2}(5\mu^2-1)\sqrt{1-\mu^2}, \quad P_{3,2}(\mu)=15\mu(1-\mu^2),$$

$$P_{3,3}(\mu)=15(1-\mu^2)\sqrt{1-\mu^2}, \quad \text{etc.}$$

Tables of these functions for $n = 1, (1), \ldots, 10$ and $m = 0, (1), \ldots, 4$ are given in [10].

When m is an integer $(0, 1, 2, \ldots)$, $n = \nu$ is a complex number, so that $\nu(\nu + 1) = 1 \pm bi$, where b is a given number, and when $-1 < \mu < 1$, we obtain

$$A(\mu) = E_{\nu m}P_{\nu m}(\mu) + F_{\nu m}Q_{\nu m}(\mu),$$

where $P_{\nu m}(\mu)$ and $Q_{\nu m}(\mu)$ are Legendre's associated functions of the first and second kind of complex degree $\nu(\nu+1) = 1 \pm bi$ and order m. The technique for tabulating these functions is given in [11]. Formula (4.2c) is finally

$$\psi=\sum_{m,n}(A_n r^n+B_n r^{-n-1})(C_m\cos m\beta+D_m\sin m\beta)\times$$
$$\times[E_{nm}P_{nm}(\mu)+F_{nm}Q_{nm}(\mu)]. \tag{4.2c'}$$

Every homogeneous function $F_n(x, y, z)$ of degree n in co-ordinates x, y, z, for which the condition $\nabla^2 F_n = 0$ is fulfilled, can be represented as a spatial spherical function

$$F_n(x, y, z) = r^n Y_n(\alpha, \beta),$$

where

$$Y_n(\alpha, \beta) = A(\alpha) B(\beta) = [E_{nm} P_{nm}(\cos\alpha) + F_{nm} Q_{nm}(\cos\alpha)] (C_m \cos m\beta + D_m \sin m\beta)$$

is a surface spherical function.

For symmetrical problems ($m = 0$),

$$Y_n(\alpha) = E_n P_n(\cos\alpha) + F_n Q_n(\cos\alpha).$$

5. Additional solutions

Additional solutions are obtained by combining already known solutions:

(a) linear combinations

$$\sum \psi_k, \quad \sum k\psi_k;$$

(b) derivatives with respect to parameters

$$\frac{\partial \psi_k}{\partial k}, \quad \frac{\partial^2 \psi_k}{\partial k^2}, \quad \frac{\partial^2 \psi_{km}}{\partial k\, \partial m}, \quad \text{etc.};$$

(c) integrals with respect to a parameter with a weight function dependent on it

$$\int f(k)\, \psi_k\, dk, \quad \text{etc.};$$

(d) derivatives and indefinite integrals with respect to α_1, α_2, α_3 if Laplace's differential equation does not contain explicitly the co-ordinates α_1, α_2, α_3 (for example, the equation in rectangular co-ordinates)

$$\frac{\partial \psi_k}{\partial x}, \quad \frac{\partial^2 \psi_k}{\partial x^2}, \quad \frac{\partial^2 \psi_k}{\partial x\, \partial y}, \quad \int \psi_k\, dx, \quad \operatorname{grad} \psi_k, \quad \text{etc.}$$

II. BIHARMONIC EQUATION

$$\nabla^2 \nabla^2 \varphi = 0. \tag{4.3}$$

Particular solutions:

(a)

$$\varphi = x\psi, \quad y\psi, \quad z\psi, \quad R^2\psi, \tag{4.4}$$

where

$$R^2 = x^2 + y^2 + z^2, \quad \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2};$$

(b) any polynomial of degree not higher than 3;

(c) polynomial of any degree with specially selected coefficients, which satisfies Eq. (4.3).

III. BOUNDARY VALUE PROBLEMS FOR THE HARMONIC AND BIHARMONIC EQUATIONS

The boundary value problems for the harmonic and biharmonic equations are treated by F. G. Tricomi in [12], Chap. IV.

Of great importance for the solution of elasticity problems are the boundary value problems for elliptic equations [$\nabla^2\psi = 0$, $\nabla^2\psi = F(x, y)$, $\nabla^2\nabla^2\varphi = 0$]; the Dirichlet, Neumann, and mixed problems.

1. Dirichlet problem (first boundary value problem)

Determine a harmonic function ψ in a closed region D from its known values on the boundary C of the region.

It has been established that for every region whose contour has no discontinuities and has a completely defined tangent everywhere and also a curvature permanently varying, except for a finite number of points of discontinuity of the first kind, the Dirichlet problem with continuous data on the contour can be reduced to a Fredholm integral equation of the second kind, with a kernel continuous everywhere [12], of the form

$$\mu(\xi) - \lambda \int_0^1 K(\xi, \eta)\,\mu(\eta)\,d\eta = \frac{1}{\pi} f(\xi), \qquad (4.5)$$
$$\lambda = 1/\pi, \quad 0 \leqslant \xi \leqslant 1,$$

and thereby, with the conditions specified above, the *existence theorem* is proved for harmonic functions and the Dirichlet problem is solved by using integral equations.

One of the most important methods of solving the Dirichlet problem on a plane is the *conformal mapping* method based on the fact that this mapping transforms Laplace's equation into itself ([12], Sec. 4.1). Almost all efficient methods of solving the Dirichlet problem, found up to now, are realized by means of conformal mappings transforming a given region D into a circle or a half-plane, the two cases in which there are explicit formulas for expressing a harmonic function assuming given values on the boundary C of the region.

The Dirichlet problem is also solved by expanding the boundary value function $f(\ldots)$ in orthogonal series taking into account the nature of the boundedness of the internal problem and subsequently determining the unknown coefficients in these series [see Eq. (4.2c′)]. The unknown harmonic function is

$$\psi(r, \mu, \beta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} r^n P_{nm}(\mu)(C_{nm}\cos m\beta + D_{nm}\sin m\beta), \tag{4.6}$$

where $\mu = \cos\alpha$, $r < a$, a is the radius of the given sphere. The unknown coefficients C_{nm} and D_{nm} are calculated from the boundary value function $f(r, \alpha, \beta)$ when $r = a$:

$$f(\alpha, \beta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} a^n P_{nm}(\cos\alpha)(C_{nm}\cos m\beta + D_{nm}\sin m\beta), \tag{4.7}$$

where, according to the orthogonality conditions,

$$C_{n0} = \frac{2n+1}{4\pi}\frac{1}{a^n}\int_0^{2\pi} d\beta \int_0^{\pi} f(\alpha, \beta) P_n(\cos\alpha)\sin\alpha\, d\alpha,$$

$$C_{nm} = \frac{2n+1}{2\pi}\frac{(n-m)!}{(n+m)!}\frac{1}{a^n}\int_0^{2\pi} d\beta \int_0^{\pi} f(\alpha, \beta) P_{nm}(\cos\alpha) \times$$

$$\times \cos m\beta \sin\alpha\, d\alpha,$$

$$D_{nm} = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \frac{1}{a^n} \int_0^{2\pi} d\beta \int_0^{\pi} f(\alpha, \beta) P_{nm}(\cos\alpha) \times$$

$$\times \sin m\beta \sin\alpha \, d\alpha,$$

$$n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots, n.$$

If we know the Green's function $g(x_0, y_0, x, y) = g(P_0, P)$ for a region D, i.e., a harmonic function in this region which assumes the same values on the boundary C as $\ln r$, where r is the distance from an arbitrary point $P(x, y)$ to a fixed point $P_0(x_0, y_0)$, it is possible at once to map conformally the region D into a circle and to give an explicit solution of the Dirichlet problem and even of a more general analogous problem for Poisson's equation

$$\nabla^2\varphi = F(x, y), \tag{4.8}$$

where F is a given continuous function; this equation is of great importance in problems relating to the torsion of prismatic and cylindrical bars [Eq. (7.9)], the analysis of membranes [13] [Eq. (5.45)], etc. The determination of the Green's function g, however, is often found no less difficult than the solution of the Dirichlet problem with the given values on the boundary C*.

For the solution of the Dirichlet problem and analogous problems (Neumann problem and mixed problems) there is a great variety of numerical methods: the finite difference method giving a system of five-term difference equations (5.37), the relaxation method of R. V. Southwell, variational methods based on the fact that the unknown function ψ minimizes the integral

$$J(\psi) = \iint\limits_D \left[\left(\frac{\partial\psi}{\partial x} \right)^2 + \left(\frac{\partial\psi}{\partial y} \right)^2 \right] dx\, dy \tag{4.9}$$

for which the Euler-Lagrange equation is Laplace's equation

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0. \tag{4.10}$$

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* One of the exceptions is an important case where the region D is a sphere or a circle of radius a centred at the origin of co-ordinates.

Considering further certain functions

$$\psi(x,\ y;\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n), \tag{4.11}$$

where α_i are the parameters, we obtain an extremum problem for the function J of n variables and write down n conditions that this function must satisfy

$$\frac{\partial J}{\partial \alpha_1}=\frac{\partial J}{\partial \alpha_2}=\ \ldots\ =\frac{\partial J}{\partial \alpha_n}=0. \tag{4.12}$$

To choose the functions ψ, use can be made of the Ritz-Timoshenko method, the Bubnov-Galerkin method, the Vlasov-Kantorovich method, etc. (see [13], Chap. I).

We present some of the results pertaining to the solution of the Dirichlet problem [14]:

(a) region D = half-space $z>0$,
boundary C = plane $z=0$,
boundary value function $= f(x,\ y)$,

$$\psi(x,\ y,\ z)=\frac{z}{2\pi}\int\int_{-\infty}^{\infty}\frac{f(\xi,\ \eta)\,d\xi\,d\eta}{[(x-\xi)^2+(y-\eta)^2+z^2]^{3/2}}=$$

$$=\text{Poisson's integral}; \tag{4.13}$$

the function $f(x,\ y)$ must be such that the reflection of the plane $z=0$ in a sphere lying outside the region D ([14], Vol. 2, Chap. IV, Sec. 1) would give a boundary value problem with continuous boundary values for the bounded region D' that is the reflection of the region D;

(b) region D = half-plane $y>0$,
boundary C = line $y=0$,
boundary value function $= f(x)$,

$$\psi(x,\ y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{(x-\xi)^2+y^2}\ ; \tag{4.14}$$

(c) region D = sphere of radius a,
boundary C = spherical surface (Fig. 19),
boundary value function $= f(a,\ \alpha,\ \beta)=f(\alpha,\ \beta)$,

$$\psi(r_0,\ \alpha_0,\ \beta)=$$

$$=-\frac{a^2-r_0^2}{4\pi a}\int_0^{2\pi}\int_0^{2\pi}\frac{f(\alpha,\ \beta)\,d\alpha\,d\beta}{[a^2+r_0^2-2ar_0\cos(\alpha-\alpha_0)]^{3/2}}= \tag{4.15}$$

= Poisson's formula;

(d) region D = circle of radius a,
boundary C = circumference,

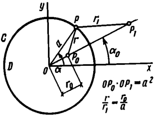

Fig. 19

boundary value function $= f(a, \alpha) = f(\alpha)$,

$$\psi(r_0, \alpha_0) = -\frac{a^2 - r_0^2}{2\pi} \int_0^{2\pi} \frac{f(\alpha)\, d\alpha}{a^2 + r_0^2 - 2ar_0 \cos(\alpha - \alpha_0)}\,. ^{*} \qquad (4.16)$$

2. Neumann problem (second boundary value problem)

Determine a harmonic function ψ (to within an arbitrary constant) in a closed region D from known values of its normal derivative on the boundary C of the region, which must satisfy the condition

$$\oint \frac{d\psi}{d\nu}\, ds = 0. \qquad (4.17)$$

Similarly to the Dirichlet problem, this problem can be solved by the methods indicated above.

In the case of a plane problem use can be made of the existing relations between two conjugate harmonic functions, i.e., between the real part u and the imaginary

* Formulas (4.15) and (4.16) are generalized to the case of Laplace's equation with any number n of variables x_i:

$$\nabla^2\psi = \sum_{i=1}^{n} \frac{\partial^2\psi}{\partial x_i^2} = 0$$

(see the monograph [14], Vol. 2, Chap. IV, Sec. 2).

part v of the same analytic function $w = u + iv$, in the form of the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x. \tag{4.18}$$

If we take an arbitrarily oriented direction ν making an angle α with the x axis, then

$$\frac{du}{d\nu} = u_x \cos\alpha + u_y \sin\alpha,$$

and for a direction ν' perpendicular to it $(\alpha + \pi/2)$ we obtain

$$\frac{dv}{d\nu'} = v_x \cos(\alpha + \pi/2) + v_y \sin(\alpha + \pi/2) =$$

$$= -v_x \sin\alpha + v_y \cos\alpha = u_x \cos\alpha + u_y \sin\alpha,$$

from which

$$\frac{du}{d\nu} = \frac{dv}{d\nu'}. \tag{4.19}$$

Since the tangential direction s and the direction of the inward normal for any closed curve described in the positive direction are related in the same way as the directions ν and ν', it follows that if the unknown function of the Neumann problem whose normal derivative assumes some given values $f^*(s)$ on the boundary is taken to be v, we obtain

$$\frac{du}{ds} = \frac{dv}{dn} = f^*(s),$$

so that the boundary values of u can be calculated, disregarding the non-essential additive constant, by the formula

$$f(s) = \int_0^s f^*(s)\, ds, \tag{4.20}$$

which, by virtue of relation (4.17), gives a uniquely defined function. We next find u by solving the Dirichlet problem with the boundary values (4.20).

After determining u, to pass to v it is only necessary to integrate the total differential

$$dv = -u_y\, dx + u_x\, dy,$$

and this is accomplished by simple quadratures ([12], Chap. IV, Sec. 4.5).

The Neumann problem occurs, for example, in the determination of a stationary temperature field $t(x, y, z)$ in the case of a steady-state thermal process that is characterized by a special form of Eq. (8.23)

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}=0 \tag{4.21}$$

and the corresponding boundary conditions.

We give an example from [15], Problem 131.

Find the stationary temperature t at the interior points of a thin rectangular plane $OACB$ (Fig. 20); heat is uniform-

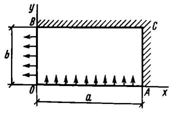

Fig. 20

ly supplied through the side OA and is uniformly removed through the side OB. The other two sides, AC and BC, are covered with thermal insulation.

Denoting by Q the quantity of heat flowing in through the side OA and flowing out through the side OB, and by k the thermal conductivity, and solving Eq. (4.21) with the boundary conditions

$$\left.\frac{\partial t}{\partial x}\right|_{x=0}=\frac{Q}{kb},\quad \left.\frac{\partial t}{\partial y}\right|_{x=a}=0,\quad \left.\frac{\partial t}{\partial y}\right|_{y=0}=$$
$$=-\frac{Q}{ka},\quad \left.\frac{\partial t}{\partial y}\right|_{y=b}=0,$$

we obtain

$$t(x, y)=\frac{Q}{2kab}\left[(y-b)^2-(x-a)^2\right]+C,$$

where C is a constant.

3. Mixed problem (third boundary value problem)

Determine a harmonic function in a closed region D when values of the function are given over a part of the contour of the region, and values of its normal derivative are given over the remainder, or when values of some linear combination

$$A(s)\,\psi(s) + B(s)\,\frac{d\psi}{d\nu} = F(s) \tag{4.22}$$

of the boundary values of the unknown harmonic function ψ and its normal derivative $d\psi/d\nu$ are given.

IV. VARIOUS FORMS OF THE GENERAL SOLUTIONS OF LAMÉ'S EQUATIONS

1. Lamé's equations in vector form

Equations (3.3) in vector form are

$$\nabla^2\left[\boldsymbol{u} + \frac{1}{2(1-2\sigma)}\,\boldsymbol{r}\theta\right] = 0 \tag{4.23}$$

or

$$\nabla^2\boldsymbol{u} + \frac{1}{1-2\sigma}\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} = 0, \tag{4.24}$$

where $\boldsymbol{u}$ = total displacement vector of a point,
$\boldsymbol{r}$ = radius vector of a point drawn from an arbitrarily chosen origin of a rectangular co-ordinate system [16],

$$\mathrm{div}\,\boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}, \quad \mathrm{grad}\,F = \frac{dF}{dr} = \boldsymbol{i}\frac{\partial F}{\partial x} + \boldsymbol{j}\frac{\partial F}{\partial y} + \boldsymbol{k}\frac{\partial F}{\partial z}.$$

2. Galerkin's solution [17, 18]

In vector form:

$$2G\boldsymbol{u} = 2(1-\sigma)\,\nabla^2\boldsymbol{\varphi} - \mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi}. \tag{4.25}$$

For rectangular co-ordinates

$\boldsymbol{\varphi} = \boldsymbol{i}\varphi_1 + \boldsymbol{j}\varphi_2 + \boldsymbol{k}\varphi_3$ = Galerkin's vector,

$$\mathrm{div}\,\boldsymbol{\varphi} = \frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y} + \frac{\partial\varphi_3}{\partial z}, \tag{4.26}$$

φ_i = arbitrary biharmonic functions.

In scalar form:

For rectangular co-ordinates

$$2G(u_x,\ u_y,\ u_z) = 2(1-\sigma)\nabla^2\varphi_{1,\,2,\,3} - \frac{\partial\varphi}{\partial x,\ y,\ z}, \tag{4.25a}$$

where

$$\varphi = \frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y} + \frac{\partial\varphi_3}{\partial z}. \tag{4.26a}$$

The stresses are determined by formulas (3.2a).

For cylindrical co-ordinates

$$\begin{aligned} 2Gu_r &= 2(1-\sigma)(\cos\beta\nabla^2\varphi_1 + \sin\beta\nabla^2\varphi_2) - \frac{\partial\varphi}{\partial r}, \\ 2Gu_\beta &= 2(1-\sigma)(\sin\beta\nabla^2\varphi_1 - \cos\beta\nabla^2\varphi_2) - \frac{1}{r}\frac{\partial\varphi}{\partial\beta}, \\ 2Gu_z &= 2(1-\sigma)\nabla^2\varphi_3 - \frac{\partial\varphi}{\partial z}, \end{aligned} \tag{4.25b}$$

where

$$\begin{aligned} \varphi &= \cos\beta\frac{\partial\varphi_1}{\partial r} - \frac{\sin\beta}{r}\frac{\partial\varphi_1}{\partial\beta} + \sin\beta\frac{\partial\varphi_2}{\partial r} + \\ &+ \frac{\cos\beta}{r}\frac{\partial\varphi_2}{\partial\beta} + \frac{\partial\varphi_3}{\partial z}, \\ \nabla^2(\ldots) &= \frac{\partial^2\ldots}{\partial r^2} + \frac{1}{r}\frac{\partial\ldots}{\partial r} + \frac{1}{r^2}\frac{\partial^2\ldots}{\partial\beta^2} + \frac{\partial^2\ldots}{\partial z^2}. \end{aligned} \tag{4.26b}$$

When $r=$ constant (shells),

$$\nabla^2(\ldots) = \frac{1}{r^2}\frac{\partial^2\ldots}{\partial\beta^2} + \frac{\partial^2\ldots}{\partial z^2};$$

when $r\to\infty$,

$$\nabla^2(\ldots) = \frac{\partial^2\ldots}{\partial r^2} + \frac{\partial^2\ldots}{\partial z^2}.$$

The stresses are determined by formulas (3.2b).

For spherical co-ordinates

$$2Gu_r = 2(1-\sigma)(\cos\beta\sin\alpha\nabla^2\varphi_1 + \sin\beta\sin\alpha\nabla^2\varphi_2 + \cos\alpha\nabla^2\varphi_3) - \frac{\partial\varphi}{\partial r},$$

$$2Gu_\beta = 2(1-\sigma)(\sin\beta\nabla^2\varphi_1 - \cos\beta\nabla^2\varphi_2) - \frac{1}{r\sin\alpha}\frac{\partial\varphi}{\partial\beta}, \qquad (4.25c)$$

$$2Gu_\alpha = 2(1-\sigma)(\cos\beta\cos\alpha\nabla^2\varphi_1 + \sin\beta\cos\alpha\nabla^2\varphi_2 - \sin\alpha\nabla^2\varphi_3) - \frac{1}{r}\frac{\partial\varphi}{\partial\alpha},$$

where

$$\varphi = \cos\beta\sin\alpha\frac{\partial\varphi_1}{\partial r} - \frac{\sin\beta}{r\sin\alpha}\frac{\partial\varphi_1}{\partial\beta} + \frac{\cos\beta\cos\alpha}{r}\frac{\partial\varphi_1}{\partial\alpha} + \sin\beta\sin\alpha\frac{\partial\varphi_2}{\partial r} + \frac{\cos\beta}{r\sin\alpha}\frac{\partial\varphi_2}{\partial\beta} + \cos\alpha\frac{\partial\varphi_3}{\partial r} + \frac{\sin\beta\cos\alpha}{r}\frac{\partial\varphi_3}{\partial\beta} - \frac{\sin\alpha}{r}\frac{\partial\varphi_3}{\partial\alpha}, \qquad (4.26c)$$

$$\nabla^2(\ldots) = \frac{\partial^2\ldots}{\partial r^2} + \frac{2}{r}\frac{\partial\ldots}{\partial r} + \frac{1}{r^2\sin^2\alpha}\frac{\partial^2\ldots}{\partial\beta^2} + \frac{1}{r^2}\frac{\partial^2\ldots}{\partial\alpha^2} + \frac{\cot\alpha}{r^2}\frac{\partial\ldots}{\partial\alpha}.$$

When $r = \text{constant}$,

$$\nabla^2(\ldots) = \frac{1}{r^2}\left(\frac{\partial^2\ldots}{\partial\alpha^2} + \cot\alpha\frac{\partial\ldots}{\partial\alpha} + \frac{1}{\sin^2\alpha}\frac{\partial^2\ldots}{\partial\beta^2}\right).$$

The stresses are determined by formulas (3.2c).

3. Papkovich's [19] and Grodskii's [20] solution

In vector form:

$$2G\boldsymbol{u} = 4(1-\sigma)\boldsymbol{\psi} - \operatorname{grad}(\boldsymbol{r}\boldsymbol{\psi} + \psi_0). \qquad (4.27)$$

For rectangular co-ordinates

$\boldsymbol{\psi} = \boldsymbol{i}\psi_1 + \boldsymbol{j}\psi_2 + \boldsymbol{k}\psi_3$ = harmonic vector,
ψ_0 = harmonic scalar,
ψ_i = arbitrary harmonic functions.

In scalar form:

$$2G(u_x, u_y, u_z) = 4(1-\sigma)\psi_{1,2,3} - \frac{\partial \psi}{\partial x,\ y,\ z}, \tag{4.27a}$$

where $\psi = x\psi_1 + y\psi_2 + z\psi_3$.

The stresses are determined by formulas (3.2a).

4. Neuber's solution [21, 22]

In vector form:

$$2Gu = 4(1-\sigma)\psi - \text{grad } F. \tag{4.28}$$

For rectangular co-ordinates

$\psi = i\psi_1 + j\psi_2 + k\psi_3 =$ harmonic vector,
$\psi_i =$ arbitrary harmonic functions,

$$\nabla^2 F = 2\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right). \tag{4.29}$$

In scalar form:

$$2G(u_x, u_y, u_z) = 4(1-\sigma)\psi_{1,2,3} - \frac{\partial F}{\partial x,\ y,\ z} \tag{4.28a}$$

with formula (4.29).

Solution (4.28) is used in problems of stress concentration.

5. Trefftz's solution [16]

$$2G(u_x, u_y, u_z) = \psi_{1,2,3} + z\frac{\partial \chi}{\partial x,\ y,\ z}, \tag{4.30}$$

where

$$\nabla^2 \psi_i = 0, \quad \frac{\partial \chi}{\partial z} = \frac{1}{4\sigma - 3}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right).$$

Solution (4.30) is used in the analysis of an infinite layer and a punch on an elastic half-space (Problem 9.7).

6. Lamé's solution

$$2G(u_x, u_y, u_z) = \psi_{1,2,3} + \frac{1}{1-2\sigma}\left(\frac{\partial \chi_{3,1,2}}{\partial y,\ z,\ x} - \frac{\partial \chi_{2,3,1}}{\partial z,\ x,\ y}\right), \tag{4.31}$$

where

$$\nabla^2\psi_i = 0, \quad \nabla^2\chi_{1,\,2,\,3} = \frac{\partial\psi_{2,\,3,\,1}}{\partial z,\ x,\ y} - \frac{\partial\psi_{3,\,1,\,2}}{\partial y,\ z,\ x}.$$

PROBLEMS

4.1. Lamé's problem for a cylinder (1852).

A circular cylinder of height h, outer radius b and inner radius a is subjected to an internal uniform pressure p_i and an external uniform pressure p_0.

Determine the stresses and displacements for the following boundary conditions:

(a) the cylinder is supported by an absolutely rigid and smooth plane,

(b) the cylinder is placed between two rigid and smooth planes a constant distance apart.

In the case of the axially symmetric deformation of solids of revolution, as shown by A. E. H. Love [5], Article 188, the stresses and strains are expressed in terms of a single biharmonic function $\varphi_3 = \varphi$ (4.25b)

$$\begin{aligned}
&2Gu_r = -\frac{\partial^2\varphi}{\partial r\,\partial z}, \quad u_\beta = 0,\\
&2Gu_z = \left[2(1-\sigma)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi + \delta,\\
&R_r = \frac{\partial}{\partial z}\left(\sigma\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\varphi, \quad R_\beta = B_r = B_z = Z_\beta = 0,\\
&B_\beta = \frac{\partial}{\partial z}\left(\sigma\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right)\varphi,\\
&Z_z = \frac{\partial}{\partial z}\left[(2-\sigma)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi,\\
&R_z = Z_r = \frac{\partial}{\partial r}\left[(1-\sigma)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi,
\end{aligned} \tag{a}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

δ is an arbitrary constant.

It can be verified by substitution that for a given stress system two equilibrium equations (1.1b) and four compatibility equations (3.4b) are satisfied if $\nabla^2\nabla^2\varphi = 0$.

To find the function φ, it is convenient to assign it as a finite power series in the variable z whose coefficients are unknown functions of r:

$$\varphi(z, r) = \sum_{h=0}^{4} f_h(r) z^h = f_0(r) + f_1(r) z + f_2(r) z^2 +$$
$$+ f_3(r) z^3 + f_4(r) z^4.$$

Substituting this expression in the biharmonic operator

$$\nabla^2\nabla^2\varphi = \left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} +\right.$$
$$\left. + 2\frac{\partial^4}{\partial r^2 \partial z^2} + \frac{2}{r}\frac{\partial^3}{\partial r \partial z^2} + \frac{\partial^4}{\partial z^4} \right) \varphi = 0,$$

and equating to zero the coefficients of like powers of z, we obtain differential equations for the determination of the unknown functions f_i. These equations are of the Euler type and are integrable in elementary functions. The result of the integration is expressed by the formulas

$$f_0(r) = A_0 + B_0 \ln r + C_0 r^2 + D_0 r^2 \ln r +$$
$$+ \frac{1}{4}\left(\frac{9}{4} B_4 + \frac{1}{2} D_2 - \frac{3}{2} A_4 - C_2 \right) r^4 -$$
$$- \frac{1}{8}(3B_4 + 2D_2) r^4 \ln r + \frac{1}{8}\left(C_4 - \frac{5}{6} D_4 \right) r^6 + \frac{D_4}{8} r^6 \ln r,$$
$$f_1(r) = A_1 + B_1 \ln r + C_1 r^2 + D_1 r^2 \ln r +$$
$$+ \frac{3}{4}\left(\frac{D_3}{2} - C_3 \right) r^4 - \frac{3}{4} D_3 r^4 \ln r,$$
$$f_2(r) = A_2 + B_2 \ln r + C_2 r^2 + D_2 r^2 \ln r +$$
$$+ \frac{3}{2}\left(\frac{D_4}{2} - C_4 \right) r^4 - \frac{3}{2} D_4 r^4 \ln r,$$
$$f_3(r) = A_3 + B_3 \ln r + C_3 r^2 + D_3 r^2 \ln r,$$
$$f_4(r) = A_4 + B_4 \ln r + C_4 r^2 + D_4 r^2 \ln r.$$

For the boundary conditions (a) we have

when $r = a$, $R_r = -p_i$, $Z_r = 0$;
when $r = b$, $R_r = -p_0$, $Z_r = 0$;
when $z = 0$, $Z_z = R_z = 0$;
when $z = h$, $u_z = 0$.

The analysis of the function φ and of the boundary conditions allows the following conclusions to be drawn:

(1) the function φ depends on the odd functions f_n. The even functions characterize a pressure proportional to the first power of z;

(2) for $Z_r = 0$ when $r = a$ and $r = b$, the constants D_1, B_3, C_3, and D_3 must be zero;

(3) the constant A_1 does not affect the state of stress and strain.

Taking into account the conditions listed above, we obtain

$$\varphi(z, r) = (B_1 \ln r + C_1 r^2) z + A_3 z^3. \qquad \text{(b)}$$

By setting up expressions for stresses and displacements, and satisfying the boundary conditions, we find

$$B_1 = \frac{p_0 - p_i}{b^2 - a^2} a^2 b^2, \quad C_1 = \frac{(1-\sigma)(p_0 b^2 - p_i a^2)}{2(1+\sigma)(b^2 - a^2)},$$

$$A_3 = -\frac{(2-\sigma)(p_0 b^2 - p_i a^2)}{3(1+\sigma)(b^2 - a^2)}, \quad \delta = -\frac{\sigma (p_0 b^2 - p_i a^2) h}{G(1+\sigma)(b^2 - a^2)}.$$

The displacements and stresses are

$$u_r = -\frac{1}{2G(b^2 - a^2)} \left[\frac{(p_0 - p_i) a^2 b^2}{r} + \frac{(1-\sigma)(p_0 b^2 - p_i a^2)}{1+\sigma} r \right],$$

$$u_z = \frac{\sigma (p_0 b^2 - p_i a^2)}{G(1+\sigma)(b^2 - a^2)} (z - h),$$

$$R_r = \frac{a^2 b^2}{b^2 - a^2} \frac{p_0 - p_i}{r^2} - \frac{p_0 b^2 - p_i a^2}{b^2 - a^2},$$

$$B_\beta = -\frac{a^2 b^2}{b^2 - a^2} \frac{p_0 - p_i}{r^2} - \frac{p_0 b^2 - p_i a^2}{b^2 - a^2},$$

$$Z_z = R_z = 0.$$

For the boundary conditions (b) it is convenient to take the height of the cylinder to be $2h$ and the function φ

according to the equation (b). The values obtained must coincide with the results of the solution of Problem 6.1 when $u_z = Z_r = 0$ and $Z_z = \sigma (R_r + B_\beta)$.

4.2. See [19].

Determine the state of stress in a rectangular plate bounded by planes $z = \pm h/2$, $x = 0$, $y = 0$, $x = a$, and $y = b$, simply supported along the edge, and subjected to a uniformly distributed normal load p (Fig. 21).

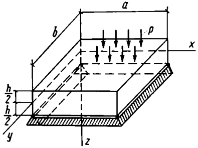

Fig. 21

For the solution of the problem, we take, according to [24], one biharmonic function, $\varphi_3 = \varphi$.

In this case, the displacements and stresses are, by (4.25a),

$$\begin{aligned}
&2Gu_x = -\frac{\partial^2\varphi}{\partial x\,\partial z}, \quad 2Gu_y = -\frac{\partial^2\varphi}{\partial y\,\partial z},\\
&2Gu_z = \left[2(1-\sigma)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi,\\
&X_x = \frac{\partial}{\partial z}\left(\sigma\nabla^2 - \frac{\partial^2}{\partial x^2}\right)\varphi, \quad X_y = -\frac{\partial^3\varphi}{\partial x\,\partial y\,\partial z},\\
&Y_y = \frac{\partial}{\partial z}\left(\sigma\nabla^2 - \frac{\partial^2}{\partial y^2}\right)\varphi,\\
&Y_z = \frac{\partial}{\partial y}\left[(1-\sigma)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi,\\
&Z_z = \frac{\partial}{\partial z}\left[(2-\sigma)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi,\\
&Z_x = \frac{\partial}{\partial x}\left[(1-\sigma)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varphi,
\end{aligned} \qquad \text{(a)}$$

where

$$\nabla^2(\ldots)=\frac{\partial^2\ldots}{\partial x^2}+\frac{\partial^2\ldots}{\partial y^2}+\frac{\partial^2\ldots}{\partial z^2}.$$

The boundary conditions of the problem are expressed as follows:

$$\begin{aligned}
&\text{when } z=-h/2, && Z_z=-p,\quad X_z=Y_z=0;\\
&\text{when } z=h/2, && Z_z=Y_z=X_z=0; \qquad \text{(b)}\\
&\text{when } x=0 \text{ and } x=a, && X_x=u_z=0;\\
&\text{when } y=0 \text{ and } y=b, && Y_y=u_z=0.
\end{aligned}$$

The biharmonic function satisfying the boundary conditions (b) for x and y is, by formulas (4.2a") and (4.4),

$$\varphi=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\sinh\left(\pi z\sqrt{m^2/a^2+n^2/b^2}\right)+\right.$$

$$+B_{mn}\cosh\left(\pi z\sqrt{m^2/a^2+n^2/b^2}\right)+$$

$$+C_{mn}z\cosh\left(\pi z\sqrt{m^2/a^2+n^2/b^2}\right)+$$

$$\left.+D_{mn}z\sinh\left(\pi z\sqrt{m^2/a^2+n^2/b^2}\right)\right]\sin m\pi x/a\sin n\pi y/b.$$

The distributed load p is also expanded in a double series according to the formula

$$p=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin m\pi x/a\sin n\pi y/b,$$

where

$$a_{mn}=\frac{4}{ab}\int_0^a\int_0^b p\sin m\pi x/a\sin n\pi y/b\,dx\,dy.$$

To determine the arbitrary constants A_{mn}, B_{mn}, C_{mn}, D_{mn}, we use four boundary conditions (b) for z, i.e., when $z=-h/2$, $Z_z=-p$, X_z or $Y_z=0$; when $z=h/2$, $Z_z=0$, X_z or $Y_z=0$ [see formulas (a)].

The investigations of the work of thick plates [24] have led to the following conclusion: the engineering theory of the analysis of thin plates [5] based on the assumptions that $e_{zz} = e_{yz} = e_{xz} = 0$ and $Z_z = 0$ (hence X_x, Y_y, and $X_y = Y_x$ vary according to a linear law) allows the solution to be simplified for plates with a ratio of the smaller size, a or b, to the thickness h equal to or greater than three, giving results closely approximating the exact values.

4.3. Boussinesq's problem (1885) [5].

The elastic half-space $z \geqslant 0$ is acted on by a force P normal to the boundary plane $z = 0$ (Fig. 22*a*).

Determine the displacements and stresses.

The problem is considered as an axially symmetric one in cylindrical co-ordinates with a logarithmic singula-

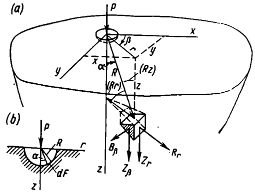

Fig. 22

rity at the origin, i.e., at the point of application of the force. The formulas to be used for the determination of displacements and stresses are the formulas (a) of Problem 4.1.

If the stresses are prescribed in accordance with the formulas (a) of Problem 4.1, they satisfy the equilibrium equations (1.1b)

$$\frac{\partial R_r}{\partial r}+\frac{\partial R_z}{\partial z}+\frac{R_r-B_\beta}{r}=0,\quad \frac{\partial Z_r}{\partial r}+\frac{\partial Z_z}{\partial z}+\frac{Z_r}{r}=0$$

and the Beltrami-Michell relations (3.4b)

$$\nabla^2 R_r-\frac{2}{r^2}(R_r-B_\beta)+\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial r^2}=0,$$

$$\nabla^2 B_\beta+\frac{2}{r^2}(R_r-B_\beta)+\frac{1}{1+\sigma}\frac{1}{r}\frac{\partial\Theta}{\partial r}=0,$$

$$\nabla^2 Z_z+\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2}=0,$$

$$\nabla^2 Z_r-\frac{Z_r}{r^2}+\frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial r\,\partial z}=0$$

if the function φ is biharmonic [see (4.1b)], i.e.,

$$\nabla^2\nabla^2\varphi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial z^2}\right)=0.$$

The problem is solved by introducing the biharmonic function

$$\varphi=\varphi_3=\frac{P}{2\pi}\{R+(1-2\sigma)[z\ln(R+z)-R]\},$$

where

$$R=\sqrt{r^2+z^2}=\sqrt{x^2+y^2+z^2}.$$

Taking into account that

$$\frac{\partial R}{\partial r}=\frac{r}{R},\quad \frac{\partial R}{\partial z}=\frac{z}{R},\quad R-z=\frac{r^2}{R+z},$$

$$\nabla^2 R=\frac{2}{R},\quad \nabla^2[z\ln(R+z)-R]=0,$$

we obtain

$$u_r=-\frac{P}{4\pi G}\frac{r}{R^2}\left[\frac{(1-2\sigma)R}{R+z}-\frac{z}{R}\right],$$

$$u_z=\frac{P}{4\pi GR}\left[2(1-\sigma)+\frac{z^2}{R^2}\right],$$

$$R_r=\frac{P}{2\pi R^2}\left[\frac{(1-2\sigma)R}{R-z}-\frac{3r^2z}{R^3}\right],$$

$$B_\beta = -\frac{(1-2\sigma)P}{3\pi R^2}\left(\frac{R}{R+z}-\frac{z}{R}\right),$$

$$Z_z = -\frac{3P}{2\pi R^2}\frac{z^3}{R^3},$$

$$R_z = Z_r = -\frac{3P}{2\pi R^2}\frac{rz^2}{R^3},$$

$$R_\beta = B_r = B_z = Z_\beta = u_\beta = 0.$$

As $R \to \infty$, all displacements and stresses tend to zero.

When $z = 0$ and $R = r$, we obtain the displacements of the boundary plane

$$u_r = -\frac{(1-2\sigma)P}{4\pi Gr} = -\frac{(1+\sigma)(1-2\sigma)P}{2\pi Er},$$

$$u_z = \frac{(1-\sigma)P}{2\pi Gr} = \frac{(1-\sigma^2)P}{\pi Er}.$$

The boundary plane $z = 0$ is free from stresses ($Z_z = R_z = 0$), with the exception of the point $R = 0$ where the stresses increase indefinitely. The latter is due to the presence of the concentrated force P applied at the origin and can be shown by the following calculations.

We cut out from the half-space at the origin a hemisphere of radius R, when R is small (Fig. 22b), loaded by forces in the form of stresses on inclined planes.

The projection of the resultant vector of these forces on the z axis is different from zero and is equal to

$$Z = \iint_F [Z_r \cos(Rr) + Z_z \cos(Rz)]\, dF = \cos(Rr) =$$

$$= -\sin\alpha = -r/R,$$

$$\cos(Rz) = -\cos\alpha = -z/R,$$

$$dF = r\, d\beta\, R\, d\alpha = \frac{3P}{2\pi}\int_0^{2\pi} d\beta \int_0^{\pi/2}\left(\frac{rz^2}{R^5}\sin\alpha + \frac{z^3}{R^5}\cos\alpha\right) rR\, d\alpha =$$

$$= 3P\int_0^{\pi/2}\sin\alpha\cos^2\alpha\, d\alpha = P.$$

We give the values of the displacements and stresses in rectangular co-ordinates x, y, z (see Fig. 22*a*):

$$u_x = \frac{P}{4\pi}\left[\frac{zx}{GR^3} - \frac{x}{(\lambda+G)R(z+R)}\right],$$

$$u_y = \frac{P}{4\pi}\left[\frac{zy}{GR^3} - \frac{y}{(\lambda+G)R(z+R)}\right],$$

$$u_z = \frac{P}{4\pi}\left[\frac{z^2}{Gr^3} + \frac{\lambda+2\mu}{G(\lambda+G)R}\right],$$

$$X_x = -\frac{P}{2\pi}\left\{\left[\frac{3x^2z}{R^5} - \frac{Gz}{(\lambda+G)R^3}\right] - \right.$$

$$\left. - \frac{G}{\lambda+G}\left[\frac{y^2+z^2}{R^3(R+z)} - \frac{x^2}{R^2(R+z)^2}\right]\right\},$$

$$Y_y = -\frac{P}{2\pi}\left\{\left[\frac{3y^2z}{R^5} - \frac{Gz}{(\lambda+G)R^3}\right] - \right.$$

$$\left. - \frac{G}{\lambda+G}\left[\frac{x^2+z^2}{R^2(R+z)} - \frac{y^2}{R^2(R+z)^2}\right]\right\},$$

$$Z_z = -\frac{P}{2\pi}\left\{\left[\frac{3z^3}{R^5} + \frac{Gz}{(\lambda+G)R^3}\right] - \frac{G}{\lambda+G}\frac{z}{R^3}\right\},$$

$$X_y = -\frac{P}{2\pi}\left[\frac{3xyz}{R^5} - \frac{Gxy(z+2R)}{(\lambda+G)R^3(R+z)^2}\right],$$

$$Y_z = -\frac{P}{2\pi}\left\{\left[\frac{3yz^2}{R^5} + \frac{Gy}{(\lambda+G)R^3}\right] - \frac{G}{\lambda+G}\frac{y}{R^3}\right\},$$

$$Z_x = -\frac{P}{2\pi}\left\{\left[\frac{3xz^2}{R^5} + \frac{Gx}{(\lambda+G)R^3}\right] - \frac{G}{\lambda+G}\frac{x}{R^3}\right\}.$$

4.4. Kelvin's problem (1848) [5].

A force P is applied at a point of an infinitely large body along the z axis (Fig. 23).

Determine the displacements and stresses.

The origin of co-ordinates is taken at the point of application of the force and is enclosed by a small spherical region. The problem is solved in rectangular co-ordinates by assigning the biharmonic function

$$\varphi = \varphi_3 = AR, \qquad \text{(a)}$$

where

$R = \sqrt{x^2 + y^2 + z^2}$,

A is an arbitrary constant.

According to the formulas (a) of Problem 4.2, we have

$$u_x = \frac{A}{2G}\frac{xz}{R^3}, \quad u_y = \frac{A}{2G}\frac{yz}{R^3},$$

$$u_z = \frac{A}{2GR}\left[\frac{z^2}{R^2} + (3-4\sigma)\right],$$

$$X_x = \frac{Az}{R^3}\left[(1-2\sigma) - \frac{3x^2}{R^2}\right],$$

$$Y_y = \frac{Az}{R^3}\left[(1-2\sigma) - \frac{3y^2}{R^2}\right], \qquad \text{(b)}$$

$$Z_z = -\frac{Az}{R^3}\left[(1-2\sigma) + \frac{3z^2}{R^2}\right],$$

$$X_y = -\frac{3Axyz}{R^5}, \quad Y_z = -\frac{Ay}{R^3}\left[(1-2\sigma) + \frac{3z^2}{R^2}\right],$$

$$Z_x = -\frac{Ax}{R^3}\left[(1-2\sigma) + \frac{3z^2}{R^2}\right].$$

To determine the arbitrary constant A, we form the projection on the z axis of all forces situated on the surface

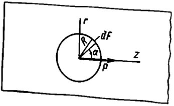

Fig. 23

of the small sphere (Fig. 23)

$$Z = \iint_F [Z_z \cos(R, z) + Z_y \cos(R, y) + Z_x \cos(R, x)]\, dF =$$

$$= |\cos(R, x) = -x/R, \quad \cos(R, y) = -y/R,$$

$$\cos(R, z) = -\cos\alpha = -z/R, \quad dF = r\, d\beta\, R\, d\alpha| =$$

$$= A\int_0^{2\pi} d\beta \int_0^{\pi} [(z^2+y^2+x^2)/R^4]\,[(1-2\sigma)+3z^2/R^2]\,rR\,d\alpha =$$

$$= 2\pi A\int_0^{\pi} \sin\alpha\,[(1-2\sigma)+3\cos^2\alpha]\,d\alpha = 8\pi(1-\sigma)A;$$

hence,

$$A = P/8\pi(1-\sigma).$$

The projections of the forces on the x and y axes are zero.

We obtain, finally,

$$u_x = \frac{(1+\sigma)P}{8\pi E(1-\sigma)}\,\frac{xz}{R^3} = \frac{(\lambda+G)P}{8\pi G(\lambda+2G)}\,\frac{\partial^2 R}{\partial z\,\partial x},$$

$$u_y = \frac{(1+\sigma)P}{8\pi E(1-\sigma)}\,\frac{yz}{R^3} = \frac{(\lambda+G)P}{8\pi G(\lambda+2G)}\,\frac{\partial^2 R}{\partial z\,\partial y},$$

$$u_z = \frac{(1+\sigma)P}{8\pi E(1-\sigma)}\,\frac{1}{R}\left[\frac{z^2}{R^2}+(3-4\sigma)\right] =$$

$$= \frac{(\lambda+G)P}{8\pi G(\lambda+2G)}\,\frac{\partial^2 R}{\partial z^2} + \frac{R}{4\pi GR}.$$

The stresses are determined by the formulas (b)

$$X_x = \frac{Pz}{8\pi(1-\sigma)R^3}\left[(1-2\sigma)-\frac{3x^2}{R^2}\right],$$

$$Y_y = \frac{Pz}{8\pi(1-\sigma)R^3}\left[(1-2\sigma)-\frac{3y^2}{R^2}\right],$$

$$X_y = -\frac{3Pxyz}{8\pi(1-\sigma)R^5},$$

$$Z_z = -\frac{Pz}{8\pi(1-\sigma)R^3}\left[(1-2\sigma)+\frac{3z^2}{R^2}\right],$$

$$Y_z = -\frac{Py}{8\pi(1-\sigma)R^3}\left[(1-2\sigma)+\frac{3z^2}{R^2}\right],$$

$$Z_x = -\frac{Px}{8\pi(1-\sigma)R^3}\left[(1-2\sigma)+\frac{3z^2}{R^2}\right].$$

4.5. Cerruti's problem (1882) [5].

The elastic half-space $z \geqslant 0$ is acted on by a force T tangential to the boundary plane $z = 0$ (Fig. 24).

Determine the displacements and stresses.

Assuming that the force T is applied at the origin along the x axis, we solve the problem by introducing

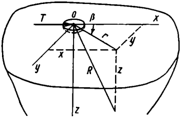

Fig. 24

two biharmonic functions (4.25a)

$$\varphi_1 = \frac{T}{4\pi(1-\sigma)} \{R + 2(1-\sigma)(1-2\sigma)[z \ln(R+z) - R]\},$$

$$\varphi_3 = \frac{T}{4\pi(1-\sigma)} (1-2\sigma) x \ln(R+z).$$

Further, according to formulas (4.25a) we obtain

$$u_x = \frac{T}{4\pi G}\left(\frac{\lambda+3G}{\lambda+G} + \frac{x^2}{R^2}\right)\frac{1}{R} - \frac{T}{2\pi(\lambda+G)R} +$$

$$+ \frac{T}{4\pi(\lambda+G)}\left[1 - \frac{x^2}{R(R+z)}\right]\frac{1}{R+z},$$

$$u_y = \frac{T}{4\pi}\frac{xy}{R}\left[\frac{1}{GR^2} - \frac{1}{(\lambda+G)(R+z)^2}\right],$$

$$u_z = \frac{T}{4\pi}\frac{x}{R}\left[\frac{z}{GR^2} + \frac{1}{(\lambda+G)(R+z)}\right];$$

at the point O

$$\iint X_z \, dx \, dy = T.$$

By formulas (3.2a), the stresses are

$$X_x = \frac{T}{2\pi}\left\{\frac{3x^3}{R^5} + \frac{Gx}{\lambda+G}\left[\frac{3}{R(R+z)^2} - \frac{x^2(3R+z)}{R^3(R+z)^3} - \frac{1}{R^3}\right]\right\},$$

$$Y_y = \frac{T}{2\pi}\left\{\frac{3xy^2}{R^5} + \frac{Gx}{\lambda+G}\left[\frac{1}{R(R+z)^2} - \frac{y^2(3R+z)}{R^3(R+z)^3} - \frac{1}{R^3}\right]\right\},$$

$$X_y = \frac{T}{2\pi}\left\{\frac{3x^2y}{R^5} + \frac{Gy}{\lambda+G}\left[\frac{1}{R(R+z)^2} - \frac{x^2(3R+z)}{R^3(R+z)^3}\right]\right\},$$

$$Z_z = \frac{3T}{2\pi}\frac{xz^2}{R^5}, \quad Y_z = \frac{3T}{2\pi}\frac{xyz}{R^5}, \quad Z_x = \frac{3T}{2\pi}\frac{x^2z}{R^5}.$$

4.6. Mindlin's problem (1936) [25].

Determine the state of stress due to a force P applied inside the elastic half-space $z \geqslant 0$ normal to the boundary plane $z = 0$ (Fig. 25).

Since the state of stress is symmetrical about the z axis, the stresses and displacements are determined from

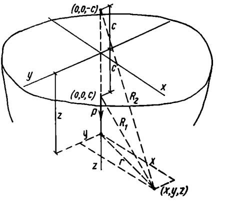

Fig. 25

the formulas (a) of Problem 4.1.

In choosing the biharmonic function φ R. D. Mindlin proceeded as follows:

(1) he applied Kelvin's solution (Problem 4.4) to the force applied at the point (0, 0, c):

$$A_1R_1 = A_1\sqrt{r^2 + (z - c)^2};$$

(2) to eliminate the stresses Z_z and R_z on the plane $z = 0$ obtained from Kelvin's solution and to satisfy the condition

$$\int_0^\infty 2\pi r Z_z\, dr = -P \quad \text{for } z > c$$

he added five strain nuclei for the point (0, 0, $-c$):

(a) force in the z direction

$$AR_2 = A\sqrt{r^2 + (z + c)^2};$$

(b) couple in the z direction

$$B(z + c)/R_2;$$

(c) centre of compression in the z direction

$$C \ln (R_2 + z + c);$$

(d) line of compression along the z axis from $z + c = 0$ to $z + c = -\infty$

$$D[(z + c)\ln (R_2 + z + c) - R_2];$$

(e) dipole (double centre of compression) with an axis parallel to the z axis

$$F/R_2;$$

thus, the biharmonic function was obtained as

$$\varphi = A_1R_1 + AR_2 + \frac{B(z+c)}{R_2} + C\ln(R_2 + z + c) + D[(z+c)\ln(R_2 + z + c) - R_2] + F/R_2. \qquad \text{(a)}$$

(3) After determining the stresses Z_z and R_z from the formulas of Problem 4.1, and satisfying three conditions of step (2), we obtain the values of all constants

$A_1 = P/8\pi(1 - \sigma)$ (Problem 4.4), $A = (3 - 4\sigma)A_1$,

$B = -2cA_1$, $C = -4(1 - 2\sigma)cA_1$,

$D = 4(1 - \sigma)(1 - 2\sigma)A_1$, $F = 2c^2A_1$.

Substituting the values obtained in the formula (a), we can find the final value of the function φ

$$\varphi=\frac{P}{8\pi(1-\sigma)}\left\{R_1+[8\sigma(1-\sigma)-1]R_2-\frac{2cz}{R_2}+\right.$$
$$\left.+4(1-2\sigma)[(1-\sigma)z-\sigma c]\ln(R_2+z+c)\right\}. \qquad \text{(b)}$$

If $c\to\infty$, all terms containing R_2 vanish, giving the solution of Kelvin's problem 4.4

$$\varphi=\frac{PR_1}{8\pi(1-\sigma)}.$$

If $c\to 0$, the solution of Boussinesq's problem 4.3 is obtained

$$\varphi=\frac{P}{2\pi}[2\sigma R-(1-2\sigma)z\ln(R+z)].$$

By using the stress function (b), the stresses are determined from the formulas (a) of Problem 4.1.

By a similar method [25] R. D. Mindlin solved the problem of a concentrated force acting inside an elastic half-space at a depth c parallel to the boundary plane $z=0$, of which Kelvin's problem ($c=\infty$) and Cerruti's problem ($c=0$) are special cases.

4.7. Michell's problem (1900) [5].

Find the stress distribution in an infinite cone $\alpha=\alpha_1$ with a force S acting at the vertex perpendicular to the axis of the cone (Fig. 26).

It is necessary to find solutions of Eqs. (3.3c) for which the displacements are inversely proportional to the radius r.

Assume u_r and u_α to be proportional to $\cos n\beta$, and u_β to $\sin n\beta$.

In this case the displacements are

$$u_r=\frac{\cos n\beta}{r}\left(-\frac{\lambda+2G}{G}\frac{r^2\theta}{\cos n\beta}+C\tan^n\frac{\alpha}{2}+D\cot^n\frac{\alpha}{2}\right),$$

$$u_\alpha=\frac{\cos n\beta}{r\sin\alpha}\left[-\frac{\lambda+3G}{2G}\sin\alpha\frac{d}{d\alpha}\left(\frac{r^2\theta}{\cos n\beta}\right)+\right.$$
$$\left.+\left(C\tan^n\frac{\alpha}{2}+D\cot^n\frac{\alpha}{2}\right)\cos\alpha+F\tan^n\frac{\alpha}{2}+H\cot^n\frac{\alpha}{2}\right],$$

$$u_\beta = \frac{\sin n\beta}{r\sin\alpha}\left[n\,\frac{\lambda+3G}{2G}\,\frac{r^2\theta}{\cos n\beta} - \right.$$
$$\left. - \left(C\tan^n\frac{\alpha}{2} - D\cot^n\frac{\alpha}{2}\right)\cos\alpha - F\tan^n\frac{\alpha}{2} + H\cot^n\frac{\alpha}{2}\right],$$

where

$$\theta = \frac{\cos n\beta}{r^2}\left[A(n+\cos\alpha)\tan^n\frac{\alpha}{2} + B(n-\cos\alpha)\cot^n\frac{\alpha}{2}\right],$$

A, B, C, D, F, H are arbitrary constants.

When $n = 0$ or $n = 1$, the solutions require special

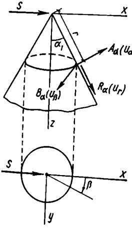

Fig. 26

investigation. To solve the problem, three types of solution are combined

(1) $$u_r = \frac{F}{4\pi G}\,\frac{\sin\alpha\cos\beta}{r},$$

$$u_\alpha = \frac{\lambda+3G}{2(\lambda+2G)}\,\frac{F}{4\pi G}\,\frac{\cos\alpha\cos\beta}{r},$$

$$u_\beta = -\frac{\lambda+3G}{2(\lambda+2G)}\,\frac{F}{4\pi G}\,\frac{\sin\beta}{r}$$

$$R_r = -\frac{3\lambda+4G}{\lambda+2G}\frac{F}{4\pi G}\frac{\sin\alpha\cos\beta}{r^2},$$

$$A_\alpha = B_\beta = \frac{G}{\lambda+2G}\frac{F}{4\pi}\frac{\sin\alpha\cos\beta}{r^2}, \quad A_\beta = 0,$$

$$B_r = \frac{G}{\lambda+2G}\frac{F}{4\pi}\frac{\sin\beta}{r^2},$$

$$A_r = -\frac{G}{\lambda+2G}\frac{F}{4\pi}\frac{\cos\alpha\cos\beta}{r^2}.$$

(2)
$$u_r = 0, \quad u_\alpha = -\frac{C}{r}\frac{\cos\beta}{1+\cos\alpha},$$

$$u_\beta = \frac{C}{r}\frac{\sin\beta}{1+\cos\alpha},$$

$$R_r = 0, \quad A_\alpha = -B_\beta = -\frac{2GC}{r^2}\frac{(1-\cos\alpha)\cos\beta}{(1+\cos\alpha)\sin\alpha},$$

$$A_r = \frac{2GC}{r^2}\frac{(1-\cos\alpha)\sin\beta}{(1+\cos\alpha)\sin\alpha}, \quad B_r = -\frac{2GC}{r^2}\frac{\sin\alpha}{1+\cos\alpha},$$

$$R_\alpha = 2G\frac{C}{r^2}\frac{\cos\beta}{1+\cos\alpha}.$$

(3)
$$u_r = \frac{D}{r}\frac{\sin\alpha\cos\beta}{1+\cos\alpha}, \quad u_\alpha = \frac{D}{r}\cos\beta, \quad u_\beta = -\frac{D}{r}\sin\beta,$$

$$R_r = -A_\alpha = -2G\frac{D}{r^2}\frac{\sin\alpha\cos\beta}{1+\cos\alpha}, \quad B_\beta = 0,$$

$$A_\beta = -G\frac{D}{r^2}\frac{\sin\alpha\sin\beta}{1+\cos\alpha},$$

$$B_r = G\frac{D}{r^2}\left(2-\frac{1}{1+\cos\alpha}\right)\sin\beta,$$

$$R_\alpha = -G\frac{D}{r^2}\left(2-\frac{1}{1+\cos\alpha}\right)\cos\beta.$$

The boundary conditions of the problem are: when $\alpha = \alpha_1$, $A_\alpha = R_\alpha = B_\alpha = 0$, giving three homogeneous equations in C, D, and F. From the resulting equations it follows that

$$C = -\frac{(1+\cos\alpha_1)^2}{8\pi(\lambda+2G)}F, \quad D = -\frac{1+\cos\alpha_1}{4\pi(\lambda+2G)}F.$$

By taking the sum of the projections on the x axis of the force S and of the stresses on a spherical surface centred at the vertex of the cone, we obtain

$$S=\frac{P}{4}\,\frac{(2+\cos\alpha_1)\,\lambda+2G}{\lambda+2G}\,(1-\cos\alpha_1)^2.$$

4.8. See the monograph [6], Chap. VI, Sec. 8.
Investigate the state of stress in a heavy sphere of radius a resting on a rigid horizontal foundation.

The acting forces are: the specific weight of material γ and the reaction of the foundation

$P = 4\pi a^3\gamma/3$

applied at the lower pole of the sphere ($\alpha = \pi$).
According to Problem 3.10, the surface load can be represented as

$$\sigma_n=(-1)^n\,\frac{2n+1}{2}\,\frac{P}{2\pi a^2}=(-1)^{n-1}\,\frac{2n+1}{3}\,\gamma a;$$

in particular,

$$\sigma_0=-\frac{1}{3}\,\gamma a,\quad \sigma_1=\gamma a. \tag{a}$$

For the specific weight with a potential Π, Eq. (4.8) is of the form

$$\nabla^2\boldsymbol{u}+\frac{1}{1-2\sigma}\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}=-\frac{\boldsymbol{P}}{G}=\frac{1}{G}\,\mathrm{grad}\,\Pi, \tag{b}$$

and it can be satisfied by introducing the displacement potential

$$\boldsymbol{u}=\frac{1}{G}\,\mathrm{grad}\,\chi, \tag{c}$$

which is determined, as the substitution of the expression (c) in the equation (b) shows, from the equation

$$\nabla^2\chi=\frac{1-2\sigma}{2(1-\sigma)}\,\Pi. \tag{d}$$

The specific weight potential is

$$\Pi=\gamma z=\gamma r\cos\alpha=\gamma r P_1(\mu), \tag{e}$$

i.e., it is a harmonic function.
Assuming in the general case

$$\Pi=\sum\Pi_n=\sum E_n r^n P_n(\mu),$$

and seeking the solution of the equation (d) in the form

$$\chi = \sum \chi_n = \sum F_n r^2 \Pi_n, \tag{f}$$

where F_n are unknown constants, we obtain

$$\chi_n = \frac{(1-2\sigma) E_n}{4(1-\sigma)(2n+3)} r^{n+2} P_n(\mu), \tag{g}$$

and according to (c),

$$u_r = \frac{1}{G}\frac{\partial \chi_n}{\partial r}, \quad u_\alpha = \frac{1}{G}\frac{\partial \chi_n}{\partial \alpha}. \tag{h}$$

In our case, taking into account the formula (e), we find:

displacements

$$u_r = \frac{3(1-2\sigma)\gamma}{20(1-\sigma)G} r^2 \cos\alpha, \quad u_\alpha = -\frac{(1-2\sigma)\gamma}{20(1-\sigma)G} r^2 \sin\alpha;$$

stresses

$$R_r = \frac{3-\sigma}{5(1-\sigma)} \gamma r \cos\alpha, \quad A_r = -\frac{(1-2\sigma)\gamma}{5(1-\sigma)} r \sin\alpha,$$
$$A_\alpha = B_\beta = \frac{1+3\sigma}{5(1-\sigma)} \gamma r \cos\alpha. \tag{i}$$

The surface of the sphere $r = a$ must be free from stresses everywhere except for the pole ($\alpha = \pi$). The particular solution (i) of the equation must be added to its general solution (c) of Problem 3.10, summing over all positive integers from $n = 0$ to $n = \infty$:

$$(n+1)(2Ga_n + K\lambda A_n) a^n + 2G_n (n-1) a^{n-2} C_n =$$
$$= (-1)^{n-1} \frac{2n+1}{3} \gamma a; \tag{j}$$
$$G[(a_n + n\bar{A}_n) + 2(n-1) a^{-2} C_n] = 0.$$

When $n = 1$ (particular solution), the right-hand sides of the equations (j) are respectively equal to [(a) + (i)]

$$\bar{\sigma}_1 = \gamma a - \frac{3-\sigma}{5(1-\sigma)} \gamma a = \frac{2(1-2\sigma)}{5(1-\sigma)} \gamma a,$$
$$\tau_1 = -\frac{1-2\sigma}{5(1-\sigma)} \gamma a \quad \left(\frac{dP_1}{d\alpha} = -\sin\alpha\right).$$

After determining the constants A_n and C_n, the stresses are found from the formulas (c) of Problem 3.10. For the technique of calculating these series, see the monograph [6], Chap. VI, Sec. 8.

4.9. See the monograph [6], Chap. VI, Sec. 5.
Investigate the state of stress in a heavy sphere of radius a rotating about the z axis with a constant angular velocity ω.

The rotation of the sphere about the z axis involves centrifugal forces having the potential

$$\Pi = -\frac{\gamma\omega^2}{2g}(x^2+y^2).$$

Transforming to spherical co-ordinates by means of the formulas

$$x = r\sin\alpha\cos\beta,\quad y = r\sin\alpha\sin\beta,$$

and taking into account that

$$P_2(\cos\alpha) = P_2(\mu) = (3\mu^2-1)/2$$

(see Chap. 4), we obtain

$$\Pi = -\frac{\gamma\omega^2}{3g}r^2+\frac{\gamma\omega^2}{3g}r^2P_2(\mu). \tag{a}$$

The first term corresponds to a potential depending only on r:

$$\Pi = \Pi(r),$$

and in this case the equation (d) of Problem 4.8 is

$$\nabla_r^2\chi = \frac{1}{r^2}\left[\frac{d}{dr}\left(r^2\frac{d\chi}{dr}\right)\right] = \frac{1-2\sigma}{2(1-\sigma)}\Pi(r),$$

from which we find, by direct integration,

$$\chi(r) = \frac{1-2\sigma}{2(1-\sigma)}\int_0^r\frac{dr}{r^2}\int_0^r r^2\Pi(r)\,dr,$$

and hence,

$$u_r = \frac{1}{G}\frac{d\chi}{dr} = \frac{1-2\sigma}{2G(1-\sigma)r^2}\int_0^r r^2\Pi(r)\,dr,$$

$$u_\alpha = 0;$$

$$R_r = \lambda\theta + 2G\frac{du_r}{dr} = \Pi(r) - \frac{2(1-2\sigma)}{(1-\sigma)r^3}\int_0^r r^2\Pi(r)\,dr,$$

$$A_\alpha = B_\beta = \lambda\theta + 2G\frac{u_r}{r} = \frac{\sigma}{1-\sigma}\Pi(r) +$$

$$+\frac{1-2\sigma}{(1-\sigma)r^3}\int_0^r r^2\Pi(r)\,dr, \tag{b}$$

$$A_r = 0.$$

For the surface of the sphere ($r = a$) to be free from stresses, to the solution (b) must be added the general solution of Eqs. (4.8) in the form [Problem 3.9, formulas (j)]

$$u_r = 2A_0(1-2\sigma)r, \quad u_\alpha = 0,$$
$$R_r = A_\alpha = B_\beta = 4G(1+\sigma)A_0, \quad A_r = 0 \tag{c}$$

with the value of A_0 determined from the relation

$$\Pi(a) - \frac{2(1-2\sigma)}{(1-\sigma)a^3}\int_0^a r^2\Pi(r)\,dr + 4G(1+\sigma)A_0 = 0$$

according to the first equation of (b).
We obtain

$$A_0 = \frac{1}{4(1+\sigma)G}\left[-\Pi(a) + \frac{2(1-2\sigma)}{(1-\sigma)a^3}\int_0^a r^2\Pi(r)\,dr\right].$$

The sum of the solutions (b) and (c) gives

$$u_r = -\frac{1-2\sigma}{2G(1+\sigma)}\left[\Pi(a) + \frac{2(1-2\sigma)}{(1-\sigma)a^3}\int_0^a r^2\Pi(r)\,dr\right]r +$$

$$+\frac{1-2\sigma}{2G(1-\sigma)}\frac{1}{r^2}\int_0^r r^2\Pi(r)\,dr, \quad u_\alpha = 0,$$

$$R_r = \Pi(r) - \Pi(a) - \frac{2(1-2\sigma)}{1-\sigma}\left[\frac{1}{r^3}\int_0^r r^2\Pi(r)\,dr -\right.$$

$$-\frac{1}{a^3}\int_0^a r^2\Pi(r)\,dr\Big],$$

$$A_r = 0,$$

$$A_\alpha = B_\beta = \frac{\sigma}{1-\sigma}\Pi(r) - \Pi(a) +$$

$$+\frac{1-2\sigma}{1-\sigma}\Big[\frac{1}{r^3}\int_0^r r^2\Pi(r)\,dr + \frac{2}{a^3}\int_0^a r^2\Pi(r)\,dr\Big].$$

If the first term of the formula (a) is taken into account, we obtain, finally,

$$u_r^{(1)} = \frac{\gamma\omega^2(1-2\sigma)r}{30Gg(1-\sigma)}\left(\frac{3-\sigma}{1+\sigma}a^2 - r^2\right), \quad u_\alpha^{(1)} = 0,$$

$$R_r^{(1)} = \frac{\gamma\omega^2}{15g}\frac{3-\sigma}{1-\sigma}(a^2-r^2), \quad A_r^{(1)} = 0,$$

$$A_\alpha^{(1)} = B_\beta^{(1)} = \frac{\gamma\omega^2}{15g}\frac{3-\sigma}{1-\sigma}\left(a^2 - \frac{1-3\sigma}{3-\sigma}r^2\right).$$

The particular solution corresponding to the second term of the formula (a) is formed by the equations (g) and (h) of Problem 4.8 for $n = 2$ and $E_n = \gamma\omega^2/3g$.

According to this solution, the stresses on the surface of the sphere ($r = a$) are

$$R_r^{(2)} = \frac{\gamma\omega^2a^2}{21g(1-\sigma)}(6-5\sigma)P_2, \quad A_r^{(2)} = \frac{(1-2\sigma)\gamma\omega^2a^2}{14g(1-\sigma)}\frac{dP_2}{d\alpha}. \qquad \text{(d)}$$

To remove the stresses (d), it is necessary to superimpose the solution (c) of Problem 3.10 with two constants, A_2 and C_2. The constants are determined from the conditions that the stresses R_r and A_r are zero when $r = a$:

$$3(2Ga_2 + K\lambda A_2)a^2 + 4GC_2 = -\frac{\gamma\omega^2a^2}{21g(1-\sigma)}(6-5\sigma),$$

$$G(a_2 + 2\overline{\overline{A}}_2)a^2 + 2GC_2 = -\frac{(1-2\sigma)\gamma\omega^2a^2}{14g(1-\sigma)},$$

where

$$a_2 = \frac{3}{7}(2K+1)A_2, \quad K = \frac{2G}{\lambda+2G} = \frac{1-2\sigma}{1-\sigma},$$

$$\overline{\overline{A}}_2 = \frac{1}{14}(3K-10)A_2.$$

4.10. See [26].

To the surface of a circular cylinder of radius a and length l is applied an axially symmetric balanced load of the form

$$\begin{aligned} &Z_z(r, 0) = f_1(r), \quad R_z(r, 0) = f_3(r), \quad 0 \leqslant r \leqslant a, \\ &Z_z(r, l) = f_2(r), \quad R_z(r, l) = f_4(r), \quad 0 \leqslant r \leqslant a, \qquad \text{(a)} \\ &R_r(a, z) = f_5(z), \quad Z_r(a, z) = f_6(z), \quad 0 \leqslant z \leqslant l, \end{aligned}$$

where f_i are piecewise continuous functions of bounded variation on the corresponding intervals.

Determine a biharmonic function φ (see Problem 4.1) satisfying the conditions (a).

The function φ is taken in the form

$$\begin{aligned} \varphi(r, z) = z(Ar^2 + Bz^2) + \sum_{k=1}^{\infty} [A_k \sinh(\mu_k z) + \\ + B_k \cosh(\mu_k z) + C_k \mu_k z \sinh(\mu_k z) + \\ + D_k \mu_k z \cosh(\mu_k z)] J_0(\mu_k r) + \sum_{k=1}^{\infty} [E_k I_0(\lambda_k r) + \\ + G_k \lambda_k r I_1(\lambda_k r)] \sin(\lambda_k z). \qquad \text{(b)} \end{aligned}$$

The function (b) satisfies the conditions (a) if

(1) $\lambda_k = k\pi/l$,

(2) the quantities μ_k are the roots of the equation $J_1(\mu_k a) = 0$ with the asymptotic formula $\mu_k a = \pi(k + 1/4) + O(1/k)$,

(3) the functions $f_1(r)$ to $f_4(r)$ on the interval $0 < r < a$ can be represented as series in Bessel functions:

$$a_0 + \sum_{k=1}^{\infty} a_k J_0(\mu_k r), \quad \sum_{k=1}^{\infty} b_k J_1(\mu_k r),$$

(4) the functions $f_5(z)$ and $f_6(z)$ on the interval $0 < z < l$ can be represented as Fourier series

$$\frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos(\lambda_k z), \quad \sum_{k=1}^{\infty} d_k \sin(\lambda_k z),$$

(5) the coefficients in the formula (b) can be represented in terms of a_i, b_i, c_i, and d_i taking into account the formulas (a) of Problem 4.1.

Chapter 5

PLANE PROBLEM IN RECTANGULAR CO-ORDINATES

I. PLANE STRESS

$$Z_z = Y_z = X_z = e_{zy} = e_{zx} = 0, \quad e_{zz} = -\frac{\sigma}{E}(X_x + Y_y).$$

1. Equilibrium equations

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + X = 0 \left(= \rho \frac{\partial^2 u_x}{\partial \tau^2} \right),$$

$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + Y = 0 \left(= \rho \frac{\partial^2 u_y}{\partial \tau^2} \right). \tag{5.1}$$

2. Geometrical equations

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y},$$

$$e_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}. \tag{5.2}$$

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - \frac{\partial^2 e_{xy}}{\partial x \, \partial y} = 0. \tag{5.3}$$

3. Physical equations

$$e_{xx}=\frac{1}{E}(X_x-\sigma Y_y),$$

$$e_{yy}=\frac{1}{E}(Y_y-\sigma X_x), \tag{5.4}$$

$$e_{xy}=\frac{2(1+\sigma)}{E}X_y.$$

$$X_x=\frac{E}{1-\sigma^2}(e_{xx}+\sigma e_{yy}),$$

$$Y_y=\frac{E}{1-\sigma^2}(e_{yy}+\sigma e_{xx}), \tag{5.5}$$

$$X_y=Y_x=\frac{E}{2(1+\sigma)}e_{xy}.$$

4. Basic equations in terms of stresses

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+X=0,\quad \frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+Y=0,$$

$$\nabla^2(X_x+Y_y)=-2\frac{\lambda+G}{\lambda+2G}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)=$$

$$=-(1+\sigma)\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right). \tag{5.6}$$

5. Basic equations in terms of displacements

$$\left[2\frac{\partial^2}{\partial x^2}+(1-\sigma)\frac{\partial^2}{\partial y^2}\right]u_x+(1+\sigma)\frac{\partial^2 u_y}{\partial x\,\partial y}+\frac{2(1-\sigma^2)}{E}X=0,$$

$$(1+\sigma)\frac{\partial^2 u_x}{\partial x\,\partial y}+\left[2\frac{\partial^2}{\partial y^2}+(1-\sigma)\frac{\partial^2}{\partial x^2}\right]u_y+$$

$$+\frac{2(1-\sigma^2)}{E}Y=0 \tag{5.7}$$

or in Lamé's form

$$G\nabla^2 u_x + G\left(\frac{2\lambda}{\lambda+2G}+1\right)\frac{\partial\theta}{\partial x} + X = 0,$$
$$G\nabla^2 u_y + G\left(\frac{2\lambda}{\lambda+2G}+1\right)\frac{\partial\theta}{\partial y} + Y = 0, \tag{5.8}$$

where

$$\theta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}, \quad \frac{2\lambda}{\lambda+2G} + 1 = \frac{1+\sigma}{1-\sigma}.$$

II. PLANE STRAIN

$$e_{zz} = e_{zx} = e_{zy} = X_z = Y_z = 0, \quad Z_z = \sigma(X_x + Y_y).$$

For the case of plane strain, in all equations of part I it is necessary to replace

$$\sigma \text{ by } \sigma_1 = \frac{\sigma}{1-\sigma}, \quad \text{and } E \text{ by } E_1 = \frac{E}{1-\sigma^2}. \tag{5.9}$$

In this case, Eqs. (5.7), for example, are

$$\left[2(1-\sigma)\frac{\partial^2}{\partial x^2} + (1-2\sigma)\frac{\partial^2}{\partial y^2}\right]u_x + \frac{\partial^2 u_y}{\partial x\,\partial y} + $$
$$+ \frac{2(1+\sigma)(1-2\sigma)}{E} X = 0,$$
$$\frac{\partial^2 u_x}{\partial x\,\partial y} + \left[2(1-\sigma)\frac{\partial^2}{\partial y^2} + (1-2\sigma)\frac{\partial^2}{\partial x^2}\right]u_y + \tag{5.7a}$$
$$+ \frac{2(1+\sigma)(1-2\sigma)}{E} Y = 0$$

and Eqs. (5.8) transform into

$$G\nabla^2 u_x + (\lambda+G)\frac{\partial\theta}{\partial x} + X = 0\left(= \rho\frac{\partial^2 u_x}{\partial \tau^2}\right),$$
$$G\nabla^2 u_y + (\lambda+G)\frac{\partial\theta}{\partial y} + Y = 0\left(= \rho\frac{\partial^2 u_y}{\partial \tau^2}\right). \tag{5.8a}$$

According to Eqs. (3.3a) we obtain

$$(\lambda+2G)\frac{\partial\theta}{\partial x} - 2G\frac{\partial\omega}{\partial y} = 0,$$
$$(\lambda+2G)\frac{\partial\theta}{\partial y} + 2G\frac{\partial\omega}{\partial x} = 0, \tag{5.8b}$$

where

$$\lambda = E_1\sigma_1/(1-\sigma_1^2), \quad E_1 = E/(1-\sigma^2), \quad \sigma_1 = \sigma/(1-\sigma)$$

or

$$\lambda = E\sigma/(1+\sigma)(1-2\sigma)$$

(see Notation).

III. SOLUTION OF BASIC EQUATIONS

1. Solution using stress-displacement functions

Equations (5.7) have the general solution

$$u_x = 2\nabla^2\varphi_1 - (1+\sigma)\frac{\partial}{\partial x}\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}\right) + ay + b,$$
$$u_y = 2\nabla^2\varphi_2 - (1+\sigma)\frac{\partial}{\partial y}\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}\right) - ax + c, \quad (5.10)$$

where φ_i = arbitrary biharmonic functions,
a, b, c = constants characterizing a rigid-body displacement.

The general solution| of Eqs. (5.7a) is of the form [see Eqs. (4.25a)]

$$u_x = 2(1-\sigma)\nabla^2\varphi_1 - \frac{\partial}{\partial x}\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}\right) + ay + b,$$
$$u_y = 2(1-\sigma)\nabla^2\varphi_2 - \frac{\partial}{\partial y}\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}\right) - ax + c. \quad (5.10a)$$

By means of the introduced stress function $\varphi(x, y)$ (Airy, 1861) in terms of which the stresses are expressed by the formulas

$$X_x = \frac{\partial^2\varphi}{\partial y^2}, \quad Y_y = \frac{\partial^2\varphi}{\partial x^2}, \quad X_y = -\frac{\partial^2\varphi}{\partial x\,\partial y} - Xy - Yx, \quad (5.11)$$

where X and Y are constant body forces,* Eqs. (5.6) are reduced to the biharmonic equation

$$\nabla^2\nabla^2\varphi = \frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2\,\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0. \quad (5.12)$$

* Formulas (5.11) are generalized in the case of variable body forces having a potential.

The biharmonic function $\varphi(x, y)$ must satisfy the stress boundary conditions of the problem

$$X_\nu = \frac{\partial^2\varphi}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2\varphi}{\partial x\,\partial y}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial\varphi}{\partial y}\right),$$
$$Y_\nu = -\frac{\partial^2\varphi}{\partial x\,\partial y}\frac{dy}{ds} - \frac{\partial^2\varphi}{\partial x^2}\frac{dx}{ds} = -\frac{d}{ds}\left(\frac{\partial\varphi}{\partial x}\right), \tag{5.13}$$

or, in the case of the second fundamental problem when displacements are given on the contour L, the conditions

$$u_x = g_1(s), \quad u_y = g_2(s), \tag{5.14}$$

where $g_i(s)$ are given displacements of the points of the contour L, which are functions of the arc length s of the contour.

2. Application of a complex variable [27, 28]

For the complex variables

$$z = x + iy \text{ and } \bar{z} = x - iy \quad (i = \sqrt{-1}) \tag{5.15}$$

Eq. (5.12) is

$$\frac{\partial^4\varphi}{\partial z^2\,\partial\bar{z}^2} = 0 \tag{5.16}$$

with the general solution

$$\varphi = \frac{1}{2}[\bar{z}\psi(z) + z\overline{\psi(z)} + \chi(z) + \overline{\chi(z)}] \tag{5.17}$$

or

$$\varphi = \mathrm{Re}[\bar{z}\psi(z) + \chi(z)], \tag{5.18}$$

where Re = real part of the bracketed expression,
$\psi(z)$, $\chi(z)$ = unknown analytic functions of the complex variable z,
$\overline{\psi(z)}$, $\overline{\chi(z)}$ = functions conjugate to $\psi(z)$ and $\chi(z)$.

The boundary conditions (5.13) for the first fundamental problem on the contour L are of the form

$$\frac{\partial\varphi}{\partial x} + i\frac{\partial\varphi}{\partial y} = \psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)} =$$
$$= i\int_0^s (X_\nu + iY_\nu)\,ds + C = f_1 + if_2 + \text{constant}. \tag{5.19}$$

For the second fundamental problem, conditions (5.14) on the contour L are

$$2G(u_x + iu_y) = \varkappa\psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)} = 2G(g_1 + ig_2), \tag{5.20}$$

where $\varkappa = \frac{3-\sigma}{1+\sigma}$ = for plane stress,

$\varkappa = 3 - 4\sigma$ = for plane strain.

The stress components are found by the Kolosov-Muskhelishvili formulas

$$\begin{aligned} X_x + Y_y &= 2[\psi'(z) + \overline{\psi'(z)}] = 4 \operatorname{Re} \psi'(z), \\ Y_y - X_x + 2iX_y &= 2[\bar{z}\psi''(z) + \chi''(z)]. \end{aligned} \tag{5.21}$$

The components of the resultant force acting over the finite arc AB (Fig. 27) are equal to [(5.13)]

$$\begin{aligned} F_x &= \int_A^B X_\nu \, ds = \int_A^B \frac{d}{ds}\left(\frac{\partial\varphi}{\partial y}\right) ds = \left[\frac{\partial\varphi}{\partial y}\right]_A^B, \\ F_y &= \int_A^B Y_\nu \, ds = -\int_A^B \frac{d}{ds}\left(\frac{\partial\varphi}{\partial x}\right) ds = -\left[\frac{\partial\varphi}{\partial x}\right]_A^B. \end{aligned} \tag{5.22}$$

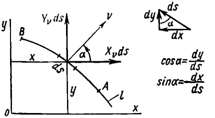

Fig. 27

The moment of the forces acting over the arc AB about the origin 0 is

$$\begin{aligned} M &= -\int_A^B (xY_\nu - yX_\nu)\, dx = -\int_A^B \left[x d\left(\frac{\partial\varphi}{\partial x}\right) + y d\left(\frac{\partial\varphi}{\partial y}\right)\right] = \\ &= -\left[x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y}\right]_A^B + [\varphi]_A^B. \end{aligned} \tag{5.23}$$

Equations (3.3a) for a plane are

$$(\lambda + 2G)\frac{\partial\theta}{\partial x} - 2G\frac{\partial\omega}{\partial y} = 0,$$
$$(\lambda + 2G)\frac{\partial\theta}{\partial y} + 2G\frac{\partial\omega}{\partial x} = 0 \tag{5.24}$$

from which it follows that θ and ω are conjugate harmonic functions of two variables, namely

$$(\lambda + 2G)\,\theta + 2iG\omega = f(x + iy) \tag{5.25}$$

(see the monograph [5], p. 204).

Equations (3.3a′) for a plane are

$$G\nabla^2 u_x + (\lambda + G)\frac{\partial\theta}{\partial x} = 0, \quad G\nabla^2 u_y + (\lambda + G)\frac{\partial\theta}{\partial y} = 0 \tag{5.24′}$$

or

$$\nabla^2(u_x + iu_y) + \frac{\lambda + G}{G}\frac{\partial\theta}{\partial \bar{z}} = 0, \tag{5.25′}$$

$$\theta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 2\,\mathrm{Re}\,\frac{\partial}{\partial z}(u_x + iu_y),$$

$$\nabla^2 \ldots = \frac{\partial^2 \ldots}{\partial x^2} + \frac{\partial^2 \ldots}{\partial y^2} = 4\frac{\partial^2 \ldots}{\partial z\,\partial \bar{z}}.$$

3. Solution by the initial function method [29]

A plane problem is solved by the mixed method: the unknowns are taken to be the displacements u_x, u_y and the stresses X_y, Y_y. The required quantities are represented as Maclaurin's series in the co-ordinate y and expressed in terms of the initial functions u_x^0, u_y^0, X_y^0, and Y_y^0, i.e., the values of the functions for $y = 0$ (Fig. 28).

For the case of plane strain the system of computing equations is of the form

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0, \quad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} = 0,$$

$$Y_y = \frac{2G}{1-2\sigma}\left[(1-\sigma)\frac{\partial u_y}{\partial y} + \sigma\frac{\partial u_x}{\partial x}\right], \tag{5.26}$$

$$X_y = Y_x = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right).$$

Introducing, for shortness, the notation

$$U = Gu_x, \quad V = Gu_y, \quad Y_y = Y, \quad X_y = X,$$
$$\frac{\partial}{\partial x} = \alpha, \quad \frac{\partial}{\partial y} = \beta, \tag{5.27}$$

we reduce Eqs. (5.26) to

$$\beta U = -\alpha V + X, \quad \beta V = -\frac{\sigma}{1-\sigma}\alpha U + \frac{1-2\sigma}{2(1-\sigma)} Y,$$
$$\beta Y = -\alpha X, \quad \beta X = -\frac{2}{1-\sigma}\alpha^2 U - \frac{\sigma}{1-\sigma}\alpha Y. \tag{5.28}$$

The stresses X_x are determined by the formula

$$X_x = \frac{2}{1-2\sigma}[(1-\sigma)\alpha U + \sigma\beta V].$$

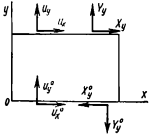

Fig. 28

The solution of Eqs. (5.28) is obtained as

$$U = L_{UU}U^0 + L_{UV}V^0 + L_{UY}Y^0 + L_{UX}X^0,$$
$$V = L_{VU}U^0 + L_{VV}V^0 + L_{VY}Y^0 + L_{VX}X^0, \tag{5.29}$$
$$Y = L_{YU}U^0 + L_{YV}V^0 + L_{YY}Y^0 + L_{YX}X^0,$$
$$X = L_{XU}U^0 + L_{XV}V^0 + L_{XY}Y^0 + L_{XX}X^0,$$

where the letter L stands for linear differential operators, which in transcendental form of the operator method are given in Table 4.1.

To transform from the symbolic to the real representation in the form of infinite series, it is necessary to expand

Table 4.1

Function	Values of linear differential operators of initial functions			
	U^0	V^0	Y^0	X^0
U	$L_{UU}=\cos\alpha y-\frac{\alpha y}{2(1-\sigma)}\sin\alpha y$	$L_{UV}=-\frac{1-2\sigma}{2(1-\sigma)}\sin\alpha y-\frac{\alpha y}{2(1-\sigma)}\cos\alpha y$	$L_{UY}=-\frac{\alpha^2 y}{4(1-\sigma)}\sin\alpha y$	$L_{UX}=\frac{1}{\alpha}\sin\alpha y-\frac{\sin\alpha y-\alpha y\cos\alpha y}{4(1-\sigma)\alpha}$
V	$L_{VU}=\frac{1-2\sigma}{2(1-\sigma)}\sin\alpha y-\frac{\alpha y}{2(1-\sigma)}\cos\alpha y$	$L_{VV}=\cos\alpha y+\frac{\alpha y}{2(1-\sigma)}\sin\alpha y$	$L_{VY}=\frac{3-4\sigma}{4(1-\sigma)\alpha}\sin\alpha y-\frac{y}{4(1-\sigma)}\cos\alpha y$	$L_{VX}=L_{UY}$
Y	$L_{YU}=\frac{\alpha^2 y}{1-\sigma}\sin\alpha y$	$L_{YV}=-\frac{\alpha}{1-\sigma}\times(\sin\alpha y-\alpha y\cos\alpha y)$	$L_{YY}=L_{VV}$	$L_{YX}=L_{UV}$
X	$L_{XU}=-\frac{\alpha}{1-\sigma}\times(\sin\alpha y+\alpha y\cos\alpha y)$	$L_{XV}=L_{YU}$	$L_{XY}=L_{VU}$	$L_{XX}=L_{UU}$
X_x	$L_U=\frac{2\alpha}{1-\sigma}\cos\alpha y-\frac{y\alpha^2}{1-\sigma}\sin\alpha y$	$L_V=-\frac{\alpha}{1-\sigma}\times(\sin\alpha y-y\alpha\cos\alpha y)$	$L_Y=\frac{\sigma}{1-\sigma}\cos\alpha y-\frac{y\alpha}{2(1-\sigma)}\sin\alpha y$	$L_X=\frac{y\alpha}{2(1-\sigma)}\cos\alpha y+\frac{3-2\sigma}{2(1-\sigma)}\sin\alpha y$

the trigonometric functions in power series of αy and to replace α by $\frac{\partial}{\partial x}$.

In plane stress, E and σ must be replaced, respectively, by

$$\frac{E(1+2\sigma)}{(1+\sigma)^2} \quad \text{and} \quad \frac{\sigma}{1+\sigma}.$$

4. Use of boundary value homogeneous solutions [30, 31, 32]

Assuming, in the homogeneous equations (5.7),

$$\begin{aligned} u_x &= (1+\sigma)\frac{\partial^2\varphi}{\partial x\,\partial y} + ay + b, \\ u_y &= -\left[2\frac{\partial^2}{\partial x^2} + (1-\sigma)\frac{\partial^2}{\partial y^2}\right]\varphi - ax + c, \end{aligned} \tag{5.30}$$

we identically satisfy the first equation, and the second equation becomes

$$\frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2\,\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0. \tag{5.31}$$

Taking

$$\varphi = \varphi(y)_{\cos}^{\sin} kx,$$

we find

$$\varphi = (A_k \cosh ky + B_k \sinh ky + C_k y \cosh ky + D_k y \sinh ky)_{\cos}^{\sin} kx. \tag{5.32}$$

Hence, for $\sigma = 0$

$$\begin{aligned} u_x &= \pm k\,[A_k k \sinh ky + B_k k \cosh ky + C_k(\cosh ky + \\ &\quad + ky \sinh ky) + D_k(\sinh ky + ky \cosh ky)]_{\sin}^{\cos} kx + ay + b, \\ u_y &= k\,[A_k k \cosh ky + B_k k \sinh ky + C_k(ky \cosh ky - \\ &\quad - 2\sinh ky) + D_k(ky \sinh ky - 2\cosh ky)]_{\cos}^{\sin} kx - \\ &\quad - ax + c, \\ X_x &= -Ek^2\,[A_k k \sinh ky + B_k k \cosh ky + \\ &\quad + C_k(\cosh ky + ky \sinh ky) + \\ &\quad + D_k(\sinh ky + ky \cosh ky)]_{\cos}^{\sin} kx, \end{aligned} \tag{5.33}$$

$$Y_y = Ek^2 [A_k k \sinh ky + B_k k \cosh ky + C_k (ky \sinh ky - \cosh ky) + D_k (ky \cosh ky - \sinh ky)]_{\cos}^{\sin} kx,$$

$$X_y = Y_x = \pm Ek^2 [A_k k \cosh ky + B_k k \sinh ky + C_k ky \cosh ky + D_k ky \sinh ky]_{\sin}^{\cos} kx.$$

If $Y_y = X_y = 0$ when $y = \pm h$, by equating to zero the determinant of the boundary conditions, we obtain

$$4\lambda^2 - \sinh^2 2\lambda = (2\lambda - \sinh 2\lambda)(2\lambda - \sinh 2\lambda) = 0, \quad (5.34)$$

where $\lambda = kh$.

For $\lambda = a \pm bi$, we obtain two governing equations

$$\frac{2a}{\sinh 2a} = \pm \cos 2b, \quad \frac{2b}{\sin 2b} = \pm \cosh 2a$$

or, for the value

$$b = \frac{1}{2} \coth 2a \sqrt{\sinh^2 2a - 4a^2}, \quad (5.35)$$

$$\frac{2a}{\sinh 2a} = \pm \cos\left(\coth 2a \sqrt{\sinh^2 2a - 4a^2}\right). \quad (5.36)$$

The graph of Eq. (5.36) is given in Fig. 29. By taking $2n$ values of a (b), we can approximately satisfy the bounda-

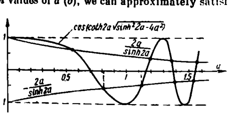

Fig. 29

ry conditions on the sides $x = c_1$ and $x = c_2$ at n points. To satisfy the boundary conditions more exactly, it is necessary to expand the particular solution due to the load in non-orthogonal functions of homogeneous solutions depending on the roots of Eqs. (5.34) (see Problem 5.9).

5. Finite-difference solution [33]

When using a rectangular net (Fig. 30*a*), successive differences $\Delta^n \varphi_k$ are obtained by the formulas

$$\frac{\partial \varphi_k}{\partial x} \cong \frac{\Delta_x \varphi_k}{2\,\Delta x} = \frac{\varphi_l - \varphi_i}{2\,\Delta x}, \quad \frac{\partial^2 \varphi_k}{\partial x^2} \cong \frac{\Delta_x^2 \varphi_k}{\Delta x^2} = \frac{\varphi_b - 2\varphi_k + \varphi_l}{\Delta x^2},$$

$$\frac{\partial \varphi_k}{\partial y} \cong \frac{\Delta_y \varphi_k}{2\,\Delta y} = \frac{\varphi_m - \varphi_n}{2\,\Delta y}, \quad \frac{\partial^2 \varphi_k}{\partial y^2} \cong \frac{\Delta_y^2 \varphi_k}{\Delta y^2} = \frac{\varphi_m - 2\varphi_k + \varphi_n}{\Delta y^2},$$

$$\frac{\partial^2 \varphi_k}{\partial x\,\partial y} \cong \frac{\dfrac{\partial}{\partial y}\left(\dfrac{\varphi_l - \varphi_i}{2\Delta x}\right)}{\dfrac{\partial}{\partial x}\left(\dfrac{\varphi_m - \varphi_n}{2\Delta y}\right)} = \frac{\varphi_p - \varphi_r - \varphi_o + \varphi_q}{4\Delta x \Delta y}, \tag{5.37}$$

$$\frac{\partial^4 \varphi_k}{\partial x^4} \cong \frac{\Delta_x^4 \varphi_k}{\Delta x^4} = \frac{6\varphi_k - 4\varphi_c - 4\varphi_i + \varphi_t + \varphi_s}{\Delta x^4},$$

$$\frac{\partial^4 \varphi_k}{\partial y^4} \approx \frac{\Delta_y^4 \varphi_k}{\Delta y^4} \quad \frac{6\varphi_k - 4\varphi_m - 4\varphi_n + \varphi_v + \varphi_u}{\Delta y^4},$$

$$\frac{\partial^4 \varphi_k}{\partial x^2\,\partial y^2} \cong \frac{4\varphi_k - 2\varphi_m - 2\varphi_n - 2\varphi_c - 2\varphi_i + \varphi_p + \varphi_q + \varphi_r + \varphi_o}{\Delta x^2\,\Delta y^2},$$

where φ_k is the value of the function at the point k.

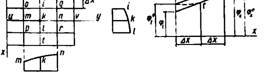

Fig. 30

In this case the harmonic equation (4.21)

$$\nabla^2 \varphi_k = 0$$

takes the form of a five-term algebraic equation

$$\alpha(\psi_l + \psi_i) + \psi_m + \psi_n - 2(1+\alpha)\psi_k = 0. \qquad (5.38)$$

The biharmonic equation (5.12)

$$\nabla^2\nabla^2\varphi_k = 0$$

takes the form of a 13-term algebraic equation

$$[6(\alpha + 1/\alpha) + 8]\varphi_k - 4[(1+\alpha)(\varphi_l + \varphi_i) +$$
$$+ (1 + 1/\alpha)(\varphi_m + \varphi_n)] + 2(\varphi_p + \varphi_q + \varphi_r + \varphi_o) +$$
$$+ \alpha(\varphi_t + \varphi_s) + \frac{1}{\alpha}(\varphi_v + \varphi_u) = 0, \qquad (5.39)$$

where

$$\alpha = (\Delta y/\Delta x)^2, \quad 1/\alpha = (\Delta x/\Delta y)^2.$$

For a square net, $\Delta x = \Delta y = h$, $a = 1$, and Eq. (5.39) becomes

$$h^4\nabla^2\nabla^2\varphi_k = 20\varphi_k - 8(\varphi_l + \varphi_i + \varphi_m + \varphi_n) +$$
$$+ 2(\varphi_p + \varphi_q + \varphi_r + \varphi_o) +$$
$$+ \varphi_t + \varphi_s + \varphi_v + \varphi_u = 0. \qquad (5.39')$$

The harmonic equation $\nabla^2\psi_k = 0$ takes the form

$$h^2\nabla^2\psi_k = \psi_m + \psi_l + \psi_n + \psi_i - 4\psi_k = 0. \qquad (5.38')$$

The stresses at the point k are determined by the formulas

$$X_x = \frac{\partial^2\varphi_k}{\partial y^2} \cong \frac{\varphi_m - 2\varphi_k + \varphi_n}{\Delta y^2},$$
$$Y_y = \frac{\partial^2\varphi_k}{\partial x^2} \cong \frac{\varphi_l - 2\varphi_k + \varphi_i}{\Delta x^2}, \qquad (5.40)$$
$$X_y = Y_x = -\frac{\partial^2\varphi_k}{\partial x\,\partial y} = \frac{(\varphi_o + \varphi_r) - (\varphi_p + \varphi_q)}{4\,\Delta x\,\Delta y}.$$

In setting up equations for the nodes nearest to the contour (m, n, o, etc.; see Fig. 30a), it is necessary to introduce the values of the function φ at the nodes on the contour and at the nearest nodes outside the contour.

The values of the function φ at the nodes on the contour are determined by the extrapolation formulas

$$\varphi_x = \varphi_l + 2\Delta x \left(\frac{\partial\varphi}{\partial x}\right)_t, \quad \varphi_y = \varphi_n + 2\Delta y \left(\frac{\partial\varphi}{\partial y}\right)_v \tag{5.41}$$

(see Fig. 30*b*).

PROBLEMS

5.1. A rectangular parallelepiped of great length ($u_z = 0$— plane strain) is subjected to a uniform pressure $-p$ and

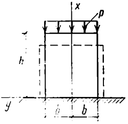

Fig. 31

supported by an absolutely rigid ($u_x = 0$) and smooth ($Y_x = 0$) foundation (Fig. 31).

Determine the state of stress and strain.

The solution of the homogeneous equations (5.7a) is taken in the form (5.10a) assuming $\varphi_1 = 0$ and $\varphi_2 = \varphi$

$$u_x = -\frac{\partial^2\varphi}{\partial x\,\partial y} + ay + b,$$

$$u_y = \left[2(1-\sigma)\frac{\partial^2}{\partial x^2} + (1-2\sigma)\frac{\partial^2}{\partial y^2}\right]\varphi - ax + c.$$

The stresses are determined by the formulas

$$X_x = \frac{E_1}{1-\sigma_1^2}(e_{xx} + \sigma_1 e_{yy}) = \frac{E(1-\sigma)}{1+\sigma}\frac{\partial}{\partial y} \times$$

$$\times \left[\frac{\sigma}{1-\sigma}\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right]\varphi,$$

$$Y_y = \frac{E_1}{1-\sigma_1^2}\,(e_{yy}+\sigma_1 e_{xx}) = \frac{E\,(1-\sigma)}{1+\sigma}\,\frac{\partial}{\partial y}\times$$

$$\times\left[\frac{\partial^2}{\partial y^2}+\frac{2-\sigma}{1-\sigma}\,\frac{\partial}{\partial x^2}\right]\varphi, \tag{a}$$

$$X_y = Y_x = \frac{E_1}{2\,(1+\sigma_1)}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) = \frac{E}{1+\sigma}\,\frac{\partial}{\partial x}\times$$

$$+\left[(1-\sigma)\,\frac{\partial^2}{\partial x^2}-\sigma\,\frac{\partial^2}{\partial y^2}\right]\varphi.$$

We assign the function φ in the form of a third-degree polynomial

$$\varphi = Ax^2y + By^3, \tag{b}$$

where A and B are unknown coefficients.

According to (a) and (b), the displacements and stresses are found by the formulas

$$u_x = -2Ax + ay + b,$$

$$u_y = 2\,[2\,(1-\sigma)\,A + 3\,(1-2\sigma)\,B]\,y - ax + c,$$

$$X_x = \frac{2E\,(1-\sigma)}{1+\sigma}\left(\frac{3\sigma}{1-\sigma}\,B - A\right),$$

$$Y_y = \frac{2E\,(1-\sigma)}{1+\sigma}\left(3B + \frac{2-\sigma}{1-\sigma}\,A\right), \quad X_y = Y_x = 0.$$

The boundary conditions of the problem are: when $x = 0$, $u_x = Y_x = 0$; when $x = h$, $X_x = -p$, $Y_x = 0$; when $y = \pm b$, $Y_y = X_y = 0$.

According to the boundary conditions, we obtain

$$A = \frac{(1-\sigma^2)\,p}{2E}, \quad B = -\frac{2-\sigma}{3\,(1-\sigma)}\,A, \quad a = b = c = 0.$$

We have, finally,

$$u_x = -\frac{(1-\sigma^2)\,p}{E}\,x, \quad u_y = \frac{\sigma\,(1+\sigma)\,p}{E}\,y,$$

$$X_x = -p, \quad X_y = Y_x = Y_y = 0.$$

The solution obtained is identical with the solution by the formulas of strength of materials.

5.2. A narrow rectangular beam ($\delta = 1$) is supported at the edges $x = \pm l/2$ and bent by a uniformly distributed load of intensity q (Fig. 32).

Investigate the state of stress in the beam.

The supporting conditions are realized by the end shearing forces, which are the resultants of the shearing

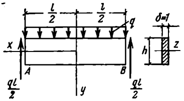

Fig. 32

stresses Y_x and are equal in magnitude to reactions that would arise at the points of support A and B. The problem is solved in terms of stresses by assigning the stress function in the form

$$\varphi(x, y) = \frac{d}{6} y^3 \left(x^2 - \frac{y^2}{5}\right) + \frac{b}{2} x^2 y + \frac{k}{6} y^3 + \frac{a}{2} x^2$$

satisfying Eq. (5.12).

According to (5.11), the stresses are

$$X_x = \frac{\partial^2 \varphi}{\partial y^2} = d\left(x^2 - \frac{2}{3} y^2\right) y + ky,$$

$$Y_y = \frac{\partial^2 \varphi}{\partial x^2} = \left(\frac{d}{3} y^2 + b\right) y + a,$$

$$X_y = Y_x = -\frac{\partial^2 \varphi}{\partial x \, \partial y} = -(dy^2 + b) x.$$

The boundary conditions of the problem are: when $y = -h/2$, $Y_y = -q$, $X_y = 0$; when $y = h/2$, $Y_y = 0$, $X_y = 0$. These boundary conditions are exact, or local (for each point of the corresponding portion). When $x =$

$= \pm l/2$, we have

$$(1)\ N_x = \int_{-h/2}^{h/2} X_x\,dy = 0, \quad (2)\ M_z = \int_{-h/2}^{h/2} X_x y\,dy = 0,$$

$$(3)\ \int_{-h/2}^{h/2} Y_x\,dy = -\frac{ql}{2}.$$

These are approximate, or integral, conditions, which are justified by Saint-Venant's principle.

According to conditions (1) and (2),

$$-dh^3/24 - bh/2 + a = -q, \quad dh^2/4 + b = 0,$$

$$dh^3/24 + bh/2 + a = 0;$$

hence, $a = -q/2, \quad b = 3q/2h, \quad d = -6q/h^3$.

Conditions (1) and (3) are identically satisfied, as can easily be verified by substitution.

According to condition (2),

$$k = \frac{q}{2I_z}\left(\frac{l^2}{4} - \frac{h^2}{10}\right),$$

where $I_z = h^3/12$ is the moment of inertia of the beam. The stresses are, finally,

$$X_x = \frac{q}{2I_z}\left(\frac{2}{3}y^2 - x^2\right)y + ky =$$

$$= \frac{q}{2I_z}\left(\frac{l^2}{4} - x^2 + \frac{2}{3}y^2 - \frac{h^2}{10}\right)y,$$

$$Y_y = -\frac{q}{2I_z}\left(\frac{1}{3}y^3 - \frac{1}{4}h^2y + I_z\right),$$

$$X_y = Y_x = -\frac{q}{2I_z}\left(\frac{h^2}{4} - y^2\right)x.$$

By the formulas of strength of materials,

$$X_x = \sigma_x = \frac{M_z}{I_z}y = \frac{q}{2I_z}\left(\frac{l^2}{4} - x^2\right)y.$$

where M_z is the bending moment,

$Y_y = \sigma_y = 0$

(the longitudinal fibres do not press on one another laterally),

$$X_y = Y_x = \tau = \frac{QS_z^{\text{cut}}}{I_z\delta} = -\frac{q}{2I_z}\left(\frac{h^2}{4} - y^2\right)x,$$

where S_z^{cut} is the static moment of the cut-off part of the cross-sectional area of the beam, Q is the shearing force.

The X_x and Y_y diagrams are given in Fig. 33.

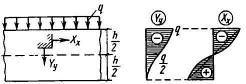

Fig. 33

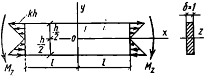

Fig. 34

▶ A thin rectangular plate ($\delta = 1$) is subjected at the ends to a load $p = 2ky$, which reduces to bending couples M_z (Fig. 34).

Investigate the state of stress in the plate by assigning the stress function in the form

$$\varphi = Ay^3,$$

where A is an arbitrary constant.

The boundary conditions of the problem are: when $y = \pm h/2$, $Y_y = X_y = 0$; when $x = \pm l$, $Y_x = 0$, $X_x = 2ky$.

Answer.

$$Y_y = X_y = 0,$$

$$X_x = 2ky = \frac{12M_z}{h^3}y = \frac{M_z}{I_z}y.$$

where $I_z = h^3/12$. This result is known from the course in strength of materials.

5.3. Zhemochkin's problem [34].

Analyse a deep and narrow ($\delta = 1$) beam of depth h having an infinite number of spans of equal length $2l$ and

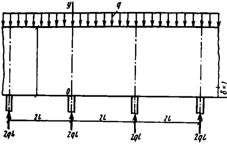

Fig. 35

supported by columns. A load of intensity q is uniformly distributed along the top of the beam (Fig. 35).

For the assumed infinitely large number of spans all of them are in identical conditions, and it is sufficient to investigate only one span. The reactions at the supports are applied along the axes of the columns and are equal to $2ql$.

Since the stresses X_x and Y_y must be symmetrical about the axis Oy, the solution is sought in the form of a trigonometric cosine series. It is impossible, however, to manage with a single trigonometric series since all conditions on the contour will not be satisfied, and hence a second-degree polynomial must be added.

The stress function is assigned in the form

$$\varphi(x, y) = A_1 x^2 + A_2 xy + A_3 y^2 +$$

$$+ \sum_{n=1}^{\infty} \cos \alpha x \, (C_{1n} \sinh \alpha y + C_{2n} \cosh \alpha y + C_{3n} y \sinh \alpha y + C_{4n} y \cosh \alpha y),$$

where

$$\alpha = n\pi/l. \tag{a}$$

It is obvious that the function $\cos(\alpha x)$ is unaltered when the quantity $2l$ is added to the argument x, since

$$\cos\left[\frac{n\pi(x+2l)}{l}\right] = \cos\left(\frac{n\pi x}{l} + 2n\pi\right) = \cos\alpha x.$$

The last equality shows that in the next spans all quantities defining the state of stress are repeated and all spans are in identical conditions.

The boundary conditions of the problem are:

(1) when $x = 0,\ l,\ X_y = 0$ by symmetry;

(2) the sum of the stresses Y_y within the limits of a half-span must be equal to the load within the same limits, i.e.,

$$\int_0^l Y_y\,dx = -ql; \tag{b}$$

(3) when $y = 0,\ X_y = 0$;

(4) when $y = 0,\ Y_y = 0$;

(5) when $y = 0,\ x = 0,\ 2l,\ Y_y \to \infty$ since the reaction is assumed to be a concentrated force;

(6) when $y = h,\ X_y = 0$;

(7) when $y = h,\ Y_y = -q$.

For the chosen stress function (a), the stresses are determined by the formulas

$$X_x = \frac{\partial^2\varphi}{\partial y^2} = 2A_3 +$$

$$+ \sum_{n=1}^{\infty} \alpha\cos\alpha x\,[\alpha\,(C_{1n}\sinh\alpha y + C_{2n}\cosh\alpha y +$$

$$+ C_{3n}y\sinh\alpha y + C_{4n}y\cosh\alpha y) + 2\,(C_{3n}\cosh\alpha y +$$

$$+ C_{4n}\sinh\alpha y)],$$

$$Y_y = \frac{\partial^2\varphi}{\partial x^2} = 2A_1 -$$

$$-\sum_{n=1}^{\infty} \alpha^2 \cos \alpha x \, (C_{1n} \sinh \alpha y + C_{2n} \cosh \alpha y +$$

$$+ C_{3n} y \sinh \alpha y + C_{4n} y \cosh \alpha y), \tag{c}$$

$$X_y = Y_x = -\frac{\partial^2 \varphi}{\partial x \, \partial y} = A_2 +$$

$$+ \sum_{n=1}^{\infty} \alpha \sin \alpha x \, [\alpha \, (C_{1n} \cosh \alpha y + C_{2n} \sinh \alpha y +$$

$$+ C_{3n} \, y \cosh \alpha y + C_{4n} y \sinh \alpha y) + C_{3n} \sinh \alpha y +$$

$$+ C_{4n} \cosh \alpha y]. \tag{d}$$

From the boundary conditions (1) it follows that

$A_2 = 0$.

From condition (2) [see the formula (b)],

$$2A_1 l - \sum_{n=1}^{\infty} \alpha \sin \alpha l \, (C_{1n} \sinh \alpha y + C_{2n} \cosh \alpha l +$$

$$+ C_{3n} y \sinh \alpha y + C_{4n} y \cosh \alpha y) = | \sin \alpha l = \sin n\pi = 0 | =$$

$$= 2A_1 l = -ql; \tag{e}$$

hence,

$A_1 = -q/2$.

From condition (3), by using the formula (d) for $y = 0$, we obtain

$$\sum_{n=1}^{\infty} \alpha \sin \alpha x \, (\alpha C_{1n} + C_{4n}) = 0;$$

hence,

$$\alpha C_{1n} + C_{4n} = 0. \tag{f}$$

From condition (4) and the formulas (c) and (e) it follows that

$$q + \sum_{n=1}^{\infty} \alpha^2 C_{2n} \cos \alpha x = 0,$$

from which, by expanding the load on the interval

$$0 < \alpha x < 2\pi$$

in a series in $\cos \alpha x$

$$q = -\sum_{n=1}^{\infty} 2q \cos \alpha x,$$

we obtain

$$\sum_{n=1}^{\infty} \cos \alpha x \, (2q - \alpha^2 C_{2n}) = 0,$$

and hence,

$$C_{2n} = 2q/\alpha^2. \tag{g}$$

When $x = 0$, $2l$, the expansion formula is not valid, but the stresses at these points are infinitely large [boundary condition (5)].

From condition (6) and the formula (d) it follows that

$$\sum_{n=1}^{\infty} \alpha \sin \alpha x \, [\alpha \, (C_{1n} \cosh \alpha h + C_{2n} \sinh \alpha h + C_{3n} h \cosh \alpha h +$$

$$+ C_{4n} h \sinh \alpha h) + C_{3n} \sinh \alpha h + C_{4n} \cosh \alpha h] = 0.$$

By equating to zero each term in the sum, and replacing C_{4n} and C_{2n} by the formulas (f) and (g), we obtain

$$\alpha \left(C_{1n} \cosh \alpha h + \frac{2q}{\alpha^2} \sinh \alpha h + \right.$$

$$\left. + C_{3n} h \cosh \alpha h - \alpha C_{1n} h \sinh \alpha h \right) +$$

$$+ C_{3n} \sinh \alpha h - \alpha C_{1n} \cosh \alpha h = 0,$$

and after rearrangement

$$-\alpha^2 C_{1n} h \sinh \alpha h + C_{3n} \, (\alpha h \cosh \alpha h + \sinh \alpha h) =$$

$$= -\frac{2q}{\alpha} \sinh \alpha h. \tag{h}$$

From condition (7) and the formula (c) it follows that

$$q+\sum_{n=1}^{\infty}\alpha^2\cos\alpha x\,(C_{1n}\sinh\alpha h+C_{2n}\cosh\alpha h+$$
$$+C_{3n}h\sinh\alpha h+C_{4n}h\cosh\alpha h)=q,$$

from which, with the formulas (f) and (g), we obtain

$$C_{1n}(\sinh\alpha h-\alpha h\cosh\alpha h)+C_{3n}h\sinh\alpha h=-\frac{2q}{\alpha^2}\cosh\alpha h. \quad \text{(i)}$$

Checking of the stresses X_x at any vertical section shows that

$$\int_0^h X_x\,dy=0 \text{ for } A_3=0 \text{ [see the equation (i)]},$$

i.e., the equilibrium condition $\sum X = 0$ is satisfied.

By solving the equations (h) and (i) simultaneously, we obtain the values of the remaining unknown coefficients in the formula (a)

$$C_{1n}=-\frac{2q}{\alpha^2}\,\frac{\alpha h+\sinh\alpha h\cosh\alpha h}{\sinh^2\alpha h-\alpha^2h^2}, \quad \text{(j)}$$

$$C_{3n}=-\frac{2q}{\alpha}\,\frac{\sinh^2\alpha h}{\sinh^2\alpha h-\alpha^2h^2}. \quad \text{(k)}$$

An analysis of the numerical coefficients in the formulas (j) and (k) in the case of a beam of sufficiently large depth ($h\geqslant l$) allows us to assume simpler expressions for the quantities C_{in}:

$$C_{1n}\cong -2q/\alpha^2,\ C_{2n}=2q/\alpha^2,\ C_{3n}\cong -2q/\alpha,\ C_{4n}\cong 2q/\alpha. \quad \text{(l)}$$

Taking into account the formulas (l) and the relation $\cosh\alpha y-\sinh\alpha y=e^{-\alpha y}$,

we finally obtain the following expressions for the stresses:

$$X_x=-2q\sum_{n=1}^{\infty}\cos\alpha x\,(1-\alpha y)\,e^{-\alpha y},$$

$$Y_y=-q-2q\sum_{n=1}^{\infty}\cos\alpha x\,(1-\alpha y)\,e^{-\alpha y}, \quad \text{(m)}$$

$$X_y=Y_x=-2q\sum_{n=1}^{\infty}\alpha y\sin\alpha x\,e^{-\alpha y}.$$

The series in the formulas (m) converge very rapidly for all points except for those near the lower edge of the deep beam (for small y).

The calculations carried out for a deep beam with $h = 2l$ and $\delta = 1$ are given in Fig. 36.

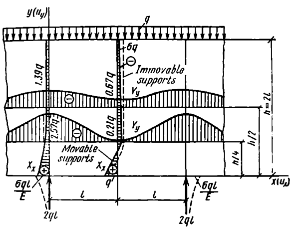

Fig. 36

The displacements are determined by integrating Eqs. (5.4)

$$\frac{\partial u_x}{\partial x} = \frac{1}{E}(X_x - \sigma Y_y), \quad \frac{\partial u_y}{\partial y} = \frac{1}{E}(Y_y - \sigma X_x),$$

$$\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{2(1+\sigma)}{E} X_y,$$

giving

$$u_x = \frac{2q}{E}\left\{\frac{\sigma x}{2} + \sum_{n=1}^{\infty} \frac{1}{\alpha} \sin \alpha x \left[(1+\sigma)\alpha y - (1-\sigma)\right] e^{-\alpha y}\right\} +$$

$$+ F_1(y),$$

$$u_y = \frac{2q}{E}\left\{-\frac{y}{2} + \sum_{n=1}^{\infty} \frac{1}{\alpha}\cos\alpha x\left[(1+\sigma)\alpha y + 2\right]e^{-\alpha y}\right\} + F_2(x),$$

$$\frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x} = 0, \tag{n}$$

where $F_1(y)$ and $F_2(x)$ are some functions satisfying the third relation of (n) and the deformation conditions.

The presence of concentrated forces introduces indeterminacy into vertical displacements, and the origin can be considered fixed only in relation to the horizontal displacements u_x.

Assuming, by convention, that when $x = l$ and $y = 0$, the displacements u_x and u_y are zero, we can determine the arbitrary functions F_i and compare the displacements of the other points with the chosen one.

Then

$$F_1(y) = -\sigma q l/E, \quad F_2(x) = -\frac{4q}{E}\sum_{n=1}^{\infty}\frac{1}{\alpha}\cos\alpha l; \tag{o}$$

as seen, the values given by (o) also satisfy the third equation of (n). When $x = 0$, $u_x = -\sigma q l/E$; when $x = 2l$, $u_x = \sigma q l/E$.

If the supports are immovable and the beam cannot extend freely, instead of the condition

$$\int_0^h X_x\,dy = 0$$

it is necessary to prescribe the condition that $u_x = 0$ when $x = 0$.

The basic equations remain the same, but the coefficient A_3 will be different from zero and

$$X_x = -\sigma q - 2q\sum_{n=1}^{\infty}\cos\alpha x(1-\alpha y)e^{-\alpha y};$$

the remaining stresses are determined from the formulas (m).

The X_x diagram at the middle section for the case of immovable supports is shown in Fig. 36 by a dashed line.

In this case the stress X_x is equal to $q(1-\sigma)$ when $y=0$, and the beam undergoes an additional uniform compression equal to σq.

5.4. Lévy's problem (1898) [5].

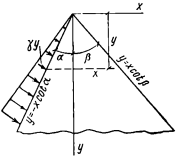

Fig. 37

Determine the stresses in an infinite thin wedge due to a fluid of specific weight γ and the specific weight of the wedge material p (Fig. 37).

The stress function is taken in the form of a third-degree homogeneous polynomial

$$\varphi(x, y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

where a, b, c, d are constants.

The boundary conditions of the problem are: when

$$y = -x\cot\alpha \quad \text{or} \quad x = -y\tan\alpha,$$

$$-X_x\cos\alpha - X_y\sin\alpha = \gamma y\cos\alpha,$$

$$-X_y\cos\alpha - Y_y\sin\alpha = \gamma y\sin\alpha;$$

when $y = x\cot\beta$ or $x = y\tan\beta$,

$$X_x\cos\beta - X_y\sin\beta = 0, \quad X_y\cos\beta - Y_y\sin\beta = 0,$$

where the stresses are determined by formulas (5.11) with $Y = p$ and $X = 0$.

By solving the equations expressing the boundary conditions, we obtain

$$a=\frac{1}{6(\tan\alpha+\tan\beta)^3}[p(\tan\beta-\tan\alpha)-$$
$$-\gamma(2-3\tan\alpha\tan\beta-\tan^2\alpha)],$$
$$b=\frac{1}{4}\left[-p-\frac{\gamma\tan\alpha}{\tan\alpha+\tan\beta}-6a(\tan\beta-\tan\alpha)\right],$$
$$c=\frac{1}{2}\tan\alpha\tan\beta\left(\frac{\gamma}{\tan\alpha+\tan\beta}-6a\right),$$
$$d=\frac{\tan^2\beta}{12}\left[-p-\frac{3\gamma\tan\alpha}{\tan\alpha+\tan\beta}+6a(3\tan\alpha+\tan\beta)\right].$$

The stresses are determined by formulas (5.11):

$$X_x=\frac{\partial^2\varphi}{\partial y^2}=2cx+6dy,\quad Y_y=\frac{\partial^2\varphi}{\partial x^2}=6ax+2by,$$
$$X_y=Y_x=-\frac{\partial^2\varphi}{\partial x\,\partial y}-px=-2cy-(2b+p)x.$$

Lévy's solution leads to a linear distribution of normal and shearing stresses, and can be used in the design of dams.

5.5. Galerkin's problem (1929) [35].

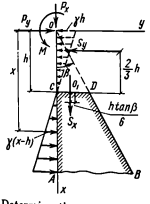

Fig. 38

Determine the stresses in an infinite trapezoidal section $ABCD$ due to the specific weight of material p (Fig. 38).

The solution is broken down into three steps:

(1) Assume the stress function φ for the triangular section AOB (see Problem 5.4).

(2) Set up the stress functions φ_1, φ_2, φ_3 for the triangular section AOB subjected at the vertex O to a horizontal force P_y, a vertical force P_x, and a moment M, respectively,

$$\varphi_1 = -\frac{y(\beta - \sin\beta\cos\beta) + x^2\sin^2\beta}{\beta^2 - \sin^2\beta} \arctan\frac{y}{x} P_y,$$

$$\varphi_2 = \frac{y\sin^2\beta + x(\beta\sin\beta\cos\beta)}{\beta^2 - \sin^2\beta} \arctan\frac{y}{x} P_x,$$

$$\varphi_3 = \frac{M}{2(\sin\beta - \beta\cos\beta)}\left(\frac{x^2-y^2}{x^2+y^2}\sin\beta - \frac{2xy}{x^2+y^2}\cos\beta + \right.$$

$$\left. + 2\cos\beta\arctan\frac{y}{x}\right).$$

(3) Choose the values of the forces P_y, P_x, and the moment M so that the stresses on the plane CD, given by the sum of the values of steps (1) and (2), will reduce to the force system prescribed on this plane.

According to Problem (5.4), with x replaced by y, we find

$$\varphi = \frac{p\cot\beta}{6}y^3 - \frac{p}{2}y^2x.$$

The gravity forces produce on the plane CD the forces (see Fig. 38)

$$S_y = 0, \quad S_x = \frac{ph^2}{2}\tan\beta, \quad M_{O_1} = S_x\frac{h\tan\beta}{6} = \frac{ph^3}{12}\tan^2\beta.$$

Since no external forces are applied to the plane CD, it follows that

$$\sum X = 0, \quad P_x + S_x = 0, \quad \sum Y = 0, \quad P_y = 0,$$

$$\sum M_{O_1} = 0, \quad P_x\frac{h\tan\beta}{2} + M + S_x\frac{h\tan\beta}{6} = 0,$$

from which we obtain

$$P_x = -\frac{ph^2}{2}\tan\beta, \quad P_y = 0, \quad M = \frac{ph^3}{6}\tan^2\beta.$$

The stress function for the trapezoidal section is

$$\Phi(x, y) = \varphi_1 + \varphi_2 + \varphi_3,$$

and the stresses are determined by the formulas

$$X_x = \frac{\partial^2 \Phi}{\partial y^2}, \quad Y_y = \frac{\partial^2 \Phi}{\partial x^2}, \quad X_y = -\frac{\partial^2 \Phi}{\partial x \, \partial y} - py.$$

▶ Determine the stresses in an infinite trapezoidal section $ABCD$ due to a fluid of specific weight γ (see Fig. 38).

The stress function φ for this case is

$$\varphi = -\frac{\gamma}{6}\left(2y^3 \cot\beta - 3y^2 x \cot\beta + 2yx^2 + x^3\right).$$

5.6. Kolosov's problem (1910) [27].

An infinite plate is weakened by an elliptical hole and subjected to a uniform extension with stresses p directed at an angle β_1 to the x axis (Fig. 39).

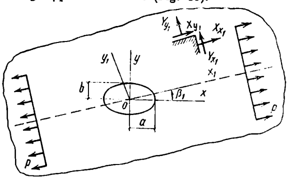

Fig. 39

Investigate the state of stress.

The region outside the elliptical hole in the complex plane $z = x + iy$ can be mapped onto the region outside the unit circle in the complex plane $\zeta = \xi + i\eta$ by the mapping function [27]

$$z = f(\zeta) = c\,(\zeta + m/\zeta), \tag{a}$$

where $0 \leqslant m \leqslant 1$, $c > 0$.

The contour of the unit circle $|\,\zeta\,| = 1$ corresponds to an ellipse centred at the origin for the z plane and having the semiaxes

$$a = c\,(1 + m), \quad b = c\,(1 - m). \tag{b}$$

According to the formulas (b),

$$c = (a + b)/2, \quad m = (a - b)/(a + b). \tag{c}$$

When the axes are rotated through the angle β_1, the relation between the stresses is, by formulas (1.5) and (1.3),

$$X_{x_1} + Y_{y_1} = X_x + Y_y,$$

$$Y_{y_1} - X_{x_1} + 2iX_{y_1} = e^{2i\beta_1}(Y_y - X_x - 2iX_y).$$

Since at infinity

$$X_{x_1}^{(\infty)} = p, \quad Y_{y_1}^{(\infty)} = X_{y_1}^{(\infty)} = 0,$$

it follows that

$$X_x + Y_y = p, \quad Y_y - X_x + 2iX_y = -pe^{-2i\beta_1}.$$

According to formulas (5.21), at infinity

$$4 \operatorname{Re} \psi'(z) = p, \quad 2[\bar{z}\psi''(z) + \chi''(z)] = -pe^{-2i\beta_1}. \tag{d}$$

For the elliptical hole, by (5.22) and (5.19),

$$\begin{aligned} &F_x + iF_y = 0, \\ &\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)} = 0. \end{aligned} \tag{e}$$

With the help of the formula (a) we find

$$\psi(z) = \psi[f(\zeta)] = \psi_1(\zeta),$$

$$\chi(z) = \chi[f(\zeta)] = \chi_1(\zeta).$$

Then

$$\begin{aligned} &\psi'(z) = \frac{d\psi}{dz} = \frac{d\psi_1}{d\zeta}\frac{d\zeta}{dz} = \frac{\psi_1'(\zeta)}{f'(\zeta)}, \\ &\chi'(z) = \frac{\partial\chi}{dz} = \frac{d\chi_1}{d\zeta}\frac{d\zeta}{dz} = \frac{\chi_1'(\zeta)}{f'(\zeta)}, \\ &\psi''(z) = \frac{d}{d\zeta}\left[\frac{\psi_1'(\zeta)}{f'(\zeta)}\right]\frac{d\zeta}{dz} = \frac{\psi_1''(\zeta)f'(\zeta) - \psi_1'(\zeta)f''(\zeta)}{[f'(\zeta)]^3}, \\ &\chi''(z) = \frac{d}{d\zeta}\left[\frac{\chi_1'(\zeta)}{f'(\zeta)}\right]\frac{d\zeta}{dz} = \frac{\chi_1''(\zeta)f'(\zeta) - \chi_1'(\zeta)f''(\zeta)}{[f'(\zeta)]^3}. \end{aligned} \tag{f}$$

Substituting the expressions (f) in formulas (5.20), (5.21), (5.22), and (5.23), we obtain

$$u_x + iu_y = \frac{3-\sigma}{E}\psi_1(\zeta) - \frac{1+\sigma}{E}\left[\frac{f(\zeta)}{f'(\zeta)}\overline{\psi_1'(\zeta)} + \overline{\chi_1'(\zeta)}\right], \tag{g}$$

$$X_x + Y_y = 4\mathrm{Re}\frac{\psi_1'(\zeta)}{f'(\zeta)}, \tag{h}$$

$$Y_y - X_x + 2iX_y = \frac{2}{[\overline{f'(\zeta)}]^3}\,[\overline{f(\zeta)}\,\psi_1''(\zeta)\,f'(\zeta) - $$
$$ - \overline{f(\zeta)}\,\psi_1'(\zeta)\,f''(\zeta) + \chi_1''(\zeta)\,f'(\zeta) - \chi_1'(\zeta)\,f''(\zeta)],$$

$$F_x + iF_y = -i\left[\psi_1(\zeta) + \frac{f(\zeta)}{\overline{f'(\zeta)}}\,\overline{\psi_1'(\zeta)} + \frac{\overline{\chi_1'(\zeta)}}{\overline{f_1'(\zeta)}}\right]_A^B, \tag{i}$$

$$M = \mathrm{Re}\left[\chi_1(\zeta) - \frac{f(\zeta)}{f'(\zeta)}\,\chi_1'(\zeta) - \frac{f(\zeta)\,\overline{f(\zeta)}}{f'(\zeta)}\,\psi_1'(\zeta)\right]_A^B.$$

For the stresses to be single valued, the functions $\psi_1'(\zeta)$ and $\chi_1''(\zeta)$ must be of the form

$$\psi_1'(\zeta) = \sum_{n=0}^{\infty} A_n\zeta^{-n}, \quad \chi_1''(\zeta) = \sum_{n=0}^{\infty} B_n\zeta^{-n}. \tag{j}$$

It is seen from the equations (e) and (i) that the following relation holds on the circle $|\zeta| = 1$:

$$\overline{f'(\zeta)}\,\psi_1(\zeta) + f(\zeta)\,\overline{\psi_1'(\zeta)} + \overline{\chi_1'(\zeta)} = 0; \tag{k}$$

also, for the conjugate functions

$$f'(\zeta)\,\overline{\psi_1(\zeta)} + \overline{f(\zeta)}\,\psi_1'(\zeta) + \chi_1'(\zeta) = 0. \tag{k'}$$

By integrating the formulas (j), we obtain

$$\psi_1(\zeta) = A_0\zeta + A_1\ln\zeta + \sum_{n=2}^{\infty}\frac{A_n\zeta^{1-n}}{1-n} + A,$$
$$\chi_1'(\zeta) = B_0\zeta + B_1\ln\zeta + \sum_{n=2}^{\infty}\frac{B_n\zeta^{1-n}}{1-n} + B, \tag{l}$$

where A and B are complex constants characterizing a rigid-body displacement.

Substituting the expressions (l) in the formula (g), we find the condition for single-valuedness of displacements, viz. that the coefficient of $\ln\zeta$ should be zero:

$$(3-\sigma)\,A_1 + (1+\sigma)\,\overline{B}_1 = 0. \tag{m}$$

For $|\zeta| = 1$ we have

$$f'(\zeta) = c\left(1 - \frac{m}{\zeta^2}\right) = c\,(1 - me^{i2\beta}),$$

$$\overline{f(\zeta)} = c\left(\bar{\zeta} + \frac{m}{\zeta}\right) = c\,(e^{-i\beta} + me^{i\beta}),$$

$$\overline{\psi_1(\zeta)} = \bar{A}_0\bar{\zeta} + \bar{A}_1 \ln \bar{\zeta} + \sum_{n=2}^{\infty} \frac{\bar{A}_n \bar{\zeta}^{1-n}}{1-n} =$$

$$= \bar{A}_0 e^{-i\beta} - iA_1\beta - \sum_{n=2}^{\infty} \frac{1}{n-1} \bar{A}_n e^{i(n-1)\beta},$$

$$\psi_1'(\zeta) = \sum_{n=0}^{\infty} A_n e^{in\beta},$$

$$\chi_1'(\zeta) = B_0 e^{i\beta} + iB_1\beta - \sum_{n=2}^{\infty} \frac{1}{n-1} B_n e^{i(1-n)\beta}.$$

Substituting these expressions in the equations (k) and (k'), and equating the coefficients of β and $e^{in\beta}$, we find

$$\begin{aligned} &c\bar{A}_1 + B_1 = 0, \\ &-c\bar{A}_2 + \frac{cm}{3}\bar{A}_4 + cmA_0 + B_0 = 0, \\ &cA_1 + cmA_3 - \frac{B_3}{3} = 0, \\ &c\bar{A}_0 + cm\bar{A}_2 + cA_0 + mA_2 - B_2 = 0, \\ &-cm\bar{A}_0 + cA_2 + cmA_4 - \frac{B_4}{3} = 0. \end{aligned} \qquad \text{(n)}$$

When $n \geqslant 3$, $A_n = 0$; when $n \geqslant 5$, $B_n = 0$. According to the formulas (d),

$$A_0 = \frac{cp}{4}, \quad B_0 = -\frac{c^2 p}{2} e^{-2i\beta_1}.$$

By solving the first equation of (n) simultaneously with the equation (m), we obtain $A_1 = B_1 = 0$. By solving the remaining equations of the system (n), we find

$$A_2 = \frac{cp}{4}(m - 2e^{2i\beta_1}),$$

$$B_2 = \frac{c^2 p}{2}(1 + m^2 - 2m\cos\beta_1),$$

$$B_3 = 0, \quad B_4 = -\frac{3cp}{2}e^{2i\beta_1}.$$

The required functions are

$$\psi'(\zeta) = \frac{cp}{4} + \frac{cp}{4}(m - 2e^{2i\beta_1})\zeta^{-2},$$

$$\chi''(\zeta) = -\frac{c^2 p}{2}e^{-2i\beta_1} + \frac{c^2 p}{2}(1 - m^2 - 2m\cos 2\beta_1)\zeta^{-2} - -\frac{3cp}{2}e^{2i\beta_1}\zeta^{-4}.$$

From this, according to the formula (h), we obtain

$$X_x + Y_y = \frac{p}{1 + m^2 - 2m\cos 2\beta}[1 - m^2 - 2\cos 2(\beta_1 - \beta) + + 2m\cos 2\beta\cos 2(\beta_1 - \beta) - 2m\sin 2\beta\sin 2(\beta_1 - \beta)].$$

The maximum stress occurs at the end of the semimajor axis ($\beta = 0$) for $\beta_1 = \pi/2$. At this point, $X_x = 0$ and

$$\max Y_y = p\frac{1 - m^2 + 2 - 2m}{1 + m^2 - 2m} = p\frac{3 + m}{1 - m}.$$

Taking into account the formulas (c), we obtain, finally,

$$\max Y_y = p\,(1 + 2a/b). \qquad \text{(o)}$$

The equation (o) expresses the results of Kolosov's problem.

▶ When $m = 0$ and $\beta_1 = 0$, the solution for a circular hole is obtained; this is Kirsch's problem (see Problem 6.7).

▶ Find the law of stress distribution in an infinite plate weakened by a square hole and subjected to an extension in two directions; the tensile forces at infinity $X_{x_1}^\infty = Y_{y_1}^\infty = p$, $X_{y_1}^\infty = 0$ are inclined to the x axis at an angle β_1. Map the square hole in the z plane onto the unit circle in the ζ plane by the mapping function

$$z \cong c\left(\frac{1}{\zeta}-\frac{1}{6}\zeta^3\right), \tag{p}$$

where $c = 3a/5$, a is the length of the side of the curvilinear square (Fig. 40).

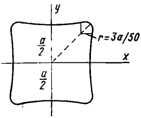

Fig. 40

The equations of the contour are obtained from the formula (p) for $|\zeta| = 1$:

$$x = c\left(\cos\beta-\frac{1}{6}\cos 3\beta\right), \quad y = -c\left(\sin\beta+\frac{1}{6}\sin 3\beta\right)$$

(see the monograph [37], p. 64).

5.7. See the monograph [28].
Solve the first fundamental problem for the half-plane $y \leqslant 0$. On the boundary of the half-plane (axis Ox) are applied normal $Y_y = N(x)$ and shearing $X_y = T(x)$ stresses, which are continuous and satisfy, for large $|x|$, the conditions $N = O(1/x)$, $T = O(1/x)$.

By adding formulas (5.21) together, we obtain

$$Y_y + iX_y = \psi'(z) + \overline{\psi'(z)} + \bar{z}\psi''(z) + \chi''(z),$$

and hence the boundary condition may be written in the form

$$N + iT = \Phi(t) + \overline{\Phi(t)} + \bar{t}\Phi'(t) + \psi(t),$$

or, what is the same thing,

$$N - iT = \overline{\Phi(t)} + \Phi(t) + t\overline{\Phi'(t)} + \overline{\psi(t)}, \tag{a}$$

where, for simplicity, we have written

$\psi'(t) = \Phi(t), \quad \chi''(t) = \psi(t).$

For the case when the functions (a) are holomorphic and vanish at infinity, it is proved ([28], p. 361) that they are equal to

$$\Phi(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{N-iT}{t-z}\,dt, \tag{b}$$

$$\psi(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{N+iT}{t-z}\,dt + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{N-iT}{(t-z)^2}\,t\,dt. \tag{c}$$

Michell's problem (1902) is solved in a similar way ([28], Sec. 93a).

Determine the state of stress in the half-plane $y \leqslant 0$ when the segment $-a \leqslant t \leqslant a$ of the axis Ox is subjected to a uniform pressure p (Fig. 41).

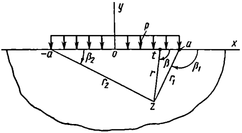

Fig. 41

In the case under consideration $T = 0$; $N = -p$ when $-a \leqslant t \leqslant a$, and $N = 0$ for the other values of t.

According to the formulas (b) and (c),

$$\Phi(z) = -\frac{p}{2\pi i}\int_{-a}^{a} \frac{dt}{z-t} = \frac{p}{2\pi i}\left[\ln(z-t)\right]_{t=-a}^{t=a} = \frac{p}{2\pi i}\ln\frac{z-a}{z+a},$$

$$\psi(z) = -\frac{zp}{2\pi i}\int_{-a}^{a} \frac{dt}{(t-z)^2} = -\frac{paz}{\pi i\,(z^2-a^2)}.$$

The expression $\ln \frac{z-a}{z+a}$ is the increment of the function $\ln(z-t)$ when t varies continuously from $-a$ to $+a$: $z-t=re^{i\beta}$; $\ln(z-t)=\ln r-i\beta$, $\ln\frac{z-a}{z+a}=\ln\frac{r_1}{r_2}-i(\beta_1-\beta_2)$.

The stresses are determined by formulas (5.21):

$$X_x+Y_y=4\,\mathrm{Re}\,\Phi(z)=-\frac{2p}{\pi}(\beta_1-\beta_2), \tag{d}$$

$$Y_y-X_x+2iX_y=2\left[\bar{z}\Phi'(z)+\psi(z)\right]=$$

$$=\frac{2pa}{\pi i}\frac{\bar{z}-z}{z^2-a^2}=\frac{4\,pay}{\pi(z^2-a^2)}=\frac{4\,pay\,(\bar{z}^2-a^2)}{\pi(z^2-a^2)(\bar{z}^2-a^2)}; \tag{e}$$

hence,

$$X_x=-\frac{p}{\pi}(\beta_1-\beta_2)+\frac{2\,pay\,(x^2-y^2-a^2)}{\pi\left[(x^2+y^2-a^2)^2+4a^2y^2\right]},$$

$$Y_y=-\frac{p}{\pi}(\beta_1-\beta_2)-\frac{2\,pay\,(x^2-y^2-a^2)}{\pi\left[(x^2+y^2-a^2)^2+4a^2y^2\right]}, \tag{f}$$

$$X_y=\frac{4\,paxy^2}{\pi\left[(x^2+y^2-a^2)^2+4a^2y^2\right]}.$$

Assuming $z^2-a^2=r_1r_2e^{-i(\beta_1+\beta_2)}$ in the formula (e), we obtain

$$Y_y-X_x+2iX_y=-\frac{4\,pay}{r_1r_2}e^{i(\beta_1+\beta_2)}, \tag{e'}$$

and taking into account the formula (d), we find

$$X_x=-\frac{p}{\pi}(\beta_1-\beta_2)+2\,pa\frac{y\cos(\beta_1+\beta_2)}{r_1r_2},$$

$$Y_y=-\frac{p}{\pi}(\beta_1-\beta_2)-2\,pa\frac{y\cos(\beta_1+\beta_2)}{r_1r_2}, \tag{f'}$$

$$X_y=-2pa\frac{y\sin(\beta_1+\beta_2)}{r_1r_2}.$$

The stresses are continuous up to the boundary. At the points $t=\pm a$, the stresses cease to be continuous, remaining bounded ($y=-r_1\sin\beta_1=-r_2\sin\beta_2$).

The displacements are continuous up to the whole boundary (including the points $t=\pm a$). When $|z|\to\infty$, the displacements increaso as $\ln|z|$.

▶ Determine the state of stress in the half-plane $y \leqslant 0$ when constant shearing stresses are applied on the segment $-a \leqslant t \leqslant a$ of the axis Ox.

5.8. See the monograph [38], Chap. VII.

A rectangular strip along the x axis ($y = 0$ and $y = h$) is placed between absolutely rigid and smooth planes;

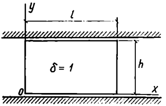

Fig. 42

arbitrary (static, geometrical, or mixed) conditions are specified at the ends of the strip ($x = 0$ and $x = l$) (Fig. 42).

Determine the state of stress and strain for the case of plane stress ($\delta = 1$).

When $y = 0$ and $y = h$,

$$u_y = X_y = 0. \tag{a}$$

In this case, for the initial functions we have

$$u_y^0 = X_y^0 = 0.$$

By satisfying the boundary conditions (a) when $y = h$, we obtain a system of two differential equations of infinitely high order in the unknown initial functions u_x^0 and Y_y^0:

$$\begin{aligned} &[(1-\sigma)\sin(\alpha h) - (1+\sigma)\,\alpha h\cos(\alpha h)]\,u_x^0 + \\ &+\frac{1}{2}\left[(3-\sigma)\frac{\sin \alpha h}{\alpha} - (1+\sigma)\,h\cos(\alpha h)\right]Y_y^0 = 0, \\ &-2(1+\sigma)\,\alpha\,[\sin(\alpha h) + \alpha h\cos(\alpha h)]\,u_x^0 + \\ &+[(1-\sigma)\sin(\alpha h) - (1+\sigma)\,\alpha h\cos(\alpha h)]\,Y_y^0 = 0. \end{aligned} \tag{b}$$

Introducing the solving function $F(x)$ by the formulas

$$u_x^0 = \left[\frac{1-\sigma}{1+\sigma} \sin(\alpha h) - \alpha h \cos(\alpha h) \right] F,$$

$$Y_y^0 = 2\alpha [\sin(\alpha h) + \alpha h \cos(\alpha h)] F, \tag{c}$$

we identically satisfy the second equation of (b), and the first equation becomes

$$[\sin^2(\alpha h)] F = 0. \tag{d}$$

The solution of the equation (d) is taken in the form

$$F = Ce^{kx}. \tag{e}$$

Substituting the expression (o) in the equation (d), we arrive at the equation

$$\sin^2(kh) = 0, \tag{f}$$

from which

$$k = k_n = n\pi/h,$$

where n is any positive integer.

Thus, the general solution of the equation (d) is

$$F = \sum_{n=0}^{\infty} A_n \cosh(k_n x) + B_n \sinh(k_n x) + C_n x \cosh(k_n x) + D_n x \sinh(k_n x), \tag{g}$$

where A_n, B_n, C_n, D_n are arbitrary constants.

The general solution does not incorporate the elementary solution in terms of polynomials corresponding to the zero roots of the equation (f).

To find the elementary solution, we represent the equations (b) as infinite series and separate the first terms in them

$$-2\sigma\alpha u_x^0 + (1-\sigma) Y_y^0 = 0, \quad 2(1+\sigma)\alpha^2 u_x^0 + \sigma\alpha Y_y^0 = 0. \tag{h}$$

We obtain, from the equations (h),

$$u_x^0 = \frac{1-\sigma}{2} B_0 x + A_0, \quad Y_y^0 = \sigma B_0, \quad X_x = B_0. \tag{i}$$

By using the expressions (i), (c), (g), and (5.29), we arrive at the following formulas:

$$u_x = A_0 + \frac{1-\sigma}{2} B_0 x - \sum_{n=1}^{\infty} (-1)^n h \Big\{ k_n \sinh(k_n x) A_n +$$

$$+ k_n \cosh(k_n x) B_n + \Big[\frac{2\sigma}{1+\sigma} \cosh(k_n x) + k_n x \sinh(k_n x) \Big] C_n +$$

$$+ \Big[\frac{2\sigma}{1+\sigma} \sinh(k_n x) + k_n x \cosh(k_n x) \Big] D_n \Big\} \cos(k_n y),$$

$$u_y = \sum_{n=1}^{\infty} (-1)^n h \Big\{ k_n \cosh(k_n x) A_n + k_n \sinh(k_n x) B_n +$$

$$+ \Big[\frac{3+\sigma}{1+\sigma} \sinh(k_n x) + k_n x \cosh(k_n x) \Big] C_n +$$

$$+ \Big[\frac{3+\sigma}{1+\sigma} \cosh(k_n x) + k_n x \sinh(k_n x) \Big] D_n \Big\} \sin(k_n y),$$

$$Y_y = \sigma B_0 + 2 \sum_{n=1}^{\infty} (-1)^n k_n h \{ k_n \cosh(k_n x) A_n + \qquad \text{(j)}$$

$$+ k_n \sinh(k_n x) B_n + [3 \sinh(k_n x) + k_n x \cosh(k_n x)] C_n +$$

$$+ [3 \cosh(k_n x) + k_n x \sinh(k_n x)] D_n \} \cos(k_n y),$$

$$X_x = B_0 - 2 \sum_{n=1}^{\infty} (-1)^n k_n h \{ k_n \cosh(k_n x) A_n +$$

$$+ k_n \sinh(k_n x) B_n + [\sinh(k_n x) + k_n x \cosh(k_n x)] C_n +$$

$$+ [\cosh(k_n x) + k_n x \sinh(k_n x)] D_n \} \cos(k_n y),$$

$$X_y = 2 \sum_{n=1}^{\infty} (-1)^n k_n h \{ k_n \sinh(k_n x) A_n +$$

$$+ k_n \cosh(k_n x) B_n + [2 \cosh(k_n x) + k_n x \sinh(k_n x)] C_n +$$

$$+ [2 \sinh(k_n x) + k_n x \cosh(k_n x)] D_n \} \sin(k_n y).$$

With the help of the arbitrary constants A_0, B_0, A_n, B_n, C_n, D_n ($n = 1, 2, 3, \ldots, \infty$) we can satisfy any boundary conditions on the sides $x = 0$ and $x = l$.

The formulas (j) are generalizations of the well-known Filon [39] and Ribière [40] solutions (see also [1], Chap. X).

▶ Solve a similar problem in the case when $u_x = Y_y = 0$ for $y = 0$ and $y = h$ (see the monograph [38], Chap. VII).

▶ Solve a similar problem in the case when $Y_y = X_y = 0$ for $y = 0$ and $y = h$ (see [41]).

5.9. See [42].

A thin rectangular strip ($\delta = 1$) is loaded by a force equal to unity at the point $x = c$, $y = h$. The edges of the

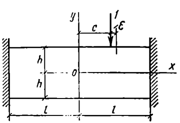

Fig. 43

strip ($x = \pm l$) are clamped ($u_x = 0$ throughout the depth $2h$, $u_y = 0$ when $y = 0$) (Fig. 43).

Formulate the boundary conditions of the problem.

The concentrated force at the point $x = c$ may be represented on the interval $-l \leqslant x \leqslant l$ as the limit of the function

$$f(x) = \begin{cases} 0 & \text{for } -l \leqslant x < c, \\ p & \text{for } c \leqslant x \leqslant c + \varepsilon, \\ 0 & \text{for } c + \varepsilon < x \leqslant l \end{cases} \tag{a}$$

as $\varepsilon \to 0$; the product $p\varepsilon$ remains finite and is equal to unity.

Taking into account the expression (a), we replace the force by the trigonometric series

$$f(x) = \frac{1}{2l} + \frac{1}{l} \sum_{n=1}^{\infty} \cos\frac{n\pi c}{l} \cos\frac{n\pi x}{l} + \frac{1}{l} \sum_{n=1}^{\infty} \sin\frac{n\pi c}{l} \sin\frac{n\pi x}{l}.$$

The boundary conditions on the longitudinal edges are: when $y = h$,

$$Y_y = \frac{1}{2l} + \frac{1}{l}\sum_{n=1}^{\infty}\cos\frac{n\pi c}{l}\cos\frac{n\pi x}{l} + \frac{1}{l}\sum_{n=1}^{\infty}\sin\frac{n\pi c}{l}\sin\frac{n\pi x}{l},$$

$X_y = 0$;

when $y = -h$,

$Y_y = X_y = 0.$

The boundary conditions on the lateral edges ($x = \pm l$) are: $u_x = 0$ for $-h < y < h$; $u_y = 0$ for $y = 0$.

The solution of the problem using the homogeneous boundary value solutions is given in [42].

5.10. See [43].

Analyse a square deep beem loaded according to Fig. 44.

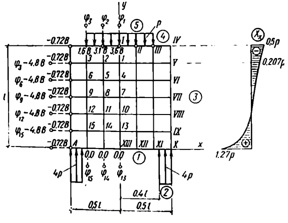

Fig. 44

Let us calculate the value of the stress function on the contour.

Portion 1:

$$Y_y = \frac{\partial^2\varphi}{\partial x^2} = 0, \quad \frac{\partial\varphi}{\partial x} = c_1, \quad \varphi = c_1 x + c_2,$$

$$X_y = -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0, \quad \frac{\partial\varphi}{\partial y} = c_3.$$

Assume $c_1 = c_2 = c_3 = 0$.

Portion 2:

$$Y_y = \frac{\partial^2\varphi}{\partial x^2} = -4p, \quad \frac{\partial\varphi}{\partial x} = -4px + c_4,$$

$$\varphi = -2px^2 + c_4 x + c_5,$$

$$X_y = -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0, \quad \frac{\partial\varphi}{\partial y} = c_6.$$

At the boundary of portions *1* and *2* the values of the function and its derivatives must coincide, and hence

$$-4px + c_4\,|_{x=0.4l} = 0, \quad \frac{\partial\varphi}{\partial y} = c_6 = 0,$$

$$-2px^2 + c_4 x + c_5\,|_{x=0.4l} = 0.$$

Consequently, $c_4 = 1.6pl$, $c_5 = -0.32pl^2$, and for portion *2*

$$\varphi = -2px^2 + 1.6plx - 0.32pl^2, \quad \frac{\partial\varphi}{\partial y} = 0.$$

When $x = l/2$, $\varphi = 0.72B$, where $B = pl^2/36$.

Portion 3:

$$X_x = \frac{\partial^2\varphi}{\partial y^2} = 0, \quad \frac{\partial\varphi}{\partial y} = c_7, \quad \varphi = c_7 y + c_8,$$

$$Y_x = -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0, \quad \frac{\partial\varphi}{\partial x} = c_9.$$

From the equality of the boundary values for portions *2* and *3* we have

$$c_7 = 0, \quad c_8 = (-2px^2 + 1.6plx - 0.32pl^2)_{x=0.6l},$$

$$c_9 = (-4px + 1.6pl)_{x=0.6l};$$

hence, $c_8 = -0.02pl^2$, $c_9 = -0.4pl$, and for portion *3*

$$\varphi = -0.02pl^2 = -0.72B, \quad \frac{\partial\varphi}{\partial x} = -0.4pl, \quad \frac{\partial\varphi}{\partial y} = 0.$$

Portion 4:

$$Y_y = \frac{\partial^2\varphi}{\partial x^2} = 0, \quad \frac{\partial\varphi}{\partial x} = c_{10}, \quad \varphi = c_{10}x + c_{11},$$

$$X_y = -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0, \quad \frac{\partial\varphi}{\partial y} = c_{12}.$$

From the equality of the boundary values for portions 3 and 4 we have

$$c_{12} = 0, \quad c_{10} = -0.4pl, \quad (c_{10}x + c_{11})_{x=0.5l} = -0.02pl^2;$$

hence, $c_{11} = 0.18pl^2$, and for portion 4

$$\varphi = -0.4plx + 0.18pl^2, \quad \frac{\partial\varphi}{\partial x} = -0.4pl, \quad \frac{\partial\varphi}{\partial y} = 0.$$

Portion 5:

$$Y_y = \frac{\partial^2\varphi}{\partial x^2} = -p, \quad \frac{\partial\varphi}{\partial x} = -px + c_{13},$$

$$\varphi = -0.5px^2 + c_{13}x + c_{14},$$

$$X_y = -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0, \quad \frac{\partial\varphi}{\partial y} = c_{15}.$$

From the equality of the boundary values for portions 4 and 5 we have

$$(-px + c_{13})_{x=0.4l} = -0.4pl,$$

$$(-0.5px^2 + c_{13}x + c_{14})_{x=0.4l} =$$

$$= (-0.4plx + 0.18pl^2)_{x=0.4l};$$

hence, $c_{13} = 0$, $c_{14} = 0.10pl^2$, $c_{15} = 0$, and for portion 5

$$\varphi = -0.5px^2 + 0.10pl^2, \quad \frac{\partial\varphi}{\partial x} = -px, \quad \frac{\partial\varphi}{\partial y} = 0.$$

When $x = 0$, $\varphi = 3.6B$; when $x = l/6$, $\varphi = 3.1B$; when $x = l/3$, $\varphi = 1.6B$.

The values of the function φ outside the contour are obtained by formulas (5.41):

along the lower edge

$$\varphi_y = \varphi_n + 2\Delta y\left(\frac{\partial\varphi}{\partial y}\right)_v = \varphi_n$$

since $(\partial\varphi/\partial y)_v = 0$, where φ_n is any lower value of φ for the inside nodal points adjacent to the contour;

along the lateral side

$$\varphi_x = \varphi_l + 2\Delta x\left(\frac{\partial\varphi}{\partial x}\right)_l = \varphi_l + 2\frac{l}{8}(-0.4pl) = \varphi_l - 4.8B,$$

where φ_l is any lateral value of φ inside the contour;
along the upper edge

$$\varphi_y = \varphi_n + 2\Delta y\left(\frac{\partial\varphi}{\partial y}\right)_v = \varphi_n,$$

where φ_n is any upper value of φ inside the contour.

The contour values of φ and the values outside the contour are given in Fig. 44.

By applying Eqs. (5.39′) to individual nodes, we find:
node 1

$$20\varphi_1 - 8(3.6B + \varphi_4 + \varphi_2 + \varphi_2) + 2(\varphi_5 + 3.1B + $$
$$+ 3.1B + \varphi_5) + \varphi_7 + \varphi_1 + \varphi_3 + \varphi_3 = 0,$$

or

$$21\varphi_1 - 16\varphi_2 + 2\varphi_3 - 8\varphi_4 + 4\varphi_5 + \varphi_7 = -16.4B;$$

node 2

$$20\varphi_2 - 8(3.1B + \varphi_5 + \varphi_3 + \varphi_1) + 2(\varphi_6 + 3.6B + $$
$$+ 1.6B + \varphi_4) + \varphi_5 - 0.72B + \varphi_8 + \varphi_2 = 0,$$

or

$$-8\varphi_1 + 22\varphi_2 - 8\varphi_3 + 2\varphi_4 - 8\varphi_5 + \varphi_8 = -15.12B,$$

etc. (for all 15 nodes shown in Fig. 44).

By solving the system of 15 equations, we obtain

$\varphi_1 = 3.356B, \quad \varphi_2 = 2.885B, \quad \varphi_3 = 1.482B,$

$\varphi_4 = 2.906B, \quad \varphi_5 = 2.512B, \quad \varphi_6 = 1.311B,$

$\varphi_7 = 2.306B, \quad \varphi_8 = 2.024B, \quad \varphi_9 = 1.097B,$

$\varphi_{10} = 1.531B, \quad \varphi_{11} = 1.381B, \quad \varphi_{12} = 0.800B,$

$\varphi_{13} = 0.634B, \quad \varphi_{14} = 0.608B, \quad \varphi_{15} = 0.396B.$

The graphical representation of the surface showing the stress function φ is given in Fig. 45.

The stresses are determined by formulas (5.40); for example:

point I

$$X_x^{I}=\frac{(3.356-2\times3.600+3.356)\,B}{l^2/36}=-0.50p \quad \text{(compression)};$$

point 1

$$X_x^{1}=\frac{(3.600-2\times3.356+2.885)\,B}{l^2/36}=-0.207p \quad \text{(compression)};$$

point XIII

$$X_x^{XIII}=\frac{(0.634-2\times0+0.634)\,B}{l^2/36}=1.27p \quad \text{(tension)}.$$

The diagram for the normal stresses X_x on the section coinciding with the y axis is given in Fig. 44.

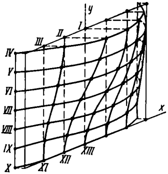

Fig. 45

In solving the finite-difference equations extensive use can be made of the modern computing technique.

Chapter 6

PLANE PROBLEM IN POLAR CO-ORDINATES

I. PLANE STRESS

$$Z_z = Z_r = Z_\beta = e_{zr} = e_{z\beta} = 0, \quad e_{zz} = -\frac{\sigma}{E}(R_r + B_\beta).$$

The equations of this chapter are obtained as a special case from the corresponding equations in cylindrical co-ordinates (1.1b, 2.1b, etc.).

1. Equilibrium equations

$$\begin{aligned} &\frac{\partial R_r}{\partial r} + \frac{1}{r}\frac{\partial R_\beta}{\partial \beta} + \frac{R_r - R_\beta}{r} + R = 0\left(= \rho\frac{\partial^2 u_r}{\partial \tau^2}\right), \\ &\frac{\partial B_r}{\partial r} + \frac{1}{r}\frac{\partial B_\beta}{\partial \beta} + \frac{2B_r}{r} + B = 0\left(= \rho\frac{\partial^2 u_\beta}{\partial \tau^2}\right). \end{aligned} \tag{6.1}$$

2. Geometrical equations

$$\begin{aligned} &e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\beta\beta} = \frac{1}{r}\frac{\partial u_\beta}{\partial \beta} + \frac{u_r}{r}, \\ &e_{r\beta} = \frac{1}{r}\frac{\partial u_r}{\partial \beta} + \frac{\partial u_\beta}{\partial r} - \frac{u_\beta}{r}. \end{aligned} \tag{6.2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial e_{\beta\beta}}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 e_{rr}}{\partial \beta^2} - \frac{\partial e_{rr}}{\partial r} = \frac{1}{r}\frac{\partial^2 (r e_{r\beta})}{\partial r\,\partial \beta}. \tag{6.3}$$

3. Physical equations

$$e_{rr}=\frac{1}{E}(R_r-\sigma B_\beta),\quad e_{\beta\beta}=\frac{1}{E}(B_\beta-\sigma R_r),$$
$$e_{r\beta}=\frac{2(1+\sigma)}{E}R_\beta. \tag{6.4}$$

$$R_r=\frac{E}{1-\sigma^2}(e_{rr}+\sigma e_{\beta\beta}),\quad B_\beta=\frac{E}{1-\sigma^2}(e_{\beta\beta}+\sigma e_{rr}),$$
$$R_\beta=\frac{E}{2(1+\sigma)}e_{r\beta}. \tag{6.5}$$

4. Basic equations in terms of stresses

$$\frac{\partial R_r}{\partial r}+\frac{1}{r}\frac{\partial R_\beta}{\partial\beta}+\frac{R_r-B_\beta}{r}+R=0,$$
$$\frac{\partial B_r}{\partial r}-\frac{1}{r}\frac{\partial B_\beta}{\partial\beta}+\frac{2B_r}{r}+B=0, \tag{6.6}$$
$$\nabla^2(R_r+B_\beta)=0,$$

where

$$\nabla^2(\ldots)=\frac{\partial^2(\ldots)}{\partial r^2}+\frac{1}{r}\frac{\partial(\ldots)}{\partial r}+\frac{1}{r^2}\frac{\partial^2(\ldots)}{\partial\beta^2}$$

5. Basic equations in terms of displacements

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{1-\sigma}{2r^2}\frac{\partial^2}{\partial\beta^2}\right)u_r+$$
$$+\frac{\partial}{\partial\beta}\left(\frac{1+\sigma}{2r}\frac{\partial}{\partial r}-\frac{3-\sigma}{2r^2}\right)u_\beta+\frac{1-\sigma^2}{E}R=0,$$
$$\frac{\partial}{\partial\beta}\left(\frac{1+\sigma}{1-\sigma}\frac{1}{r}\frac{\partial}{\partial r}+\frac{3-\sigma}{1-\sigma}\frac{1}{r^2}\right)u_r+ \tag{6.7}$$
$$+\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{2}{1-\sigma}\frac{1}{r^2}\frac{\partial^2}{\partial\beta^2}\right)u_\beta+\frac{2(1+\sigma)}{E}B=0.$$

By making a change of the variables according to the formula

$$r=e^t\quad(t=\ln r), \tag{6.8}$$

Eqs. (6.7) can be reduced to equations with constant coefficients

$$\left(\frac{\partial^2}{\partial t^2}-1+\frac{1-\sigma}{2}\frac{\partial^2}{\partial\beta^2}\right)u_r+\frac{\partial}{\partial\beta}\left(\frac{1+\sigma}{2}\frac{\partial}{\partial t}-\frac{3-\sigma}{2}\right)u_\beta+$$
$$+\frac{1-\sigma^2}{E}e^{2t}R(t)=0, \tag{6.9}$$
$$\frac{\partial}{\partial\beta}\left(\frac{1+\sigma}{1-\sigma}\frac{\partial}{\partial t}+\frac{3-\sigma}{1-\sigma}\right)u_r+\left(\frac{\partial^2}{\partial t^2}-1+\frac{2}{1-\sigma}\frac{\partial^2}{\partial\beta^2}\right)u_\beta+$$
$$+\frac{2(1+\sigma)}{E}e^{2t}B(t)=0.$$

II. PLANE STRAIN

$$e_{zz}=e_{zr}=e_{r\beta}=Z_r=Z_\beta=0,\quad Z_z=\sigma(R_r+B_\beta).$$

For the case of plane strain, it is necessary to replace in all equations of part I,

$$\sigma \text{ by } \sigma_1=\frac{\sigma}{1-\sigma},\quad \text{and}\quad E \text{ by } E_1=\frac{E}{1-\sigma^2}.$$

III. SOLUTION OF BASIC EQUATIONS

1. Solution using displacement and stress functions

The solution of the homogeneous equations (6.7) can be taken in the form suggested by B. G. Galerkin:

$$u_r=-\frac{1}{2G}\left[(1+\sigma)\frac{\partial\varphi}{\partial r}-2(\cos\beta\nabla^2\varphi_1+\sin\beta\nabla^2\varphi_2)\right]+$$
$$+a\sin\beta+b\cos\beta, \tag{6.10}$$
$$u_\beta=-\frac{1}{2G}\left[\frac{1+\sigma}{r}\frac{\partial\varphi}{\partial\beta}-2(\sin\beta\nabla^2\varphi_1-\cos\beta\nabla^2\varphi_2)\right]+$$
$$+a\cos\beta-b\sin\beta+cr,$$

where φ_i are arbitrary biharmonic functions,

$$\varphi=\cos\beta\frac{\partial\varphi_1}{\partial r}-\frac{\sin\beta}{r}\frac{\partial\varphi_1}{\partial\beta}+\sin\beta\frac{\partial\varphi_2}{\partial r}+\frac{\cos\beta}{r}\frac{\partial\varphi_2}{\partial\beta},$$

a, b, and c are constants characterizing a rigid-body displacement.

The general solution ($R = B = 0$) of Eqs. (6.6) with the help of the stress function $\varphi(r, \beta)$ is obtained by taking the stresses according to the formulas

$$
\begin{aligned}
R_r &= \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \beta^2}, \quad B_\beta = \frac{\partial^2 \varphi}{\partial r^2}, \\
B_r &= R_\beta = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \beta}\right) = \frac{1}{r^2}\frac{\partial \varphi}{\partial \beta} - \frac{1}{r}\frac{\partial^2 \varphi}{\partial r\,\partial \beta}.
\end{aligned}
\tag{6.11}
$$

When the stresses are given in the form of (6.11), the system of equations (6.6) is reduced to the biharmonic equation

$$
\nabla^2\nabla^2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \beta^2}\right)^2 \varphi = 0. \tag{6.12}
$$

A wide class of biharmonic functions φ is obtained from harmonic functions ψ by means of transformations

$$\varphi = r\psi\cos\beta, \quad r\psi\sin\beta, \quad r^2\psi.$$

Of the known particular solutions of the biharmonic equation (6.12) we mention the following:

$$
\begin{aligned}
\varphi(r, \beta) &= A_0{}^{*} + B_0\beta + A\ln r + Br^2\ln r + Cr^2 + Dr\beta{}^{\cos}_{\sin}\beta + \\
&+ \left[(A_1 r^{*} + B_1 r^3 + C_1 r^{-1} + D_1 r\ln r){}^{\sin}_{\cos}\beta - \frac{2D_1}{1-\sigma} r\beta{}^{\cos}_{\sin}\beta^{**}\right] + \\
&+ \sum_{m=2,3}^{\infty}(A_m r^m + B_m r^{m+2} + C_m r^{-m} + D_m r^{-m+2}){}^{\sin}_{\cos} m\beta + \\
&+ \sum_{m=2,3}^{\infty} r^m[A_m\cos m\beta + B_m\sin m\beta + \\
&+ C_m\cos(m-2)\beta + D_m\sin(m-2)\beta].
\end{aligned}
\tag{6.13}
$$

In polary symmetric problems, instead of Eq. (6.12) we have the equation

$$
\begin{aligned}
&\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 \varphi = \frac{d^4\varphi}{dr^4} + \frac{2}{r}\frac{d^3\varphi}{dr^3} - \frac{1}{r^2}\frac{d^2\varphi}{dr^2} + \\
&+ \frac{1}{r^3}\frac{d\varphi}{dr} = 0
\end{aligned}
\tag{6.14}
$$

---

* The quantities A_0 and A_1 do not affect the state of stress and may be omitted.

** This term is taken in the case of a closed ring to obtain single-valued displacements; in Eq. (6.13) it is written for plane stress.

whose general solution is

$$\varphi = A \ln r + Br^2 \ln r + Cr^2 + D. \qquad (6.15)$$

The stresses are determined by the formulas

$$R_r = \frac{1}{r}\frac{d\varphi}{dr} = \frac{A}{r^2} + B(1+2\ln r) + 2C,$$
$$B_\beta = \frac{d^2\varphi}{dr^2} = -\frac{A}{r^2} + B(3+2\ln r) + 2C, \qquad (6.16)$$
$$B_r = R_\beta = 0.$$

2. Application of a complex variable [27]

Referring to Fig. 46, we have

$$u_x = u_r \cos\beta - u_\beta \sin\beta, \quad u_y = u_r \sin\beta + u_\beta \cos\beta, \qquad (6.17)$$

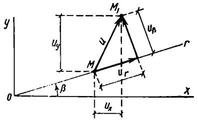

Fig. 46

from which

$$u_x + iu_y = u_r(\cos\beta + i\sin\beta) + iu_\beta(\cos\beta + i\sin\beta) = (u_r + iu_\beta)\,e^{i\beta}.$$

For plane stress, according to conditions (5.20) we find

$$u_r + iu_\beta = e^{-i\beta}\left\{\frac{3-\sigma}{E}\psi(z) - \frac{1+\sigma}{E}\left[z\overline{\psi'(z)} + \overline{\chi'(z)}\right]\right\}. \qquad (6.18)$$

Substituting $z = re^{i\beta}$ and $\bar{z} = re^{-i\beta}$ on the right-hand side of formula (6.18), and separating the real and imaginary parts, we obtain expressions for u_r and u_β in polar coordinates.

According to the formulas

$$R_r = X_x \cos^2 \beta + Y_y \sin^2 \beta + X_y \sin 2\beta,$$
$$B_\beta = X_x \sin^2 \beta + Y_y \cos^2 \beta - X_y \sin 2\beta,$$
$$R_\beta = (Y_y - X_x) \sin \beta \cos \beta + X_y (\cos^2 \beta - \sin^2 \beta),$$

we find the relations

$$R_r + B_\beta = X_x + Y_y,$$
$$B_\beta - R_r + 2iR_\beta = (Y_y - X_x + 2iX_y)\, e^{2i\beta},$$

from which, taking into account (5.21), we obtain

$$R_r + B_\beta = 2\,[\psi'(z) + \overline{\psi'(z)}] = 4\mathrm{Re}\,\psi'(z), \tag{6.19}$$

$$B_\beta - R_r + 2iR_\beta = 2\,[\bar{z}\psi''(z) + \chi''(z)]\, e^{2i\beta}. \tag{6.20}$$

By subtracting (6.20) from (6.19), we find

$$R_r - iR_\beta = \psi'(z) + \overline{\psi'(z)} - [\bar{z}\psi''(z) + \chi''(z)]\, e^{2i\beta}. \tag{6.21}$$

3. Use of boundary value homogeneous solutions

$$u_r = \frac{\partial}{\partial\beta}\left(\frac{1+\sigma}{2}\frac{\partial}{\partial t} - \frac{3-\sigma}{2}\right) f_1 + \left(\frac{\partial^2}{\partial t^2} - 1 + \frac{2}{1-\sigma}\frac{\partial^2}{\partial\beta^2}\right) f_2 + a\sin\beta + b\cos\beta,$$
$$u_\beta = \left(1 - \frac{\partial^2}{\partial t^2} - \frac{1-\sigma}{2}\frac{\partial^2}{\partial\beta^2}\right) f_1 - \frac{\partial}{\partial\beta}\left(\frac{1+\sigma}{1-\sigma}\frac{\partial}{\partial t} + \frac{3-\sigma}{1-\sigma}\right) f_2 + a\cos\beta - b\sin\beta + ce^t, \tag{6.22}$$

where $f_i = f_i(\beta, t)$ are functions satisfying the equation

$$\frac{\partial^4 f}{\partial t^4} + 2\frac{\partial^4 f}{\partial t^2\,\partial\beta^2} + \frac{\partial^4 f}{\partial\beta^4} + 2\frac{\partial^2 f}{\partial\beta^2} - 2\frac{\partial^2 f}{\partial t^2} + f = 0. \tag{6.23}$$

A particular solution of Eq. (6.23) may be taken in the form:

for a wedge-shaped region

$$f = B(\beta)\, e^{\lambda t},$$

from which

$$f(\beta, r) = r^k [A_k \cos(k-1)\beta + B_k \sin(k-1)\beta + C_k \cos(k+1)\beta + D_k \sin(k+1)\beta]; \qquad (6.24)$$

for an unclosed annular region

$$f = T(t)_{\cos}^{\sin} k\beta,$$

from which

$$f(\beta, r) = (A_k r^{k+1} + B_k r^{-k-1} + C_k r^{k-1} + D_k r^{-k+1})_{\cos}^{\sin} k\beta, \quad (6.25)$$

where k is an undetermined parameter, A_k, B_k, C_k, D_k are arbitrary constants.

The displacements are:

in the first case

$$u_r = \frac{r^k}{2} [(1+\sigma)k - (3-\sigma)] \{(k-1)[B_k \cos(k-1)\beta - A_k \sin(k-1)\beta] + (k+1)[D_k \cos(k+1)\beta - C_k \sin(k+1)\beta]\} + a \sin\beta + b\cos\beta,$$

$$u_\beta = -\frac{r^k}{2} \{(k-1)[(1+\sigma)k + (3-\sigma)][A_k \cos(k-1)\beta + B_k \sin(k-1)\beta] + (k+1)[(1+\sigma)k - (3-\sigma)] \times [C_k \cos(k+1)\beta + D_k \sin(k+1)\beta]\} + a\cos\beta - b\sin\beta - cr; \qquad (6.26)$$

in the second case

$$u_r = \pm \frac{k}{2} \{(1+\sigma)(k-2) A_k r^{k+1} - [(1+\sigma)k + 4] B_k r^{-k-1} + [(1+\sigma)k - 4] C_k r^{k-1} - (1+\sigma)(k+2) D_k r^{-k+1}\}_{\sin}^{\cos} k\beta + a\sin\beta + b\cos\beta,$$

$$u_\beta = -\frac{k}{2} \{[(1+\sigma)k + 4] A_k r^{k+1} + [(1+\sigma)k + 4] \times B_k r^{-k-1} + [(1+\sigma)k - 4] C_k r^{k-1} + [(1+\sigma)k - 4] \times D_k r^{-k+1}\}_{\cos}^{\sin} k\beta + a\cos\beta - b\sin\beta - cr. \qquad (6.27)$$

Further, formulas (6.5) are used to set up expressions for stresses. Thus, for the first case

$$\begin{aligned}
R_r &= Gkr^{k-1}\{(1+\sigma)(k-3)(k-1)[B_k\cos(k-1)\beta - \\
&- A_k\sin(k-1)\beta] + (k+1)[(1+\sigma)k-(3-\sigma)]\times \\
&\times [D_k\cos(k+1)\beta - C_k\sin(k+1)\beta]\}, \\
B_\beta &= -Gk(k+1)r^{k-1}\{(1+\sigma)(k-1)\times \\
&\times [B_k\cos(k-1)\beta - A_k\sin(k-1)\beta] + \\
&+ [(1+\sigma)k-(3-\sigma)][D_k\cos(k+1)\beta - C_k\sin(k+1)\beta]\}, \\
B_r &= R_\beta = -Gkr^{k-1}\{(1+\sigma)(k-1)^2[A_k\cos(k-1)\beta + \\
&+ B_k\sin(k-1)\beta] + (k+1)[(1+\sigma)k-(3-\sigma)]\times \\
&\times [C_k\cos(k+1)\beta + D_k\sin(k+1)\beta]\}.
\end{aligned} \tag{6.28}$$

Having the values of displacements (6.26) and (6.27) and of stresses (6.28), it is possible to formulate any boundary value homogeneous solutions.

PROBLEMS

6.1. Lamé's problem (1852).

A long hollow cylinder is subjected to a normal external pressure p_0 and a normal internal pressure p_i uniformly distributed over the lateral surface (Fig. 47).

Find the stresses and displacements.

In view of polar symmetry and two boundary conditions

$$R_{r=a} = -p_i, \quad R_{r=b} = -p_0 \tag{a}$$

we take the stress function (6.15) in the form

$\varphi = A \ln r + Cr^2$*.

According to formulas (6.16), the stresses are

$$R_r = \frac{A}{r^2} + 2C, \quad B_\beta = -\frac{A}{r^2} + 2C, \quad R_\beta = B_r = 0.$$

By using the boundary conditions (a), we obtain

$$A = \frac{a^2b^2(p_0-p_i)}{b^2-a^2}, \quad 2C = \frac{a^2p_i - b^2p_0}{b^2-a^2},$$

* For a solid cylinder, $\varphi = Cr^2$, $R_r = B_\beta = -p_0$, $R_\beta = B_r = 0$.

from which

$$R_r = \frac{1}{b^2 - a^2}\left[a^2 p_i - b^2 p_0 + \frac{a^2 b^2}{r^2}(p_0 - p_i)\right],$$

$$B_\beta = \frac{1}{b^2 - a^2}\left[a^2 p_i - b^2 p_0 - \frac{a^2 b^2}{r^2}(p_0 - p_i)\right].$$

To determine the displacements u_r ($u_\beta = 0$), it is necessary to integrate Eqs. (6.4), with Eqs. (6.2) and

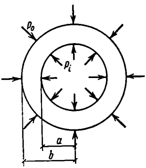

Fig. 47

relations (5.9). Equations (6.4) for the given problem are of the form

$$E_1 \frac{du_r}{dr} = R_r - \sigma_1 B_\beta, \quad E_1 \frac{u_r}{r} = B_\beta - \sigma_1 R_r. \tag{b}$$

From the equations (*b*) we obtain

$$u_r = \frac{1}{2G(b^2 - a^2)}\left[(1 - 2\sigma)(a^2 p_i - b^2 p_0) r + \frac{a^2 b^2}{r}(p_i - p_0)\right].$$

6.2. Golovin's problem (1881) [44, 45].

A flat circular bar of outer radius b and inner radius a with a section in the form of a narrow rectangle ($\delta = 1$) is bent by moments M applied at the ends (Fig. 48).

Determine the stresses and displacements.

In view of the fact that the stresses are independent of the polar angle β, we take, according to formula (6.15), the stress function in the form $\varphi = A \ln r + Br^2 \ln r + Cr^2$.

The boundary conditions of the problem are

$$R_{r=a} = 0, \quad R_{r=b} = 0, \quad \int_a^b B_\beta \, dr = 0, \quad \int_a^b B_\beta r \, dr = M. \qquad \text{(a)}$$

By writing out in full the equations (a), we obtain

$$\begin{aligned} &A/a^2 + B(1 + 2\ln a) + 2C = 0, \\ &A/b^2 + B(1 + 2\ln b) + 2C = 0, \\ &A \ln \frac{b}{a} + B(b^2 \ln b - a^2 \ln a) + C(b^2 - a^2) = M. \end{aligned} \qquad \text{(b)}$$

The third condition of (a) is satisfied if the first two are fulfilled.

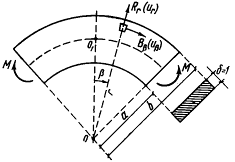

Fig. 48

We solve the equations (b):

$$A = -\frac{4M}{N} a^2 b^2 \ln \frac{b}{a}, \quad B = -\frac{2M}{N}(b^2 - a^2),$$

$$C = \frac{M}{N}[b^2 - a^2 + 2(b^2 \ln b - a^2 \ln a)],$$

where

$$N = (b^2 - a^2) - 4a^2 b^2 \ln^2 \frac{b}{a}.$$

The stresses are

$$R_r = -\frac{4M}{N}\left(\frac{a^2 b^2}{r^2} \ln \frac{b}{a} + b^2 \ln \frac{r}{b} + a^2 \ln \frac{a}{r}\right),$$

$$B_\beta = -\frac{4M}{N}\left(-\frac{a^2 b^2}{r^2} \ln \frac{b}{a} + b^2 \ln \frac{r}{b} + a^2 \ln \frac{a}{r} + b^2 - a^2\right).$$

The approximate solution by the formulas of strength of materials, where the stresses B_β vary according to a hyperbolic law, is in good agreement with the foregoing exact solution.

To determine the displacements, it is necessary to integrate Eqs. (6.4):

$$\frac{\partial u_r}{\partial r} = \frac{1}{E}\left\{\frac{1+\sigma}{r^2}A + [2(1-\sigma)\ln r + 1 - 3\sigma]B + 2(1-\sigma)C\right\},$$

$$\frac{1}{r}\frac{\partial u_\beta}{\partial \beta} + \frac{u_r}{r} = \frac{1}{E}\left\{-\frac{1+\sigma}{r^2}A + [2(1-\sigma)\ln r + 3 - \sigma]\times\right.$$

$$\left.\times B + 2(1-\sigma)C\right\}, \qquad \text{(c)}$$

$$\frac{1}{r}\frac{\partial u_r}{\partial \beta} + \frac{\partial u_\beta}{\partial r} - \frac{u_\beta}{r} = 0.$$

We integrate the first and second equations of (c) successively

$$u_r = \frac{1}{E}\left\{-\frac{1+\sigma}{r}A + [2(1-\sigma)\ln r - 1 - \sigma]rB + \right.$$

$$\left. + 2(1-\sigma)rC\right\} + f_1'(\beta),$$

$$u_\beta = \frac{4B}{E}r\beta - f_1(\beta) + f_2(r).$$

Substituting the values of displacements thus found in the third equation of (c), we obtain two equations

$$f_2'(r) - \frac{1}{r}f_2(r) = 0, \quad f_2''(\beta) + f_1(\beta) = 0,$$

from which

$$f_1(\beta) = b\sin\beta - a\cos\beta, \quad f_2(r) = cr.$$

The displacements are finally

$$u_r = \frac{1}{E}\left\{-\frac{1+\sigma}{r}A + [2(1-\sigma)\ln r - 1 - \sigma]rB + \right.$$

$$\left. + 2(1-\sigma)rC\right\} + a\sin\beta + b\cos\beta,$$

$$u_\beta = \frac{4B}{E}r\beta + a\cos\beta - b\sin\beta + cr$$

[cf. formula (6.10)].

To determine the arbitrary constants a, b, and c, it is necessary to fix the bar so as to eliminate its motion as a rigid body; for example, to assume, at the point O_1,

$$u_r = u_\beta = \frac{\partial u_\beta}{\partial r} = 0.$$

The displacement u_β is made up of two components: the displacement and the rotation of the section through an angle $4B\beta/E$ about the centre C as a rigid body. Consequently, in pure bending the cross sections remain plane (Bernoulli's hypothesis).

6.3. Michell's problem (1900) [45].

Determine the state of stress in a thin ($\delta = 1$) infinite wedge of angle 2α to whose vertex is applied a force P making an angle β_1 with the axis of the wedge (Fig. 49).

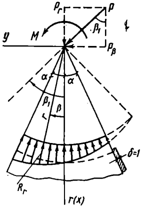

Fig. 49

We assign the stress function in the form

$\varphi = Ar\beta \sin \beta + Br\beta \cos \beta.$

The boundary conditions of the problem are: when $\beta = \pm\alpha$, $B_\beta = R_\beta = 0$; when $r \to 0$, $\sum X = \sum Y = 0$.

According to Eqs. (6.11), the stresses are

$R_r = (2A \cos \beta)/r - (2B \sin \beta)/r, \quad B_\beta = R_\beta = 0.$

Thus, the boundary conditions on the sides of the wedge are satisfied, and the conditions for determining the arbitrary constants are

$$\sum X = \int_{-\alpha}^{\alpha} R_r \cos \beta r \, d\beta + P \cos \beta_1 = 0,$$

$$\sum X = \int_{-\alpha}^{\alpha} R_r \sin \beta r \, d\beta + P \sin \beta_1 = 0,$$

from which

$$A = -\frac{2P\cos\beta_1}{2\alpha + \sin 2\alpha} = -\frac{2P_r}{2\alpha + \sin 2\alpha},$$

$$B = \frac{2P\sin\beta_1}{2\alpha - \sin 2\alpha} = \frac{2P_\beta}{2\alpha - \sin 2\alpha}.$$

We obtain, finally,

$$R_r = -\frac{2P}{r}\left(\frac{\cos\beta_1 \cos\beta}{2\alpha + \sin 2\alpha} + \frac{\sin\beta_1 \sin\beta}{2\alpha - \sin 2\alpha}\right).$$

As $r \to 0$, the stresses $R_r \to \infty$ since it has been assumed that the force is applied at the point.

When the angle 2α is small, the stresses determined by the formulas of strength of materials are close to those obtained in the problem considered.

▶ Determine the state of stress if a moment M is applied to the vertex of the wedge (see Fig. 49).

Hint. Take the stress function in the form $\varphi = A\beta + B \sin 2\beta$.

Answer.

$$R_r = -\frac{2C}{r^2}\sin 2\beta, \quad B_\beta = 0, \quad R_\beta = \frac{C}{r^2}(\cos 2\beta - \cos 2\alpha),$$

where C is determined from the condition

$$\int_{-\alpha}^{\alpha} rBr \, d\beta = M$$

and is equal to
$C = M/(\sin 2\alpha - 2\alpha \cos 2\alpha)$.

▶ Determine the state of stress in a thin infinite wedge of angle α with a uniformly distributed vertical load of intensity q applied over the inclined face (Fig. 50).

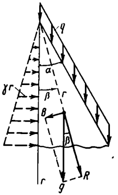

Fig. 50

Hint. Take the stress function in the form [see (6.13)]
$\varphi = r^2 (A_2 \cos 2\beta + B_2 \sin 2\beta + C_2\beta + D_2)$.

Answer.

$$R_r = \frac{q}{\sin\alpha}\cos^2\beta, \quad B_\beta = -\frac{q}{\sin\alpha}\sin^2\beta,$$

$$B_r = R_\beta = \frac{q}{2\sin\alpha}\sin 2\beta.$$

▶ Determine the state of stress in a thin infinite wedge of angle α for two loading conditions (Fig. 50):

(1) the pressure of a fluid of specific weight γ on the vertical face;

(2) the specific weight of the wedge material g.

Hint. In both cases take the stress function in the form [see (6.13)]

$$\varphi = r^3 (A_3 \cos 3\beta + B_3 \sin 3\beta + C_3 \cos\beta + D_3 \sin\beta),$$

determine the stresses by formulas (6.11) and, to take account of the action of the specific weight, add to the expressions obtained the particular values of the stresses by the formulas

$$\overline{R}_r = -gr\cos\beta,\quad \overline{B}_\beta = -gr\cos\beta,\quad \overline{R}_\beta = \overline{B}_r = 0,$$

which are derived from the equilibrium equations (6.6) when

$$R = g\cos\beta,\quad B = -g\sin\beta.$$

6.4. Flamant's problem (1892).

Determine the state of stress and strain in the elastic half-space $x \geqslant 0$ loaded by a concentrated force P perpendicular to the boundary $x = 0$ (Fig. 51).

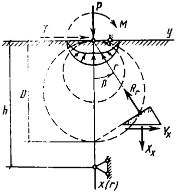

Fig. 51

This problem is a special case of Problem 6.3 if we put $\alpha = \pi/2$ and $\beta_1 = 0$. Taking this into account, we obtain

$$R_r = -\frac{2P\cos\beta}{\pi r},\quad B_\beta = R_\beta = B_r = 0,$$

i.e., the case of an elementary radial distribution of stresses (Michell, 1900).

On any circumference of arbitrary diameter D tangential to the boundary line at the point of application of the force (see Fig. 51) we have

$D = r/\cos\beta, \quad R_r = -2P/\pi D = \text{constant}.$

The principal shearing stress at all points of the circumference is

$\tau_{max} = |R_r - B_\beta|/2 = P/\pi D = \text{constant};$

hence, in the photoelastic analysis of plane models, similar circumferences (isobars, i.e., lines of equal stress) show on the screen near the points of application of forces.

Equations (6.4) for plane strain are

$$\begin{aligned}
&\frac{\partial u_r}{\partial r} = -\frac{2(1-\sigma^2)P}{\pi E}\frac{\cos\beta}{r},\\
&\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\beta}{\partial\beta} = \frac{2\sigma(1+\sigma)P}{\pi E}\frac{\cos\beta}{r}, \qquad \text{(a)}\\
&\frac{1}{r}\frac{\partial u_r}{\partial\beta} + \frac{\partial u_\beta}{\partial r} - \frac{u_\beta}{r} = 0.
\end{aligned}$$

By integrating the first and second equations successively, we obtain

$$\begin{aligned}
&u_r = -\frac{2(1-\sigma^2)P}{\pi E}\cos\beta\ln r + f'(\beta),\\
&u_\beta = \frac{2\sigma(1+\sigma)P}{\pi E}\sin\beta + \frac{2(1-\sigma^2)P}{\pi E}\sin\beta\ln r - f(\beta) + f_1(r).
\end{aligned}$$

To find the unknown functions $f(\beta)$ and $f_1(r)$, we substitute the values of displacements in the third equation of (a), which after some manipulation becomes

$$f''(\beta) + f(\beta) + \frac{2(1+\sigma)(1-2\sigma)P}{\pi E}\sin\beta = f_1(r) - rf_1'(r)$$

and breaks up in two equations

$$\begin{aligned}
&f''(\beta) + f(\beta) = -\frac{2(1+\sigma)(1-2\sigma)P}{\pi E}\sin\beta,\\
&f_1'(r) - \frac{1}{r}f_1(r) = 0^*.
\end{aligned}$$

---

* Equating both sides of the equation to zero and not to an arbitrary constant is due to the fact that this constant does not enter into the expression for displacements.

The unknown functions are

$$f(\beta) = C_1 \cos\beta + C_2 \sin\beta + \frac{(1+\sigma)(1-2\sigma)P}{\pi E}\beta\cos\beta,$$

$$f_1(r) = C_3 r.$$

The displacements are finally

$$u_r = -\frac{2(1-\sigma^2)P}{\pi E}\ln r\cos\beta - \frac{(1+\sigma)(1-2\sigma)P}{\pi E}\beta\sin\beta +$$

$$+\left[C_2 + \frac{(1+\sigma)(1-2\sigma)P}{\pi E}\right]\cos\beta - C_1\sin\beta,$$

$$u_\beta = \frac{2(1-\sigma^2)P}{\pi E}\ln r\sin\beta - \frac{(1+\sigma)(1-2\sigma)P}{\pi E}\beta\cos\beta -$$

$$-C_1\cos\beta - \left[C_2 - \frac{2\sigma(1+\sigma)P}{\pi E}\right]\sin\beta + C_3 r.$$

To determine the arbitrary constants C_i, it is necessary to fix the half-plane so as to eliminate its motion as a rigid body; for example (see Fig. 51), when $\beta = 0$, $u_\beta = 0$; when $r = h$ and $\beta = 0$, $u_r = 0$.

In this case [45]

$$C_1 = C_3 = 0,$$

$$C_2 = \frac{2(1-\sigma^2)P}{\pi E}\ln h - \frac{(1+\sigma)(1-2\sigma)P}{\pi E}$$

and the displacements are determined from the formulas

$$u_r = -\frac{2(1-\sigma^2)P}{\pi E}\left[\ln\frac{r}{h}\cos\beta + \frac{1-2\sigma}{2(1-\sigma)}\beta\sin\beta\right],$$

$$u_\beta = \frac{2(1-\sigma^2)P}{\pi E}\left\{\left[\ln\frac{r}{h} + \frac{1}{2(1-\sigma)}\right]\sin\beta - \right.$$

$$\left. - \frac{1-2\sigma}{2(1-\sigma)}\beta\cos\beta\right\}.$$

The points of the boundary line ($\beta = \pm\pi/2$) have the displacements

$$u_r|_{\beta=\pm\pi/2} = u_y = -\frac{(1+\sigma)(1-2\sigma)P}{2E},$$

$$u_\beta|_{\beta=\pm\pi/2} = u_x = \pm\frac{2(1-\sigma^2)P}{\pi E}\ln\frac{y}{h} \pm \frac{(1+\sigma)P}{\pi E}.$$

Referring to Fig. 51, the stresses on planes parallel to the x and y axes are

$$X_x = R_r \cos^2\beta = -\frac{2P}{\pi}\frac{x^3}{(x^2+y^2)^2} = -\frac{2P}{\pi}\frac{\cos^3\beta}{r},$$

$$Y_y = R_r \sin^2\beta = -\frac{2P}{\pi}\frac{xy^2}{(x^2+y^2)^2} = -\frac{2P}{\pi}\frac{\cos\beta\sin^2\beta}{r},$$

$$Y_x = X_y = \frac{R_r}{2}\sin 2\beta = -\frac{2P}{\pi}\frac{x^2y}{(x^2+y^2)^2} =$$

$$= -\frac{2P}{\pi}\frac{\cos^2\beta\sin\beta}{r}.$$

▶ Determine the stresses in the half-plane due to a moment (see Fig. 51).

Answer.

$$R_r = -\frac{2M}{\pi r^2}\sin 2\beta, \quad B_\beta = 0,$$

$$R_\beta = B_r = \frac{2M}{\pi r^2}\cos^2\beta$$

(see Problem 6.3).

▶ Cerruti's problem (1882).

Determine the stresses and displacements in an elastic half-plane loaded at the origin by a force T directed along the y axis (see Fig. 51).

Answer.

$$R_r = -\frac{2T}{\pi\delta}\frac{\sin\beta}{r}, \quad B_\beta = R_\beta = B_r = 0,$$

$$u_r = \frac{T}{\pi\delta E}\left[2\ln\frac{h}{r}\sin\beta - (1-\sigma)\beta\cos\beta\right],$$

$$u_\beta = \frac{T}{\pi\delta E}\left[(1+\sigma)\cos\beta - 2\ln\frac{h}{r}\cos\beta - (1-\sigma)\beta\sin\beta\right].$$

6.5. Hertz's problem (1883).

Determine the state of stress in a circular disk compressed by two forces P not passing through the centre (Fig. 52).

The problem is solved by superimposing three states ([1], p. 522):

(1) the point A is a pole from which a radial compression emanates producing the stresses

$$R_{r_1} = -\frac{2P}{\pi}\frac{\cos\beta_1}{r_1}, \quad B_{\beta_1} = R_{\beta_1} = 0;$$

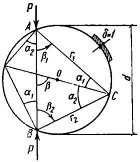

Fig. 52

(2) the point B is another pole from which a radial compression emanates producing the stresses

$$R_{r_2} = -\frac{2P}{\pi}\frac{\cos\beta_2}{r_2}, \quad B_{\beta_2} = R_{\beta_2} = 0;$$

(3) to the disk is applied a uniform all-round tension p for which

$$R_{r_3} = B_{\beta_3} = p, \quad R_{\beta_3} = 0$$

(see Problem 6.1).

When all three loads are acting simultaneously, the stresses at the point C are

$$R_r = p - \frac{2P}{\pi}\left(\frac{1}{r_1}\cos\beta_1\cos^2\alpha_1 + \frac{1}{r_2}\cos\beta_2\cos^2\alpha_2\right),$$

$$B_\beta = p - \frac{2P}{\pi}\left(\frac{1}{r_1}\cos\beta_1\sin^2\alpha_1 + \frac{1}{r_2}\cos\beta_2\sin^2\alpha_2\right),$$

$$B_r = -\frac{2P}{\pi}\left(\frac{1}{r_1}\cos\beta_1\sin\alpha_1\cos\alpha_1 - \right.$$
$$\left. - \frac{1}{r_2}\cos\beta_2\sin\alpha_2\cos\alpha_2\right).$$

Taking into account that

$$\alpha_1 + \beta_2 = \pi/2, \quad \alpha_2 + \beta_1 = \pi/2,$$
$$r_1 = d\cos\alpha_1 = d\sin\beta_2, \quad r_2 = d\cos\alpha_2 = d\sin\beta_1,$$

where d is the diameter of the circumference, we obtain, finally,

$$R_r = p - \frac{2P}{\pi d}\sin(\beta_1 + \beta_2),$$
$$B_\beta = p - \frac{2P}{\pi d}\cot\beta_1\cot\beta_2\sin(\beta_1 + \beta_2),$$
$$R_\beta = B_r = 0.$$

For the circumference ABC to be free from radial stresses R_r at all points, except at A and B, we have to put

$$p = \frac{2P}{\pi d}\sin(\beta_1 + \beta_2),$$

where $\sin(\beta_1 + \beta_2)$ is a constant.

▶ Determine the state of stress in a circular disk compressed by two forces P passing through the centre of the disk, and construct the normal stress diagram at a diametral section perpendicular to the forces (see Problem 3.1).

6.6. See [46].

Determine the state of stress and strain in a hollow circular semicylinder of large length, resting on an absolutely

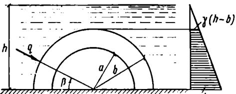

Fig. 53

rigid ($u_\beta = 0$) and smooth ($R_\beta = 0$) foundation, due to a hydrostatic load $q = \gamma(h - b\sin\beta)$, where γ is the specific weight of the fluid (Fig. 53).

We assign the stress function in the form of (6.13)

$$\varphi = A_0 r^2 + B_0 \ln r + \sum_{m=2,\,4}^{\infty} (A_m r^m + B_m r^{m+2} + C_m r^{-m} + D_m r^{-m+2}) \cos m\beta. \qquad \text{(a)}$$

The boundary conditions of the problem are: when $r = b$, $R_r = -\gamma(h - b\sin\beta)$ and $B_r = 0$; when $r = a$, $R_r = 0$ and $B_r = 0$; when $\beta = 0$ or π, $R_\beta = 0$ and $u_\beta = 0$.

Expanding $\sin\beta$ on the interval $0 \leqslant \beta \leqslant \pi$ by the formula

$$\sin\beta = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=2,\,4,\,\ldots}^{\infty} \frac{\cos m\beta}{(m-1)(m+1)},$$

and assuming the stresses by formulas (6.11)

$$R_r = 2A_0 + \frac{B_0}{r^2} - \sum_{2,\,4,\,\ldots}^{\infty} [m(m-1) A_m r^{m-2} + (m-2)(m+1) B_m r^m + m(m+1) C_m r^{-m-2} + (m+2)(m-1) D_m r^{-m}] \cos m\beta,$$

$$B_\beta = 2A_0 - \frac{B_0}{r^2} + \sum_{2,\,4,\,\ldots}^{\infty} [m(m-1) A_m r^{m-2} + (m+2)(m+1) B_m r^m + m(m+1) C_m r^{-m-2} + (m+2)(m-1) D_m r^{-m}] \cos m\beta,$$

$$R_\beta = B_r = \sum_{2,\,4,\,\ldots}^{\infty} m [(m-1) A_m r^{m-2} + (m+1) B_m r^m - (m+1) C_m r^{-m-2} - (m-1) D_m r^{-m}] \sin m\beta,$$

we obtain the values of all arbitrary constants in the formula (a) from the first four boundary conditions

$$A_0 = \frac{\gamma(2b - \pi h) b^2}{2(b^2 - a^2)}, \quad B_0 = -\frac{\gamma(2b - \pi h) a^2 b^2}{b^2 - a^2},$$

$$A_m = \frac{4\gamma}{\Delta}(m+1)[mC - (m-1)D]\alpha^{-m+1}a^{-m+3},$$

$$B_m = \frac{4\gamma}{\Delta}(m-1)[mD - (m+1)C]\alpha^{-m+1}a^{-m+1},$$

$$C_m = \frac{4\gamma}{\Delta}(m-1)D\alpha^{-m+1}a^{m+3},$$

$$D_m = -\frac{4\gamma}{\Delta}(m+1)D\alpha^{-m+1}a^{m+1},$$

where

$$\alpha = b/a,$$

$$\Delta = \pi(CB - mAD)(m-1)^2(m+1)^2,$$

$$A = (m-1)\alpha^2 - (m-2) - \alpha^{-2(m+1)},$$

$$B = m^2\alpha^{-2} - (m-2) - (m+2)\alpha^{-2m},$$

$$C = -(m-1)\alpha^{-2} + m - \alpha^{-2(m+1)},$$

$$D = -m\alpha^{-2} + (m+1) + \alpha^{-2m}.$$

The displacements are determined from Eqs. (6.4) with (6.2). By integrating the first two equations of (6.4), and equating to zero the arbitrary functions*, we find

$$E_1 u_r = 2(1-\sigma_1)A_0 r - (1+\sigma_1)\frac{B_0}{r} -$$

$$- \sum_{2,4,\ldots}^{\infty} \{(1+\sigma_1)mA_m r^{m-1} + [m-2+\sigma_1(m+2)]B_m r^{m+1} -$$

$$-(1+\sigma_1)mC_m r^{-m-1} - [m+2+\sigma_1(m-2)]D_m r^{-m+1}\}\cos m\beta,$$

$$E_1 u_\beta = \sum_{2,4,\ldots}^{\infty} \{(1+\sigma)mA_m r^{m-1} + [(1+\sigma_1)m+4]B_m r^{m+1} +$$

$$+(1+\sigma_1)mC_m r^{-m-1} + [(1+\sigma_1)m-4]D_m r^{-m+1}\}\sin m\beta.$$

* The third equation of (6.4) $e_{r\beta} = \frac{2(1+\sigma_1)}{E_1} R_\beta$ with the stress function in the form (a) is satisfied identically.

Thus, the boundary conditions in β are satisfied.

In the case of a uniform external pressure p_0 and a uniform internal pressure p_i on the semicylinder, provided that $p_0 b \geqslant p_i a$, the well-known solution of Lamé's problem is obtained.

▶Find the state of stress and strain in a solid semicylinder ($a = 0$) of radius b, resting on an absolutely rigid and smooth foundation, due to a hydrostatic load (Fig. 53).

Hint. Take the stress function in the form

$$\varphi = A_0 r^2 + \sum_{m=2,\,4,\,\ldots}^{\infty} (A_m r^m + B_m r^{m+2}) \cos m\beta.$$

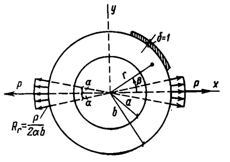

Fig. 54

▶Investigate the state of stress in a thin ($\delta = 1$) circular ring extended by two forces P (Fig. 54) (see the monograph [11, p. 512).

Hints. (1) Replace the concentrated force by a uniformly distributed load $q = P/2\alpha b$ in the portion $2\alpha b$, where α is a small angle.

(2) Assume the stress function in the form (a).

6.7. Kirsch's problem (1898).

Investigate the state of stress in a thin ($\delta = 1$) uniformly extended rectangular plate with a small circular hole (Fig. 55).

The stresses caused by the small hole are local stresses rapidly decaying with the distance from it.

The stresses in a solid plate can be determined in terms of the stress function

$$\varphi_0 = \frac{1}{2} py^2 = \frac{1}{2} pr^2 \sin^2 \beta = \frac{1}{4} pr^2 (1 - \cos 2\beta)$$

for which

$$X_x = \frac{\partial^2 \varphi_0}{\partial y^2} = p, \quad Y_y = X_y = 0.$$

In using the stress function φ for the solution of the present problem, the resulting stresses for large values of r must be the same as with the function φ_0.

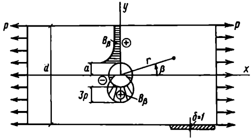

Fig. 55

According to formulas (6.13), we assign the stress function in the form

$$\varphi = A \ln r + Br^2 \ln r + Cr^2 +$$
$$+ (A_2 r^2 + B_2 r^4 + C_2 r^{-2} + D_2) \cos 2\beta.$$

To avoid increasing stresses with increasing r, we set $B = B_2 = 0$, and to obtain the same stresses for large values of r as with the function φ_0, it is necessary that $C = p/4$ and $A_2 = -p/4$. Thus, the stress function is

$$\varphi = A \ln r + pr^2/4 + (-pr^2/4 + C_2 r^{-2} + D_2) \cos 2\beta,$$

and the stresses are

$$R_r = A/r^2 + p/2 + (p/2 - 6C_2/r^4 - 4D_2/r^2)\cos 2\beta,$$

$$B_\beta = -A/r^2 + p/2 + (-p/2 + 6C_2/r^4)\cos 2\beta,$$

$$R_\beta = B_r = -(p/2 + 6C_2/r^4 + 2D_2/r^2)\sin 2\beta.$$

The arbitrary constants are determined from the conditions: when $r = a$, $R_r = B_r = 0$*.

On setting up these conditions, we obtain

$$A = -pa^2/2, \quad C_2 = -pa^4/4, \quad D_2 = pa^2/2.$$

The stresses are finally

$$R_r = \frac{p}{2}\left[1 - \frac{a^2}{r^2} + \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\beta\right],$$

$$B_\beta = \frac{p}{2}\left[1 + \frac{a^2}{r^2} - \left(1 + \frac{3a^4}{r^4}\right)\cos 2\beta\right],$$

$$R_\beta = B_r = -\frac{p}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\beta.$$

Since the body is not simply connected, it is necessary to check the single-valuedness of displacements. Calculations show that the displacements are single valued; this proves the validity of the solution.

Figure 55 shows the B_β diagram: when $\beta = \pm\pi/2$, $B_\beta = 3p$; when $\beta = 0$ or π, $B_\beta = -p$.

Thus, an increase in stresses (concentration) occurs at the edge of the hole.

If the width d of the plate is not very great compared with the diameter $2a$ of the hole, but still greater than $4a$, the maximum value of B_β can be determined by the formula

$$\max B_\beta \cong 3pd/(d - a).$$

6.8. Michell's problem (1900) [5].

Determine the state of stress in a thin ($\delta = 1$) circular heavy disk resting on an absolutely rigid horizontal plane (Fig. 56).

---

* It follows from Eqs. (6.11) that we can also set up conditions for the function φ: when $r = a$, $\varphi = 0$ and $\frac{\partial \varphi}{\partial r} = 0$.

The force at the point of support of the disk is

$$P = \pi R^2 \gamma,$$

where γ is the specific weight of material.

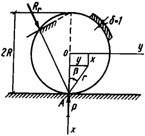

Fig. 56

The problem is solved by superimposing two states: (1) the point A is a pole from which a radial compression emanates producing the stresses

$$R_r = -\frac{2P}{\pi}\frac{\cos\beta}{r} = -\frac{2\gamma R^2}{r}\cos\beta, \quad B_\beta = R_\beta = B_r = 0,$$

and the stresses at the edge of the disk ($r = 2R\cos\beta$) are (Problem 6.4)

$$R_r = -\gamma R, \quad X_x = -\gamma R\cos^2\beta, \quad Y_y = -\gamma R\sin^2\beta,$$
$$X_y = -\frac{\gamma R}{2}\sin 2\beta;$$

(2) to eliminate the stresses thus found, it is necessary to add the system of stresses

$$X_x = \frac{\partial^2\varphi}{\partial y^2} = \frac{\gamma}{2}(R-x), \quad X_y = -\frac{\partial^2\varphi}{\partial x\,\partial y} = \frac{\gamma}{2}y,$$
$$Y_y = \frac{\partial^2\varphi}{\partial x^2} = \frac{\gamma}{2}(R+x)$$

determined by the stress function

$$\varphi = \frac{\gamma}{4}\left[\frac{x^3}{3} + R(x^2+y^2) - y^2x\right].$$

Noting that $y = r \sin \beta$, $x = R - r \cos \beta$, it is easy to obtain stresses at the edge of the disk ($r = 2R \cos \beta$), which are reciprocal to those indicated in part (1).

At each horizontal section there is a radial pressure acting in the direction of the point A and equal to

$$\frac{\gamma}{2r}(4R^2 \cos^2 \beta - r^2).$$

6.9. See [27].

An infinite plate is weakened by a circular hole with known stresses R_r and B_r applied on the contour of the hole for $z = ae^{i\beta}$, where a is the radius of the hole.

Investigate the state of stress.

The analytic functions $\psi'(z)$ and $\chi''(z)$ of solution (6.21) must be expanded in power series so that they will be finite when $r = \infty$. These functions are of the form

$$\psi'(z) = \sum_{n=0}^{\infty} A_n z^{-n}, \quad \chi''(z) = \sum_{n=0}^{\infty} B_n z^{-n}, \tag{a}$$

where A_n and B_n are complex constants.

It is seen from formulas (6.19) and (6.21) that the stresses at infinity are determined by the constant B_0 and the real part of the constant A_0. The imaginary part of the constant A_0 does not affect the state of stress.

By integrating (a) with respect to z, we find

$$\psi(z) = A_0 z + A_1 \ln z - \sum_{n=2}^{\infty} \frac{A_n z^{-n+1}}{n-1} + C_1,$$

$$\chi'(z) = B_0 z + B_1 \ln z - \sum_{n=2}^{\infty} \frac{B_n z^{-n+1}}{n-1} + C_2,$$

where C_i are complex constants.

Taking into account the relations

$$\overline{\psi'(z)} = \sum_{n=0}^{\infty} \overline{A}_n \overline{z}^{-n}, \quad \overline{\chi'(z)} = \overline{B}_0 \overline{z} + \overline{B}_1 \ln \overline{z} -$$

$$- \sum_{n=2}^{\infty} \frac{\overline{B}_n \overline{z}^{-n+1}}{n-1} + \overline{C}_2,$$

from formula (6.18) we obtain

$$u_r + iu_\beta = e^{-i\beta}\Big[\frac{3-\sigma}{E}\Big(A_0 z + A_1 \ln z - \sum_{n=2}^{\infty}\frac{A_n z^{-n+1}}{n-1} + C_1\Big) -$$

$$-\frac{1+\sigma}{E}\Big(\bar{A}_0 z + r^2\sum_{n=1}^{\infty}\bar{A}_n \bar{z}^{-(n+1)}\Big) -$$

$$-\frac{1+\sigma}{E}\Big(\bar{B}_0\bar{z} + \bar{B}_1 \ln \bar{z} - \sum_{n=2}^{\infty}\frac{\bar{B}_n \bar{z}^{-n+1}}{n-1} + \bar{C}_2\Big)\Big]. \qquad \text{(b)}$$

Setting $z = re^{i\beta}$ gives $\ln z = \ln r + i\beta$.

This function is not single valued in β. The increment of $u_r + iu_\beta$ on passing once round the hole is

$$2\pi i e^{-i\beta}\Big(\frac{3-\sigma}{E}A_1 + \frac{1+\sigma}{E}\bar{B}_1\Big),$$

and the condition for single-valuedness of displacements is

$$(3-\sigma)A_1 + (1+\sigma)\bar{B}_1 = 0,$$

from which

$$A_1 = -\frac{1+\sigma}{3-\sigma}\bar{B}_1. \qquad \text{(c)}$$

Since the stresses R_r and B_r are given at $r = a$, the expression $(R_r - iB_r)_{r=a}$ can be expanded in a complex Fourier series [47]

$$(R_r - iB_r)_{r=a} = \sum_{n=-\infty}^{\infty} C_n e^{in\beta}, \qquad \text{(d)}$$

where the coefficients C_n are determined by the formula

$$C_n = \frac{1}{2\pi}\int_0^{2\pi}[R_r(\beta) - B_r(\beta)]_{r=a}\, e^{-in\beta}\, d\beta,$$

$$n = 0,\ 1,\ -1,\ -2,\ \ldots$$

Substituting the series (a) and (d) in solutions (6.21), and taking into account the conditions on the contour

$(r = a)$, we find

$$\sum_{n=-\infty}^{\infty} C_n e^{in\beta} = \sum_{n=0}^{\infty} \frac{A_n}{a^n} e^{-in\beta} + \sum_{n=0}^{\infty} \frac{\overline{A}_n}{a^n} e^{in\beta} +$$

$$+ \sum_{n=0}^{\infty} \frac{nA_n}{a^n} e^{-in\beta} - \sum_{n=0}^{\infty} \frac{B_n}{a^n} e^{-i(n-2)\beta} =$$

$$= \sum_{n=0}^{\infty} \left[(1+n) A_n - \frac{B_{n+2}}{a^2} \right] \frac{e^{-in\beta}}{a^n} -$$

$$- \frac{B_1}{a} e^{i\beta} - B_0 e^{i2\beta} + \sum_{n=0}^{\infty} \frac{\overline{A}_n}{a^n} e^{in\beta}. \qquad \text{(e)}$$

By comparing the coefficients of like powers of $e^{in\beta}$ on both sides of the equation (e), we obtain

$$A_0 + \overline{A}_0 - \frac{B_2}{a^2} = C_0, \quad \frac{\overline{A}_1}{a} - \frac{B_1}{a} = C_1,$$

$$\frac{\overline{A}^2}{a^2} - B_0 = C_2, \quad \frac{\overline{A}_n}{a^n} = C_n \quad \text{for } n \geqslant 3, \qquad \text{(f)}$$

$$\frac{1+n}{a^n} A_n - \frac{B_{n+2}}{a^{n+2}} = C_{-n} \quad \text{for } n \geqslant 1.$$

Noting that $A_0 + \overline{A}_0$ and B_0 characterize the state of stress at infinity, we consider them known. The magnitude of the imaginary part of the constant A_0 corresponds to the displacement of an absolutely rigid body (b), and it can be taken to be unity. The constants are determined by the formulas

$A_0 + \overline{A}_0 = 2A_0, \quad \overline{A}_1 = -\frac{1+\sigma}{3-\sigma} B_1$ [see formula (c)].

Further, from the equations (f) and (c) we find

$$B_1 = -\frac{3-\sigma}{4} aC_1, \quad A_1 = \frac{1+\sigma}{4} a\overline{C}_1,$$

$$B_2 = 2A_0 a^2 - C_0 a^2, \quad A_2 = \overline{B}_0 a^2 + \overline{C}_2 a^2;$$

for $n \geqslant 3$

$$B_n = (n-1) a^2 A_{n-2} - a^n C_{-n+2}, \quad A_n = \overline{C}_n a^n.$$

The complete solution of the problem is possible if the stress distribution along the circular contour and the conditions at infinity are given.

▶ Determine, by using the method considered, the state of stress and strain in a wide plate weakened at the middle part by a small circular hole of radius a and subjected to a uniform extension with stresses p directed along the x axis (Fig. 55).

Answer. (1) For the stresses, see the formula (b) in Problem 6.7.

(2) The displacements for $C_1 = C_2 = 0$ are, by the formula (b),

$$u_r = \frac{p(1+\sigma)}{2Er}\left[\frac{1-\sigma}{1+\sigma}r^2 + a^2 + \left(\frac{4a^2}{1+\sigma} + r^2 - \frac{a^4}{r^2}\right)\cos 2\beta\right],$$

$$u_\beta = -\frac{p(1+\sigma)}{2Er}\left(\frac{1-\sigma}{1+\sigma}2a^2 + r^2 + \frac{a^4}{r^2}\right)\sin 2\beta.$$

▶ Find the stress distribution in an infinite plate with a circular hole if $R_r = -p$ and $B_r = 0$ at the edge of the hole ($r = a$). The stresses at infinity are zero.

Answer.

$$\psi'(z) = 0, \quad \chi''(z) = pa^2/z^2,$$

$$R_r = -pa^2/r^2, \quad B_\beta = pa^2/r^2, \quad B_r = R_\beta = 0,$$

$$u_r = pa^2(1+\sigma)/Er, \quad u_\beta = 0.$$

▶ Find the stresses in a thin circular ring of outer radius b and inner radius a compressed across the thickness by two forces (Fig. 57).

For the solution, see [48].

6.10. See [49].

Find the homogeneous solutions for a thin wedge fixed at a finite number of points of the base if the following data are given (Fig. 58): the wedge is acted on by a fluid of specific weight γ, $\alpha = \pi/6 = 0.524$, $\delta = \pi/12 = 0.262$, $h = 4$ m.

To obtain a particular solution satisfying the nonhomogeneous boundary conditions on the faces OA and OB, we assume $k = 2$ in Eqs. (6.26) and (6.28) and, to simplify the expressions, take $\sigma = 0$.

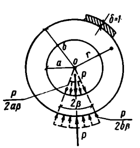

Fig. 57

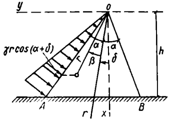

Fig. 58

The arbitrary constants of the particular solution are determined from the conditions: when $\beta = \alpha$, $B_\beta = -\gamma r \cos(\alpha + \delta)$ and $R_\beta = 0$; when $\beta = -\alpha$, $B_\beta = R_\beta = 0$.

The general homogeneous solution is obtained for the homogeneous boundary conditions on the faces OA and OB: when $\beta = \alpha$, $B_\beta = R_\beta = 0$; when $\beta = -\alpha$, $B_\beta = R_\beta = 0$, which in expanded form give four equations

$$(k-1)\sin(k-1)\alpha A_k - (k-1)\cos(k-1)\alpha B_k + (k-3)\sin(k+1)\alpha C_k - (k-3)\cos(k+1)\alpha D_k = 0,$$

$$(k-1)^2\cos(k-1)\alpha A_k + (k-1)^2\sin(k-1)\alpha B_k +$$

$$+ (k-3)(k+1)\cos(k+1)\alpha C_k + (k-3)(k+1) \times$$
$$\times \sin(k+1)\alpha D_k = 0, \tag{a}$$
$$(k-1)\sin(k-1)\alpha A_k + (k-1)\cos(k-1)\alpha B_k +$$
$$+ (k-3)\sin(k+1)\alpha C_k + (k-3)\cos(k+1)\alpha D_k = 0,$$
$$(k-1)^2\cos(k-1)\alpha A_k - (k-1)^2\sin(k-1)\alpha B_k +$$
$$+ (k-3)(k+1)\cos(k+1)\alpha C_k -$$
$$- (k-3)(k+1)\sin(k+1)\alpha D_k = 0.$$

In order to have the constants A_k, B_k, C_k, and D_k different from zero, the determinant $\Delta(k)$ of the system (a) must be zero. By expanding the determinant, we obtain a transcendental equation for determining k

$$\sin 2k\alpha = \pm k \sin 2\alpha. \tag{b}$$

Setting

$$k = a \pm ib,$$

substituting in the equation (b), and separating the real and imaginary parts, we obtain equations for determining a and b:

$$\sin 2a\alpha \cosh 2b\alpha = \pm a \sin 2\alpha, \quad \cos 2a\alpha \sinh 2b\alpha =$$
$$= \pm b \sin 2\alpha, \text{ or, for the value}$$
$$a = \pm \frac{\coth|2b\alpha|}{\sin 2\alpha}\sqrt{\sinh^2 2b\alpha - b^2\sin^2 2\alpha}, \tag{c}$$
$$\cos\frac{2}{\sin 2\alpha}\left(\coth|2b\alpha|\sqrt{\sinh^2 2b\alpha - b^2\sin^2 2\alpha}\right)\alpha =$$
$$= \pm \sin 2\alpha \frac{b}{\sinh 2b\alpha}. \tag{d}$$

For the numerical values of the problem ($\alpha = 0.524$), the equations (d) and (c) are

$$a = \pm 1.155 \coth|1.05b|\sqrt{\sinh^3 1.05b - 0.75b^2}, \tag{e}$$
$$\cos 1.21\left(\coth|1.05b|\sqrt{\sinh^3 1.05b - 0.75b^2}\right) =$$
$$= \pm 0.866b/\sinh 1.05b. \tag{f}$$

The graph of the equation (f) is shown in Fig. 59. The values of some of its roots are given below:

m	$k_m = a_m \pm ib_m$
1-4	$\pm 4.051 \pm i1.94$
5-8	$\pm 7.178 \pm i2.45$
. . . . . .	. . . .
. . . . . .	. . . .

For the set of the parameters $k_m = a_m \pm ib_m$ found from the equations (a) and (f), the corresponding constants are determined by the formulas

$$A_k = F_k \Delta_1(k), \quad B_k = F_k \Delta_2(k), \quad C_k = F_k \Delta_3(k),$$
$$D_k = F_k \Delta_4(k),$$

where $\Delta_i(k)$ are the cofactors of the elements of a row or column of the determinant*; F_k are arbitrary proportionality factors.

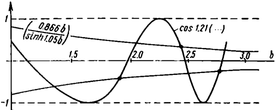

Fig. 59

By taking any finite number of terms in the general solution, it is possible to satisfy the boundary conditions on the supporting plane AB at any finite number of points.

The numerical calculations for an elastic wedge on a rigid and an elastic foundation by the foregoing method using computers are given by N. E. Borisov in [50, 51].

* For simplicity, the subscript m on k is omitted.

Chapter 7

TORSION OF PRISMATIC AND CYLINDRICAL BARS

I. PURE TORSION OF BARS OF CONSTANT SECTION

1. Assumptions

In solving problems of the pure torsion of bars use is made of Saint-Venant's semi-inverse method assuming

$X_x = Y_y = Z_z = X_y = 0,$

where z is the axis of a bar.

2. Basic equations

With the assumptions adopted, the computing equations are:

Static equations [see (1.1a)]

$$\frac{\partial X_z}{\partial z} = 0, \quad \frac{\partial Y_z}{\partial z} = 0, \quad \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} = 0. \tag{7.1}$$

Boundary conditions [(1.2)]

on the lateral surface

$$Z_x \cos(\nu,\ x) + Z_y \cos(\nu,\ y) = 0, \tag{7.2}$$

at the ends ($z = 0$ and $z = l$)

$$\iint_F X_z\, dF = 0, \quad \iint_F Y_z\, dF = 0, \quad \iint_F (Y_z x - X_z y)\, dF = M_z, \tag{7.3}$$

where M_z is the torque.

The geometrical equations (2.1a), with (3.1a), take the form

$$e_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad e_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0,$$
$$e_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad e_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \frac{X_z}{G}, \tag{7.4}$$
$$e_{zz} = \frac{\partial u_z}{\partial z} = 0, \quad e_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = \frac{Y_z}{G},$$

and Eqs. (3.4a) become

$$\nabla^2 Y_z = 0, \quad \nabla^2 X_z = 0. \tag{7.5}$$

3. Solution of problems by means of Prandtl's function (1903)

The stresses are expressed in terms of the function $\Phi = \Phi(x, y)$ by the formulas

$$X_z = Z_x = \frac{\partial \Phi}{\partial y}, \quad Y_z = Z_y = -\frac{\partial \Phi}{\partial x}. \tag{7.6}$$

According to Eqs. (7.5),

$$\nabla^2 \Phi = C. \tag{7.7}$$

By integrating Eqs. (7.4), we find, omitting terms representing rigid-body displacements of a bar,

$$u_x = -\alpha yz, \quad u_y = \alpha xz, \quad u_z = u_z(x, y), \tag{7.8}$$

where α is the angle of twist per unit length of the bar.

From the last two equations of (7.4) we obtain the values of the shearing stresses

$$X_z = G\left(\frac{\partial u_z}{\partial x} - \alpha y\right), \quad Y_z = G\left(\frac{\partial u_z}{\partial y} + \alpha x\right), \tag{7.6'}$$

and, on comparing Eqs. (7.6) and (7.6′), we find Poisson's equation (4.6) for Prandtl's function

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2\alpha G, \tag{7.7'}$$

from which, by Eq. (7.7),

$$C = -2\alpha G.$$

The relation between the displacement u_z and the stress function Φ is determined by equalities (7.6) and (7.6′)

from which it follows that

$$\frac{\partial u_z}{\partial x} = \alpha y + \frac{1}{G}\frac{\partial \Phi}{\partial y}, \quad \frac{\partial u_z}{\partial y} = -\alpha x - \frac{1}{G}\frac{\partial \Phi}{\partial x} \tag{7.9}$$

and

$$\nabla^2 u_z = 0.$$

Equations (7.9), with (7.7), are always integrable [see Eqs. (4.17) to (4.21)].

The quantity u_z characterizes the warping of the cross section, i.e., its distortion out of the plane during the deformation of the bar.

4. Properties of Prandtl's function [1]

From Eq. (7.2) (Fig. 60) it follows that

$$\frac{\partial \Phi}{\partial y}\frac{dy}{ds} + \frac{\partial \Phi}{\partial x}\frac{dx}{ds} = \frac{d\Phi}{ds} = 0,$$

and hence on the contour of a solid bar we have

$$\Phi(x, y) = 0. \tag{7.10}$$

The shearing stress at any point in the section is directed along the tangent to the line Φ = constant passing through

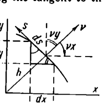

Fig. 60

this point and is proportional to the rate of change of Φ along the normal to this line

$$S_z = -\frac{\partial \Phi}{\partial \nu}, \quad V_z = 0. \tag{7.11}$$

By the shear circulation theorem (R. Bredt, 1896),

$$\oint_{\Phi=C} S_z\, ds = 2\alpha G F_{\Phi}, \tag{7.12}$$

where $F_\Phi = \oint\limits_{\Phi=C} h\, ds/2$ is the area of the solid section enclosed by the curve under consideration.

By the third equation of (7.3),

$$M_z = 2\int F_\Phi\, d\Phi = 2\iint\limits_F \Phi\, dF, \tag{7.13}$$

where $d\Phi = S_z\, dv$ is the differential of the stress function (7.11), F is the cross-sectional area (including holes).

II. PURE TORSION OF CIRCULAR BARS (SHAFTS) OF VARIABLE SECTION [36]

1. Assumptions

In the case of the torsion of a shaft of variable section (Fig. 61) the problem is solved in cylindrical co-ordinates under the following assumptions:

$$u_r = u_z = 0, \quad u_\beta = u_\beta\,(r,\, z), \tag{7.14}$$

$$R_r = B_\beta = Z_z = R_z = 0. \tag{7.14'}$$

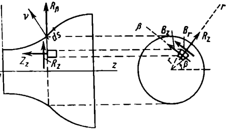

Fig. 61

2. Basic equations

Under the above assumptions, (7.14), the computing equations are:

Geometrical equations [(2.1b)]

$$e_{rr} = \frac{\partial u_r}{\partial r} = 0, \quad e_{zz} = \frac{\partial u_z}{\partial z} = 0,$$

$$e_{\beta\beta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\beta}{\partial\beta}=0,$$
$$e_{zr}=\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}=0, \tag{7.15}$$
$$e_{r\beta}=\frac{1}{r}\frac{\partial u_r}{\partial\beta}+\frac{\partial u_\beta}{\partial r}-\frac{u_\beta}{r}=\frac{\partial u_\beta}{\partial r}-\frac{u_\beta}{r},$$
$$e_{\beta z}=\frac{1}{r}\frac{\partial u_z}{\partial\beta}+\frac{\partial u_\beta}{\partial z}=\frac{\partial u_\beta}{\partial z}.$$

Equations of Hooke's law

$$B_r=R_\beta=G\left(\frac{\partial u_\beta}{\partial r}-\frac{u_\beta}{r}\right),\quad B_z=Z_\beta=G\frac{\partial u_\beta}{\partial z}. \tag{7.16}$$

Static equations [(1.1b)]

In the absence of body forces, there is only one equilibrium equation left:

$$\frac{\partial B_r}{\partial r}+\frac{\partial B_z}{\partial z}+\frac{2B_r}{r}=0,$$

and the remaining equations are satisfied identically.

The last equation can be written in the form

$$\frac{\partial}{\partial r}(r^2B_r)+\frac{\partial}{\partial z}(r^2B_z)=0 \tag{7.17}$$

and satisfied identically by introducing the stress function ψ by the formulas

$$B_r=-\frac{1}{r^2}\frac{\partial\psi}{\partial z},\quad B_z=\frac{1}{r^2}\frac{\partial\psi}{\partial r}. \tag{7.18}$$

By solving Eqs. (7.16) and (7.18) or the fifth equation of (2.4b) simultaneously (the remaining strain compatibility equations are satisfied identically), we obtain

$$\frac{\partial^2\psi}{\partial r^2}-\frac{3}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=0. \tag{7.19}$$

If the lateral surface is free from external forces, the resulting shearing stress is directed along the tangent to the contour of the axial section, and its projection on the normal ν is zero. In this case we have

$$B_r\cos(\nu, r)+B_z\cos(\nu, z)=0,$$

where

$$\cos(\nu,\ r) = \frac{dz}{ds}, \quad \cos(\nu,\ z) = -\frac{dr}{ds}.$$

Taking into account formulas (7.18), we obtain

$$\frac{\partial\psi}{\partial z}\frac{dz}{ds} + \frac{\partial\psi}{\partial r}\frac{dr}{ds} = \frac{d\psi}{ds} = 0,$$

from which it follows that on the contour

$$\psi = \text{constant}, \tag{7.20}$$

and at the ends ($z = 0,\ z = l$)

$$M_z = \int_F\int B_z r\,dF = \int_0^a\int_0^{2\pi} B_z r^2\,dr\,d\beta = 2\pi\int_0^a r^2 B_z\,dr =$$

$$= 2\pi\int_0^a \frac{\partial\psi}{\partial r}\,dr = 2\pi\psi\Big|_0^a, \tag{7.21}$$

where a is the radius of the cross section under consideration, determined by the equation of the generator.

If the lateral surface is acted on by a load p, then

$$\frac{1}{r^2}\frac{\partial\psi}{\partial z}\frac{dz}{ds} + \frac{1}{r^2}\frac{\partial\psi}{\partial r}\frac{dr}{ds} = -p,$$

from which

$$\frac{d\psi}{ds} = -r^2 p,$$

and instead of formula (7.20) we obtain

$$\psi = -\int_0^s r^2 p\,ds. \tag{7.22}$$

3. Solution of the differential equation for the torsion of a shaft [52]

There are several possible forms of solution of Eq. (7.19).

In terms of power functions

Assume

$$\psi = r^n z^m. \tag{7.23}$$

Substituting the value of ψ in Eq. (7.19), we find $n = 4$ and $m = 1$, from which

$$\psi = (Az + B)(Cr^4 + D), \tag{7.24}$$

and the stresses become

$$B_r = -\frac{A}{r^3}(Cr^4 + D), \quad B_z = 4Cr(Az + B). \tag{7.25}$$

From formulas (7.25) we obtain several special cases; for example, when $A = D = 0$ and $B = 1$, we have the elementary solution of the torsion problem for a circular shaft. In this case

$$\psi = Cr^4, \quad B_r = 0, \quad B_z = 4Cr,$$

and, from formula (7.21),

$$C = M_z/2\pi a^4.$$

In terms of Bessel functions
Assuming

$$\psi = R(r)\, Z(z),$$

where $R(r)$ is a function of the variable r, and $Z(z)$ is a function of the variable z, and substituting in Eq. (7.19), we obtain

$$\frac{d^2R}{dr^2} - \frac{3}{r}\frac{dR}{dr} \pm \lambda^2 R = 0, \quad \frac{d^2Z}{dz^2} \mp \lambda^2 Z = 0, \tag{7.26}$$

where λ is some number.

Equations (7.26) have the following two solutions [53]:

$$\psi = (A \sinh \lambda z + B \cosh \lambda z)\,[Cr^2 J_2(\lambda r) + Dr^2 Y_2(\lambda r)], \tag{7.27}$$

$$\psi = (A \sin \lambda z + B \cos \lambda z)\,[Cr^2 I_2(\lambda r) + Dr^2 K_2(\lambda r)], \tag{7.28}$$

where $J_2(\lambda r)$ and $Y_2(\lambda r)$ are the Bessel functions of a real argument of the second order of the first and second kind, respectively; $I_2(\lambda r)$ and $K_2(\lambda r)$ are the Bessel functions of an imaginary argument of the second order of the first and second kind, respectively. The stresses are determined by the formulas

$$B_r = -\lambda\,(A \cosh \lambda z + B \sinh \lambda z)\,[CJ_2(\lambda r) + DY_2(\lambda r)],$$

$$B_z = \lambda\,(A \sinh \lambda z + B \cosh \lambda z)\,[CJ_1(\lambda r) + DY_1(\lambda r)] \tag{7.29}$$

and

$$B_r = -\lambda (A \cos \lambda z - B \sin \lambda z) [CI_2 (\lambda r) + DK_2 (\lambda r)],$$
$$B_z = \lambda (A \sin \lambda z + B \cos \lambda z) [CI_1 (\lambda r) + DK_1 (\lambda r)], \quad (7.30)$$

where J_1, Y_1, I_1, K_1 are the Bessel functions of the first order.

In terms of Legendre functions

The differential equation for the torsion of shafts of variable section (7.19) in curvilinear orthogonal isothermal co-ordinates* is of the form

$$\frac{\partial}{\partial \xi}\left(\frac{1}{r^3}\frac{\partial \psi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{r^3}\frac{\partial \psi}{\partial \eta}\right) + 0, \quad (7.31)$$

where $\xi (r, z)$ and $\eta (r, z)$ are curvilinear orthogonal isothermal co-ordinates in the plane of the axial section of a shaft.

The co-ordinates ξ and η in the plane rOz (see Fig. 61) are related to the co-ordinates r and z by the equalities

$$r = Ce^{\xi} \sin \eta, \quad z = Ce^{\xi} \cos \eta, \quad (7.32)$$

and inversely

$$\xi = \ln \sqrt{r^2 + z^2}, \quad \eta = \text{arc tan } r/z.$$

Assuming

$$\psi = f_1 (\xi) f_2 (\eta),$$

where $f_1 (\xi)$ is a function of ξ, and $f_2 (\eta)$ is a function of η, and substituting in Eq. (7.31), we obtain, with (7.32), two equations

$$\begin{aligned} &\frac{d^2 f_1}{d\xi^2} - 3\frac{df_1}{d\xi} - (n-1)(n+2) f_1 = 0, \\ &\frac{d^2 f_2}{d\eta^2} - 3 \cot \eta \frac{df_2}{d\eta} + (n-1)(n+2) f_2 = 0, \end{aligned} \quad (7.33)$$

where n is some constant number.

---

* The isothermal co-ordinates $\xi (r, z)$ and $\eta (r, z)$ satisfy the relations $\frac{\partial \xi}{\partial z} = -\frac{\partial \eta}{\partial r}$ and $\frac{\partial \xi}{\partial r} = \frac{\partial \eta}{\partial z}$.

From the first equation of (7.33), assuming $f_1(\xi) = e^{m\xi}$, we find

$$f_1(\xi) = A_n e^{(n+2)\xi} + B_n e^{(-n+1)\xi}. \tag{7.34}$$

The solution of the second equation of (7.33) is sought in the form

$$f_2(\eta) = \sin^2 \eta Y(\cos \eta) = (1 - \mu^2) Y(\mu), \tag{7.35}$$

where $\mu = \cos \eta$.

Substituting the value of $f_2(\eta)$ in the second equation of (7.33), we arrive at the Legendre equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dY}{d\mu}\right] + \left[n(n+1) - \frac{4}{1-\mu^2}\right] Y = 0, \tag{7.36}$$

from which

$$Y(\mu) = (1-\mu^2)\frac{d^2 P_n(\mu)}{d\mu^2}, \tag{7.37}$$

where $P_n(\mu)$ are Legendre functions of the first kind, or Legendre's polynomials if n is an integer.

The first solution of Eq. (7.31) is

$$\psi_n = [A_n e^{(n+2)\xi} + B_n e^{(-n+1)\xi}] \sin^4 \eta \frac{d^2 P_n(\mu)}{d\mu^2}. \tag{7.38}$$

The second solution is of the form

$$\psi_n = [A_n e^{(n+2)\xi} + B_n e^{(-n+1)\xi}] \sin^4 \eta \frac{d^2 Q_n(\mu)}{d\mu^2}, \tag{7.39}$$

where $Q_n(\mu)$ are Legendre functions of the second kind.

When $n = 0$ and $n = 1$, the solutions are obtained directly from the second equation of (7.33):

when $n = 0$,

$$f_2 = C_0 \cos \eta + D_0 (1 + \cos^2 \eta);$$

when $n = 1$,

$$f_2 = C_1 + D_1 (3 \cos \eta - \cos^3 \eta).$$

Thus, solutions (7.38) and (7.39) are supplemented by two values of the function ψ:

$$\psi_0 = (A_0 e^{2\xi} + B_0 e^{\xi}) [C_0 \cos \eta + D_0 (1 + \cos^2 \eta)], \tag{7.40}$$

$$\psi_1 = (A_1 e^{3\xi} + B_1) [C_1 + D_1 (3 \cos \eta - \cos^3 \eta)].$$

In the case of elliptic co-ordinates ξ and η, which are related to the co-ordinates r and z by the equalities

$$r = a \sinh \xi \sin \eta, \quad z = a \cosh \xi \cos \eta, \tag{7.41}$$

assuming

$$\psi = f_1(\xi) f_2(\eta),$$

we arrive at a solution in the form (see the monograph [52], p. 92)

$$\begin{aligned} \psi_0 &= [A_0 \cosh \xi + B_0 (1 + \cosh^2 \xi)] [C_0 \cos \eta + D_0 (1 + \\ &+ \cos^2 \eta)], \\ \psi_1 &= [A_1 + B_1 (3 \cosh \xi - \cosh^3 \xi] [C_1 + D_1 (3 \cos \eta - \\ &- \cos^3 \eta)], \\ \psi_n &= \sinh^4 \xi \sin^4 \eta \left[A_n \frac{d^2 P_n(\theta)}{d\theta^2} + B_n \frac{d^2 Q_n(\theta)}{d\theta^2} \right] \times \\ &\times \left[C_n \frac{d^2 P_n(\mu)}{d\mu^2} + D_n \frac{d^2 Q_n(\mu)}{d\mu^2} \right], \end{aligned} \tag{7.42}$$

where $\mu = \cos \eta$, $\theta = \cosh \xi$,

$P_n (\ldots)$ = Legendre functions of the first kind,

$Q_n (\ldots)$ = Legendre functions of the second kind.

If the co-ordinates r and z are interchanged, i.e., the poles of the elliptical co-ordinate system are placed not on the axis of the shaft Oz, but on the axis Or, the relation between r, z and ξ, η is

$$r = a \cosh \xi \cos \eta, \quad z = a \sinh \xi \sin \eta, \tag{7.43}$$

and solution (7.42) becomes

$$\begin{aligned} \psi_0 &= [A_0 \sinh \xi + B_0 (1 - \sinh^2 \xi)] [C_0 \sin \eta + D_0 (1 + \\ &+ \sin^2 \eta)], \\ \psi_1 &= [A_1 + B_1 (3 \sinh \xi + \sinh^3 \xi)] [C_1 + D_1 (3 \sin \eta - \\ &- \sin^3 \eta)], \\ \psi_n &= i^n \cosh^4 \xi \cos^4 \eta \left[A_n \frac{d^2 P_n(0)}{d0^2} + B_n \frac{d^2 Q_n(0)}{d0^2} \right] \times \\ &\times \left[C_n \frac{d^2 P_n(\mu)}{d\mu^2} + D_n \frac{d^2 Q_n(\mu)}{d\mu^2} \right], \end{aligned} \tag{7.44}$$

where $0 = i \sinh \xi$, $\mu = \sin \eta$.

PROBLEMS

7.1. A bar of elliptical section $f(x, y) = x^2/a^2 + y^2/b^2 - 1 = 0$ is twisted by a torque M_z.

Investigate the state of stress in the bar.

We assign the stress function in the form

$$\Phi = Af(x, y) = A(x^2/a^2 + y^2/b^2 - 1),$$

where A is an unknown factor.

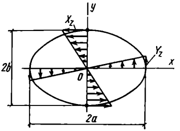

Fig. 62

Substituting the function Φ in Eq. (7.7), we obtain

$$2A/a^2 + 2A/b^2 = -2\alpha G,$$

from which

$$A = -\alpha G a^2 b^2/(a^2 + b^2),$$

and the stress function is

$$\Phi = \frac{\alpha G a^2 b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{a}$$

The stresses are determined by formulas (7.6)

$$X_z = \frac{\partial \Phi}{\partial y} = \frac{2\alpha G a^2}{a^2+b^2}\, y, \quad Y_z = -\frac{\partial \Phi}{\partial x} = \frac{2\alpha G b^2}{a^2+b^2}\, x.$$

The stress diagrams are given in Fig. 62.

To determine M_z, we use formula (7.13). According to the formula (a), the area of the ellipse is

$$F_\Phi = \pi ab\left(1 - \frac{a^2+b^2}{\alpha G a^2 b^2}\right) = \pi ab\left(1 - \frac{\Phi}{\Phi_{max}}\right),$$

where, for $x = y = 0$,

$\Phi_{max} = \alpha G a^2 b^2/(a^2 + b^2)$.

By (7.13),

$$M_z = 2 \int_0^{\Phi_{max}} F_\Phi \, d\Phi = 2\pi ab \int_0^{\Phi_{max}} (1 - \Phi/\Phi_{max}) \, d\Phi =$$

$$= \pi ab \Phi_{max} = \pi \alpha G a^3 b^3/(a^2 + b^2).$$

The maximum stress occurs at the points $(0, \pm b)$

$\max X_z = 2M_z/(\pi a b^2)$.

From Eqs. (7.9) we find

$$u_z = -\alpha \frac{a^2 - b^2}{a^2 + b^2} xy. \quad \text{(b)}$$

Thus, the cross sections of a bar of elliptical section do not remain plane in torsion, but transform into surfaces

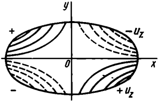

Fig. 63

whose horizontals are equilateral hyperbolas having the axes Ox and Oy as asymptotes (Fig. 63) [1].

7.2. A bar of circular section $f(x, y) = x^2 + y^2 - a^2 = 0$ is twisted by a torque M_z.
Investigate the state of stress in the bar.

We assume the following expression for the stress function

$\Phi = Af(x, y) = A(x^2 + y^2 - a^2)$,

where A is an unknown factor.

By Eq. (7.7),

$$2A + 2A = -2\alpha G,$$

from which

$$A = -\alpha G/2,$$

and the stress function is

$$\Phi = \frac{\alpha G}{2}(a^2 - x^2 - y^2).$$

The stresses are determined by formulas (7.6):

$$X_z = \frac{\partial \Phi}{\partial y} = -\alpha G y, \quad Y_z = -\frac{\partial \Phi}{\partial x} = \alpha G x.$$

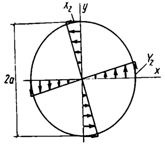

Fig. 64

The stress diagrams are given in Fig. 64.

By (7.13),

$$M_z = \pi \alpha G a^4/2.$$

The maximum stresses are

$$\max X_z = \max Y_z = M_z/W_p,$$

where $W_p = \pi a^3/2$ is the polar section modulus.

All formulas of the present problem are a special case of the formulas of Problem 7.1 when $a = b$, i.e., when the ellipse transforms into a circle. For a bar of circular section, $u_z = 0$, according to the formula (b) of Problem 7.1, i.e., the cross sections remain plane in torsion.

7.3. Webor's problem (1921).

A circular bar of diameter b with a semicircular notch of radius a is twisted by a torque M_z (Fig. 65).

Find the state of stress in the bar.

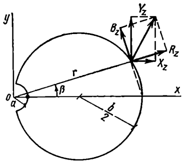

Fig. 65

The equations of the contours of the section in polar co-ordinates are of the form

$$f_1(r, \beta) = r - b\cos\beta = 0, \quad f_2(r, \beta) = r^2 - a^2 = 0.$$

The stress function is taken in the form

$$\Phi(r, \beta) = A\frac{f_1 f_2}{r} = A\left(r^2 - a^2 - br\cos\beta + \frac{ba^2}{r}\cos\beta\right),$$

where A is an unknown factor.

The function Φ is zero on the contour.

In Cartesian co-ordinates when

$$r\cos\beta = x, \quad r\sin\beta = y, \quad r^2 = x^2 + y^2,$$

the stress function is

$$\Phi = A\left(x^2 + y^2 - a^2 - bx + \frac{ba^2x}{x^2+y^2}\right).$$

By (7.7),

$$A = -\alpha G/2,$$

and the stress function is

$$\Phi = \frac{\alpha G}{2}\left[a^2 - r^2 + b\cos\beta\left(r - \frac{a^2}{r}\right)\right].$$

Referring to Fig. 65, the shearing stresses in polar co-ordinates are

$$R_z = X_z \cos\beta + Y_z \sin\beta = \frac{\partial\Phi}{\partial y}\frac{1}{r}\frac{\partial y}{\partial\beta} - \frac{\partial\Phi}{\partial x}\left(-\frac{1}{r}\frac{\partial x}{\partial\beta}\right) = \frac{1}{r}\frac{\partial\Phi}{\partial\beta},$$

$$B_z = Y_z \cos\beta - X_z \sin\beta = -\frac{\partial\Phi}{\partial x}\frac{\partial x}{\partial r} - \frac{\partial\Phi}{\partial y}\frac{\partial y}{\partial r} = -\frac{\partial\Phi}{\partial r}.$$

By differentiating the function Φ, we obtain

$$R_z = -\alpha G b\,(1 - a^2/r^2)\sin\beta,$$

$$B_z = \alpha G\,[r - b\,(1 + a^2/r^2)\cos\beta].$$

The shearing stress assumes a maximum value at the point of the contour located at the root of the notch

$$(\max B_z)_{\substack{\beta=0\\ r=a}} = -\alpha G\,(2b - a).$$

When $b \gg a$, it is twice as high as that on the unnotched contour (stress concentration at notches).

7.4. Saint-Venant's problem.

A rectangular bar of sides a and b ($a > b$) is twisted by a torque M_z (Fig. 66).

Investigate the state of stress in the bar.

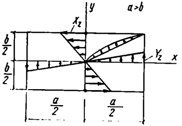

Fig. 66

The stress function is taken in the form

$$\Phi = \alpha G\,(b^2/4 - y^2) + F, \tag{a}$$

where F is an unknown function.

Substituting the expression (a) in Eq. (7.7), we find that the function F must satisfy the harmonic equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=0 \qquad \text{(b)}$$

and the boundary conditions: when $x = \pm a/2$, $F = \alpha G\,(y^2 - b^2/4)$; when $y = \pm b/2$, $F = 0$.

Following the Fourier method, we seek a particular solution of the equation (b) in the form

$$F\,(x,\ y) = X\,(x)\,Y\,(y),$$

where $X\,(x)$ is a function of x, and $Y\,(y)$ is a function of y.

Substituting the function $F\,(x,\ y)$ in the equation (b), and separating the variables, we arrive at the equations

$$\frac{d^2X}{dx^2}-\lambda^2 X=0,\quad \frac{d^2Y}{dy^2}+\lambda^2 y=0, \qquad \text{(c)}$$

where λ^2 is a constant.

In view of the symmetry of the problem the solution of the equations (c) is taken in the form of even functions $X = \cosh \lambda x$, $Y = \cos \lambda y$, from which

$$F=\sum_k A_k \cosh \lambda_k x \cos \lambda_k y.$$

When $y = \pm b/2$, $F = 0$; hence, $\cos \lambda_k b/2 = 0$ and $\lambda_k = (2k + 1)\,\pi/b$ $(k = 0,\ 1,\ 2,\ \ldots)$.

When $x = \pm a/2$, $F = \alpha G\,(y^2 - b^2/4)$, i.e.,

$$\sum_{k=0}^{\infty} A_k \cosh\frac{(2k+1)\,\pi a}{2b}\cos\frac{(2k+1)\,\pi y}{b}=\alpha G\,(y^2-b^2/4). \qquad \text{(d)}$$

The right-hand side of the equality (d) on the interval $-b/2 \leqslant y \leqslant b/2$ is expanded in a trigonometric cosine series

$$\alpha G\,(y^2-b^2/4)=\sum_{k=0}^{\infty} B_k \cos\frac{(2k+1)\,\pi}{b}\,y, \qquad \text{(e)}$$

where

$$B_k = \frac{2}{b}\int_{-b/2}^{b/2} \alpha G\,(y^2 - b^2/4)\cos\frac{(2k+1)\,\pi y}{b}\,dy = -\frac{8\alpha G b^2 (-1)^k}{\pi^3 (2k+1)^3}.$$

On comparing the coefficients A_k and B_k in the expressions (d) and (e), we obtain

$$A_k = B_k/\cosh\frac{(2k+1)\,\pi a}{2b}.$$

The stress function is finally

$$\Phi = \alpha G\left[\frac{b^2}{4} - y^2 - \frac{8b^2}{\pi^3}\sum_{k=0}^{\infty}\frac{(-1)^k\cosh\frac{(2k+1)\,\pi x}{b}\cos\frac{(2k+1)\,\pi y}{b}}{(2k+1)^3\cosh\frac{(2k+1)\,\pi a}{2b}}\right].$$

The maximum shearing stress occurs at the middle of the longer sides, i.e., at $x = 0$ and $y = \pm b/2$

$$\max X_z = \frac{\partial F}{\partial y}\bigg|_{\substack{x=0\\ y=\pm b/2}} = \alpha G b\left[1 - \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2\cosh\frac{(2k+1)\,\pi a}{2b}}\right].$$

The stress diagrams are given in Fig. 66.
By (7.13),

$$M_z = \alpha G a b^3\left[\frac{1}{3} - \frac{64}{\pi^5}\,\frac{b}{a}\sum_{k=0}^{\infty}\frac{\tanh\frac{(2k+1)\,\pi a}{2b}}{(2k+1)^5}\right].$$

The infinite series converge rapidly when $a/b \gg 1$.

For practical calculations, it is convenient to use the formulas

$$\max X_z = M_z/q_1ab^2, \quad \max Y_z = q_2 (\max X_z),$$

$$\alpha = M_z/C_t,$$

where $C_t = q_3Gab^3$ is the torsional rigidity.

The values of the coefficients q_i are given in the table.

a/b	q_1	q_2	q_3	a/b	q_1	q_2	q_3
1	0.208	1.000	0.140	6	0.298	0.743	0.298
1.5	0.230	0.860	0.196	8	0.307	0.743	0.307
2	0.246	0.795	0.229	10	0.312	0.743	0.312
3	0.267	0.753	0.263	∞	0.333	0.743	0.333
4	0.282	0.745	0.281		(1/3)		(1/3)

7.5. Saint-Venant's problem.

A bar with a cross section in the form of an equilateral triangle of height a (Fig. 67) is twisted by a torque M_z.

Investigate the state of stress in the bar.

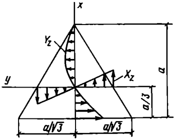

Fig. 67

The stress function is taken in the form

$$\Phi = A\,[x^2 + y^2 - (x^3 - 3xy^2)/a - 4a^2/27].$$

It can easily be verified that the function Φ vanishes on the contour of the section

$$x = -\frac{a}{3}, \quad y = \pm\frac{x}{\sqrt{3}} \pm \frac{2}{3}\frac{a}{\sqrt{3}}.$$

From (7.7),

$$A = -\alpha G/2,$$

and the stress function (a) is

$$\Phi = -\frac{\alpha G}{2}\left[x^2 + y^2 - \frac{1}{a}(x^3 - 3xy^2) - \frac{4a^2}{27}\right].$$

By (7.6), the stresses are

$$X_z = -\alpha G(y + 3xy/a), \quad Y_z = \alpha G\left[x - \frac{3}{2a}(x^2 - y^2)\right].$$

The stress diagrams are given in Fig. 67.
From (7.9),

$$u_z = \frac{\alpha}{2a}(3xy^2 - x^3);$$

u_z vanishes when $x = 0$ and $x = \pm\sqrt{3}\,y$, i.e., on all three perpendiculars dropped from the vertices of the triangle bounding the cross section to its sides. The lines

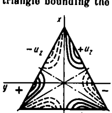

Fig. 68

u_z = constant are third-order algebraic curves whose general character is shown in Fig. 68 [1].

7.6. Leibenzon's problem [54].

A bar with a cross section in the form of a semiring (Fig. 69) is twisted by a torque M_z.

Investigate the state of stress in the bar.

Poisson's equation (7.7) in polar co-ordinates is

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\beta^2}=-2\alpha G. \qquad \text{(a)}$$

Let us find a solution of the equation (a) satisfying condition (7.10) for the stress function on the contour $\Phi = 0$.

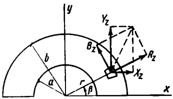

Fig. 69

We expand the right-hand side of the equation (a) on the interval $0 \leqslant \beta \leqslant \pi$ in a Fourier series

$$-2\alpha G=-\frac{8\alpha G}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)\beta \qquad \text{(b)}$$

and seek the solution of the equation (a) in the form of a series

$$\Phi(r,\beta)=\sum_{n=0}^{\infty}f_n(r)\sin(2n+1)\beta. \qquad \text{(c)}$$

Substituting the expressions (b) and (c) in the formula (a), we obtain an equation for determining $f_n(r)$

$$\frac{d^2f_n}{dr^2}+\frac{1}{r}\frac{df_n}{dr}-\frac{(2n+1)^2}{r^2}f_n=-\frac{8}{\pi}\frac{\alpha G}{2n+1}. \qquad \text{(d)}$$

By solving the equation (d), we find

$$f_n(r)=A_nr^{2n+1}+B_nr^{-2n-1}+C_nr^2, \qquad \text{(e}$$

where A_n and B_n are integration constants,

$$C_n = \frac{8\alpha G}{\pi (2n-1)(2n+1)(2n+3)}$$

is a constant in the particular solution.

The series (c) satisfies the condition $\Phi = 0$ on the straight sections ($\beta = 0$ and $\beta = \pi$). We determine A_n and B_n from the other two conditions

$$f_n(a) = f_n(b) = 0.$$

We obtain, finally,

$$f_n(r) = C_n b^2 (\rho^2 - a_n \rho^{2n+1} - b_n \rho^{-2n-1}),$$

where

$$a_n = \frac{1-k^{2n+3}}{1-k^{4n+2}}, \quad b_n = k^{2n+3} \frac{1-k^{2n-1}}{1-k^{4n+2}},$$

$$k = a/b, \quad \rho = r/b.$$

The torsion function (c) is

$$\Phi(\rho, \beta) = b^2 \sum_{n=0}^{\infty} C_n (\rho^2 - a_n \rho^{2n+1} - b_n \rho^{-2n-1}) \sin(2n+1)\beta.$$

By (7.6),

$$X_z = \frac{\partial \Phi}{\partial y} = \frac{1}{b}\left(\frac{\partial \Phi}{\partial \rho} \sin\beta + \frac{\partial \Phi}{\partial \beta} \frac{\cos\beta}{\rho}\right),$$

$$Y_z = -\frac{\partial \Phi}{\partial x} = -\frac{1}{b}\left(\frac{\partial \Phi}{\partial \rho} \cos\beta - \frac{\partial \Phi}{\partial \beta} \frac{\sin\beta}{\rho}\right).$$

Hence, referring to Fig. 69,

$$R_z = Y_z \sin\beta + X_z \cos\beta = \frac{1}{b}\frac{1}{\rho}\frac{\partial \Phi}{\partial \beta},$$

$$B_z = Y_z \cos\beta - X_z \sin\beta = -\frac{1}{b}\frac{\partial \Phi}{\partial \rho}.$$

We obtain, finally,

$$R_z = b \sum_{n=0}^{\infty} C_n (2n+1) [\rho - a_n \rho^{2n} - b_n \rho^{-2(n+1)}] \cos(2n+1)\beta,$$

$$B_z = -b \sum_{n=0}^{\infty} C_n (2n+1) \left[\frac{2}{2n+1} \rho - a_n \rho^{2n} + \right.$$
$$\left. + b_n \rho^{-2(n+1)} \right] \sin (2n+1) \beta.$$

The resulting shearing stress attains a maximum value when $\rho = 1$ and $\beta = \pi/2$ (at the middle of the semicircular arc of longer radius).

Determine the relation between the angle of twist α and the torque M_z.

▶Consider the torsion of a bar of semicircular section when $a = 0$ (Timoshenko's problem).

7.7. Föppl's problem (1905) [36].

A conical shaft is twisted by a torque M_z applied to its vertex (Fig. 70).

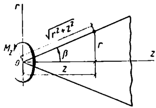

Fig. 70

Determine the shearing stresses.

Any function of the ratio

$$\frac{z}{\sqrt{r^2+z^2}} = \cos \beta = \text{constant}$$

satisfies Eq. (7.19).

The function ψ is taken in the form

$$\psi = C \left[\frac{z}{\sqrt{r^2+z^2}} - \frac{1}{3} \left(\frac{z}{\sqrt{r^2+z^2}} \right)^3 \right].$$

The constant C is deremined from Eq. (7.21):

$$C = \frac{3M_z}{2\pi (2 - 3\cos\beta + \cos^3\beta)}.$$

The shearing stresses are, by (7.18),

$$B_r = -Cr^3/(r^2 + z^2)^{5/2}, \quad B_z = -Crz/(r^2 + z^2)^5$$

7.8. Melan's problem (1920) [55].

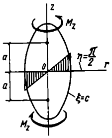

Fig. 71

A shaft in the form of an ellipsoid of revolution is twisted by torques M_z applied to its vertices (Fig. 71).

Find the shearing stresses.

The elliptical co-ordinates ξ and η are related to the co-ordinates r and z by the equalities

$$r = a \sinh \xi \sin \eta, \quad z = a \sinh \xi \cos \eta.$$

The lines η = constant represent a family of hyperbolas, orthogonal to the ellipses ξ = constant, whose real axis is Oz, and the conjugate axis is Or.

The lines ξ = constant are a family of confocal ellipses with an interfocal distance $2a$ and a major axis coinciding with the axis Ox (see Fig. 13; in Fig. 71 this is the Oz axis).

The function ψ is taken in the form

$$\psi = C(2 - 3\cosh \xi + \cosh^2 \xi).$$

This function vanishes on the axis of the shaft ($\xi = 0$). It is constant on a surface of ξ = constant corresponding to some ellipsoid of revolution.

The stresses are

$$\tau_\xi = 0, \quad \tau_\eta = \frac{3C \sinh \xi}{a^3 \sin^3 \eta \, (\cosh^2 \xi - \cos^2 \xi)^{1/2}}.$$

For $\eta = \pi/2$, i.e., on the axis Or

$$\tau_\eta = \frac{3C}{a^3} \tanh \xi = \frac{3C}{a^3} \frac{r}{\sqrt{r^2 + a^2}}.$$

7.9. Melan's problem (1920) [55].

An infinite shaft, having the form of one of the parts of a hyperboloid of revolution of two sheets, is twisted by a torque M_z applied to its vertex (Fig. 72).

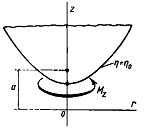

Fig. 72

Find the shearing stresses.

We assign the stress function in the form

$\psi = C\,(2 - 3 \cos \eta + \cos^3 \eta)$,

where the constant C is determined by formula (7.21).

The stresses are

$$\tau_\xi = \frac{3C \sin \eta}{a^3 \sinh^2 \xi \, (\cosh^2 \xi - \cos^2 \eta)^{1/2}}, \quad \tau_\eta = 0.$$

▶Investigate the torsion of a shaft with a lateral surface generated by revolving a hyperbolic arc about the conjugate axis (Fig. 73).

Hint. Take the stress function in the form

$\psi = C\,(2 - 3 \sin \eta + \sin^3 \eta)$.

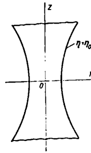

Fig. 73

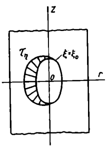

Fig. 74

7.10. See [52], p. 98.
Investigate the torsion of a cylindrical shaft weakened by a hollow in the form of an ellipsoid of revolution whose major axis is situated on the axis Oz (Fig. 74).

Assuming in the third equality of (7.42)

$$n = 2, \quad A_2 = C_2a^4, \quad B_2 = 0, \quad C_2 = 1, \quad D_2 = -D,$$

we obtain

$$\psi = Cr^4\left\{1 - D\left[\frac{\cosh\xi\,(2-3\sinh^2\xi)}{8\sinh^4\xi} - \frac{3}{8}\ln\tanh\frac{\xi}{2}\right]\right\}, \qquad \text{(a)}$$

where

$$D = \frac{1}{\dfrac{\cosh\xi_0\,(2-3\sinh^2\xi_0)}{8\sinh^4\xi_0} - \dfrac{3}{8}\ln\tanh\dfrac{\xi_0}{2}}.$$

The function (a) is zero when $r = 0$ and $\xi = \xi_0$, i.e., on the axis Oz and on the contour of an ellipse in the axial section of the shaft. As $z \to \infty$, the function ψ tends to the value Cr^4, i.e., to the solution for a solid cylindrical shaft.

The shearing stresses are

$$\tau_\xi = -\frac{2Ca\sin^2\eta\sinh^2\xi}{(\cosh^2\xi-\cos^2\eta)^{1/2}}\left\{1-D\left[\frac{\cosh\xi\,(2-3\sinh^2\xi)}{8\sinh^4\xi}-\right.\right.$$

$$\left.\left.-\frac{3}{8}\ln\tanh\frac{\xi}{2}\right]\right\},$$

$$\tau_\eta = \frac{Ca\sin^2\eta\sinh^2\xi}{(\cosh^2\xi-\cos^2\eta)^{1/2}}\left\{4\coth\xi+D\left[\frac{\cosh^2\xi+2\sinh^2\xi}{2\sinh^2\xi}+\right.\right.$$

$$\left.\left.+\frac{3}{2}\coth\xi\ln\tanh\frac{\xi}{2}\right]\right\}.$$

On an inner curved contour ($\xi=\xi_0$)

$$\tau_\xi=0,\quad \tau_\eta=\frac{CDa\sin^2\eta}{\cosh^3\xi_0(\cosh^2\xi_0-\cos^2\eta)^{1/2}}.$$

▶Investigate the torsion of a cylindrical shaft with an elliptical hollow whose major axis is perpendicular to the axis of the shaft.

Hint. In choosing the function ψ assume in the third equality of (7.44)

$$n=2,\quad A_2=C_2a^4,\quad B_1=0,\quad C_2=1,\quad D_2=D.$$

▶Investigate the torsion of a cylindrical shaft with a spherical hollow situated on its axis.

Hint. In choosing the function ψ assume in Eq. (7.38)

$$n=2,\quad A_2=C_2,\quad B_2=-Ce^{5\xi_0}.$$

Chapter 8

THERMAL PROBLEM

It is assumed that the temperature varies over a range in which the elastic coefficients do not change; the extensions are proportional to temperature and are the same in all directions, and hence the shearing strains are zero during the heating of an element of volume. Steady-state and transient thermal processes are considered.

I. STEADY-STATE THERMAL PROCESS

A steady-state thermal process is that in which $t = t(x, y, z)$ is a known function of position.

To determine a stationary temperature field $t = t(x, y, z)$ use is made of the heat conduction equation [56]

$$\nabla^2 t = 0$$

with the corresponding boundary conditions [see Eqs. (4.21) and (8.23)].

In the design of structures on a frozen soil, where $k = k(t)$, the heat conduction equation is of the form [57]

$$\frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\right)=0,$$

where k is the Maxwell thermal conductivity [see Eq. (8.23)].

1. Static and geometrical equations

They are taken in the form of (1.1a) and (2.1a), (1.1b) and (2.1b), (1.1c) and (2.1c), depending on the type of coordinate axes.

2. Physical equations

Direct form of Hooke's law: the Duhamel-Neumann equations (1838, 1885)

$$e_{xx} = \frac{1}{E}[X_x - \sigma(Y_y + Z_z)] + \alpha t, \quad e_{xy} = \frac{1}{G}X_y,$$
$$e_{yy} = \frac{1}{E}[Y_y - \sigma(Z_z + X_x)] + \alpha t, \quad e_{yz} = \frac{1}{G}Y_z, \tag{8.1}$$
$$e_{zz} = \frac{1}{E}[Z_z - \sigma(X_x + Y_y)] + \alpha t, \quad e_{zx} = \frac{1}{G}Z_x,$$

where α is the coefficient of linear thermal expansion (1/°C), αt is the unit thermal expansion.

Inverse form of Hooke's law

$$X_x = \lambda\theta + 2Ge_{xx} - K\alpha t, \quad X_y = Ge_{xy},$$
$$Y_y = \lambda\theta + 2Ge_{yy} - K\alpha t, \quad Y_z = Ge_{yz}, \tag{8.2}$$
$$Z_z = \lambda\theta + 2Ge_{zz} - K\alpha t, \quad Z_x = Ge_{zx},$$

where $K = 2G + 3\lambda = E/(1 - 2\sigma)$ is thrice the bulk modulus (see Problem 1.3).

3. Duhamel-Neumann thermal equations

$$(\lambda + G)\frac{\partial\theta}{\partial x} + G\nabla^2 u_x - K\alpha\frac{\partial t}{\partial x} = 0,$$
$$(\lambda + G)\frac{\partial\theta}{\partial y} + G\nabla^2 u_y - K\alpha\frac{\partial t}{\partial y} = 0, \tag{8.3}$$
$$(\lambda + G)\frac{\partial\theta}{\partial z} + G\nabla^2 u_z - K\alpha\frac{\partial t}{\partial z} = 0.$$

The surface conditions (1.2) expressed in terms of the components of the displacement vector u are of the form

$$\left(\lambda\theta + 2G\frac{\partial u_x}{\partial x}\right)l + G\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)m +$$
$$+ G\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)n = K\alpha\, tl,$$
$$G\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)l + \left(\lambda\theta + 2G\frac{\partial u_y}{\partial y}\right)m +$$
$$+ G\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)n = K\alpha\, tm, \tag{8.4}$$

$$G\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right) l+G\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right) m+$$
$$+\left(\lambda\theta+2G\frac{\partial u_z}{\partial z}\right) n=K\alpha tn.$$

From Eqs. (8.3) and (8.4) it follows that the thermal problem reduces to the usual elastic problem involving body forces

$$X,\ Y,\ Z=\text{grad}\ (-K\alpha t)=-K\alpha\left(\frac{\partial t}{\partial x},\ \frac{\partial t}{\partial y},\ \frac{\partial t}{\partial z}\right)$$

and an external normal surface pressure
$p=-K\alpha t.$

4. Beltrami-Michell equations

$$\begin{aligned}
&\nabla^2 X_x+\frac{2(\lambda+G)}{K}\frac{\partial^2\Theta'}{\partial x^2}+\frac{2GK\alpha}{\lambda+2G}\nabla^2 t+2G\alpha\frac{\partial^2 t}{\partial x^2}=0,\\
&\nabla^2 Y_y+\frac{2(\lambda+G)}{K}\frac{\partial^2\Theta'}{\partial y^2}+\frac{2GK\alpha}{\lambda+2G}\nabla^2 t+2G\alpha\frac{\partial^2 t}{\partial y^2}=0,\\
&\nabla^2 Z_z+\frac{2(\lambda+G)}{K}\frac{\partial^2\Theta'}{\partial z^2}+\frac{2GK\alpha}{\lambda+2G}\nabla^2 t+2G\alpha\frac{\partial^2 t}{\partial z^2}=0, \qquad (8.5)\\
&\nabla^2 X_y+\frac{2(\lambda+G)}{K}\frac{\partial^2\Theta'}{\partial x\,\partial y}+2G\alpha\frac{\partial^2 t}{\partial x\,\partial y}=0,\\
&\nabla^2 Y_z+\frac{2(\lambda+G)}{K}\frac{\partial^2\Theta'}{\partial y\,\partial z}+2G\alpha\frac{\partial^2 t}{\partial y\,\partial z}=0,\\
&\nabla^2 Z_x+\frac{2(\lambda+G)}{K}\frac{\partial^2\Theta'}{\partial z\,\partial x}+2G\alpha\frac{\partial^2 t}{\partial z\,\partial x}=0,
\end{aligned}$$

where

$$\Theta'=X_x+Y_y+Z_z=K\theta-3K\alpha t,$$
$$\frac{2(\lambda+G)}{K}=\frac{2}{1+\sigma},\quad \frac{2GK}{\lambda+2G}=\frac{E}{1-\sigma}.$$

5. Solution of Lamé's equations

The solution of Eqs. (8.3) is taken in the form

$$u_x=u_x^{(1)}+u_x^{(2)},\quad u_y=u_y^{(1)}+u_y^{(2)},\quad u_z=u_z^{(1)}+u_z^{(2)}, \qquad (8.6)$$

where $u^{(1)}$ is the general solution, $u^{(2)}$ is a particular solution.

The general solution is assumed according to Chap. 4, for example, in the form proposed by P. F. Papkovich (4.27)

$$2G\,(u_x^{(1)},\ u_y^{(1)},\ u_z^{(1)}) = 4\,(1-\sigma)\,\psi_{1,\,2,\,3} - \frac{\partial\psi}{\partial x,\ y,\ z},$$

where ψ_i are arbitrary harmonic functions,

$$\psi = x\psi_1 + y\psi_2 + z\psi_3.$$

Assuming, in finding a particular solution,

$$u_x^{(2)} = \frac{\partial F}{\partial x}, \quad u_y^{(2)} = \frac{\partial F}{\partial y}, \quad u_z^{(2)} = \frac{\partial F}{\partial z}, \tag{8.7}$$

we obtain, from Eqs. (8.3), Poisson's equation for the function F

$$\nabla^2 F = \frac{K\alpha}{\lambda + 2G}\,t = \frac{1+\sigma}{1-\sigma}\,\alpha t, \tag{8.8}$$

from which

$$F\,(x,\ y,\ z) = -\frac{K\alpha}{4\pi\,(\lambda + 2G)} \int\limits_V \frac{t\,(\xi,\ \eta,\ \zeta)\,dV}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}\,. \tag{8.9}$$

where ξ, η, ζ are the co-ordinates of an element of volume dV, V is the volume of the whole body.

The boundary conditions (8.4) are

$$\begin{aligned}
&\left(\lambda\theta^{(1)} + 2G\,\frac{\partial u_x^{(1)}}{\partial x}\right) l + G\left(\frac{\partial u_y^{(1)}}{\partial x} + \frac{\partial u_x^{(1)}}{\partial y}\right) m + \\
&+ G\left(\frac{\partial u_z^{(1)}}{\partial x} + \frac{\partial u_x^{(1)}}{\partial z}\right) n = \frac{2GK\alpha t}{\lambda + 2G}\,l - 2G\,\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \nu}\right), \\
&G\left(\frac{\partial u_y^{(1)}}{\partial x} + \frac{\partial u_x^{(1)}}{\partial y}\right) l + \left(\lambda\theta^{(1)} + 2G\,\frac{\partial u_y^{(1)}}{\partial y}\right) m + \\
&+ G\left(\frac{\partial u_z^{(1)}}{\partial y} + \frac{\partial u_y^{(1)}}{\partial z}\right) n = \frac{2GK\alpha t}{\lambda + 2G}\,m - 2G\,\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \nu}\right), \\
&G\left(\frac{\partial u_z^{(1)}}{\partial x} + \frac{\partial u_x^{(1)}}{\partial z}\right) l + G\left(\frac{\partial u_z^{(1)}}{\partial y} + \frac{\partial u_y^{(1)}}{\partial z}\right) m + \\
&+ \left(\lambda\theta^{(1)} + 2G\,\frac{\partial u_z^{(1)}}{\partial z}\right) n = \frac{2GK\alpha t}{\lambda + 2G}\,n - 2G\,\frac{\partial}{\partial z}\left(\frac{\partial F}{\partial \nu}\right).
\end{aligned} \tag{8.10}$$

6. Plane strain

Basic conditions

$$\frac{\partial u_z}{\partial x}=\frac{\partial u_z}{\partial y}=0,\quad \frac{\partial u_z}{\partial z}=a,$$
$$u_x=u_x(x,\ y),\quad u_y=u_y(x,\ y),\quad u_z=az,$$
$$X_z=Y_z=0,\quad Z_z=\frac{\lambda}{2(\lambda+G)}(X_x-Y_y)-\tag{8.11}$$
$$-\frac{GK}{\lambda+G}(a-\alpha t),$$
$$X_x=X_x(x,\ y),\quad Y_y=Y_y(x,\ y),\quad X_y=X_y(x,\ y).$$

Equilibrium equations

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0,\quad \frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}=0.\tag{8.12}$$

Strain compatibility equation

$$\nabla^2(X_x+Y_y)+\frac{2GK\alpha}{\lambda+2G}\nabla^2 t=0.\tag{8.13}$$

By integrating Eqs. (8.12) and (8.13), we obtain the stresses

$$X_x=\frac{\partial^2(U-T)}{\partial y^2},\quad Y_y=\frac{\partial^2(U-T)}{\partial x^2},\quad X_y=-\frac{\partial^2(U-T)}{\partial x\,\partial y},\tag{8.14}$$

where U is a function satisfying the biharmonic equation

$$\nabla^2\nabla^2 U=0,\tag{8.15}$$

T is a function satisfying Poisson's equation

$$\nabla^2 T=\frac{2GK\alpha}{\lambda+2G}t=t_1.\tag{8.16}$$

To determine the displacements, it is necessary to integrate Eqs. (8.1), with (8.11),

$$2G(\lambda+G)\frac{\partial u_x}{\partial x}=\frac{\lambda+2G}{2}X_x-\frac{\lambda}{2}Y_y+GK\alpha t-\lambda Ga,$$
$$2G(\lambda+G)\frac{\partial u_y}{\partial y}=\frac{\lambda+2G}{2}Y_y-\frac{\lambda}{2}X_x+GK\alpha t-\lambda Ga;$$

by integrating, we obtain

$$2G(\lambda+G)\,u_x=\frac{\lambda+2G}{2}\int\nabla^2U\,dx-(\lambda+G)\frac{\partial(U-T)}{\partial x}-$$
$$-\lambda Gax+f_1(y),$$
$$2G(\lambda+G)\,u_y=\frac{\lambda+2G}{2}\int\nabla^2U\,dy-(\lambda+G)\frac{\partial(U-T)}{\partial y}-$$
$$-\lambda Gay+f_2(x), \tag{8.17}$$

where $f_1(y) = Ay + B$, $f_2(x) = -Ax + C$ are functions corresponding to a rigid-body displacement.

In solving the problem in polar co-ordinates, formulas (8.14) are transformed into

$$B_r=\frac{1}{r}\frac{\partial(U-T)}{\partial r}+\frac{1}{r^2}\frac{\partial^2(U-T)}{\partial\beta^2},$$
$$B_\beta=\frac{\partial^2(U-T)}{\partial r^2}, \tag{8.18}$$
$$B_{r\beta}=B_r=-\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(U-T)}{\partial\beta}\right],$$

and Eq. (8.16) is

$$\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2T}{\partial\beta^2}=\frac{2GK\alpha t}{\lambda+2G}=t_1. \tag{8.19}$$

The displacement components are found from the equations

$$2G(\lambda+G)\frac{\partial u_r}{\partial r}=\frac{\lambda+2G}{2}B_r-\frac{\lambda}{2}B_\beta+GK\alpha t-\lambda Ga,$$
$$2G(\lambda+G)\left(\frac{1}{r}\frac{\partial u_\beta}{\partial\beta}+\frac{u_r}{r}\right)=$$
$$=\frac{\lambda+2G}{2}B_\beta-\frac{\lambda}{2}B_r+GK\alpha t-\lambda Ga.$$

By integrating these equations, we obtain, finally,

$$
\begin{aligned}
&2G(\lambda+G)\,u_r = \frac{\lambda+2G}{2}\int \nabla^2 U\,dr - (\lambda+G)\frac{\partial (U-T)}{\partial r} - \\
&-\lambda G a r + F_1(\beta), \\
&2G(\lambda+G)\,u_\beta = -\frac{\lambda+2G}{2}\iint \nabla^2 U\,dr\,d\beta - \\
&-\frac{\lambda r}{2}\int \nabla^2 U\,d\beta + (\lambda+G)\frac{\partial}{\partial r}\left[r\int \frac{\partial (U-T)}{\partial r}\,d\beta\right] + \\
&+\frac{2G(\lambda+G)K\alpha}{\lambda+2G}\,r\int t\,d\beta - \int F_1(\beta)\,d\beta + F_2(r),
\end{aligned}
\tag{8.20}
$$

where $F_1(\beta) = A\sin\beta + B\cos\beta$, $F_2(r) = Cr + A$ are functions corresponding to a rigid-body displacement.

7. Plane stress

Basic conditions

$$
\begin{aligned}
&Z_z = X_z = Y_z = 0, \\
&X_x = X_x(x, y),\quad Y_y = Y_y(x, y),\quad X_y = X_y(x, y).
\end{aligned}
\tag{8.21}
$$

Formulas (8.14) and (8.15) remain unchanged, and Eq. (8.16) reduces to

$$\nabla^2 T = \frac{GK\alpha}{\lambda+G}\,t = t_2. \tag{8.22}$$

To determine the displacements, it is necessary to integrate Eqs. (8.1) with (8.21).

II. TRANSIENT THERMAL PROCESS

A transient thermal process is that in which $t = t(x, y, z, \tau)$ is an unknown function of position and time τ.

To determine the temperature, use is made of the heat conduction equation

$$\varkappa\nabla^2 t + \frac{W}{c\rho} = \frac{\partial t}{\partial \tau}, \tag{8.23}$$

where $\varkappa = k/c\rho$ (cm²/s) is the thermal diffusivity, k (cal·s/cm²·°C) is the Maxwell thermal conductivity, c (cal/g·°C) is the specific heat, $\rho = \gamma/g$ (g·s²/cm⁴) is the density, W (cal·s/cm⁴) is the quantity of heat generated in unit volume per unit time by a heat source situated within an element of volume dV.

Equation (8.23) is integrated taking into account various surface conditions. The following situations are most frequently encountered in the solution of problems:

(1) The temperature on the surface is a given function of position and time.

(2) The heat flow through the surface of the body is zero, i.e., at all points of the surface with normal ν we have

$$\frac{\partial t}{\partial \nu} = 0. \tag{8.24}$$

(3) The heat flow through the surface of the body is a given function of position and time.

(4) There is radiation from the surface. If the heat flow through the surface is proportional to the temperature difference at the boundary between the body (t) and the surrounding medium (t_0), i.e., if the heat flow is determined by the expression

$$H\,(t - t_0),$$

where H is the surface heat transfer coefficient, the boundary condition is of the form

$$k\,\frac{\partial t}{\partial \nu} + H\,(t - t_0) = 0. \tag{8.25}$$

(5) At the boundary between two layers

$$k_1\,\frac{\partial t_1}{\partial \nu} = k_2\,\frac{\partial t_2}{\partial \nu}.$$

PROBLEMS

8.1. Determine the stresses in a symmetrically heated $[t = t\,(r)]$ long tube (plane strain) of outer radius b and inner radius a.

According to formulas (8.18)

$$R_r = \frac{1}{r}\frac{d(U-T)}{dr}, \quad R_\beta = B_r = 0, \quad B_\beta = \frac{d^2(U-T)}{dr^2}. \tag{a}$$

From Eq. (8.19),

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = t_1$$

or

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = t_1 r,$$

from which

$$T = C_0 + C_1 \ln r + \int_c^r \frac{dr}{r}\int_c^r t_1 r\,dr.$$

The function U should be taken in the form

$U = C_2 \ln r + C_3 r^2.$

According to the formulas (a), the stresses are

$$R_r = \frac{C_2 - C_1}{r^2} + 2C_3 - \frac{1}{r^2}\int_c^r t_1 r\,dr,$$

$$B_\beta = -\frac{C_2 - C_1}{r^2} + 2C_3 + \frac{1}{r^2}\int_c^r t_1 r\,dr - t_1.$$

The arbitrary constants $(C_2 - C_1)$ and C_3 are determined from the conditions: when $r = a$ and $r = b$, $R_r = 0$, from which

$$C_3 = \frac{1}{2(b^2 - a^2)}\int_a^b t_1 r\,dr,$$

$$C_2 - C_1 = \frac{1}{b^2 - a^2}\left[b^2\int_c^a t_1 r\,dr - a^2\int_c^b t_1 r\,dr\right].$$

The stress Z_z is determined by formula (8.11) since $X_x + Y_y = R_r + B_\beta$.

For a very short tube (plane stress), t_1 is replaced by t_2 according to formula (8.22).

8.2. Gadolin's problem (1858) [58].

Determine the stresses in a long cylinder made up of two tubes of different material fitted on each other (Fig. 75). The temperature of the cylinder is $t = t(r)$, i.e., it is symmetrical about an axis passing through the centre. The contact between the tubes is maintained throughout.

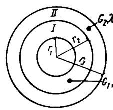

Fig. 75

Denote the pressure between the tubes during heating by X.

The value of X is determined from the condition that the displacements of the points of the outer radius of the inner tube $u_r^{(1)}$ and of the inner radius of the outer tube $u_r^{(2)}$ are the same.

According to formulas (8.20) and Problem 6.1, the displacements of the tubes in the radial direction are

$$2G_1u_r^{(1)} = \frac{r_1^2 \int\limits_0^{r_1} tr\,dr - r_2^2 \int\limits_0^{r_1} tr\,dr}{(r_2^2 - r_1^2)\,r} + \frac{1}{r}\int\limits_0^r tr\,dr +$$

$$+ \frac{G_1 r}{(\lambda_1 + G_1)(r_2^2 - r_1^2)} \int\limits_{r_1}^{r_2} tr\,dr - X\left(\frac{\lambda_1 + 2G_1}{K_1}\frac{r_2^2}{r_2^2 - r_1^2} r +\right.$$

$$\left. + \frac{r_1^2 r_2^2}{r_2^2 - r_1^2}\frac{1}{r}\right), \tag{a}$$

$$2G_2u_r^{(2)} = \frac{r_2^2 \int\limits_0^{r_2} tr\,dr - r_3^2 \int\limits_0^{r_2} tr\,dr}{(r_3^2 - r_2^2)\,r} + \frac{1}{r}\int\limits_0^r tr\,dr +$$

$$+\frac{G_3 r}{(\lambda_2+G_2)(r_3^2-r_2^2)}\int_{r_2}^{r_3} tr\,dr + X\left(\frac{\lambda_2+2G_2}{K_2}\frac{r_3^2}{r_3^2-r_2^2}r+\right.$$

$$\left.+\frac{r_2^2 r_3^2}{r_3^2-r_2^2}\frac{1}{r}\right).$$

The equations (a) may be written more compactly as

$$2G_1u_r^{(1)}=\varphi_1(r)-X\psi_1(r).\quad 2G_2u_r^{(2)}=\varphi_2(r)+X\psi_2(r), \qquad \text{(b)}$$

where the values of φ_i and ψ_i are evident if we compare the equations (a) and (b).

Since when $r=r_2$, $u_r^{(1)}=u_r^{(2)}$, it follows that

$$X=\frac{G_2\varphi_1(r_2)-G_1\varphi_2(r_2)}{G_1\psi_1(r_2)-G_2\psi_2(r_2)}. \qquad \text{(c)}$$

8.3. Determine the stresses in a non-symmetrically heated long tube of inner radius a and outer radius b. The temperature in the tube is expressed by a known function $t=t(r,\ \beta)$.

Since the case of plane strain is considered, we apply formulas (8.18). The function T is calculated from Eq. (8.19)

$$\nabla^2 T = t_1(r,\ \beta), \qquad \text{(a)}$$

where

$$t_1(r,\ \beta)=\frac{2GK\alpha}{\lambda+2G}t(r,\ \beta).$$

We expand the function $t_1(r,\ \beta)$ in a trigonometric series

$$t_1(r,\ \beta)=\varphi_0(r)+\sum_{n=1}^{\infty}[\varphi_n(r)\cos n\beta+\psi_n(r)\sin n\beta], \qquad \text{(b)}$$

where

$$\varphi_n(r)=\frac{1}{\pi}\int_{-\pi}^{\pi} t_1(r,\ \beta)\cos n\beta\,d\beta,$$

$$\psi_n(r)=\frac{1}{\pi}\int_{-\pi}^{\pi} t_1(r,\ \beta)\sin n\beta\,d\beta.$$

The term $\varphi_0(r)$ is excluded from the consideration since this solution is given in Problem 8.1.

The function T is sought in the form of a series

$$T = \sum_{n=1}^{\infty} [f_n(r)\cos n\beta + g_n(r)\sin n\beta]. \tag{c}$$

Substituting the expressions (b) and (c) in the equation (a), we find the following equations for the unknown functions $f_n(r)$ and $g_n(r)$:

$$\begin{aligned} f_n''(r) + \frac{1}{r} f_n'(r) - \frac{n^2}{r^2} f_n(r) = \varphi_n(r), \\ g_n''(r) + \frac{1}{r} g_n'(r) - \frac{n^2}{r^2} g_n(r) = \psi_n(r). \end{aligned} \tag{d}$$

Since the equations (d) are of the Euler type, by introducing a new variable $z = \ln r$ $(r = e^z)$, we obtain the first equation of (d) as

$$f_n''(z) - n^2 f_n(z) = e^{2z}\varphi_n(e^z) = \omega_n(z). \tag{e}$$

The general solution of the equation (e) is

$$f_n(z) = a_n e^{nz} + b_n e^{-nz}.$$

A particular solution is found by the method of variation of the arbitrary constants a_n and b_n (see Problem 3.3).

We obtain, finally, for $f_n(z)$

$$f_n(z) = a_n e^{nz} + b_n e^{-nz} + \frac{e^{nz}}{2n} \int_{k_n}^{e^z} e^{-nz}\omega_n(z)\,dz - \\ - \frac{e^{-nz}}{2n} \int_{l_n}^{e^z} e^{nz}\omega_n(z)\,dz, \tag{f}$$

where k_n and l_n are arbitrarily chosen numbers.

After the corresponding replacement of the function $\omega_n(z)$ on the basis of the equation (e) we obtain a similar expression for $g_n(z)$. As a result of the substitution of the expression (f) in the formula (c) and the inverse transformation to the variable r we find

$$T=\sum_{n=1}^{\infty}\left[\frac{r^{n}}{2n}\int_{k_n}^{r} r^{-n+1}\varphi_n(r)\,dr-\frac{r^{-n}}{2n}\int_{l_n}^{r} r^{n+1}\varphi_n(r)\,dr\right]\cos n\beta+$$

$$+\sum_{n=1}^{\infty}\left[\frac{r^{n}}{2n}\int_{h_n}^{r} r^{-n+1}\psi_n(r)\,dr-\frac{r^{-n}}{2n}\int_{l_n}^{r} r^{n+1}\psi_n(r)\,dr\right]\sin n\beta.$$

The terms of the general solution will enter into the expression for the function U.

The biharmonic function U is taken, by (6.13), in the form

$$U=\left[(B_1r^3+C_1r^{-1}+D_1r\ln r)_{\cos}^{\sin}\beta-\frac{2D_1}{1-\sigma_1}\,r\beta_{\sin}^{\cos}\beta\right]+$$

$$+\sum_{n=2}^{\infty}(A_nr^n+B_nr^{n+2}+C_nr^{-n}+D_nr^{-n+2})_{\cos}^{\sin}n\beta.$$

The stresses are determined from formulas (8.18), after which the arbitrary constants of the general solution are found from the conditions: when $r=a$ and $r=b$, $R_r=B_r=0$.

8.4. See [59].

Determine the state of stress in an infinite triangular dam due to a temperature $t_1=\sum_{k=0}^{\infty} r^k\varphi_k(\beta)$ (Fig. 76).

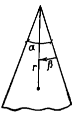

Fig. 76

From Eq. (8.19) we have

$$T = \sum_{k=0}^{n} \frac{r^{k+2}}{k+2} \Big[\sin (k+2) \beta \int_0^\beta \varphi_k (\beta) \cos (k+2) \beta \, d\beta - $$
$$- \cos (k+2) \beta \int_0^\beta \varphi_k (\beta) \sin (k+2) \beta \, d\beta \Big] =$$
$$= \sum_{k=0}^{n} \frac{r^{k+2}}{k+2} g_k (\beta).$$

The function U is taken, by (6.13), in the form

$$U = r^2 (A_2 \cos 2\beta + B_2 \sin 2\beta + C_2 + D_2\beta) +$$
$$+ \sum_{k=1}^{n} r^{k+2} [A_k \cos (k+2) \beta + B_k \sin (k+2) \beta +$$
$$+ C_k \cos k\beta + D_k \sin k\beta].$$

Further, the stresses are determined from formulas (8.18):

$$R_r = 2C_2 + 2D_2\beta - 2A_2 \cos 2\beta + 2B_2 \sin 2\beta - g_0 (\beta) +$$
$$+ \sum_{k=1}^{n} r^k (k+1) [-(k+2) A_k \cos (k+2) \beta -$$
$$- (k+2) B_k \sin (k+2) \beta + (2-k) C_k \cos k\beta +$$
$$+ (2-k) D_k \sin k\beta + g_k (\beta)],$$
$$B_\beta = 2C_2 + 2D_2\beta + 2A_2 \cos 2\beta - 2B_2 \sin 2\beta - g_0 (\beta) +$$
$$+ \sum_{k=1}^{n} (k+1)(k+2) r^k \Big[A_k \cos (k+2) \beta +$$
$$+ B_k \sin (k+2) \beta + C_k \cos k\beta + D_k \sin k\beta - \frac{g_k (\beta)}{k+2} \Big],$$

$$R_\beta = B_r = -D_2 + 2A_2 \sin 2\beta + 2B_2 \cos 2\beta + \frac{1}{2} g_0'(\beta) + \\ + \sum_{k=1}^{n} (k+1) r^k \Big[(k+2) A_k \sin (k+2)\beta - \\ - (k+2) B_k \cos (k+2)\beta + kC_k \sin k\beta - kD_k \cos k\beta + \frac{g_k'(\beta)}{k+1} \Big].$$

The arbitrary constants are found from the boundary conditions: when $\beta = 0$ and $\beta = \alpha$, $R_\beta = B_\beta = 0$.

In [59] G. N. Maslov examined the cases when the temperature varied according to the laws

$$t = \varphi(\beta), \quad t = \sum_{k=2}^{n} r^k \varphi_k(\beta), \quad t = \sum_{k=2}^{n} r^{-k} \varphi_k(\beta), \quad t = r\varphi(\beta)$$

and plotted graphs of stress variation.

8.5. Determine the state of stress in a hollow sphere of inner radius a and outer radius b due to a temperature $t = t(r)$.

For reasons of polar symmetry it is obvious that the only non-vanishing displacement is $u_r = u_r(r)$. According to Eqs. (8.3) and (3.3b), we have

$$\frac{d\theta}{dr} = \frac{d}{dr}\left(\frac{du_r}{dr} + \frac{2u_r}{r}\right) = \frac{K\alpha}{\lambda + 2G} \frac{dt}{dr}.$$

The general solution of this equation is

$$u_r = C_1 r + \frac{C_2}{r^2} + \frac{K\alpha}{\lambda+2G} \frac{1}{r^2} \int_0^r r^2 t \, dr.$$

From Eqs. (8.2) for polar co-ordinates we obtain

$$\begin{aligned} R_r &= \lambda\theta + 2G\frac{du_r}{dr} - K\alpha t = KC_1 - 4G\frac{C_2}{r^3} - \\ &- \frac{4GK\alpha}{\lambda+2G} \frac{1}{r^3} \int_0^r r^2 t \, dr, \\ B_\beta &= A_\alpha = \lambda\theta + 2G\frac{u_r}{r} - K\alpha t = KC_1 + 2G\frac{C_2}{r^3} + \\ &+ \frac{2GK\alpha}{\lambda+2G} \frac{1}{r^3} \int_0^r r^2 t \, dr - \frac{2GK\alpha t}{\lambda+2G}. \end{aligned} \quad \text{(a)}$$

From the boundary conditions of the problem (when $r = a$ and $r = b$, $R_r = 0$) we find

$$C_1 = \frac{4G\alpha \int_a^b r^2 t\,dr}{(\lambda+2G)(b^3-a^3)},$$

$$C_2 = \frac{K\alpha}{\lambda+2G}\left[\frac{a^3}{b^3-a^3}\int_a^b r^2 t\,dr - \int_0^a r^2 t\,dr\right].$$

The equations (a) are finally

$$R_r = \frac{4GK\alpha}{\lambda+2G}\left[\frac{r^3-a^3}{(b^3-a^3)r^3}\int_a^b r^2 t\,dr - \frac{1}{r^3}\int_a^r r^2 t\,dr\right],$$

$$B_\beta = A_\alpha = \frac{2GK\alpha}{\lambda+2G}\left[\frac{2r^3+a^3}{(b^3-a^3)r^3}\int_a^b r^2 t\,dr + \frac{1}{r^3}\int_a^r r^2 t\,dr\right].$$

By the equation (b),

$$u_r = \frac{\alpha}{\lambda+2G}\left[\frac{4Gr^3+Ka^3}{(b^3-a^3)r^2}\int_a^b r^2 t\,dr - \frac{K}{r^2}\int_a^r r^2 t\,dr\right].$$

The solution for a solid sphere is obtained assuming $C_2 = a = 0$.

8.6. Determine the stresses in a cylindrical body of revolution due to $t = t(z, r)$, where the z axis coincides with the axis of revolution.

In this case, in the absence of body forces the stresses are calculated by the formulas of axially symmetric deformation (Problem 4.1) for bodies loaded on the surface by a normal pressure $K\alpha t$ whose intensity varies along the axis of the body, and with fictitious mass forces (8.3).

Assuming

$$\nabla^2\nabla^2 \frac{\partial U}{\partial z} = 0$$

and

$$\nabla^2 \frac{\partial T}{\partial z} = \frac{2GK\alpha}{\lambda+2G} t(z, r) = t_1(z, r),$$

where

$$\nabla^2(\dots) = \frac{\partial^2 \dots}{\partial r^2} + \frac{1}{r}\frac{\partial \dots}{\partial r} + \frac{\partial^2 \dots}{\partial z^2},$$

we obtain the stresses by the formulas

$$R_r = \frac{\partial}{\partial z}\left[\frac{\lambda}{2(\lambda+G)}\nabla^2 U - \frac{\partial^2 (U-T)}{\partial r^2} - \nabla^2 T\right],$$

$$B_\beta = \frac{\partial}{\partial z}\left[\frac{\lambda}{2(\lambda+G)}\nabla^2 U - \frac{1}{r}\frac{\partial (U-T)}{\partial r} - \nabla^2 T\right],$$

$$Z_z = \frac{\partial}{\partial z}\left[\frac{3\lambda+4G}{2(\lambda+G)}\nabla^2 U - \frac{\partial^2 (U-T)}{\partial z^2} - \nabla^2 T\right], \qquad \text{(a)}$$

$$R_z = Z_r = \frac{\partial}{\partial r}\left[\frac{\lambda+2G}{2(\lambda+G)}\nabla^2 U - \frac{\partial^2 (U-T)}{\partial z^2}\right]^*.$$

Here

$$\frac{\lambda}{2(\lambda+G)} = \sigma, \quad \frac{3\lambda+4G}{2(\lambda+G)} = 2-\sigma, \quad \frac{\lambda+2G}{2(\lambda+G)} = 1-\sigma.$$

The equations (a) satisfy the equilibrium equations (8.3) and the Beltrami-Michell equations [Eqs. (8.5), see Problem 3.1].

The function U is chosen in one of the forms satisfying the biharmonic equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 U = 0. \qquad \text{(b)}$$

Below are given some particular solutions of the equation (b):

$$U(z, r) = \sum_{k=0}^{n} f_k(r) z^k,$$

---

* The formulas (a) are obtained from the general solution of the nonhomogeneous equations (8.3) in cylindrical co-ordinates.

where $f_k(r)$ are functions of r (Problem 4.1);

$U(z, r) = (C_1 + C_2 z + C_3 z^2 + C_4 z^3) \ln r$,

$U(z, r) = C(r^2 + z^2)^n$,

where $n = -1/2, 1, 1/2$;

$U(z, r) = C(r^2 + z^2)^n z$,

where $n = -3/2, -1/2, 1$;

$U(z, r) = C\left[(r^2 + z^2)^{-5/2} z^2 - \frac{1}{3}(r^2 + z^2)^{-3/2}\right]$,

etc. (see the textbook [60]).

The arbitrary constants of the solution are determined from the boundary conditions on the surface of the body:

$Z_z = R_r = -K\alpha t, \quad Z_r = 0.$

8.7. See the monograph [58].

Determine the state of stress in the half-plane $x \geqslant 0$ due to a non-uniform temperature $t_1(x, y) = \frac{2GK\alpha}{\lambda + 2G} t(x, y)$ (Fig. 77).

To solve the problem, the functions t_1, T, and U are represented as Fourier integrals. As is known, a function

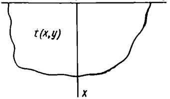

Fig. 77

$f(x, y)$ given in an infinite region can be represented as a Fourier integral if this function satisfies a Dirichlet condition and, in addition, the condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| \, dx \, dy = A,$$

where A is a finite quantity.

It the function $t(x, y)$ satisfies the above conditions and is absolutely integrable, then

$$t_1(x, y) =$$

$$= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty d\alpha_1 \, d\alpha_2 \int_{-\infty}^\infty \int_{-\infty}^\infty t(\xi, \eta) \cos \alpha_1 (\xi - x) \times$$

$$\times \cos \alpha_2 (\eta - y) \, d\xi \, d\eta.$$

The function T satisfying Poisson's equation is of the form

$$T(x, y) = -\frac{1}{\pi^2} \int_0^\infty \int_0^\infty \frac{1}{\alpha_1^2 + \alpha_2^2} \times$$

$$\times [\omega_1(x, \alpha_1, \alpha_2) \cos \alpha_2 y + \omega_2(x, \alpha_1, \alpha_2) \sin \alpha_2 y] \, d\alpha_1 \, d\alpha_2,$$

where

$$\omega_1(x, \alpha_1, \alpha_2) = \int_{-\infty}^\infty \int_{-\infty}^\infty t(\xi, \eta) \cos \alpha_1 (\xi - x) \cos \alpha_2 \eta \, d\eta \, d\xi,$$

$$\omega_2(x, \alpha_1, \alpha_2) = \int_{-\infty}^\infty \int_{-\infty}^\infty t(\xi, \eta) \cos \alpha_1 (\xi - x) \sin \alpha_2 \eta \, d\eta \, d\xi.$$

The function U is taken in the form

$$U = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty [\psi_1(x, \alpha_1, \alpha_2) \cos \alpha_2 y +$$

$$+ \psi_2(x, \alpha_1, \alpha_2) \sin \alpha_2 y] \, d\alpha_1 \, d\alpha_2,$$

where

$$\psi_1(x, \alpha_1, \alpha_2) = (A_1 + B_1 x) e^{-\alpha_1 x},$$

$$\psi_2(x, \alpha_1, \alpha_2) = (A_2 + B_2 x) e^{-\alpha_1 x}.$$

The stresses are, by (8.14),

$$X_x = -\frac{1}{\pi^2}\int_0^\infty\int_0^\infty \alpha_2^2\Big\{\psi_1(x, \alpha_1, \alpha_2)\cos\alpha_2 y +$$

$$+\psi_2(x, \alpha_1, \alpha_2)\sin\alpha_2 y + \frac{1}{\alpha_1^2+\alpha_2^2}[\omega_1(x, \alpha_1, \alpha_2)\cos\alpha_2 y +$$

$$+\omega_2(x, \alpha_1, \alpha_2)\sin\alpha_2 y]\Big\}\, d\alpha_1\, d\alpha_2,$$

$$Y_y = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \Big\{\psi_1''(x, \alpha_1, \alpha_2)\cos\alpha_2 y +$$

$$+\psi_2''(x, \alpha_1, \alpha_2)\sin\alpha_2 y + \frac{1}{\alpha_1^2+\alpha_2^2}[\omega_1''(x, \alpha_1, \alpha_2)\cos\alpha_2 y +$$

$$+\omega_2''(x, \alpha_1, \alpha_2)\sin\alpha_2 y]\Big\}\, d\alpha_1\, d\alpha_2,$$

$$X_y = -\frac{1}{\pi^2}\int_0^\infty\int_0^\infty \alpha_2\Big\{-\psi_1'(x, \alpha_1, \alpha_2)\sin\alpha_2 y +$$

$$+\psi_2'(x, \alpha_1, \alpha_2)\cos\alpha_2 y + \frac{1}{\alpha_1^2+\alpha_2^2}[\omega_1'(x, \alpha_1, \alpha_2)\sin\alpha_2 y +$$

$$+\omega_2'(x, \alpha_1, \alpha_2)\cos\alpha_2 y]\Big\}\, d\alpha_1\, d\alpha_2.$$

From the boundary conditions (when $x=0$, $Y_x=0$ and $X_x=0$) we find

$$A_1 = \frac{\omega_1(0, \alpha_1, \alpha_2)}{\alpha_1^2+\alpha_2^2}, \quad A_2 = \frac{\omega_2(0, \alpha_1, \alpha_2)}{\alpha_1^2+\alpha_2^2},$$

$$B_1 = \frac{\omega_1'(0, \alpha_1, \alpha_2)+\alpha_2\omega_1(0, \alpha_1, \alpha_2)}{\alpha_1^2+\alpha_2^2},$$

$$B_2 = \frac{\omega_2'(0, \alpha_1, \alpha_2)+\alpha_2\omega_2(0, \alpha_1, \alpha_2)}{\alpha_1^2+\alpha_2^2}.$$

The foregoing algorithm enables one to obtain the general solution of the problem, but the calculations involve great difficulties due to the integration of complicated functions.

8.8. See the monograph [56], Chap. VIII, Sec. 2.

Determine the stresses in the half-space $z \geqslant 0$ in the presence of a point heat source of intensity W_M on the surface.

We place the origin of co-ordinates at the point heat source. The analytic expression for the temperature field produced by this source in a body filling the whole space is determined by solving Eq. (8.23) in the form of (8.9):

$$t = W_M/4\pi kR, \tag{a}$$

where $R = \sqrt{r^2 + z^2}$, W_M is the intensity of the heat source, $k = \varkappa\rho c$.

The temperature field (a) can be used for the case of a half-space if the surface bounding the half-space ($z = 0$) is assumed to be perfectly thermally insulated. Indeed, the temperature gradient

$$\frac{\partial t}{\partial z} = -\frac{W_M}{4\pi k}\frac{z}{R^3}$$

is zero when $z = 0$, i.e., the heat flow in a direction normal to the surface of the half-space vanishes.

To find the state of stress, we use the thermoelastic displacement potential F of Eq. (8.8).

For the present problem, Eq. (8.8) is taken in the form

$$\nabla^2 F = \frac{c}{R}, \tag{b}$$

where

$$c = \frac{1+\sigma}{1-\sigma}\frac{\alpha W_M}{4\pi k}.$$

A particular integral of the equation (b) is the expression

$$F = \frac{c}{2}R;$$

this can easily be verified by taking into account the equalities

$$\frac{\partial R}{\partial z} = \frac{z}{R}, \quad \frac{\partial R}{\partial r} = \frac{r}{R}.$$

The stresses are determined by the formulas

$$\begin{aligned}
\overline{R}_r &= 2G\left(\frac{\partial^2 F}{\partial r^2} - \nabla^2 F\right) = cG\left(\frac{z^2}{R^3} - \frac{2}{R}\right),\\
\overline{Z}_z &= 2G\left(\frac{\partial^2 F}{\partial z^2} - \nabla^2 F\right) = cG\left(\frac{r^2}{R^3} - \frac{2}{R}\right),\\
\overline{B}_\beta &= 2G\left(\frac{1}{r}\frac{\partial F}{\partial r} - \nabla^2 F\right) = -c\frac{G}{R},\\
\overline{R}_z &= \overline{Z}_r = 2G\frac{\partial^2 F}{\partial r\,\partial z} = -\frac{cGrz}{R^3}.
\end{aligned} \tag{c}$$

All stresses become zero as $R \to \infty$. On the surface ($z = 0$), $\overline{R}_z = 0$, but the normal stress remains, $\overline{Z}_z = -cG/r$.

To eliminate this stress, we superimpose on the solution obtained a second stress field defined by Love's displacement function [5] given in the form

$$\varphi = A\,[r^2 \ln (R + z) + Rz] + B\,[z^2 \ln (R + z) - Rz],$$

where A and B are as yet arbitrary constants.

The stresses of the second field are determined by the formulas

$$\begin{aligned}
\overline{\overline{R}}_r &= \frac{2G}{1-2\sigma}\frac{\partial}{\partial z}\left(\sigma\nabla^2\varphi - \frac{\partial^2\varphi}{\partial r^2}\right) =\\
&= \frac{2G}{1-2\sigma}\frac{1}{R}\left[\left(2\sigma - \frac{z^2}{R^2}\right)(2A+B) + \frac{2Bz}{R+z}\right],\\
\overline{\overline{Z}}_z &= \frac{2G}{1-2\sigma}\frac{\partial}{\partial z}\left[(2-\sigma)\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right] =\\
&= \frac{2G}{1-2\sigma}\frac{1}{R}\left[\left(3 - 2\sigma + \frac{z^2}{R^2}\right)(2A+B) - 2B\right],\\
\overline{\overline{B}}_\beta &= \frac{2G}{1-2\sigma}\frac{\partial}{\partial z}\left[\sigma\nabla^2\varphi - \frac{1}{r}\frac{\partial\varphi}{\partial r}\right] =\\
&= \frac{2G}{1-2\sigma}\frac{1}{R}\left[(2\sigma-1)(2A+B) + \frac{2BR}{R+z}\right],\\
\overline{\overline{R}}_z &= \overline{\overline{Z}}_r = \frac{2G}{1-2\sigma}\frac{\partial}{\partial r}\left[(1-\sigma)\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right] =\\
&= \frac{2G}{1-2\sigma}\frac{r}{R}\left\{\left[\frac{2(1-\sigma)}{R+z} + \frac{z}{R^2}\right](2A+B) - \frac{2B}{R+z}\right\}.
\end{aligned} \tag{d}$$

The integration constants A and B are determined from the conditions: when $z = 0$, and hence $R = r$,

$$Z_z = \bar{Z}_z + \bar{\bar{Z}}_z = 0, \quad R_z = \bar{R}_z + \bar{\bar{R}}_z = 0.$$

By writing out the last equalities, we find

$$(3 - 2\sigma)(2A + B) - 2B - (1 - 2\sigma)\, c/2 = 0,$$

$$2(1 - \sigma)(2A + B) - 2B = 0,$$

from which

$$2A + B = (1 - \sigma)\, c/2, \quad B = (1 - \sigma)(1 - 2\sigma)\, c/2.$$

Now, by the formulas (d), we obtain

$$\bar{\bar{R}}_r = cG \frac{1}{R}\left(2\frac{z+\sigma R}{R+z} - \frac{z^2}{R^2}\right), \quad \bar{\bar{Z}}_z = cG\frac{1}{R}\left(1 + \frac{z^2}{R^2}\right),$$
$$\bar{\bar{B}}_\beta = \frac{cG}{R(R+z)}[R - (1-2\sigma)z], \quad \bar{\bar{R}}_z = \bar{\bar{Z}}_r = cG\frac{rz}{R^3}, \tag{e}$$

and finally, by adding the two stress fields, (c) and (e), we find

$$R_r = -2(1-\sigma)\frac{cG}{R+z},$$
$$B_\beta = 2(1-\sigma)\, cG\left(\frac{1}{R+z} - \frac{1}{R}\right), \tag{f}$$
$$Z_z = 0, \quad R_z = Z_r = 0.$$

The displacements are determined by the formulas

$$u_r = \frac{\partial F}{\partial r} - \frac{1}{1-2\sigma}\frac{\partial^2 \varphi}{\partial r\,\partial z} = c(1-\sigma)\frac{r}{R+z},$$
$$u_z = \frac{\partial F}{\partial z} + \frac{1}{1-2\sigma}\left[2(1-\sigma)\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2}\right] = \tag{g}$$
$$= c(1-\sigma)\ln(R+z).$$

It is seen from the expressions (g) that the displacement u_r remains bounded at infinity, and the displacement u_z increases indefinitely. At the origin (heat source), both displacements have a singularity.

▶By differentiating the expressions (f) and (g) with respect to z, it is possible to obtain the stress and displacement fields in the half-space subjected to a heat dipole (source and sink), located at the origin, whose axis coincides with the z axis.

8.9. See [15], Problem 97.

The initial temperature of a sphere of radius a is polarly symmetric and is determined by a function $f(r)$. On the surface of the sphere, 0°C temperature is maintained.

Determine the temperature at any point inside the sphere for $\tau > 0$.

The heat conduction equation (8.23) for this case is of the form

$$\varkappa\left(\frac{\partial^2 t}{\partial r^2}+\frac{2}{r}\frac{\partial t}{\partial r}\right)=\frac{\partial t}{\partial \tau}.$$

Assuming

$$t = v/r,$$

where

$$v = v(r,\ \tau),$$

we obtain the equation

$$\varkappa = \frac{\partial^2 v}{\partial r^2} = \frac{\partial v}{\partial \tau} \tag{a}$$

with the boundary conditions

$$v(0,\ \tau) = 0, \quad v(a,\ \tau) = 0 \tag{b}$$

and the initial condition

$$v(r,\ 0) = f(r). \tag{c}$$

By solving the equation (a), with the expressions (b) and (c), we obtain

$$t(r,\ \tau)=\frac{2}{ar}\sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2\varkappa}{a^2}\tau}\sin\frac{n\pi r}{a}\int_0^a rf(r)\sin\frac{n\pi r}{a}\,dr.$$

Further one can follow Problem 8.5 for the thermostatic case, and Problem 10.6 for the thermodynamic case taking into account, in addition to the temperature, the inertia forces.

► See [15], Problem 106.

The initial temperature of a sphere of radius a and of the surrounding medium is 0°. From the time $\tau = 0$, the surrounding temperature rises linearly with time, so that $t = b\tau$, where b is a constant. The heat exchange between the sphere and the medium takes place according to Newton's law [Eq. (8.25)].

Find the temperature distribution inside the sphere considering the problem to be polarly symmetric.

8.10. See [15], Problem 116.

A cylinder of radius a and length l with temperature $t = f(r, z)$ when $\tau = 0$ is placed in a medium with 0°C temperature. The heat exchange of the lateral surface and the bases of the cylinder with the surrounding medium takes place according to Newton's law.

Find the temperature distribution inside the cylinder at any instant.

The heat conduction equation (8.23) in cylindrical coordinates in the case of axial symmetry is of the form

$$\varkappa\left(\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{\partial^2 t}{\partial z^2}\right)=\frac{\partial t}{\partial \tau}. \tag{a}$$

The boundary conditions are represented, according to Eq. (8.25), as

$$\begin{aligned} &k\frac{\partial t}{\partial z}-Ht\,|_{z=0}=0, \quad k\frac{\partial t}{\partial z}+Ht\,|_{z=l}=0,\\ &k\frac{\partial t}{\partial r}+Ht\,|_{r=a}=0, \quad t(0, z, \tau)<\infty, \end{aligned} \tag{b}$$

and the initial condition is

$$t(r, z, 0) = f(r, z). \tag{c}$$

By integrating the equation (a), with the expressions (b) and (c), we obtain

$$t(r, z, \tau)=\sum_{k,\,n=1}^{\infty} A_{kn}J_0\left(\mu_k\frac{r}{a}\right)\left(\cos\frac{\nu_n z}{l}+\right.$$

$$\left.+\frac{p}{\nu_n}\sin\frac{\nu_n z}{l}\right)e^{-\varkappa\left(\frac{\nu_n^2}{l^2}+\frac{\bar{\mu}_k^2}{a^2}\right)\tau},$$

where

$$A_{kn} = \frac{4\mu_k^2 \nu_n^2}{la^2 (\mu_k^2 + H_1 a) [p(p+2) + \nu_n^2]} \int_0^l \int_0^a r f(r, z) \times$$

$$\times J_0\left(\frac{\mu_k r}{a}\right)\left(\cos\frac{\nu_n z}{l} + \frac{p}{\nu_n}\sin\frac{\nu_n z}{l}\right) dz\, dr,$$

$H_1 = H/k, \quad p = Hl/k,$

μ_k are the positive roots of the equation

$\mu J_0'(\mu) + H_1 a J_0(\mu) = 0,$

ν_n are the positive roots of the equation

$2 \tan \nu = \nu/p - p/\nu.$

Comprehensive information on the application of Bessel functions to problems of elasticity and heat conduction is contained in the monograph [61].

Chapter 9

CONTACT PROBLEM

Contact problems are concerned with the determination of the state of stress and strain in the region of contact between two bodies.

Besides the general relations of the theory of elasticity, in the solution of contact problems extensive use is made of the formulas given below.

1. THE ACTION OF PUNCHES ON AN ELASTIC HALF-PLANE

In studying the action of an absolutely rigid body (punch) on an elastic half-plane ($y \geqslant 0$) under plane strain condi-

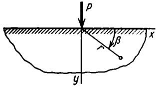

Fig. 78

tions use is made of Flamant's formula (Problem 6.4) (Fig. 78)

$$u_y(x, 0) = -\frac{2(1-\sigma^2)P}{\pi E}\ln\frac{x}{h} - \frac{(1+\sigma)P}{\pi E} =$$

$$= -\frac{2(1-\sigma^2)P}{\pi E}\ln|x| + C, \tag{9.1}$$

$$u_x(x, 0)_{x \gtrless 0} = \mp\frac{(1+\sigma)(1-2\sigma)}{2E}P.$$

If a force P is applied to a punch having a flat base with eccentricity e, the vertical displacement diagram for the points of the base must be trapezoidal and is determined by the expression

$$u_y(x,\ 0) = A + Bx \tag{9.2}$$

(Fig. 79).

The equilibrium equations are of the form

$$\int_{-a}^{a} p(\xi)\,d\xi = P, \quad \int_{-a}^{a} p(\xi)\,\xi\,d\xi = Pe. \tag{9.3}$$

By neglecting the frictional forces over the base of the punch, we obtain the following conditions for the determina-

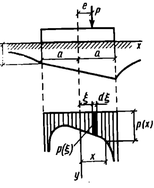

Fig. 79

tion of the normal stresses on the base: when $y = 0$, $X_y = 0$; when $y = 0$, and $-a \leqslant x \leqslant a$,

$$u_y(x,\ 0) = -\frac{2(1-\sigma^2)}{\pi E}\int_{-a}^{a} p(\xi)\ln|x-\xi|\,d\xi = A + Bx^*; \tag{9.4}$$

when $y = 0$ and $-a > x > a$, $p(x) = 0$.

If the base of the punch is bounded by a curve $y_1 = -f_1(x)$, the pressure $p(x)$ on the given segment of contact $-a \leqslant x \leqslant a$ is found as the solution of the integral equa-

* The constant C of formula (9.1) is incorporated in A.

tion

$$\int_{-a}^{a} p(\xi) \ln |1 - x/\xi| \, d\xi + f(x) = C, \tag{9.5}$$

where

$$f(x) = -y_1(x)/\vartheta, \quad \vartheta = \pi E/2(1 - \sigma^2).$$

The solution of Eq. (9.5) given by I. Ya. Shtaerman [62] is of the form

$$p(x) = -\frac{1}{\pi^2} \sqrt{a^2 - x^2} \int_{-a}^{a} \frac{f'(\xi)}{\sqrt{a^2 - \xi^2}} \frac{d\xi}{\xi - x} \tag{9.6}$$

under the condition

$$\int_{-a}^{a} \frac{f'(\xi)\, d\xi}{\sqrt{a^2 - \xi^2}} = 0 \tag{9.7}$$

expressing that $p(x)$ has no other than logarithmic singularities.

In considering the problem of a punch acting on an elastic half-plane, I. Ya. Shtaerman [62] proposed a new design model of a foundation, which generalizes the Winkler-Fuss model in the theory of elasticity, stating that additional displacements resulting from local deformations of the surface of the foundation according to Winkler's law are added to the displacements of formula (9.4).

Thus,

$$u_y(x, 0) = \frac{1}{k} p(x) - \frac{2(1 - \sigma^2)}{\pi E} \int_{-a}^{a} p(\xi) \ln |x - \xi| \, d\xi =$$
$$= A + Bx. \tag{9.8}$$

In the case under consideration the determination of the stresses $p(x)$ over the base of the punch is reduced to the solution of the Fredholm integral equation of the second kind

$$p(x) - \lambda \int_{-a}^{a} p(\xi) \ln |x - \xi| \, d\xi = A + Bx,$$

where $\lambda = 2(1 - \sigma^2)k/\pi E$, k (g-f/cm³) is the modulus of the elastic foundation.

The solution of Eq. (9.8) is given in [62]. For the design model considered, the boundary stresses are finite. When $k = \infty$, Eq. (9.8) transforms into Eq. (9.4); when $E = \infty$, we obtain the compliance method, i.e., Winkler's method, which is widely used in the analysis of elastic beams and punches (absolutely rigid foundations).

II. THE ACTION OF PUNCHES ON AN ELASTIC HALF-SPACE

In studying the action of a punch on an elastic half-space ($z \geqslant 0$) use is made of Boussinesq's formulas (Problem 4.3): when $z = 0$ and $R = r$ (Fig. 22),

$$u_z(x,\ y,\ 0) = \frac{(1-\sigma)\,P}{2\pi G r} = \frac{(1-\sigma^2)\,P}{\pi E r}, \tag{9.9}$$

$$u_r(x,\ y,\ 0) = -\frac{(1-2\sigma)\,P}{4\pi G r} = -\frac{(1+\sigma)(1-2\sigma)\,P}{2\pi E r}. \tag{9.9'}$$

If a force P is applied to a punch having a flat base, with eccentricities e_x and e_y about the y and x axes, respectively, the vertical displacement diagram for the points of the base must be bounded by the plane

$$u_z(x,\ y,\ 0) = A + Bx + Cy. \tag{9.10}$$

The equilibrium equations are

$$\begin{aligned} &\iint_F p(\xi,\ \eta)\,d\xi\,d\eta = P,\\ &\iint_F p(\xi,\ \eta)\,\xi\,d\xi\,d\eta = Pe_x = M_y,\\ &\iint_F p(\xi,\ \eta)\,\eta\,d\xi\,d\eta = Pe_y = M_x, \end{aligned} \tag{9.11}$$

where F is the area of the base of the punch.

By neglecting the frictional forces over the base of the punch, we obtain the following conditions for the determination of the normal stresses on the base: when $z = 0$, $X_z =$

$= Y_z = 0$; when $z = 0$, inside the area (region) F

$$u_z(x,\ y,\ 0) = \frac{1-\sigma^2}{\pi E} \iint_F \frac{p(\xi,\ \eta)\, d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta)^2]^{1/2}} =$$

$$= A + Bx + Cy; \tag{9.12}$$

when $z = 0$, outside the area F

$$p(x,\ y) = 0,$$

where A, B, C are coefficients determining the position of the plane of the punch base during the deformation.

Closed solutions for the problem thus stated are available only for the cases when the area of the base of the punch is bounded by an ellipse or a circle (Problem 9.3).

To take account of both the normal pressure $p(x,\ y) = Z_z(x,\ y,\ 0)$ and the tangential pressures $t_x(x,\ y) = X_z(x,\ y,\ 0)$ and $t_y(x,\ y) = Y_z(x,\ y,\ 0)$ on the plane of contact between the punch and the half-space, recourse must be made to the general solutions of Lamé's equations (4.23) to (4.31) (see Problem 9.7).

III. CONTACT BETWEEN TWO ELASTIC BODIES

In the case of contact between two elastic bodies bounded by surfaces $z_1 = f_1(x,\ y)$ and $z_2 = f_2(x,\ y)$ use is made of the following integral equation for determining the pressure $p(\xi,\ \eta)$ in the region of contact [63]:

$$\delta - f_1(x,\ y) + f_2(x,\ y) = -u_z^{(1)}(x,\ y,\ 0) + u_z^{(2)}(x,\ y,\ 0) =$$

$$= \frac{1}{\pi}\left[\frac{1-\sigma_2^2}{E_2} - \frac{1-\sigma_1^2}{E_1}\right] \iint_F \frac{p(\xi,\ \eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}, \tag{9.13}$$

where δ is the approach of the axes of the elastic bodies due to local compression in the region of contact, F is the area of contact, $u_z^{(i)}(x,\ y,\ 0)$ are the elastic displacements of points A_i of the bodies (Problem 9.1) calculated by formula (9.12).

PROBLEMS

9.1. Hertz's problem (1881) [45].

Consider the state of stress in two long contacting cylinders with parallel axes pressed against each other along their length by distributed forces of intensity p.

We take two points, A_1 and A_2, on the surfaces of the cylinders, which are at a distance x from a plane passing through the axes of the cylinders (Fig. 80a). The distance between these points before deformation is

$$y_1 + y_2 \cong x^2/2R_1 + x^2/2R_2 = \beta x^2,$$

where

$$\beta = 1/2R_1 + 1/2R_2 = (R_1 + R_2)/2R_1R_2.$$

Under the load p, the cylinders are flattened in the region of contact forming a plane of contact in the shape of

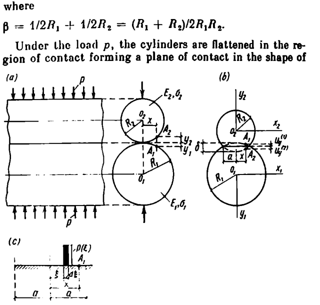

Fig. 80

a straight strip of width $2a$, and their axes come closer together by an amount δ (Fig. 80b).

If $a \gg x$, the points A_1 and A_2 will coincide and

$$\delta - u_y^{(1)} - u_y^{(2)} = y_1 + y_2 = \beta x^2,$$

or

$$u_y^{(1)} + u_y^{(2)} = \delta - \beta x^2,$$

where $u_y^{(1)}$ and $u_y^{(2)}$ are the vertical projections of the displacements of the points A_1 and A_2, respectively.

If the width of the plane of contact is small compared with the radii of the cylinders, each of the cylinders may be approximately considered as an elastic half-plane using formula (9.1) for calculating the displacements $u_y^{(1)}$ and $u_y^{(2)}$.

Assuming the pressure on the contact area to be variable, we obtain

$$\int_{-a}^{a} p(\xi)\, d\xi = p.$$

Under the load $p(\xi)$ acting on a strip of width $d\xi$, the point A_1 (Fig. 80c) is displaced in the vertical direction by the amount [see (9.1)]

$$du_y^{(1)} = -\frac{2(1-\sigma_1^2)}{\pi E_1} p(\xi)\, d\xi \ln \frac{|x-\xi|}{R_1} - \frac{1+\sigma_1}{\pi E_1} p(\xi)\, d\xi =$$

$$= -\frac{2(1-\sigma_1^2)}{\pi E_1}\left[\ln|x-\xi| + \frac{1}{2(1-\sigma_1)} - \ln R_1\right] p(\xi)\, d\xi;$$

the total displacement is

$$u_y^{(1)} = -\frac{2(1-\sigma_1^2)}{\pi E_1}\left\{\int_{-a}^{a} p(\xi) \ln|x-\xi|\, d\xi + \right.$$

$$\left. + \left[\frac{1}{2(1-\sigma_1)} - \ln R_1\right] p\right\}.$$

A similar expression is obtained for $u_y^{(2)}$ by replacing the index 1 by 2.

By adding $u_y^{(1)}$ and $u_y^{(2)}$ together, we find

$$\frac{2}{\pi}\left(\frac{1-\sigma_1^2}{E_1} + \frac{1-\sigma_2^2}{E_2}\right) \int_{-a}^{a} p(\xi) \ln|x-\xi|\, d\xi = \beta x^2 + C,$$

where C stands for the sum of terms independent of x.

By differentiating with respect to x, and eliminating the interval of integration $x - \varepsilon_1 < \xi < x + \varepsilon_2$, when ε_1 and ε_2 tend to zero, $\lim(\varepsilon_2/\varepsilon_1) = 1$, and the integrand tends to infinity [45], we can obtain an integral equation of the form

$$\frac{2}{\pi}\left(\frac{1-\sigma_1^2}{E_1}+\frac{1-\sigma_2^2}{E_2}\right)\int_{-a}^{a}\frac{p(\xi)}{x-\xi}\,d\xi=2\beta x. \qquad \text{(a)}$$

By solving equation (a), Hertz found that

$$p(x)=\frac{p_{max}}{a}\sqrt{a^2-x^2},$$

where

$$p_{max}=2p/\pi a, \quad a=\sqrt{\frac{4pR_1R_2}{\pi(R_1+R_2)}\left(\frac{1-\sigma_1^2}{E_1}+\frac{1-\sigma_2^2}{E_2}\right)}.$$

The maximum compressive stress p_{max} occurs at the middle of the strip of contact.

9.2. Hertz's problem (1881) [45].

Consider the strains and stresses in the region of contact between two spheres pressed against each other by forces P.

We take two points, A_1 and A_2, on the surfaces of the spheres, which are at a distance ρ from an axis z passing through the centres of the spheres (Fig. 81*a*). The distance between these points before deformation is

$$\zeta_1+\zeta_2=\rho^2/2R_1+\rho^2/2R_2=\beta\rho^2,$$

where

$$\beta=1/2R_1+1/2R_2=(R_1+R_2)/2R_1R_2.$$

Under the load P, the spheres are flattened in the region of contact forming a plane of contact in the shape of a circle of radius a, and their centres come closer together by an amount δ.

If $a \gg \rho$, the points A_1 and A_2 will coincide giving (Problem 9.1)

$$u_z^{(1)}+u_z^{(2)}=\delta-\beta\rho^2, \qquad \text{(a)}$$

where $u_z^{(1)}$ and $u_z^{(2)}$ are the vertical projections of the displacements of the points A_1 and A_2, respectively.

If the spheres are approximately considered as elastic half-spaces, we can find the vertical projections of the displacements by formula (9.9):

$$u_z^{(i)} = \iint_F \frac{1-\sigma_i^2}{\pi E_i} \frac{p(r)}{s} dF,$$

where $p(r)$ is the pressure at a distance r from the z axis for which

$$\iint_F p(r)\, dF = P,$$

s is the distance from the point A_i, where the deflection is being determined, to the point of application of the load $p(r)\, dF$, $dF = s\, ds\, d\psi$, $r = \rho \sin \psi$ (Fig. 81*b*).

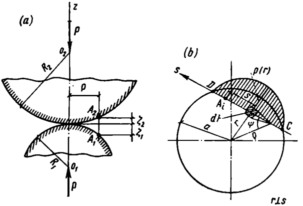

Fig. 81

By taking the sum of the vertical projections of the displacements (a), we obtain the integral equation

$$\frac{1}{\pi}\left(\frac{1-\sigma_1^2}{E_1} + \frac{1-\sigma_2^2}{E_2}\right) \iint_F \frac{p(r)}{s} dF = \delta - \beta\rho^2,$$

or

$$\frac{1}{\pi}\left(\frac{1-\sigma_1^2}{E_1}+\frac{1-\sigma_2^2}{E_2}\right)\int\int\limits_F p(r)\,ds\,d\psi=\delta-\beta\rho^2.$$

By solving this equation, Hertz found [45] that

$$p(r)=\frac{p_{max}}{a}\sqrt{a^2-r^2},$$

where

$$p_{max}=3P/2\pi a^2,$$

$$a=\sqrt[3]{\frac{3PR_1R_2}{4(R_1+R_2)}\left(\frac{1-\sigma_1^2}{E_1}+\frac{1-\sigma_2^2}{E_2}\right)}.$$

The approach of the centres of the spheres is

$$\delta=\left(\frac{1-\sigma_1^2}{E_1}+\frac{1-\sigma_2^2}{E_2}\right)\frac{\pi a}{2}p_{max}.$$

The maximum compressive stress occurs at the centre of the contact area ($r = 0$); here the material acts under conditions of uniform compression. The maximum tensile stress occurs on the contour of the contact area; when $E_1 = E_2 = E$ and $\sigma_1 = \sigma_2 = 0.3$, it is equal to

$$Z_{z=0}=0.133p_{max}$$

9.3. Schleicher's problem.

Determine the state of stress produced in the elastic half-space $z \geqslant 0$ under a circular punch of radius a whose force of gravity is P (Fig. 82).

According to Problem 4.3,

$$u_z=\frac{P}{4\pi GR}\left[2(1-\sigma)+\frac{z^2}{R^2}\right].$$

On the surface ($z = 0$)

$$u_{z=0}=(1-\sigma)P/2\pi Gr=(1-\sigma^2)P/\pi Er,$$

where $r=\sqrt{x^2+y^2}$.

Since the deflections of all points of the punch are the same, it follows that

$$u_{z=0}=\text{constant}=\frac{1}{\pi E_1}\int\limits_F\frac{p(x,y)\,dx\,dy}{\sqrt{x^2+y^2}},\tag{a}$$

where $E_1 = E/(1-\sigma^2)$, $p(x, y)$ is the unknown pressure at a point of co-ordinates x, y.

By solving the equation (a) simultaneously with the equation

$$P = \int_F p(x, y)\,dx\,dy,$$

we obtain

$$p(x, y) = \frac{p_0}{2\sqrt{1 - r^2/a^2}},$$

where

$p_0 = P/\pi a^2$.

The pressure diagram is given in Fig. 82.

▶ Derive the solution of Boussinesq's problem (1885)

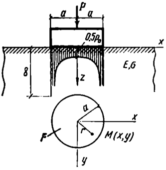

Fig. 82

for an elliptical punch loaded at the centre by a force P.

Answer.

$$p(x, y) = \frac{P}{2\pi ab\sqrt{1 - (x/a)^2 - (y/b)^2}},$$

where a and b are the semiaxes of the ellipse.

9.4. Egorov's problem (1938) [64].

Determine the state of stress produced in the elastic half-space $z \geqslant 0$ under a punch having the shape of a circle of radius a and loaded by a force P applied with eccentricity e (Fig. 83).

The conditions for the solution of the problem are:

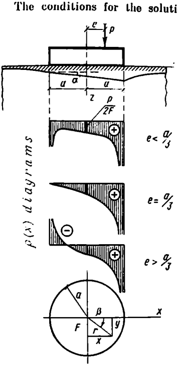

Fig. 83

when $z = 0$, $Y_z = X_z = 0$; when $z = 0$ and $r = \sqrt{x^2+y^2} \leqslant a$,

$$u_{z=0} = \frac{1}{\pi E_1} \int_F \frac{p(x, y)\, dx\, dy}{\sqrt{x^2+y^2}} = A + Bx;$$

when $z = 0$ and $r = \sqrt{x^2+y^2} > a$,

$p(x, y) = 0$.

To solve the problem, we set up the equations of equilibrium

$$\sum Z=0, \quad \int_F p(x, y)\,dx\,dy = P,$$

$$\sum M_y=0, \quad \int_F p(x, y)\,x\,dx\,dy = Pe,$$

$$\sum M_x=0, \quad \int_F p(x, y)\,y\,dx\,dy = 0.$$

The solution satisfying the conditions of the problem is

$$p(r)=\frac{1+3\frac{er}{a^2}\cos\beta}{2\pi a\sqrt{a^2-r^2}}P,$$

$$u_{z=0}(r)=\frac{1-\sigma^2}{2Ea}\left(\frac{3}{2}\frac{er}{a^2}\cos\beta+1\right)P,$$

$$\text{angle } \alpha=\frac{3}{4}\frac{1-\sigma^2}{Ea^3}Pe.$$

When $\beta=0$ and $r=x$, the stresses are determined from the formula

$$p(x)=\frac{P}{\pi a^2}\frac{1+3\frac{xe}{a^2}}{2\sqrt{1-\left(\frac{x}{a}\right)^2}}. \tag{a}$$

By the formula for eccentric compression,

$$p(x)=\frac{P}{\pi a^2}\left(1+4\frac{xe}{a^2}\right).$$

According to the formula (a), when $x=-a$ and $e=a/3$, the tensile stresses under the punch are zero since

$$\lim_{x\to(-a)}\frac{1+x/a}{\sqrt{1-(x/a)^2}}=\lim_{x\to(-a)}\frac{a+x}{\sqrt{a^2-x^2}}=\lim_{x\to(-a)}\sqrt{\frac{a+x}{a-x}}=0.$$

▶By using Gorbunov-Posadov's solution [65], investigate the action of a strip punch ($a/b=5$) on the elastic half-space $z\geqslant 0$ (Fig. 84).

Hint. Use the formulas

$$u_z(0,\ y,\ 0)=\frac{1-\sigma^2}{E}k_0\left(\frac{a}{b}\right)\frac{P}{\sqrt{F}},\quad \alpha=\frac{1-\sigma^2}{E}k_1\left(\frac{a}{b}\right)\frac{M_y}{a^3}$$

and tables for the quantities $k_0\,(a/b)$ and $k_1\,(a/b)$ when $a/b = 1$ to 10.

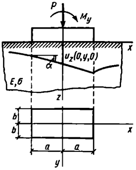

Fig. 84

9.5. Sadowsky's problem (1928).

Determine the state of stress produced in the elastic half-plane $y \geqslant 0$ by a punch of width $2a$ whose force of gravity is P (Fig. 85).

The frictional forces over the base of the punch are neglected in the determination of normal pressures.

In using formula (9.1), it is necessary to take into account that h is a variable quantity and to determine the deflection due to the distributed load by the formula

$$u_y(x,\ 0)=-\frac{2(1-\sigma)^2}{\pi E}\int_{-a}^{a}p(\xi)\ln|x-\xi|\,d\xi.$$

The conditions for the solution of the problem are: when $y = 0$, $X_y = 0$; when $y = 0$ and $-a \leqslant x \leqslant a$,

$$-\frac{2(1-\sigma)^2}{\pi E}\int_{-a}^{a}p(\xi)\ln|x-\xi|\,d\xi=A;$$

when $y = 0$ and $-a > x > a$,

$p(\xi) = 0.$

By setting up the equation of equilibrium for the punch

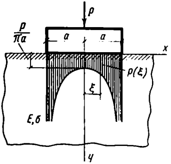

Fig. 85

($\sum Y = 0$), we obtain

$$\int_{-a}^{a} p(\xi)\, d\xi = P.$$

The solution satisfying the conditions of the problem is

$$p(x) = \frac{P}{\pi \sqrt{a^2 - x^2}} = \frac{2q_0}{\pi} \frac{1}{\sqrt{1 - \bar{x}^2}},$$

where

$$q_0 = P/2a, \quad \bar{x} = x/a.$$

9.6. Florin's problem (1936) [66].

Determine the state of stress produced in the elastic half-plane $y \geqslant 0$ under a punch of width $2a$ to which a moment $M = Pe$ is applied (Fig. 86).

The conditions for the solution of the problem are: when $y = 0$, $X_y = 0$; when $y = 0$ and $-a \leqslant x \leqslant a$,

$$-\frac{2(1-\sigma^2)}{\pi E} \int_{-a}^{a} p(\xi) \ln|x - \xi|\, d\xi = Bx;$$

when $y = 0$ and $\infty > x > a$, $-a > x > -\infty$,

$$p(\xi) = 0.$$

By setting up the equation of equilibrium for the punch ($\sum M_z = 0$), we obtain

$$\int_{-a}^{a} p(\xi)\,\xi\,d\xi = M.$$

The solution satisfying the conditions of the problem is

$$p(x) = \frac{2M}{\pi a^2}\,\frac{x}{\sqrt{a^2-x^2}}, \quad \tan\alpha = \frac{4(1-\sigma^2)}{\pi E a^2}\,M.$$

9.7. See [67, 68].

Determine the pressure under a punch having the shape of a circle of radius a in the elastic half-space $z \geqslant 0$ for the

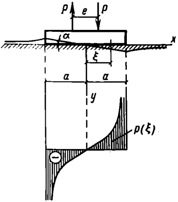

Fig. 86

case of known tangential forces in the region of contact $t(r)$ directed along a radius drawn from the origin to the point under consideration.

According to Eq. (4.30), the elastic displacements are

$$u_{x,y,z} = \psi_{1,2,3} + z\frac{\partial \chi}{\partial x, y, z}, \tag{a}$$

where $\nabla^2 \psi_i = 0$,

$$\frac{\partial \chi}{\partial z} = \frac{1}{4\sigma - 3}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right). \tag{b}$$

The stresses on the plane $z = 0$ are denoted by

$$X_z(x, y, 0) = X(x, y), \quad Y_z(x, y, 0) = Y(x, y),$$

$$Z_z = (x, y, 0) = -p(x, y).$$

By Hooke's law and the equations (a),

$$\frac{\partial \psi_1}{\partial z} + \frac{\partial(\psi_3 + \chi)}{\partial x}\Big|_{z=0} = \frac{X}{G}, \quad \frac{\partial \psi_2}{\partial z} + \frac{\partial(\psi_1 + \chi)}{\partial y}\Big|_{z=0} = \frac{Y}{G}. \tag{c}$$

By differentiating and adding the equations (c) together, we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}\right) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\psi_3 + \chi) = \frac{1}{G}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right),$$

or, noting that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\psi_3 + \chi) = -\frac{\partial^2(\psi_3 + \chi)}{\partial z^2},$$

we arrive at the equation

$$\frac{\partial}{\partial z}\left[\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} - \frac{\partial(\psi_3 + \chi)}{\partial z}\right] = \frac{1}{G}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right).$$

This equation enables one to determine the harmonic function in the square brackets, ψ, from its normal derivative on the plane $z = 0$ by using the formula

$$\psi = -\frac{1}{2\pi}\int\int_{\infty} \frac{\psi_z'(\xi, \eta, 0)}{R}\, d\xi d\eta, \tag{d}$$

where

$$R = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}.$$

Assuming all functions to vanish at infinity, we obtain, by the formula (d),

$$\psi = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} - \frac{\partial (\psi_3 + \chi)}{\partial z} =$$

$$= -\frac{1}{2\pi G} \iint\limits_{\infty} \frac{X'_\xi(\xi, \eta) + Y'_\eta(\xi, \eta)}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}} \, d\xi d\eta. \qquad \text{(e)}$$

On the basis of the expression (b) we obtain the following equation instead of (e):

$$\frac{\partial \psi_3}{\partial z} + 2(1-\sigma)\frac{\partial \chi}{\partial z} = \frac{1+\sigma}{2\pi E} \iint\limits_{\infty} \frac{X'_\xi + Y'_\eta}{R} \, d\xi \, d\eta. \qquad \text{(f)}$$

The normal stress on the surface is, by Hooke's law and the equation (b),

$$p(x, y) = -\frac{E}{1+\sigma}\left[\frac{\partial \psi_3}{\partial z} + (1-2\sigma)\frac{\partial \chi}{\partial z}\right]_{z=0}. \qquad \text{(g)}$$

Eliminating the function $\frac{\partial \chi}{\partial z}$ from the formulas (f) and (g) gives

$$\left.\frac{\partial \psi_3}{\partial z}\right|_{z=0} = -\frac{2(1-\sigma^2)}{E} p(x, y) -$$

$$-\frac{(1-2\sigma)(1+\sigma)}{2\pi E} \iint\limits_{\infty} \frac{X'_\xi + Y'_\eta}{r} \, d\xi \, d\eta, \qquad \text{(h)}$$

where

$r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$.

According to the expression (a), the surface displacements are

$u_x^0 = \psi_1(x, y, 0), \quad u_y^0 = \psi_2(x, y, 0), \quad u_z^0 = \psi_3(x, y, 0).$

We determine the deformation of the surface of the elastic half-space due to the given tangential forces $t(r)$.

Introduce the function

$$T(r) = \int_r^\infty t(r)\, dr.$$

In this case

$$\frac{\partial T}{\partial x} = -t(r)\cos\beta = -X, \quad \frac{\partial T}{\partial y} = -t(r)\sin\beta = -Y,$$

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = -\nabla_{xy}^2 T.$$

The formula (h) with $p(x, y) = 0$ is

$$\left.\frac{\partial \psi_3}{\partial z}\right|_{z=0} = \frac{(1-2\sigma)(1+\sigma)}{2\pi E} \int\int_{\infty} \frac{\nabla_{\xi\eta}^2 T}{r} \, d\xi \, d\eta = \frac{\partial u_z^0}{\partial z}. \tag{i}$$

In the following discussion it is assumed that the functions T and ψ_3 vanish at infinity. By applying the formula (d) to ψ_3, and differentiating with respect to z, we find

$$\frac{\partial \psi_3}{\partial z} = -\frac{1}{2\pi} \frac{\partial}{\partial z} \int\int_{\infty} \frac{1}{R} \frac{\partial \psi_3(\xi, \eta, 0)}{\partial z} d\xi \, d\eta,$$

from which we arrive at the formula for the solution of the Dirichlet problem for the half-space $z \geqslant 0$

$$\psi_3(x, y, z) = -\frac{1}{2\pi} \lim_{s\to\infty} \frac{\partial^2}{\partial z^2} \int\int_{s} \frac{\psi_3(\xi, \eta, 0)}{R} \, d\xi \, d\eta. \tag{j}$$

It follows from the formula (j) that

$$\left.\frac{\partial \psi_3}{\partial z}\right|_{z=0} = -\frac{1}{2\pi} \lim_{s\to\infty} \frac{\partial^2}{\partial z^2} \int\int_{s} \frac{\psi_3(\xi, \eta, 0)}{r} \, d\xi \, d\eta =$$

$$= \frac{1}{2\pi} \lim_{s\to\infty} \nabla_{xy}^2 \int\int_{s} \frac{\psi_3(\xi, \eta, 0)}{r} \, d\xi \, d\eta =$$

$$= \frac{1}{2\pi} \int\int_{\infty} \frac{1}{r} \nabla_{\eta\xi}^2 \psi_3(\xi, \eta, 0) \, d\xi \, d\eta. \tag{k}$$

On comparing the formulas (i) and (k), we find the deflection of the surface of the half-space under the tangential forces only

$$u_z^0(t) = \frac{(1-2\sigma)(1+\sigma)}{E} T(r).$$

Denoting the deflection due to the normal pressure $p(x, y)$ by $u_z^0(p)$, we obtain

$$u_z^0(p) + u_z^0(t) = Z(r),$$

where $Z(r)$ is the equation of the surface of the punch being indented, or

$$u_z^0(p) = Z(r) - u_z^0(t).$$

The pressure $p(r)$ is calculated by the formula

$$p(r) = \frac{C}{2\pi\sqrt{a^2-r^2}} - \frac{1}{\pi}\int_0^{\pi/2} d\beta \int_0^{\Phi} \left\{ \frac{E}{1-\sigma^2} \times \right.$$

$$\times \left[Z''\left(\sqrt{r^2\sin^2\beta+u^2}\right) + \frac{Z'\left(\sqrt{r^2\sin^2\beta+u^2}\right)}{\sqrt{r^2\sin^2\beta+u^2}} \right] +$$

$$\left. + \frac{1-2\sigma}{1-\sigma}\left[t'\left(\sqrt{r^2\sin^2\beta+u^2}\right) + \frac{t\left(\sqrt{r^2\sin^2\beta+u^2}\right)}{\sqrt{r^2\sin^2\beta+u^2}} \right] \right\} du.$$

For a smooth punch having no corners at the contact boundary, $C = 0$.

9.8. See [69].

Investigate the state of stress in a circular bar of constant section ($\delta = 1$), which rests on an absolutely rigid and smooth foundation and is symmetrically loaded by bending moments M and normal forces N (Fig. 87).

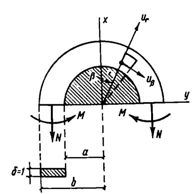

Fig. 87

The stresses in the bar are expressed by the formulas

$$R_r = \frac{1}{r}\frac{d\varphi}{\partial r} + \frac{1}{r}\frac{\partial^2\varphi}{\partial\beta^2}, \quad B_\beta = \frac{\partial^2\varphi}{\partial r^2},$$
$$R_\beta = B_r = -\frac{1}{r}\frac{\partial}{\partial\beta}\left(\frac{\partial\varphi}{\partial r} - \frac{\varphi}{r}\right), \tag{a}$$

where φ is a biharmonic function.

The boundary conditions of the problem are:

$$\left.\begin{array}{ll} \text{when } r = a, & u_r = B_r = 0; \\ \text{when } r = b, & R_r = B_r = 0; \end{array}\right\} \tag{b}$$

when $\beta = \pm\pi/2$,

$$\int_a^b B_\beta\, dr = N, \quad \int_a^b B_\beta r\, dr = M, \quad R_\beta = 0. \tag{c}$$

We assume the function φ in the form of one of the known particular integrals of the biharmonic equation (6.13):

$$\varphi = (Ar^\lambda + Br^{\lambda+2} + Cr^{-\lambda} + Dr^{-\lambda+2}) \cos\lambda\beta, \tag{d}$$

where λ is an undetermined parameter.

By the formulas (a)

$$R_r = [\lambda(1-\lambda)Ar^{\lambda-2} - (\lambda+1)(\lambda-2)Br^\lambda -$$
$$-\lambda(\lambda+1)Cr^{-\lambda-2} + (1-\lambda)(\lambda+2)Dr^{-\lambda}]\cos\lambda\beta,$$
$$B_\beta = [\lambda(\lambda-1)Ar^{\lambda-2} + (\lambda+1)(\lambda+2)Br^\lambda +$$
$$+\lambda(\lambda+1)Cr^{-\lambda-2} + (\lambda-1)(\lambda-2)Dr^{-\lambda}]\cos\lambda\beta, \tag{e}$$
$$R_\beta = B_r = \lambda[(\lambda-1)Ar^{\lambda-2} + (\lambda+1)Br^\lambda -$$
$$-(\lambda+1)Cr^{-\lambda-2} + (1-\lambda)Dr^{-\lambda}]\sin\lambda\beta.$$

The displacements are determined from the equations

$$\frac{\partial u_r}{\partial r} = \frac{1}{E}(R_r - \sigma B_\beta),$$
$$\frac{1}{r}\frac{\partial u_\beta}{\partial\beta} + \frac{u_r}{r} = \frac{1}{E}(B_\beta - \sigma R_r), \tag{f}$$
$$\frac{1}{r}\frac{\partial u_r}{\partial\beta} + \frac{\partial u_\beta}{\partial r} - \frac{u_\beta}{r} = \frac{2(1+\sigma)}{E}R_\beta.$$

By integrating the equations (f), and equating to zero the arbitrary functions, we find

$$Eu_r = \{-(1+\sigma)\lambda A r^{\lambda-1} - [\lambda - 2 + \sigma(\lambda+2)]Br^{\lambda+1} +$$
$$+ (1+\sigma)\lambda C r^{-\lambda-1} + [\lambda + 2 + \sigma(\lambda-2)]Dr^{-\lambda+1}\}\cos\lambda\beta,$$
$$Eu_\beta = \{(1+\sigma)\lambda A r^{\lambda-1} + [(1+\sigma)\lambda + 4]\,Br^{\lambda+1} + \qquad \text{(g)}$$
$$+ (1+\sigma)\lambda C r^{-\lambda-1} + [(1+\sigma)\lambda - 4]\,Dr^{-\lambda+1}\}\sin\lambda\beta.$$

By setting up the conditions (b), and equating to zero the determinant $\Delta(\lambda)$ of these equations, we obtain a transcendental equation for the determination of λ

$$\Delta(\lambda) \equiv \lambda(\lambda^2-1)b^{-3}[-2\lambda(3-\sigma)(\lambda+1)\alpha + 6(1-\sigma)\times$$
$$\times(\lambda^2-1)\alpha^{-1} + (1+\sigma)(\lambda^3+5\lambda-2)\alpha^{-3} +$$
$$+ 2(\lambda^2+4\lambda+2-2\sigma)\,\alpha^{-2\lambda-1} - \qquad \text{(h)}$$
$$- 2(\lambda+1)(2\lambda-1+\sigma)\,\alpha^{2\lambda-1}] = 0,$$

where $\alpha = a/b$.

When $\lambda = 0$, we obtain Lamé's solution; when $\lambda = \pm 1$, we obtain particular integrals of the biharmonic equation different from the solution (d). The remaining roots of the equation (h) give a set of parameters λ_m.

For each parameter λ_m, the corresponding constants are determined by the formulas

$$A_m = \Delta_1(\lambda_m)\,F_m, \quad B_m = \Delta_2(\lambda_m)\,F_m,$$
$$C_m = \Delta_3(\lambda_m)\,F_m, \quad D_m = \Delta_4(\lambda_m)\,F_m, \qquad \text{(i)}$$

where $\Delta_i(\lambda_m)$ are the cofactors of the elements of a row or column of the determinant $\Delta(\lambda_m)$, F_m is an arbitrary factor of proportionality.

By summing the solutions over m, we obtain the general expression for the stress function for a state of stress symmetrical in the β co-ordinate

$$\varphi = \sum_m [\Delta_1(\lambda_m) r^{\lambda m} + \Delta_2(\lambda_m) r^{\lambda m+2} + \Delta_3(\lambda_m) r^{-\lambda m} + \Delta_4(\lambda_m) r^{-\lambda m+2}] F_m \cos \lambda_m \beta. \quad \text{(j)}$$

Substituting the series (j) in the formulas (e) and (g), we find the values of the stresses and displacements.

By leaving the necessary number of terms in the series (j), it is possible to satisfy two integral conditions of (c), and to require that R_β should be zero at a finite number of points $r = r_i$ for the local condition of (c); at the points $r = a$ and $r = b$ this condition is already fulfilled [the equations (b)].

▶Solve Problem 9.8 in the absence of displacements u_r and u_β along the line of contact $r = a$.

9.9. See [70].

Determine the pressure when the half-plane $y \leq 0$ is indented by a punch, taking into account the frictional forces

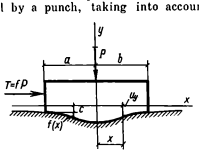

Fig. 88

between them. Assume that the punch is in a state of limiting shear equilibrium* when the horizontal shearing force $T = fP$, where f is the coefficient of friction and P is the force pressing the punch (Fig. 88).

By taking $f(x)$ to be the equation of the contour of the punch, we obtain the boundary conditions in the form $Y_{y=0} = X_{y=0}$ free surface,
$X_{y=0} + fY_{y=0} = 0$, $u_{y=0} = f(x) + c$ surface under (a) punch.

* The force T produces no moment (the punch does not turn).

Integrating the strain equations for the half-plane gives

$$\frac{\pi E_1}{2} u_x + C_1 = \frac{1-2\sigma}{2(1-\sigma)} \pi \int\limits_a^x Y_{y=0}\, dx + \int\limits_a^b X_{y=0} \ln(\xi - x)\, d\xi,$$

$$\frac{\pi E_1}{2} u_y + C_2 = \int\limits_a^b Y_{y=0} \ln(\xi - x)\, d\xi - \frac{1-2\sigma}{2(1-\sigma)} \pi \int\limits_a^x X_{y=0}\, dx, \tag{b}$$

where $E_1 = E/(1-\sigma^2)$, C_i are constants.

The last equation of (b) is represented as

$$\frac{\pi E_1}{2} u_y + C_2 = \int\limits_a^b Y_{y=0} \ln|\xi - x|\, d\xi - \frac{1-2\sigma}{2(1-\sigma)} \pi \int\limits_0^x X_{y=0}\, dt.$$

The derivative of the displacement is

$$\frac{\pi E_1}{2} \left(\frac{\partial u_y}{\partial x}\right)_{y=0} = \int\limits_a^b Y_{y=0} \frac{d\xi}{\xi - x} - \frac{1-2\sigma}{2(1-\sigma)} \pi X_{y=0}. \tag{c}$$

On the basis of the condition (a) the expression (c) takes the form

$$\frac{\pi E_1}{2} \left(\frac{\partial u_y}{\partial x}\right)_{y=0} = \int\limits_a^b Y_{y=0} \frac{d\xi}{\xi - x} + \frac{1-2\sigma}{2(1-\sigma)} \pi f Y_{y=0}. \tag{d}$$

We introduce a function of the complex variable $z = x + iy$:

$$w_1(z) = \bar{u}_x - i\bar{u}_y = \int\limits_a^b Y_{y=0} \frac{dx}{x - z}. \tag{e}$$

The right-hand side of the formula (d) is expressed in terms of the real and imaginary parts of the function (e) for $y = 0$:

$$\frac{\pi E_1}{2} \left(\frac{\partial u_y}{\partial x}\right)_{y=0} = \bar{u}_x + f \frac{1-2\sigma}{2(1-\sigma)} \bar{u}_y.$$

By taking the condition (a), we obtain

$$\frac{\pi E_1}{2} f'(x) = \bar{u}_x + f \frac{1-2\sigma}{2(1-\sigma)} \bar{u}_y.$$

On the free surface $Y_{y=0} = 0$ and the imaginary part of the function for $y = 0$ is zero.

The boundary conditions on the sections outside the punch are: when $y = 0$ and $-\infty < x < a$, $b < x < \infty$,

$$\bar{u}_y = 0;$$

the boundary condition in the region of contact between the punch and the elastic half-plane is: when $y = 0$ and $a < x < b$,

$$\bar{u}_x + f\frac{1-2\sigma}{2(1-\sigma)}\bar{u}_y = \frac{\pi E_1}{2} f'(x). \tag{f}$$

Besides, the function $w_1(z)$:

(1) may have singularities of the form $z^{-\alpha}$, where $0 < \alpha < 1$, on the real axis because of the absence of concentrated forces under the punch;

(2) must behave at infinity as Pz^{-1}, where P is the force pressing the punch.

The constant lost in differentiating the expression (c) must enter into the expression for the function (f) and is determined from additional data.

Thus, the problem has reduced to a special case of the mixed boundary value problem of finding a function of a complex variable (Riemann-Hilbert problem).

It is necessary to find a function that satisfies, on the x axis, the condition

$$a(x)\, u_x + b(x)\, u_y = F(x),$$

where

$$a(x) = 0, \quad b(x) = 1 \text{ when } -\infty < x < a, \quad b < x < \infty;$$

$$a(x) = 1, \quad b(x) = f\frac{1-2\sigma}{2(1-\sigma)} \text{ when } a < x < b;$$

$$F(x) = \frac{\pi E_1}{2} f'(x).$$

The solutions of the homogeneous problem are of the form

$$w_1(z) = \exp\left[\frac{1}{\pi}\arctan\left(f\frac{1-2\sigma}{2-2\sigma}\right)\int_a^b \frac{dx}{x-z}\right]\frac{iP(z)}{(z-a)(z-b)},$$

where $P(z)$ is a polynomial.

By taking different polynomials $P(z)$, we obtain different solutions of the homogeneous problem:

$$w_1^0=\left(\frac{z-b}{z-a}\right)^{\theta},$$

$$w_2^0=\left(\frac{z-b}{z-a}\right)^{1-\theta}, \tag{g}$$

$$w_1^0=\left(\frac{z-b}{z-a}\right)^{\theta}\frac{1}{z-b}=\frac{1}{(z-a)^{\theta}(z-b)^{1-\theta}},$$

where

$$\theta=\frac{1}{\pi}\arctan\frac{2(1-\sigma)}{f(1-2\sigma)};$$

here $|\theta|<1$.

For the first polynomial of (g), we obtain

$$w_1(z)=\frac{E_1}{2\sqrt{1+f^2\left(\frac{1-2\sigma}{2-2\sigma}\right)^2}}\left(\frac{z-b}{z-a}\right)^{1-\theta}\times$$

$$\times\int_a^b f'(x)\left(\frac{b-x}{x-a}\right)^{-\theta}\frac{dx}{x-z}. \tag{h}$$

For the solution (h) to become general, it is necessary to add to it a function satisfying the homogeneous conditions

$$\frac{C}{(z-a)^{\theta}(z-b)^{1-\theta}}. \tag{i}$$

The pressure exerted on the punch is

$$p(x)=-\operatorname{Im}[w_1(z)]_{z=x-i\cdot 0}.$$

If there is no friction between the punch and the elastic body, then

$$f=0 \quad \text{and} \quad \theta=\frac{1}{\pi}\arctan\frac{2(1-\sigma)}{f(1-2\sigma)}=\frac{1}{2}.$$

Substituting the values $f=0$ and $\theta=1/2$ in the expression (h), with the function (i) taken into account, letting

$z \to x$, and separating the imaginary part, we obtain

$$p(x) = -Y_{y=0} =$$

$$= -\frac{E_1}{2} \frac{1}{\sqrt{(x-a)(b-x)}} \int_a^b f'(x) \sqrt{(\xi-a)(b-\xi)} \times$$

$$\times \frac{d\xi}{\xi - x} - \frac{1}{\sqrt{(x-a)(b-x)}}.$$

▶Determine the pressure, taking into account friction, when a half-plane is indented by a punch with a flat base [$f(x)$ = constant] under a force P (see the monograph [63], Chap. I, Sec. 8).

Answer.

$$w_1(z) = -\frac{P}{\sqrt{(z-a)(z-b)}} \left(\frac{z-a}{z-b}\right)^{1/2-\theta}.$$

When $f = 0$ and $\theta = 1/2$,

$$w_1(z) = -\frac{P}{\sqrt{(z-a)(z-b)}}.$$

The normal pressure on the surface of the half-plane is

$$p(x) = -\frac{1}{\pi} \operatorname{Im} [w_1(z)]_{z=x+i0} = \frac{1}{\pi} \frac{P}{\sqrt{(a+x)(b-x)}}.$$

Cf. the results of Problem 9.5.

9.10. See [71].

Determine the state of stress and strain in the elastic half-plane $y \leqslant 0$ if on the segment AB

$$u_x = g_1 + C_1, \quad u_y = g_2 + C_2, \tag{a}$$

on the axis Ox outside the segment AB

$$Y_y = N = 0, \quad X_y = T = 0, \tag{b}$$

and the resultant vector (X, Y) of forces acting on the segment AB is known (Fig. 89).

According to the conditions of the problem,

$$\int_{AB} T\,dx = X, \quad \int_{AB} N\,dx = -Y. \tag{c}$$

If N and T were known on the segment AB, the problem would be solved.

By applying the complex-variable method ($z = x + iy$), we find expressions for the auxiliary functions [28]:

$$\psi(z) = -\frac{1}{2\pi i}\int_{AB}(N + iT)\ln(t - z)\,dt + C_3,$$
$$\chi'(z) = -\frac{1}{2\pi i}\int_{AB}(N - iT)\ln(t - z)\,dt - z\psi'(z) + C_4. \qquad \text{(d)}$$

Substituting the expressions (d) in the displacement formula (5.20)

$$2G(u_x + iu_y) = \varkappa\psi(z) - \overline{z\psi'(z)} - \overline{\chi'(z)},$$

where for plane strain

$$\varkappa = \frac{\lambda + 3G}{\lambda + G} = 3 - 4\sigma,$$

and passing to the limit as $z \to \tau$, where τ is a point of the segment AB, we obtain, taking into account the for-

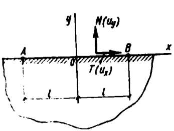

Fig 89

mulas (a), integral equations for the determination of N and T:

$$\int_{-l}^{l} T(t)\ln|t - \tau|\,dt + \alpha\pi\int_{0}^{\tau} N(t)\,dt =$$
$$= -\frac{4G\pi}{\varkappa + 1}g_1(\tau) + C_1 = f_1(\tau) + C_1, \qquad \text{(e)}$$

$$\int_{-l}^{l} N(t)\ln|t-\tau|\,dt-\alpha\pi\int_{0}^{\tau} T(t)\,dt=$$
$$=\frac{4G\pi}{\varkappa+1}g_2(\tau)+C_2=f_2(\tau)+C_2,$$

where

$$\alpha=\frac{\varkappa-1}{\varkappa+1}<1.$$

We introduce into consideration functions U_1+iV_1 and U_2+iV_2 of the complex variable z

$$U_1+iV_1=\int_{-l}^{l} T(t)\ln(t-z)\,dt,$$
$$U_2+iV_2=\int_{-l}^{l} N(t)\ln(t-z)\,dt. \tag{f}$$

It is seen from the formulas (f) that U_1 and U_2 are logarithmic potentials for simple layers distributed on AB with densities T and N.

By symmetry, $U_1(x, y)$ and $U_2(x, y)$ are even functions in y. By the well-known formula of potential theory, we obtain

$$T=\frac{1}{2\pi}\left[\left(\frac{\partial U_1}{\partial y}\right)_+-\left(\frac{\partial U_1}{\partial y}\right)_-\right]=\frac{1}{\pi}\left(\frac{\partial U_1}{\partial y}\right)_+ \tag{g}$$

and a similar formula for N.

The plus and minus signs refer to the values of the derivative obtained when approaching points of the segment AB from above $(y>0)$ and from below $(y<0)$, respectively.

From the formula (g), by using the Cauchy-Riemann relations, we derive

$$\int_0^{\tau} T(x)\,dx=\frac{1}{\pi}\int_0^{\tau}\left(\frac{\partial U_1}{\partial y}\right)_+dx=-\frac{1}{\pi}\int_0^{\tau}\left(\frac{\partial V_1}{\partial x}\right)_+dx=$$
$$=-\frac{1}{\pi}V_1(x,+0)+C, \tag{h}$$

and similarly for N.

On the basis of the formulas (h), the solution of the equations (e) reduces to finding two real functions, U_1 and U_2, continuous in the whole plane, harmonic outside the segment AB, which behave at infinity as [from the expressions (f) and (c)] $X \ln |z|$ and $Y \ln |z|$, respectively, and satisfy at the upper edge of the segment AB the conditions

$$U_1 - \alpha V_2 = f_1 + C_1, \quad U_2 + \alpha V_1 = f_2 + C_2, \tag{i}$$

where V_1 and V_2 are functions conjugate to U_1 and U_2, respectively.

To solve the problem, we map the z plane cut along AB onto the outside of the circle $|\zeta| = 1$ of the plane $\zeta = \xi + i\eta$ by the well-known relation

$$z = \frac{l}{2}\left(\zeta + \frac{1}{\zeta}\right).$$

On the basis of the foregoing properties of the unknown functions we have, for $|\zeta| > 1$,

$$\begin{aligned} U_1 + iV_1 &= X \ln \zeta + \sum_1^\infty a_n \zeta^{-n} + C_1, \\ U_2 + iV_2 &= Y \ln \zeta + \sum_1^\infty b_n \zeta^{-n} + C_2, \end{aligned} \tag{j}$$

where a_n and b_n are unknown real coefficients.

Putting $\zeta = re^{i\beta}$ $(r \geqslant 1)$, and assuming that the functions U_1, U_2 and the one-signed parts of the functions V_1, V_2 are continuous up to the contour of the circle, and also that the above expansions are valid for $r = 1$ (it follows that the series $\sum a_n^2$ and $\sum b_n^2$ converge), we obtain, by the formulas (i), for $0 \leqslant \beta \leqslant \pi$:

$$\begin{aligned} \sum_1^\infty a_n \cos n\beta + \alpha \sum_1^\infty b_n \sin n\beta &= F_1 + C_1, \\ \sum_1^\infty b_n \cos n\beta - \alpha \sum_1^\infty a_n \sin n\beta &= F_2 + C_2, \end{aligned} \tag{k}$$

where

$$F_1 = f_1 + \alpha Y\beta, \quad F_2 = f_2 - \alpha X\beta.$$

By multiplying both sides of the equations (k) by $\frac{2}{\pi}\cos m\beta$ ($m = 1, 2, \ldots$), and integrating between the limits 0 and π, we arrive at an infinite system of linear equations with an infinite number of unknowns of the form

$$a_m + \alpha \sum_{n=1}^{\infty} a_{mn} b_n = C'_m, \quad b_m - \alpha \sum_{n=1}^{\infty} a_{mn} a_n = C''_m,$$

where

$$a_{mn} = \frac{2}{\pi}\int_0^{\pi} \cos m\beta \sin n\beta \, d\beta =$$

$$= \begin{cases} \dfrac{4n}{\pi(n^2 - m^2)} & \text{if } (n-m) \text{ is odd,} \\ 0 & \text{if } (n-m) \text{ is even,} \end{cases}$$

C'_m and C''_m are the Fourier coefficients of the functions F_1 and F_2 in their cosine-series expansions on the interval $[0, \pi]$.

If f_1 and f_2 have, for example, bounded derivatives with respect to β on the interval $[0, \pi]$, the series on the left-hand sides of the expressions (k), and the more so the series in the expressions (j), converge absolutely and uniformly. After finding $U_1 + iV_1$ and $U_2 + iV_2$, we determine $\psi(z)$ and $\chi'(z)$ directly from the formulas (d) and (*f*).

Chapter 10

DYNAMIC PROBLEM

Assuming that the motion of an elastic isotropic body (medium) is characterized by infinitesimal strains, we can write the equations of motion by applying D'Alembert's principle:

$$(\lambda + G)\left(\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}\right) + G\nabla^2(u_x, u_y, u_z) =$$

$$= \rho\left(\frac{\partial^2 u_x}{\partial\tau^2}, \frac{\partial^2 u_y}{\partial\tau^2}, \frac{\partial^2 u_z}{\partial\tau^2}\right), \tag{10.1}$$

or in vector form

$$(\lambda + G)\,\mathrm{grad}\,\theta + G\nabla^2\boldsymbol{u} = \rho\frac{\partial^2\boldsymbol{u}}{\partial\tau^2}. \tag{10.2}$$

By using the vector identity

$$\mathrm{rot}\ \ \mathrm{rot} = \mathrm{grad}\ \ \mathrm{div} - \nabla^2,$$

we obtain, from (10.2),

$$(\lambda + 2G)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - G\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u} = \rho\frac{\partial^2\boldsymbol{u}}{\partial\tau^2}. \tag{10.2'}$$

Under the conditions of motion adopted, the formulas for strains and Hooke's law remain unchanged.

1. SIMPLE HARMONIC MOTION

For a simple harmonic motion with period $T = 2\pi/p$, where p is the circular frequency of vibration, the displacements can be expressed as

$$u_x = A\cos(p\tau + \varepsilon)\,u'_x,\ u_y = A\cos(p\tau + \varepsilon)\,u'_y,$$
$$u_z = A\cos(p\tau + \varepsilon)\,u'_z,$$

and Eqs. (10.1) are obtained in the form

$$(\lambda + G)\left(\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}\right) + (G\nabla^2 + \rho p^2)(u_x, u_y, u_z)^* = 0. \quad (10.3)$$

By differentiating these equations with respect to x, y, z, respectively, and adding the results together, we find

$$(\nabla^2 + h^2)\theta = 0, \quad (10.4)$$

where

$$h^2 = \rho p^2/(\lambda + 2G). \quad (10.5)$$

On putting

$$\varkappa^2 = \rho p^2/G, \quad (10.6)$$

we obtain Eqs. (10.3) in the form

$$(\nabla^2 + \varkappa^2)(u_x, u_y, u_z) = \left(1 - \frac{\varkappa^2}{h^2}\right)\left(\frac{\partial\theta}{\partial x}\,\frac{\partial\theta}{\partial y}\,\frac{\partial\theta}{\partial z}\right). \quad (10.7)$$

If θ satisfies Eq. (10.4), i.e., $\nabla^2\theta = -h^2\theta$, the solution of Eqs. (10.7) is

$$u_x, u_y, u_z = u_y^{(1)} + u_x^{(2)},\ u_y^{(1)} + u_y^{(2)},\ u_z^{(1)} + u_z^{(2)}, \quad (10.8)$$

where $u_x^{(1)}$, $u_y^{(1)}$, $u_z^{(1)}$ is the general solution of Eqs. (10.7) without the right-hand sides provided that

$$\theta^{(1)} = \operatorname{div} u = \frac{\partial u_x^{(1)}}{\partial x} + \frac{\partial u_y^{(1)}}{\partial y} + \frac{\partial u_z^{(1)}}{\partial z} = 0 \quad (10.9)$$

since it is determined from Eq. (10.4);

$$u_x^{(2)}, u_y^{(2)}, u_z^{(2)} = -\frac{1}{h^2}\left(\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}\right) \quad (10.10)$$

is a particular solution of Eqs. (10.7) with the right-hand sides.

Equations (10.4) and (10.7) without the right-hand sides are called the equations of vibration, or the standing wave equations.

Their particular solutions are:

rectangular co-ordinates

$$u_{lm}^{(1)} = e^{i(kx+ly+mz)} \quad (10.11)$$

(plane wave), where $k^2 + l^2 + m^2 = \varkappa^2$;

---

* In Eqs. (10.3) primes on the displacements are omitted for simplicity.

cylindrical co-ordinates

$$u_{hm}^{(1)} = e^{\pm i(mz+h\beta)} R_h \left(r \sqrt{\varkappa^2 - m^2}\right)^{*} \tag{10.12}$$

(cylindrical wave); when $k = m = 0$, $u_{00}^{(1)} = (a_1\beta + a_2)(b_1 z + b_2) \times \times R_0(\varkappa r)$;

spherical co-ordinates

$$u_n^{(1)} = \frac{1}{\sqrt{r}} R_{n+1/2}(\varkappa r) Y_n(\alpha, \beta)^{**} \tag{10.13}$$

(spherical wave); when $n = 0$, $u_0^{(1)} = e^{i\varkappa r}/r$.

For vibrations in a plane ($m = 0$), a plane wave transforms into a line wave, and a cylindrical wave into a circular wave.

II. PROPAGATION OF VOLUME WAVES IN AN ELASTIC ISOTROPIC MEDIUM

When $\theta = 0$, Eqs. (10.1) become

$$G\nabla^2 (u_x, u_y, u_z) = \rho \frac{\partial^2}{\partial \tau^2} (u_x, u_y, u_z). \tag{10.14}$$

If ω_x, ω_y, ω_z [Eqs. (2.3a)] are zero, so that the vector u is the gradient of a potential φ, i.e.,

$$u = \text{grad}\,\varphi = i\frac{\partial\varphi}{\partial x} + j\frac{\partial\varphi}{\partial y} + k\frac{\partial\varphi}{\partial z},$$

or

$$u_x = \frac{\partial\varphi}{\partial x}, \quad u_y = \frac{\partial\varphi}{\partial y}, \quad u_z = \frac{\partial\varphi}{\partial z},$$

then

$$\theta = \nabla^2\varphi,$$

$$\left(\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}\right) = \nabla^2 (u_x, u_y, u_z),$$

---

* $R_k(pr) = E_k J_k(pr) + F_k N_k(pr)$ is the cylindrical function of order k (see Chap. 4).

** $Y_n(\alpha, \beta) = \sum_m C_{mn} P_{nm}(\cos\alpha)\, e^{im\beta}$ is the spherical function of degree n (see Chap. 4).

and Eqs. (10.1) take the form

$$(\lambda + 2G)\nabla^2 (u_x,\ u_y,\ u_z) = \rho \frac{\partial^2}{\partial \tau^2}(u_x,\ u_y,\ u_z). \qquad (10.15)$$

Equations (10.14) and (10.15) are wave equations in space

$$c_i^2 \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial \tau^2}, \qquad (10.16)$$

where c_i is the velocity of wave propagation, $\varphi = u_x, u_y, u_z$.

Shear waves, or waves of distortion (S-waves), involving no dilatation ($\theta = 0$) have a velocity $c_2 = \sqrt{G/\rho}$; irrotational waves ($\omega_x = \omega_y = \omega_z = 0$), or compression-dilatation waves (P-waves), involving a change of volume, travel with a velocity $c_1 = \sqrt{\lambda + 2G/\rho}$.

If $\varphi = \varphi(x,\ y,\ \tau)$, Eq. (10.16) assumes the form of the wave equation in a plane

$$c_i^2 = \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = \frac{\partial^2 \varphi}{\partial \tau^2}\,^{*}. \qquad (10.17)$$

When $\varphi = \varphi(x,\ \tau)$ [similarly, $\varphi = \varphi(y,\ \tau)$], we obtain a one-dimensional wave equation

$$c^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial \tau^2}. \qquad (10.18)$$

The general solution of Eq. (10.18) given by D'Alembert is of the form

$$\varphi = f_1(x - c\tau) + f_2(x + c\tau), \qquad (10.19)$$

where f_i are arbitrary functions.

The process expressed by Eq. (10.19) involves two waves travelling with a velocity c.

If $\varphi = \varphi(r,\ \tau)$, where r is the radius vector of a point, Eq. (10.18) takes the form

$$\frac{c^2}{r}\frac{\partial^2}{\partial r^2}(r\varphi) = \frac{\partial^2 \varphi}{\partial \tau^2}. \qquad (10.20)$$

* For $u_x = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}$, $\quad u_y = \frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x}$, we obtain

$$c_1^2 \nabla^2 \psi_1 = \frac{\partial^2 \psi_1}{\partial \tau^2} \quad \text{and} \quad c_2^2 \nabla^2 \psi_2 = \frac{\partial^2 \psi_2}{\partial \tau^2}.$$

The general solution of Eq. (10.20) is

$$\varphi = f_1(r - c\tau)/r + f_2(r + c\tau)/r, \qquad (10.21)$$

where f_i are arbitrary functions.

The process expressed by Eq. (10.21) involves two circular waves, with a source at the fixed point, travelling with a velocity c.

The surface s of the disturbed part of the medium moves in the direction of its normal ν with a velocity c.

The kinematic conditions (nine equations) for points of the surface s ([5], Article 205) are of the form

$$\begin{aligned}
\frac{\partial u_x}{\partial \nu} &= \frac{\partial u_x}{\partial x}\Big/ l = \frac{\partial u_x}{\partial y}\Big/ m = \frac{\partial u_x}{\partial z}\Big/ n = -\frac{1}{c}\frac{\partial u_x}{\partial \tau},\\
\frac{\partial u_y}{\partial \nu} &= \frac{\partial u_y}{\partial x}\Big/ l = \frac{\partial u_y}{\partial y}\Big/ m = \frac{\partial u_y}{\partial z}\Big/ n = -\frac{1}{c}\frac{\partial u_y}{\partial \tau}, \qquad (10.22)\\
\frac{\partial u_z}{\partial \nu} &= \frac{\partial u_z}{\partial x}\Big/ l = \frac{\partial u_z}{\partial y}\Big/ m = \frac{\partial u_z}{\partial z}\Big/ n = -\frac{1}{c}\frac{\partial u_z}{\partial \tau},
\end{aligned}$$

where $l = \cos(x, \nu)$, $m = \cos(y, \nu)$, $n = \cos(z, \nu)$ are the direction cosines of the surface s.

The dynamic conditions (three equations) for points of the surface s are

$$\rho c\left(\frac{\partial u_x}{\partial \tau}, \ \frac{\partial u_y}{\partial \tau}, \ \frac{\partial u_z}{\partial \tau}\right) = -(X_\nu, Y_\nu, Z_\nu), \qquad (10.23)$$

where X_ν, Y_ν, Z_ν are the projections on the x, y, z axes of the stress on the surface s.

The right-hand sides of Eqs. (10.23) may be written as

$$\begin{aligned}
-\Big[(\lambda + G)\, l\frac{\partial u_x}{\partial x} + G\left(l\frac{\partial u_x}{\partial x} + m\frac{\partial u_x}{\partial y} + n\frac{\partial u_x}{\partial z}\right) +\\
+ \left(\lambda l\frac{\partial u_y}{\partial y} + Gm\frac{\partial u_y}{\partial x}\right) + \left(\lambda l\frac{\partial u_z}{\partial z} + Gn\frac{\partial u_z}{\partial x}\right)\Big], \qquad (10.24)
\end{aligned}$$

. . . .

. . . .

The next two expressions of (10.24) are obtained by cyclic permutation of the letters x, y, z, l, m, n.

III. WAVE PROPAGATION OVER THE SURFACE OF AN ELASTIC ISOTROPIC BODY

1. Rayleigh waves [72]

Plane waves of a simple harmonic nature travelling over the boundary plane $z = 0$ of the medium $z \geqslant 0$, for which the disturbance penetrates only a very short distance into the medium, are called Rayleigh waves (R-waves). The displacements for wave propagation in the x direction are of the form

$$u_x,\ u_y,\ u_z = u_x^{(1)} + u_x^{(2)},\quad u_y^{(1)} + u_y^{(2)},\quad u_z^{(1)} + u_z^{(2)}, \tag{10.25}$$

where

$$\begin{aligned} u_x^{(1)},\ u_y^{(1)},\ u_z^{(1)} &= (fs,\ k,\ f)\,\varkappa^{-2} Q e^{-sz+i(p\tau - fx)},\\ u_x^{(2)},\ u_y^{(2)},\ u_z^{(2)} &= (if,\ 0,\ d)\,h^{-2} P e^{-dz+i(p\tau - fx)} \end{aligned} \tag{10.26}$$

and, by Eqs. (10.4), (10.5), (10.7),

$$d^2 = f^2 - h^2,\quad s^2 = f^2 - \varkappa^2. \tag{10.27}$$

On the boundary plane $z = 0$, the following conditions must be fulfilled:

$$\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} = 0,\quad \frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y} = 0,\quad \lambda\theta + 2G\frac{\partial u_z}{\partial z} = 0, \tag{10.28}$$

or, in expanded form,

$$\begin{aligned} &\frac{2dfP}{h^2} + (s^2 + f^2)\frac{Q}{\varkappa^2} = 0,\quad k = 0,\\ &(\varkappa^2 - 2h^2)P/h^2 - 2d^2P/h^2 - 2sfQ/\varkappa^2 = 0, \end{aligned} \tag{10.29}$$

where λ/G has been replaced by $\varkappa^2/h^2 - 2$.

The condition $k = 0$ shows that the motion takes place in the xz plane ($u_y = 0$).

The elimination of P and Q from Eqs. (10.29) leads to the equation

$$\bar{\varkappa}^8 - 8\bar{\varkappa}^6 + 24\bar{\varkappa}^4 - 16\,(1 + \bar{h}^2)\bar{\varkappa}^2 + 16\bar{h}^2 = 0, \tag{10.30}$$

where $\bar{\varkappa}^2 = \varkappa^2/f^2$ and $\bar{h}^2 = h^2/f^2$.

For an incompressible material ($\lambda = \infty$), $\bar{h}^2/\bar{\varkappa}^2 = 0$, and from Eq. (10.30) we obtain

$$\bar{\varkappa}^6 - 8\bar{\varkappa}^4 + 24\bar{\varkappa}^2 - 16 = 0.$$

For the real value of the root $\varkappa^2 = 0.91262...$,

$$d^2 = f^2, \quad s^2 = 0.08737...f^2,$$

and the velocity of wave propagation (the wavelength is $2\pi/f$) is determined by the formula

$$c_R = p/f = 0.955^*...\sqrt{G/\rho}, \tag{10.31}$$

i.e., it is close to the velocity of shear waves.

The Rayleigh surface waves (R) are formed on the surface of a half-space as a result of the superposition of the longitudinal (P) and transverse (S) waves at the moment of reflection from the boundary of the half-space $z \geqslant 0$.

If the disturbance centre is at the origin, the displacements of points of the half-space coinciding in direction with their radius vectors produce longitudinal P-waves, and the displacements normal to the radius vectors produce transverse S-waves.

The front of Rayleigh waves away from the disturbance source has a large radius, and hence these waves may be regarded as plane. They decay with depth according to the exponential law (10.26), and predominate near the surface ($z = 0$). According to the investigations of G. F. Miller and H. Pursey [73], the energy transferred by various waves is partitioned as follows: the Rayleigh (R) waves — 67 per cent, the transverse (S) waves—26 per cent, the longitudinal (P) waves—7 per cent; hence, in the design of the foundations of machines and buildings primary attention must be given to the study of the Rayleigh surface waves.

Figure 90 shows to scale the positions of the fronts of different (R, S, and P) waves and the displacements of particles of the medium produced by each wave at a fixed

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* The factor in front of the root ranges from 0.874 to 0.956 when σ varies from 0 to 0.5, respectively (when $\sigma = 0.25$, it is equal to 0.92).

instant. The arrows indicate the directions of displacements of particles at the front of the corresponding waves (see [74]).

The practical application of the foregoing theory of wave propagation in an elastic half-space to the design of foun-

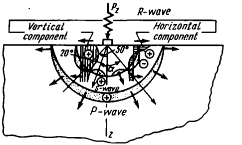

Fig. 90

dations is given in the monograph [75], and the description of experiments may be found in [76].

2. Love waves [77]

If a body is composed of two physically different regions, $z \geqslant 0$ and $-h \leqslant z \leqslant 0$, another type of waves occurs, known as Love waves. The displacements in this case are of the form

$$u_x, u_y, u_z = (0, v, 0)e^{iq(x-c\tau)} \quad (-h \leqslant z \leqslant 0),$$

$$u_x, u_y, u_z = (0, v', 0)\, e^{iq(x-c\tau)} \quad (z \geqslant 0), \tag{10.32}$$

where v and v' are functions of z*.

By securing the continuity of v and $Y_z = Z_y$ across the plane $z = 0$, and assuming $Y_z = 0$ when $z = -h$, we obtain

$$v = v_0 \cos [\sigma q (z + h)] \quad (-h \leqslant z \leqslant 0),$$

$$v' = v_0 \cos (\sigma q h) e^{-\sigma' q z} \; (z \geqslant 0),$$

where v_0 is an arbitrary quantity,

$$\sigma^2 = c^2/c_2^2 - 1, \quad (\sigma')^2 = 1 - c^2/(c_2')^2,$$

$$c_2^2 = G/\rho, \quad (c_2')^2 = G'/\rho',$$

* Here and henceforth, the unprimed symbols refer to the upper layer, and the primed symbols to the lower layer.

and the velocity of propagation of Love waves c, provided $c_2' > c_2$, is determined from the equation

$$G'\left[1-\frac{c^2}{(c_2')^2}\right]^{1/2}=G\left(\frac{c^2}{c_2^2}-1\right)^{1/2}\tan\left[\left(\frac{c^2}{c_2^2}-1\right)^{1/2}gh\right]. \tag{10.33}$$

It follows that the velocity of propagation of Love waves depends on the wavelength $2\pi/q$, and thus wave dispersion will take place.

IV. EXCITATION OF ELASTIC WAVES BY BODY FORCES [5]

If the motion takes place under the action of body forces X, Y, Z dependent on the co-ordinates x, y, z and the time τ, the equations of motion (10.1) are

$$(\lambda+G)\left(\frac{\partial\theta}{\partial x},\ \frac{\partial\theta}{\partial y},\ \frac{\partial\theta}{\partial z}\right)+G\nabla^2(u_x,\ u_y,\ u_z)+$$
$$+\rho(X,\ Y,\ Z)=\rho\left(\frac{\partial^2u_x}{\partial\tau^2},\ \frac{\partial^2u_y}{\partial\tau^2},\ \frac{\partial^2u_z}{\partial\tau^2}\right). \tag{10.34}$$

By representing the body forces as*

$$X,\ Y,\ Z=\text{grad}\ \Phi+\text{rot}\ (L,\ M,\ N)=\left(\frac{\partial\Phi}{\partial x},\frac{\partial\Phi}{\partial y},\frac{\partial\Phi}{\partial z}\right)+$$
$$+\left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z},\ \frac{\partial L}{\partial z}-\frac{\partial N}{\partial x},\ \frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right), \tag{10.35}$$

and the displacements as

$$u_x,\ u_y,\ u_z=\text{grad}\ \varphi+\text{rot}\ (F,\ G,\ H), \tag{10.36}$$

we satisfy Eqs. (10.34) if the functions φ, F, G, H satisfy the equations

$$\frac{\partial^2\varphi}{\partial\tau^2}-c_1^2\nabla^2\varphi=\Phi,\quad \frac{\partial^2F}{\partial\tau^2}-c_2^2\nabla^2F=L,$$
$$\frac{\partial^2G}{\partial\tau^2}-c_2^2\nabla^2G=M,\quad \frac{\partial^2H}{\partial\tau^2}-c_2^2\nabla^2H=N, \tag{10.37}$$

where $c_1^2=(\lambda+2G)/\rho$ is the square of the velocity of irrotational (P) waves, $c_2^2=G/\rho$ is the square of the velocity of shear (S) waves.

* Kelvin's solution (1848) for the static action of forces.

Particular solutions of Eqs. (10.37) may be represented in the form

$$\begin{aligned} \varphi &= \frac{1}{4\pi c_1^2} \iiint \frac{1}{r} \Phi' (\tau - r/c_1)\, dx'\, dy'\, dz', \\ F &= \frac{1}{4\pi c_2^2} \iiint \frac{1}{r} L' (\tau - r/c_2)\, dx'\, dy'\, dz', \\ G &= \frac{1}{4\pi c_2^2} \iiint \frac{1}{r} M' (\tau - r/c_2)\, dx'\, dy'\, dz', \\ H &= \frac{1}{4\pi c_2^2} \iiint \frac{1}{r} N' (\tau - r/c_2)\, dx'\, dy'\, dz'. \end{aligned} \tag{10.38}$$

By expressing Φ', L', ... in terms of X', Y', Z' by the formulas

$$\begin{aligned} \Phi' &= -\frac{1}{4\pi} \iiint_T \left(X' \frac{\partial r^{-1}}{\partial x} + Y' \frac{\partial r^{-1}}{\partial y} + Z' \frac{\partial r^{-1}}{\partial z} \right) dx' dy' dz', \\ L' &= \frac{1}{4\pi} \iiint_T \left(Z' \frac{\partial r^{-1}}{\partial y} - Y' \frac{\partial r^{-1}}{\partial z} \right) dx'\, dy'\, dz', \\ M' &= \frac{1}{4\pi} \iiint_T \left(X' \frac{\partial r^{-1}}{\partial z} - Z' \frac{\partial r^{-1}}{\partial x} \right) dx'\, dy'\, dz', \\ N' &= \frac{1}{4\pi} \iiint_T \left(Y' \frac{\partial r^{-1}}{\partial x} - X' \frac{\partial r^{-1}}{\partial y} \right) dx'\, dy'\, dz', \end{aligned} \tag{10.39}$$

where X', Y', Z' are the values of X, Y, Z at a point (x', y', z') inside a region T when the body forces are different from zero, r is the distance of the point (x, y, z) from the point (x', y', z'), we can perform the integration of formulas (10.38).

V. DEFORMATION OF SOLIDS UNDER CENTRIFUGAL FORCES

The equations for displacements symmetrical about the z axis ($u_\beta = \partial u_r/\partial\beta = \partial u_z/\partial\beta = 0$) are, by Eqs. (3.3b),

$$\begin{aligned} (\lambda + 2G) \frac{\partial \theta}{\partial r} + 2G \frac{\partial \omega_\beta}{\partial z} &= -p^2 \rho r, \\ (\lambda + 2G) \frac{\partial \theta}{\partial z} - \frac{2G}{r} \frac{\partial}{\partial r} (r\omega_\beta) &= 0, \end{aligned} \tag{10.40}$$

or, by Eqs. (3.3b′),

$$(\lambda + G)\frac{\partial\theta}{\partial r} + G\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) = -p^2\rho r,$$

$$(\lambda + G)\frac{\partial\theta}{\partial z} + G\nabla^2 u_z = 0,$$

where

$$\theta = e_{rr} + e_{\beta\beta} + e_{zz} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z},$$

$$2\omega_\beta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \quad \omega_r = \omega_z = 0,$$

p is the angular velocity of rotation.

In a rectangular co-ordinate system we obtain, from Eqs. (3.3a′),

$$\begin{aligned}(\lambda + G)\frac{\partial\theta}{\partial x} + G\nabla^2 u_x + p^2\rho x &= 0,\\ (\lambda + G)\frac{\partial\theta}{\partial y} + G\nabla^2 u_y + p^2\rho y &= 0,\\ (\lambda + G)\frac{\partial\theta}{\partial z} + G\nabla^2 u_z &= 0.\end{aligned} \tag{10.41}$$

VI. PLANE DYNAMIC PROBLEMS

1. Equations of motion

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = \rho\,\frac{\partial^2 u_x}{\partial \tau^2}, \quad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} = \rho\,\frac{\partial^2 u_y}{\partial \tau^2}. \tag{10.42}$$

2. Equations of Hooke's law

$$\begin{aligned}X_x &= (\lambda + 2G)\,\frac{\partial u_x}{\partial x} + \lambda\,\frac{\partial u_y}{\partial y},\\ Y_y &= (\lambda + 2G)\,\frac{\partial u_y}{\partial y} + \lambda\,\frac{\partial u_x}{\partial x},\\ X_y &= Y_x = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right).\end{aligned} \tag{10.43}$$

3. Strain compatibility equation

$$\nabla^2 (X_x + Y_y) - \frac{2(\lambda + G)}{\lambda + 2G} \rho \frac{\partial^2}{\partial \tau^2} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) = 0,$$

or (10.44)

$$\left(\nabla^2 - \frac{1}{c_1^2} \frac{\partial^2}{\partial \tau^2} \right) (X_x + Y_y) = 0.$$

4. Complex-variable method [78, 79]

By differentiating the first equation of (10.42) with respect to x, the second with respect to y, and subtracting one result from the other, we obtain

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{2c_2^2} \frac{\partial^2}{\partial \tau^2} \right) X_x = \left(\frac{\partial^2}{\partial y^2} - \frac{1}{2c_2^2} \frac{\partial^2}{\partial \tau^2} \right) Y_y. \qquad (10.45)$$

Equation (10.45) is satisfied identically if the stresses are equal to

$$X_x = \frac{\partial^2 \Phi}{\partial y^2} - \frac{1}{2c_2^2} \frac{\partial^2 \Phi}{\partial \tau^2}, \quad Y_y = \frac{\partial^2 \Phi}{\partial x^2} - \frac{1}{2c_2^2} \frac{\partial^2 \Phi}{\partial \tau^2}, \qquad (10.46)$$

where the function Φ is the dynamic analogue of Airy's function, satisfying the equation

$$\left(\nabla^2 - \frac{1}{c_1^2} \frac{\partial^2}{\partial \tau^2} \right) \left(\nabla^2 - \frac{1}{c_2^2} \frac{\partial^2}{\partial \tau^2} \right) \Phi(x, y, \tau) = 0. \qquad (10.47)$$

If the disturbances propagate with a velocity c parallel to the x axis, by applying the transformation

$$\xi = x - c\tau, \quad \eta = y,$$

we obtain Eq. (10.47) in the form

$$\left(\frac{\partial^2}{\partial \xi^2} - \frac{1}{s_1^2} \frac{\partial^2}{\partial \eta^2} \right) \left(\frac{\partial^2}{\partial \xi^2} - \frac{1}{s_2^2} \frac{\partial^2}{\partial \eta^2} \right) \Phi = 0, \qquad (10.48)$$

where

$$s_1 = i\beta_1 = i \left(1 - \frac{c^2}{c_1^2} \right)^{1/2}, \quad s_2 = i\beta_2 = i \left(1 - \frac{c^2}{c_2^2} \right)^{1/2}. \qquad (10.49)$$

Formulas (10.46) become

$$X_x = \frac{\partial^2 \Phi}{\partial \eta^2} - \frac{c^2}{c_1^2} \frac{\partial^2 \Phi}{\partial \xi^2} = \frac{\partial^2 \Phi}{\partial \eta^2} - \frac{1}{2} (1 - \beta_1^2) \frac{\partial^2 \Phi}{\partial \xi^2}, \qquad (10.50)$$

$$Y_y = \frac{1}{2} (1 + \beta_2^2) \frac{\partial^2 \Phi}{\partial \xi^2}$$

and

$$X_y + Y_y = \frac{\partial^2\Phi}{\partial\eta^2} + \beta_2^2\frac{\partial^2\Phi}{\partial\xi^2}. \tag{10.51}$$

Assuming

$$z_1 = \xi + s_1\eta, \quad z_2 = \xi + s_2\eta, \tag{10.52}$$

we obtain the solution of Eq. (10.48):

$$\begin{aligned}\Phi &= F_1(z_1) + \overline{F_1(z_1)} + F_2(z_2) + \overline{F_2(z_2)} = \\ &= 2\,\mathrm{Re}\,[F_1(z_1) + F_2(z_2)],\end{aligned} \tag{10.53}$$

where $F_1(z_1)$, $\overline{F_1(z_1)}$ and $F_2(z_2)$, $\overline{F_2(z_2)}$ are conjugate analytic functions of the complex variables z_1 and z_2.

Substituting expression (10.53) in formulas (10.50) and (10.51), we find

$$\begin{aligned}X_x &= -2\,\mathrm{Re}\left[\left(\frac{1}{2}+\beta_1^2-\frac{1}{2}\beta_2^2\right)F_1''(z_1) + \right.\\ &\left. + \frac{1}{2}(1+\beta_2^2)F_2''(z_2)\right],\\ Y_y &= (1+\beta_2^2)\,\mathrm{Re}\,[F_1''(z_1)+F_2''(z_2)],\\ X_x + Y_y &= -2(\beta_1^2-\beta_2^2)\,\mathrm{Re}\,[F_1''(z_1)].\end{aligned} \tag{10.54}$$

After substituting expressions (10.54) in formulas (10.43) and integrating, we obtain

$$\begin{aligned}Gu_x &= -\mathrm{Re}\left[F_1'(z_1) + \frac{1}{2}(1+\beta_2^2)F_2'(z_2)\right],\\ Gu_y &= \mathrm{Im}\left[\beta_1 F_1'(z_1) + \frac{1+\beta_2^2}{2\beta_2}F_2'(z_2)\right].\end{aligned} \tag{10.55}$$

Inserting these expressions in the third formula of (10.43), we find

$$X_y = 2\,\mathrm{Im}\left[\beta_1 F_1''(z_1) + \frac{(1+\beta_2^2)^2}{4\beta_2}F_2''(z_2)\right]. \tag{10.56}$$

The problem can also be solved in terms of displacements. In this case, substituting formulas (10.43) in Eqs. (10.42), we obtain

$$\begin{aligned}&\left[\frac{\lambda+2G}{G}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]u_x + \frac{\lambda+G}{G}\frac{\partial u_y}{\partial x\,\partial y} = \frac{\rho}{G}\frac{\partial^2 u_x}{\partial\tau^2},\\ &\frac{\lambda+G}{G}\frac{\partial^2 u_x}{\partial x\,\partial y} + \left[\frac{\lambda+2G}{G}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right]u_y = \frac{\rho}{G}\frac{\partial^2 u_y}{\partial\tau^2}.\end{aligned} \tag{10.57}$$

Assuming

$$u_x = -\frac{\lambda+G}{G}\frac{\partial^2\Phi}{\partial x\,\partial y},$$
$$u_y = \frac{\lambda+2G}{G}\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} - \frac{\rho}{G}\frac{\partial^2\Phi}{\partial \tau^2}, \qquad (10.58)$$

where $\Phi = \Phi(x, y, \tau)$, we identically satisfy the first equation of (10.57), and the second equation takes the form of Eq. (10.47)

$$\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial\tau^2}\right)\left(\nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial\tau^2}\right)\Phi = 0, \qquad (10.59)$$

which, for the case considered above, is solved in a similar way.

5. Integral transformation method [79 to 81]

Various forms of the integral transformation method based on Fourier and Laplace transformations are in use. Below is given the procedure of solution by one of the forms of this method:

(a) for each function f appearing in Eqs. (10.42) and (10.43) or in Eqs. (10.57), apply an integral transformation converting the given equations into simpler equations containing the functions $\bar{f}$;

(b) determine the functions $\bar{f}$; the constants entering into these functions are found by considering the boundary conditions;

(c) by using Fourier's integral theorem, invert the function $\bar{f}$ into f.

6. Method of functionally invariant solutions [82 to 84]

According to this method, the solution of the wave equation

$$\nabla^2 u - \frac{1}{c_1^2}\frac{\partial^2 u}{\partial\tau^2} = 0 \qquad (10.60)$$

is sought in the form

$$u = \operatorname{Re}[f(\zeta)], \qquad (10.61)$$

where $f(\zeta)$ is an arbitrary complex function of the argument $\zeta(x, y, \tau)$ satisfying the wave equation [see solution (10.19)],

$\zeta(x, y, \tau)$ is a functionally invariant solution of Eq. (10.60) satisfying the condition

$$\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2-\frac{1}{c_1^2}\left(\frac{\partial\zeta}{\partial\tau}\right)^2=0. \qquad (10.62)$$

The general integral of the system of differential equations for ζ is taken in implicit form:

$$l(\zeta)\,\tau+m(\zeta)x+n(\zeta)\,y-k(\zeta)=0, \qquad (10.63)$$

where l, m, n, k are coefficients.

For a plane wave, the coefficients of the variables x, y, τ must be real. By assigning two coefficients, say l and m, we obtain

$$k(\zeta)=l\tau-mx\pm\sqrt{1/c_1^2-m^2y}, \qquad (10.64)$$

where the plus sign corresponds to the motion of the wave towards the boundary, and the minus sign away from it.

The foregoing method is used to investigate the laws of reflection of elastic vibrations from the free surface of a body.

It is found that the boundary conditions cannot be satisfied by any one type of wave.

When a dilatation (shear) wave is incident on a free boundary

$$\varphi_1=f\left(l\tau-mx+\sqrt{1/c_1^2-m^2y}\right), \qquad (10.65)$$

both types of wave are reflected:

$$\varphi_2=Af\left(l\tau+mx-\sqrt{1/c_1^2-m^2y}\right)$$

and

$$\psi_2=Bf\left(l\tau-mx-\sqrt{1/c_2^2-m^2y}\right).$$

The constants A and B are determined from the boundary conditions by substituting for $\varphi=\varphi_1+\varphi_2$ and ψ_2.

VII. THERMODYNAMIC PROBLEM

In the case of a high rate of time variation of a temperature field, its pulsation or an instantaneous action of a thermal impulse corresponding to thermal shock effect [85], it is necessary to set up the differential equations of motion (10.1) with the right-hand sides involving the temperature propagation as a function of position and time. In this case

Eqs. (8.3) take the form

$$(\lambda+G)\frac{\partial\theta}{\partial x}+G\nabla^2u_x=\rho\frac{\partial^2u_x}{\partial\tau^2}+K\alpha\frac{\partial t}{\partial x},$$
$$(\lambda+G)\frac{\partial\theta}{\partial y}+G\nabla^2u_y=\rho\frac{\partial^2u_y}{\partial\tau^2}+K\alpha\frac{\partial t}{\partial y}, \qquad (10.66)$$
$$(\lambda+G)\frac{\partial\theta}{\partial z}+G\nabla^2u_z=\rho\frac{\partial^2u_z}{\partial\tau^2}+K\alpha\frac{\partial t}{\partial z}.$$

The stresses are determined by formulas (8.2).

When there is no mutual transformation of the thermal and mechanical energies, the temperature $t = t(x, y, z, \tau)$ is determined from the heat conduction equation (8.23) subject to surface conditions.

If the mutual transformation of the energies is taken into account, it is necessary to consider the refined heat conduction equation (see the monograph [58]).

If, at a point $M_1(x_1, y_1, z_1)$ of an infinite body, there is an instantaneous heat source of intensity

$$b_1 = \overline{W}/(c\rho) \quad (^\circ\mathrm{C\cdot cm^3}),$$

where $\overline{W}$ (cal·s²/cm) is the quantity of heat (in calories) generated at the point, divided by g, Eq. (8.23) becomes

$$\varkappa\nabla^2t=\frac{\partial t}{\partial\tau},$$

and the temperature distribution at any instant is determined by its solution [57]

$$t(x, y, z, \tau)=$$
$$=\frac{b_1}{(2\sqrt{\pi\varkappa\tau})^3}\exp\left[-\frac{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}{4\varkappa\tau}\right]. \quad (10.67)$$

By integrating solution (10.67) throughout the volume, we obtain

$$\int\int\limits_{-\infty}^{\infty}\int t(x, y, z, \tau)\,dx\,dy\,dz=b_1.$$

It is seen from solution (10.67) that as $\tau\to 0$, $t\to 0$ at all points except at a single point, $M_1(x_1, y_1, z_1)$, where it becomes infinite. Relation (10.67) is the analogue of the Green's function.

The temperature distribution in an infinite body under a two-dimensional heat flow produced by an instantaneous heat source on a straight line passing through a point M_1 (x_1, y_1) parallel to the z axis at the time $\tau = 0$ is determined by the expression

$$t(x, y, \tau) = \frac{b_2}{4\pi\varkappa\tau} \exp\left[-\frac{(x-x_1)^2+(y-y_1)^2}{4\varkappa\tau}\right] \qquad (10.68)$$

since

$$\int\int_{\infty}^{\infty} t(x, y, \tau)\, dx\, dy = b_2 \quad (^\circ\text{C}\cdot\text{cm}^2).$$

The solution of the thermodynamic problem of a thermal shock on the surface of a half-space is given in [86].

PROBLEMS

10.1. Chree's problem (1892) [87].

Determine the state of stress in a cylinder of radius b and length $2l$ rotating with an angular velocity p.

The computing equations are Eqs. (10.40).

The boundary conditions of the problem are: when $r = b$, $R_r = Z_r = 0$; when $z = \pm l$,

$$R_z = 0 \quad \text{and} \quad \int_0^b Z_z 2\pi r\, dr = 0.$$

The last condition is an integral one (instead of $Z_z = 0$), but according to Saint-Venant's principle, the resulting state of stress is sufficiently accurate at points remote from the ends of the cylinder.

In terms of E and σ Eqs. (10.40) are

$$\frac{(1-\sigma)E}{(1+\sigma)(1-2\sigma)}\frac{\partial\theta}{\partial r} + \frac{E}{1+\sigma}\frac{\partial\omega_\beta}{\partial z} = -p^2\rho r,$$

$$\frac{(1-\sigma)E}{(1+\sigma)(1-2\sigma)}\frac{\partial\theta}{\partial z} - \frac{E}{1+\sigma}\frac{1}{r}\frac{\partial}{\partial r}(r\omega_\beta) = 0.$$

The solution of these equations satisfying the above boundary conditions is of the form

$$u_r = Ar - \frac{p^2\rho}{8E}\frac{(1+\sigma)(1-2\sigma)}{1-\sigma}r^3, \quad u_z = Bz,$$

where

$$A = \frac{p^2\rho b^3}{8E}\frac{3-5\sigma}{1-\sigma}, \quad B = -\frac{p^2\rho b^3\sigma}{2E}.$$

The stresses are equal to

$$R_r = \frac{p^2\rho(b^2-r^2)}{8}\frac{3-2\sigma}{1-\sigma}, \quad Z_r = R_z = 0,$$

$$B_\beta = \frac{p^2\rho}{8}\left(\frac{3-2\sigma}{1-\sigma}b^2 - \frac{1+2\sigma}{1-\sigma}r^2\right),$$

$$Z_z = \frac{p^2\rho(b^2-2r^2)}{4}\frac{\sigma}{1-\sigma}.$$

10.2. See [45].

Determine the state of stress in a thin annular disk of outer radius b and inner radius a rotating with a constant angular velocity p.

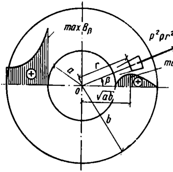

Fig. 91

The disk is subjected to centrifugal inertia forces $p^2\rho r$ (Fig. 91).

The equation of motion is, by (6.1),

$$\frac{dR_r}{dr} + \frac{R_r - B_\beta}{r} + p^2\rho r = 0^*. \tag{a}$$

* When $h = h(r)$, the equation (a) becomes

$$\frac{d}{dr}(hrR_r) - hB_\beta + p^2\rho h r^2 = 0.$$

Substituting the stress values (6.5) in the equation (a), and cancelling out the constant factor, we obtain

$$\frac{d^2u_r}{dr^2}+\frac{1}{r}\frac{du_r}{dr}-\frac{u_r}{r^2}=\frac{\rho(1-\sigma^2)p^2r}{E}. \qquad \text{(b)}$$

By solving the equation (b), we find

$$u_r=C_1r+\frac{C_2}{r}-\frac{\rho(1-\sigma^2)p^2r^3}{8E}.^*$$

By (6.5),

$$R_r=A-\frac{B}{r^2}-\frac{(3+\sigma)\rho p^2r^2}{8}, \quad B_\theta=A+$$
$$+\frac{B}{r^2}-\frac{(1+3\sigma)\rho p^2r^3}{8},$$

where the new unknowns

$A = EC_1/(1-\sigma)$ and $B = EC_2/(1+\sigma)$

are determined from the boundary conditions: when $r = a$ and $r = b$, $R_r = 0$.

The stresses are, finally,

$$R_r=\frac{(3+\sigma)\rho p^2}{8}(b^2-r^2)\left(1-\frac{a^2}{r^2}\right),$$
$$B_\theta=\frac{(3+\sigma)\rho p^2}{8}\left[b^2-\frac{1+3\sigma}{3+\sigma}r^2-a^2\left(1-\frac{b^2}{r^2}\right)\right].$$

The R_r and B_θ diagrams are given in Fig. 91.

10.3. Chree's problem (1889) [88].
Determine the displacements of a sphere of radius a rotating about the z axis with an angular velocity p.

The boundary conditions (1.2) for the surface of the sphere ($r = a$) are

$X_r = Y_r = Z_r = 0.$

The body forces in Eqs. (10.41) may be considered as the gradient of the potential

$\varphi = \rho p^2 (x^2 + y^2)/2,$

* A particular solution is taken in the form $\bar{u}_r = C_3r^3$.

which may be represented in the form

$$\varphi = \rho p^2 r^2/3 + \rho p^2 (x^2 + y^2 - 2z^2)/6, \tag{a}$$

where

$$r^2 = x^2 + y^2 + z^2.$$

The first term in the formula (a) gives a purely radial body force $2\rho p^2 r/3$ with a purely radial displacement u_r ($u_\beta = u_\alpha = 0$), which can be determined from the first equation of (10.40):

$$(\lambda + 2G) \frac{d}{dr} \left(\frac{du_r}{dr} + 2 \frac{u_r}{r} \right) + \frac{2}{3} \rho p^2 r = 0. \tag{b}$$

By solving the equation (b), we obtain

$$u_r = \frac{a^2 \rho p^2}{15(\lambda + 2G)} r \left(\frac{5\lambda + 6G}{3\lambda + 2G} - \frac{r^2}{a^2} \right),$$

from which

$$u_x, u_y, u_z = \frac{a^2 \rho p^2}{15(\lambda + 2G)} \left(\frac{5\lambda + 6G}{3\lambda + 2G} - \frac{r^2}{a^2} \right) (x, y, z).$$

In a similar way it is possible to solve the problem of the deformation of a sphere due to the mutual attraction of its particles (the earth).

The second term of the potential φ [see the formula (a)] represents a spatial spherical function of the second order

$$V_2 = r^2 Y_2(\alpha) = r^2 P_2(\cos\alpha) = r^2 (3\cos^2\alpha - 1)/2$$

[see Chap. 4].

Equations (10.41) for this case are

$$(\lambda + G) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \Theta + G\nabla^2 (u_x, u_y, u_z) +$$

$$+ \frac{\rho p^2}{6} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) V_2 = 0.$$

By using the properties of spherical functions [see Chap. 4], we obtain the solution

$$u_x, u_y, u_z = A [(5\lambda + 7G) r^2 (x, y, -2z) -$$

$$-(2\lambda + 7G)(x^2 + y^2 - 2z^2)(x, y, z)] +$$

$$+ B(x, y, -2z) - \frac{\rho p^2}{42(\lambda + 2G)} [r^2 (x, y, -2z) +$$

$$+ (x^2 + y^2 - 2z^2)(x, y, z)],$$

where the arbitrary constants A and B determined from the boundary conditions are

$$A = -\frac{\rho p^2 (7\lambda + 6G)}{42(\lambda + 2G)(19\lambda + 14G)G}, \quad B = \frac{\rho p^2 (4\lambda + 3G)}{3G(19\lambda + 14G)}.$$

10.4. Pochhammer's problem (1876) [89].

Investigate the vibrations of an isotropic circular cylinder of radius a whose surface is free from stresses.

To solve the problem, we make use of Eqs. (3.3b) substituting inertia forces for the body forces R, B, Z; we thus obtain

$$\begin{aligned}
&(\lambda + 2G)\frac{\partial \theta}{\partial r} - \frac{2G}{r}\frac{\partial \omega_z}{\partial \beta} + 2G\frac{\partial \omega_\beta}{\partial r} = \rho\frac{\partial^2 u_r}{\partial \tau^2},\\
&(\lambda + 2G)\frac{1}{r}\frac{\partial \theta}{\partial \beta} - 2G\frac{\partial \omega_r}{\partial z} + 2G\frac{\partial \omega_z}{\partial r} = \rho\frac{\partial^2 u_\beta}{\partial \tau^2}, \qquad \text{(a)}\\
&(\lambda + 2G)\frac{\partial \theta}{\partial z} - \frac{2G}{r}\frac{\partial}{\partial r}(r\omega_\beta) + \frac{2G}{r}\frac{\partial \omega_r}{\partial \beta} = \rho\frac{\partial^2 u_z}{\partial \tau^2}.
\end{aligned}$$

Assuming that the displacements are harmonic functions of z and τ of the form

$$u_r = Ue^{i(\gamma z + p\tau)}, \quad u_\beta = Ve^{i(\gamma z + p\tau)}, \quad u_z = We^{i(\gamma z + p\tau)},$$

where U, V, W are functions of r and β, we arrive at a number of solutions.

Torsional vibrations.

When $U = W = 0$ and $V = V(r)$, the first and third equations of (a) are satisfied identically, and the second equation takes the form of a Bessel equation

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} + \left(\varkappa^2 - \frac{1}{r^2}\right)V = 0, \qquad \text{(b)}$$

where $\varkappa^2 = \rho p^2/G - \gamma^2$.

By integrating the equation (b), we obtain

$$V = AJ_1(\varkappa, r),$$

where $J_1(\varkappa, r)$ is the Bessel function of the first order.

The surface conditions are satisfied if $\varkappa$ is the root of the equation

$$\frac{d}{da}\left[\frac{J_1(\varkappa, a)}{a}\right]=0.$$

When $\varkappa = 0$, $\gamma^2 = \rho p^2/G$ and $V = Ar$; the displacements are

$$u_r=u_z=0, \quad u_\beta=Are^{ip\left(\sqrt{\frac{\rho}{G}}z-\tau\right)}. \tag{c}$$

The solution (c) represents a torsional wave, which propagates along the axis of the cylinder with a velocity $c_2 = \sqrt{G/\rho}$.

For a cylinder of length l with stress-free ends, we obtain

$$u_\beta=A_n r\cos\frac{n\pi r}{l}\cos\left(\frac{n\pi}{l}\sqrt{\frac{G}{\rho}}\tau+\varepsilon\right),$$

where n is an integer, ε is the phase of vibration.

Longitudinal vibrations.

When $V = 0$, $U = U(r)$, $W = W(r)$, the second equation of (a) is satisfied identically, and the first and third equations take the form of Bessel equations

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+h^2\theta=0, \quad \frac{\partial^2\omega_\beta}{\partial r^2}+\frac{1}{r}\frac{\partial\omega_\beta}{\partial r}+\left(\varkappa^2-\frac{1}{r^2}\right)\omega_\beta=0, \tag{d}$$

where

$$h^2 = \rho p^2/(\lambda + 2G) - \gamma^2, \quad \varkappa^2 = \rho p^2/G - \gamma^2.$$

By solving the equations (d), we find

$\theta \propto J_0(hr)$ and $\omega_\beta \propto J_1(\varkappa r)$.

By satisfying the equations

$$\theta=\left(\frac{dU}{dr}+\frac{U}{r}+i\gamma W\right)e^{i(\gamma z+p\tau)},$$

$$2\omega_\beta=\left(i\gamma U-\frac{dW}{dr}\right)e^{i(\gamma z+p\tau)},$$

we obtain

$$U = A\frac{d}{dr}J_0(hr) + B\gamma J_1(\varkappa r),$$

$$W = Ai\gamma J_0(hr) + \frac{B}{r}\frac{d}{dr}[rJ_1(\varkappa r)].$$

The stresses on the surface of the cylinder ($r = a$) are zero if the constants A and B are related by the equalities

$$\begin{aligned}&\left[2G\frac{d^2J_0(ha)}{da^2} - \frac{\rho p^2\lambda}{\lambda+2G}J_0(ha)\right]A + 2G\gamma\frac{dJ_1(\varkappa a)}{da}B = 0,\\ &2\gamma\frac{dJ_0(ha)}{da}A + \left(2\gamma^2 - \frac{\rho p^2}{G}\right)J_1(\varkappa a)B = 0.\end{aligned} \tag{e}$$

By setting the determinant of the equations (e) equal to zero, we obtain an equation for the determination of frequencies.

Transverse vibrations.

Assuming

$$U = U(r)\cos\beta,\quad V = V(r)\sin\beta,\quad W = W(r)\cos\beta,$$

and substituting these quantities in the equations (a), we obtain three differential equations in the functions $U(r)$, $V(r)$, $W(r)$ whose solution is

$$U(r) = A\frac{dJ_1(hr)}{dr} + B\gamma\frac{dJ_1(\varkappa r)}{dr} + C\frac{J_1(\varkappa r)}{r},$$

$$V(r) = -A\frac{J_1(hr)}{r} - B\gamma\frac{J_1(\varkappa r)}{r} - C\frac{dJ_1(\varkappa r)}{dr},$$

$$W(r) = iA\gamma J_1(hr) - iB\varkappa^2 J_1(\varkappa r).$$

The conditions of zero stresses on the surface are too complicated to be given here [5].

10.5. Investigate the radial vibrations of a thin ($\delta = 1$) annular disk of inner radius a and outer radius b whose edges are free from stresses.

Since the problem is polarly symmetric, the computing equation in terms of displacements is, by Eq. (6.7),

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)u_r - \frac{1-\sigma^2}{E}\rho\frac{\partial^2 u_r}{\partial\tau^2} = 0.$$

Assuming

$u_r = W \cos(p\tau + \varepsilon)$*,

we obtain a Bessel equation

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} + \left(\varkappa^2 - \frac{1}{r^2}\right) W = 0, \tag{a}$$

where

$$\varkappa^2 = \frac{1-\sigma^2}{E} \rho p^2.$$

By solving the equation (a), we find

$W = AJ_1(\varkappa r) + BN_1(\varkappa r)$,

and the displacement becomes, finally,

$u_r = [AJ_1(\varkappa r) + BN_1(\varkappa r)] \cos(p\tau + \varepsilon)$.

The stresses are determined by formulas (6.5).

At the edges of the disk $r = a$ and $r = b$ the stresses R_r are zero; this leads to two equations

$$A\left[\frac{dJ_1(\varkappa a)}{da} + \frac{\sigma}{a} J_1(\varkappa a)\right] + B\left[\frac{dN_1(\varkappa a)}{da} + \frac{\sigma}{a} N_1(\varkappa a)\right] = 0,$$

$$A\left[\frac{dJ_1(\varkappa b)}{db} + \frac{\sigma}{b} J_1(\varkappa b)\right] + B\left[\frac{dN_1(\varkappa b)}{db} + \frac{\sigma}{b} N_1(\varkappa b)\right] = 0.$$

By eliminating A and B from these equations, we obtain the frequency equation.

10.6. Poisson's problem (1828) [5].

Consider the radial vibrations of a hollow sphere of outer radius b and inner radius a.

The problem is polarly symmetric, i.e., all quantities depend only on r and τ, and $u_\alpha = u_\beta = \omega_r = \omega_\beta = \omega_\alpha = 0$.

According to the first equation of (3.3b), we have

$$(\lambda + 2G)\frac{\partial\theta}{\partial r} - \rho\frac{\partial^2 u_r}{\partial \tau^2} = 0,$$

* The problem can also be solved by the method of separation of variables assuming $u_r = R(r)\,T(\tau)$.

where

$$0=\frac{\partial u_r}{\partial r}+2\frac{u_r}{r} \quad \text{[see formula (2.2b)].}$$

Assuming

$$u_r = A W(r) \cos(p\tau + \varepsilon),$$

we obtain the equation

$$\frac{d^2W}{dr^2}+\frac{2}{r}\frac{dW}{dr}+\left(h^2-\frac{2}{r^2}\right)W=0, \tag{a}$$

where

$$h^2 = \rho p^2/(\lambda + 2G).$$

The integral of the equation (a) is

$$W(r)=\frac{d}{d(hr)}\left(\frac{A\sin hr+B\cos hr}{hr}\right).$$

The boundary conditions of the problem are: when $r = a$ and $r = b$,

$$R_r=\lambda\theta+2G\frac{\partial u_r}{\partial r}=(\lambda+2G)\frac{\partial u_r}{\partial r}+2\lambda\frac{u_r}{r}=0. \tag{b}$$

The condition (b) for a sphere of radius r is of the form

$$\{(\lambda + 2G)\,[(2 - h^2r^2)\sin hr - 2hr\cos hr] +$$
$$+ 2\lambda\,(hr\cos hr - \sin hr)\}\,A + \{(\lambda + 2G)\,[(2 -$$
$$- h^2r^2)\cos hr + 2hr\sin hr] -$$
$$-2\lambda\,(hr\sin hr + \cos hr)\}\,B = 0.$$

By writing the last equation for the values $r = a$ and $r = b$, and eliminating A and B from these equations, we obtain the frequency equation

$$\frac{Kha+(h^2a^2-K)\tan ha}{(h^2a^2-K)-Kha\tan ha}=\frac{Khb+(h^2b^2-K)\tan hb}{(h^2b^3-K)-Khb\tan hb},$$

where

$$2 - K = 2\lambda/(\lambda + 2G).$$

For a very thin layer, the period is

$$T=\pi a\sqrt{\frac{\rho}{G}\frac{1-\sigma}{1+\sigma}}.$$

For a solid sphere, $B = 0$.

▶Lamb's problem (1882) [5].

Investigate the vibrations of a solid sphere of radius a for which $\theta = 0$.

Hints.

(1) Assume the solution of the homogeneous equations (10.7) in the form

$$u_{x,y,z} = R(r)\, Y_{x,y,z}(\alpha, \beta).$$

(2) To determine the relations between the functions $Y_{x,y,z}(\alpha, \beta)$, use condition (10.9).

10.7. Determine the displacements in an infinite body due to a concentrated force $\chi(\tau)$ applied at the origin and acting in the x direction.

In solving the problem we assume that a region D where the body forces are different from zero decreases indefinitely, and

$$\rho \iiint_D X'\, dx'\, dy'\, dz' = X_0,$$

where X_0 is a force acting on a point (x', y', z') in the x direction.

In the case under consideration we assume

$$X_0 = \chi(\tau - r/c_1),$$

where r is the distance of a point (x, y, z) from the origin.

According to formulas (10.39), we determine the quantities

$$\Phi'(\tau - r/c_1) = -\frac{1}{4\pi\rho}\chi(\tau - r/c_1)\frac{\partial r^{-1}}{\partial x'}, \quad L' = 0,$$

$$M'(\tau - r/c_2) = \frac{1}{4\pi\rho}\chi(\tau - r/c_2)\frac{\partial r^{-1}}{\partial z'},$$

$$N'(\tau - r/c_2) = -\frac{1}{4\pi\rho}\chi(\tau - r/c_2)\frac{\partial r^{-1}}{\partial y'}.$$

By dividing the space around a point (x, y, z) into thin slices by spherical surfaces centred at this point, integrals

(10.38) may be expressed by the formulas

$$\int\int\int \frac{1}{r}\Phi'\left(\tau-\frac{r}{c_1}\right) dx'\,dy'\,dz' =$$

$$= \int_0^\infty -\frac{1}{4\pi\rho}\chi\left(\tau-\frac{r}{c_1}\right)\frac{dr}{r}\int\int\frac{\partial r^{-1}}{\partial x'}\,ds,$$

where ds is a surface element of a sphere of radius r,

$$\int\int \frac{\partial r^{-1}}{\partial x'}\,ds = 0$$

if the origin is inside s,

$$\int\int \frac{\partial r^{-1}}{\partial x'}\,ds = 4\pi r^2\frac{\partial r_0^{-1}}{\partial x}$$

if the origin is outside s, r_0 is the distance of the point (x, y, z) from the origin.

In the first case $r_0 < r$, in the second $r_0 > r$.

By integrating with respect to r (the upper limit may be replaced by r_0), we obtain

$$\varphi = -\frac{1}{4\pi c_1^2\rho}\frac{\partial r_0^{-1}}{\partial x}\int_0^{r_0} r\chi(\tau - r/c_1)\,dr =$$

$$= -\frac{1}{4\pi\rho}\frac{\partial r^{-1}}{\partial x}\int_0^{r/c_1}\tau'\chi(\tau-\tau')\,d\tau', \tag{a}$$

where $\tau' = r/c_1$, and r has been used for r_0.

In a similar way we find

$$F = 0,$$

$$G = \frac{1}{4\pi\rho}\frac{\partial r^{-1}}{\partial z}\int_0^{r/c_2}\tau'\chi(\tau-\tau')\,d\tau', \tag{b}$$

$$H = -\frac{1}{4\pi\rho}\frac{\partial r^{-1}}{\partial y}\int_0^{r/c_2}\tau'\chi(\tau-\tau')\,d\tau'.$$

By determining the displacements by formulas (10.36), with the formulas (a) and (b), we obtain

$$
\begin{aligned}
u_x &= \frac{1}{4\pi\rho}\frac{\partial^2 r^{-1}}{\partial x^2}\int_{r/c_1}^{r/c_2}\tau'\chi(\tau-\tau')\,d\tau' + \\
&+ \frac{1}{4\pi\rho r}\left(\frac{\partial r}{\partial x}\right)^2\left[\frac{1}{c_1^2}\chi(\tau-r/c_1)-\right. \\
&\left.-\frac{1}{c_2^2}\chi\left(\tau-\frac{r}{c_2}\right)\right]+\frac{\chi(\tau-r/c_2)}{4\pi\rho c_2^2 r}, \\
u_y &= \frac{1}{4\pi\rho}\frac{\partial^2 r^{-1}}{\partial x\,\partial y}\int_{r/c_1}^{r/c_2}\tau'\chi(\tau-\tau')\,d\tau' + \\
&+ \frac{1}{4\pi\rho r}\frac{\partial r}{\partial x}\frac{\partial r}{\partial y}\left[\frac{1}{c_1^2}\chi\left(\tau-\frac{r}{c_1}\right)-\frac{1}{c_2^2}\chi\left(\tau-\frac{r}{c_2}\right)\right], \\
u_z &= \frac{1}{4\pi\rho}\frac{\partial^2 r^{-1}}{\partial x\,\partial z}\int_{r/c_1}^{r/c_2}\tau'\chi(\tau-\tau')\,d\tau' + \qquad (c) \\
&+ \frac{1}{4\pi\rho r}\frac{\partial r}{\partial x}\frac{\partial r}{\partial z}\left[\frac{1}{c_1^2}\chi\left(\tau-\frac{r}{c_1}\right)-\frac{1}{c_2^2}\chi\left(\tau-\frac{r}{c_2}\right)\right], \\
0 &= \frac{1}{4\pi c_1^2\rho}\frac{\partial}{\partial x}\left[\frac{1}{r}\chi\left(\tau-\frac{r}{c_1}\right)\right], \\
\omega_{x,y,z} &= \frac{1}{8\pi c_2^2\rho}\left(0,\ \frac{\partial}{\partial z},\ -\frac{\partial}{\partial y}\right)\left[\frac{1}{r}\chi\left(\tau-\frac{r}{c_2}\right)\right].
\end{aligned}
$$

►Calculate the displacements when $\chi(\tau) = A\cos p\tau$.

Note.

$$
\int_{r/c_1}^{r/c_2}\tau'\chi(\tau-\tau')\,d\tau' = \frac{A}{p^2}\Big[\cos p(\tau-r/c_2)-\cos p(\tau-r/c_1)-
$$

$$
-\frac{pr}{c_2}\sin p(\tau-r/c_2)+\frac{pr}{c_1}\sin p(\tau-r/c_1)\Big].
$$

The process involves simple harmonic waves of two kinds travelling with a velocity $c_1 = \sqrt{(\lambda + 2G)/\rho}$ (dilatation wave) and a velocity $c_2 = \sqrt{G/\rho}$ (shear wave).

▶Calculate the displacements (c) when $\chi(\tau) = \text{constant}$ and compare with the results of Problem 4.4.

10.8. See the monograph [79], Sec. 76.

Determine the displacements of the half-plane $y \geqslant 0$ when a pulse of normal pressure moving with a velocity c is applied to the boundary $y = 0$.

The boundary conditions are: when $y = 0$,

$$Y_y = -[P''(x - c\tau) + P''(x - c\tau)]/2, \tag{a}$$

$$X_y = 0 \tag{b}$$

On putting $\eta = 0$ in expression (10.56), we find that the equation (b) is satisfied if we assume

$$\beta_1 F_1''(\xi) + \frac{(1+\beta_2^2)^2}{4\beta_2} F_2''(\xi) = 0,$$

from which

$$F_2''(z_2) = -\frac{4\beta_1\beta_2}{(1+\beta_2^2)^2} F_1''(z_2). \tag{c}$$

Substituting the last relation in formula (10.54), we obtain the following expression for Y_y:

$$Y_y = (1+\beta_2^2)^{-1} \operatorname{Re}[(1+\beta_2^2)^2 F_1''(z_1) + 4\beta_1\beta_2 F_1''(z_2)].$$

Thus, the boundary condition (a) is satisfied if we assume

$$F_1(\xi) = -\frac{(1+\beta_2^2)^3 P(\xi)}{(1+\beta_2^2)^3 - 4\beta_1\beta_2}. \tag{d}$$

By expressions (10.55) and the equations (c) and (d),

$$\begin{aligned} Cu_x &= \frac{1+\beta_2^2}{(1+\beta_2^2)^2 - 4\beta_1\beta_2} \operatorname{Re}\left[R'(z_1) - \frac{8\beta_1\beta_2}{(1+\beta_2^2)^3} P'(z_2)\right], \\ Cu_y &= -\frac{\beta_1}{(1+\beta_2^2)^2 - 4\beta_1\beta_2} \operatorname{Im}[(1+\beta_2^2) P'(z_1) - 2P'(z_2)]. \end{aligned} \tag{e}$$

The formulas (e) express the solution due to I. N. Sneddon.

▶Sneddon's problem [90]

Solve the preceding problem for the case of a moving pulse of tangential pressure.

Hint. Take the boundary conditions in the form: when $x = 0$,

$$Y_y = 0, \quad X_y = -[T''(x - c\tau) + \overline{T}''(x - c\tau)]/2.$$

▶Galin's problem [63; 79, Sec. 76].

Investigate the state of stress in the half-plane $y \geqslant 0$ produced by a punch moving over its surface with a constant velocity c.

10.9. See the monograph [79], Sec. 77.

Determine the displacements in the half-plane $y \geqslant 0$ when a varying pressure $p(x, \tau)$ is applied to its boundary $y = 0$.

We introduce a variable $\tau' = c_1\tau$.

The boundary conditions of the problem are: when $y = 0$,

$$Y_y = -p(x, \tau'), \quad X_y = 0. \tag{a}$$

To solve the problem, we apply to all quantities appearing in expressions (10.42) and (10.43) a two-dimensional Fourier transformation defined by the formula

$$\bar{f}(\xi, y, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, \tau') e^{i(\xi x + \omega\tau')} dx\, d\tau'.$$

If both sides of each of the equations in systems (10.42) and (10.43) are multiplied by $\exp[i(\xi x + \omega\tau'')]$ and integrated with respect to each of the variables x and τ' from $-\infty$ to ∞*, we obtain a system of simultaneous differential equations

$$\begin{aligned} i\xi\overline{X}_x - \frac{d\overline{Y}_x}{dy} &= (\lambda + 2G)\,\omega^2\bar{u}_x, \\ i\xi\overline{Y}_x - \frac{d\overline{Y}_y}{dy} &= (\lambda + 2G)\,\omega^2 u_y. \end{aligned} \tag{b}$$

The stresses are determined by the formulas

$$\overline{X}_x = -i\xi(\lambda + 2G)\,\bar{u}_x + \lambda\frac{d\bar{u}_y}{dy},$$

* It is assumed that the unknowns in Eqs. (10.42) and (10.43) tend to zero as $\sqrt{x^2 + y^2} \to \infty$.

$$\overline{Y}_y = -i\xi\lambda\overline{u}_x + (\lambda + 2G)\frac{d\overline{u}_y}{dy},$$
$$\overline{Y}_x = G\left(\frac{d\overline{u}_x}{dy} - i\xi\overline{u}_y\right). \tag{c}$$

If the formulas (c) are substituted in the equations (b), we obtain two simultaneous differential equations for the determination of the Fourier transforms of the functions $\overline{u}_x$ and $\overline{u}_y$:

$$(\beta^2 - 1)\, i\xi\frac{d\overline{u}_x}{dy} + \left[\xi^2 - \beta^2\left(\omega^2 + \frac{d^2}{dy^2}\right)\right]\overline{u}_y = 0,$$
$$\left[\beta^2(\xi^2 - \omega^2) - \frac{d^2}{dy^2}\right]\overline{u}_x + (\beta^2 - 1)\, i\xi\frac{d\overline{u}_y}{dy} = 0, \tag{d}$$

where

$\beta^2 = (\lambda + 2G)/G.$

By elimination, the equations (d) can be simplified to

$$\left(\frac{d^2}{dy^2} - n_1^2\right)\left(\frac{d^2}{dy^2} - n_2^2\right)(\overline{u}_x, \overline{u}_y) = 0, \tag{e}$$

where

$n_1^2 = \xi^2 - \omega^2, \quad n_2^2 = \xi^2 - \beta^2\omega^2.$

From the equations (e) we obtain

$$\overline{u}_x = A_1e^{-n_1y} + A_2e^{-n_2y},$$
$$\overline{u}_y = B_1e^{-n_1y} + B_2e^{-n_2y}, \tag{f}$$

where the integration constants A_i and B_i depend on ξ and ω.

Substituting the expressions (f) in the equations (d), we find the following relations between the integration constants:

$$\xi A_2 = in_2B_1, \; n_1A_1 = i\xi B_1. \tag{g}$$

By rearranging the boundary conditions (a), we obtain two more equations

$$G\left[(\beta^2 - 2)\frac{d\overline{u}_y}{dy} - i\xi\beta^2\overline{u}_x\right]_{y=0} = -\overline{p}(\xi, \omega),$$
$$\left[\frac{d\overline{u}_x}{dy} - i\xi\overline{u}_y\right]_{y=0} = 0. \tag{h}$$

By solving the equations (g) and (h), we find

$$A_1 = \frac{i\xi\bar{p}\,(\xi^2-\beta^2\omega^2/2)}{2Gc}, \quad A_2 = -\frac{i\xi n_1 n_2 \bar{p}}{2Gc},$$

$$B_1 = \frac{n_1\bar{p}\,(\xi^2-\beta^2\omega^2/2)}{2Gc}, \quad B_2 = -\frac{\xi^2 n_1 \bar{p}}{2Gc},$$

where

$$c = c\,(\xi,\ \omega) = (\xi^2 - \beta^2\omega^2/2)^2 - n_1 n_2 \xi^2.$$

Substituting these constants in the equations (f), and inverting the resulting expressions by the two-dimensional Fourier integral theorem, we obtain the following expressions for the components of the displacement vector:

$$u_x = \frac{1}{4\pi G} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\bar{p}}{c} \Big[i\xi \left(\xi^2 - \frac{1}{2}\beta^2\omega^2\right) e^{-n_1 y} -$$

$$- i\xi n_1 n_2 e^{-n_2 y} \Big] e^{-i(\xi x+\omega\tau')}\, d\xi\, d\omega,$$

$$u_y = \frac{1}{4\pi G} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\bar{p}}{c} \Big[n_1 \left(\xi^2 - \frac{1}{2}\beta^2\omega^2\right) e^{-n_1 y} - \xi^2 n_1 e^{-n_2 y} \Big] \times$$

$$\times e^{-i(\xi x-\omega\tau')}\, d\xi\, d\omega.$$

The solution of this problem by the integral transformation method is given in the monograph [80].

10.10. See [91].

Investigate the vibrations of the elastic half-plane $y \geqslant 0$ when a source of elastic displacements begins to act at its boundary. At $\tau \leqslant 0$, the half-plane is at rest.

In plane strain and in the absence of body forces Lamé's equations are of the form

$$(\lambda + G)\frac{\partial\theta}{\partial x} + G\nabla^2 u_x = \rho\frac{\partial^2 u_x}{\partial\tau^2}, \quad (\lambda + G)\frac{\partial\theta}{\partial y} + G\nabla^2 u_y =$$

$$= \rho\frac{\partial^2 u_y}{\partial\tau^2}.$$

In plane stress λ must be replaced by

$$\lambda^* = 2\lambda G/(\lambda + 2G).$$

The stresses are determined by the formulas

$$X_x = \lambda\theta + 2G\frac{\partial u_x}{\partial x}, \quad Y_y = \lambda\theta + 2G\frac{\partial u_y}{\partial y},$$

$$X_y = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right).$$

Assume that as $x \to \infty$ and $|y| \to \infty$, all components of the displacement vector and of the stress tensor tend to zero. We introduce a function $\Phi(x, y, \tau)$ and express the displacements in terms of this function

$$u_x = -\frac{\lambda+G}{G}\frac{\partial^2\Phi}{\partial x\,\partial y}, \quad u_y = \frac{\lambda+2G}{G}\frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial x^2} - \frac{\rho}{G}\frac{\partial^2\Phi}{\partial\tau^2}. \quad \text{(b)}$$

Substituting the expressions (b) in the equations (a), we arrive at Eq. (10.59). The stresses are expressed in terms of the function Φ as follows:

$$X_x = -(\lambda+2G)\frac{\partial^3\Phi}{\partial x^2\,\partial y} + \lambda\frac{\partial^3\Phi}{\partial x^3} - \frac{\rho\lambda}{G}\frac{\partial^3\Phi}{\partial y\,\partial\tau^2},$$

$$Y_y = (3\lambda+4G)\frac{\partial^3\Phi}{\partial x^2\,\partial y} + (\lambda+2G)\frac{\partial^3\Phi}{\partial y^3} - (\lambda+2G)\frac{\rho}{G}\frac{\partial^3\Phi}{\partial y\,\partial\tau^2}, \quad \text{(c)}$$

$$X_y = Y_x = -\lambda\frac{\partial^3\Phi}{\partial x\,\partial y^2} + (\lambda+2G)\frac{\partial^3\Phi}{\partial x^3} - \rho\frac{\partial^3\Phi}{\partial x\,\partial\tau^2}.$$

To solve Eq. (10.59), we apply two integral transformations, namely the Laplace transformation in the variable τ and the Fourier complex transformation in the variable y.

Since we are considering the case of zero initial conditions, we have to put

$$\Phi(x, y, 0) = \frac{\partial\Phi(x, y, 0)}{\partial\tau} = \frac{\partial^2\Phi(x, y, 0)}{\partial\tau^2} =$$

$$= \frac{\partial^3\Phi(x, y, 0)}{\partial\tau^3} = 0.$$

By multiplying Eq. (10.59) by $e^{-p\tau}$, and integrating with respect to τ from 0 to ∞, we find an auxiliary differential equation for $\overline{\Phi}$ with two independent variables, x and y,

$$(\nabla^2 - p^2/c_1^2)(\nabla^2 - p^2/c_2^2)\,\overline{\Phi}(x, y, p) = 0 \quad \text{(d)}$$

where

$$\overline{\Phi}(x, y, p) = \int_0^\infty e^{-p\tau}\Phi(x, y, \tau)\, d\tau.$$

By multiplying the equation (d) by $e^{i\alpha x}$, and integrating with respect to x from $-\infty$ to $+\infty$, we obtain a differential equation of the form

$$\left(\frac{d^2}{dy^2} - \alpha^2 - \frac{p^2}{c_1^2}\right)\left(\frac{d^2}{dy^2} - \alpha^2 - \frac{p^2}{c_2^2}\right)\overline{F}(\alpha, y, p) = 0, \qquad \text{(e)}$$

where

$$\overline{F}(\alpha, y, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x}\overline{\Phi}(x, y, p)\, dx.$$

The solution of the equation (e) bounded at infinity is of the form

$$\overline{F}(\alpha, y, p) = A\exp\left(-y\sqrt{\alpha^2 + p^2/c_1^2}\right) + B\exp\left(-y\left(\sqrt{\alpha^2 + p^2/c_2^2}\right)\right),$$

where A and B are determined from the conditions on the boundary of the half-plane $y = 0$.

By making use of the inversion formula for the Fourier complex transformation

$$\overline{\Phi}(x, y, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\alpha x}\overline{F}(\alpha, y, p)\, d\alpha,$$

we find that

$$\overline{\Phi}(x, y, p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\alpha x}\left[A(\alpha, p)\exp\left(-y\sqrt{\alpha^2 + p^2/c_1^2}\right) + B(\alpha, p)\exp\left(-y\sqrt{\alpha^2 + p^2/c_2^2}\right)\right] d\alpha.$$

By using next the inversion formula for the Laplace transformation

$$\Phi(x, y, \tau) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{p\tau}\overline{\Phi}(x, y, p)\, dp,$$

we obtain, finally,

$$\Phi(x, y, \tau) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{p\tau} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} \times \right.$$

$$\times [A(\alpha, p) \exp\left(-y \sqrt{\alpha^2 + p^2/c_1^2}\right) +$$

$$\left. + B(\alpha, p) \exp\left(-y \sqrt{\alpha^2 + p^2/c_2^2}\right)] \, d\alpha \right\} dp, \qquad \text{(f)}$$

where $A(\alpha, p)$ and $B(\alpha, p)$ are determined from the boundary conditions.

After finding $\Phi(x, y, \tau)$ from the expression (f), we calculate the displacement components by the formulas (b), and the stresses by the formulas (c).

REFERENCES

1. Papkovich P. F., "Theory of Elasticity," Oborongiz, Leningrad-Moscow, 1939 (in Russian).
2. Krutkov Yu. A., "Stress Function Tensor and General Solutions in Elastostatics," Izd. Akad. Nauk SSSR, Moscow-Leningrad, 1949 (in Russian).
3. Vlasov V. Z., "Selected Works", Vol. 1, Izd. Akad. Nauk SSSR, Moscow, 1962 (in Russian).
4. Vinokurov L. P., "Direct Methods for Solving Three-dimensional and Contact Problems for Blocks and Foundations", Izd. Khar'kov. Univ., Khar'kov, 1956 (in Russian).
5. Love A. E. H., "A Treatise on the Mathematical Theory of Elasticity," 4th ed., Cambridge Univ. Press, 1927.
6. Lurie A. I., "Three-dimensional Problems of Elasticity," Gostekhizdat, Moscow, 1955 (in Russian).
7. Jahnke E., Emde F., "Tables of Functions with Formulae and Curves," 4th ed., Dover Publ., New York, 1945 (German and English).
8. Neuber H., "Beiträge für den achssymetrischen Spannungszustand", Diss., München, 1932; see also [6].
9. Goursat E., "Cours d'analyse mathématique," 5-e ed., T. 3, P. 1, Gauthier-Villars, Paris, 1956. Transl.—"A Course in Mathematical Analysis," Vol. 3, Part 1, Dover Publ., New York, 1959.
10. Tallqvist Hj., "Sechsstellige Tafeln der 32 ersten Kugelfunktionen P_n (cos θ)," Helsingfors, 1938.
11. Rekach V. G., *Analysis of Thin Spherical Shells*, Sbornik Trudov Mosk. Inzhenerno-stroit. Inst., Vyp. 34, 148-194 (1963) (in Russian).
12. Tricomi F. G., "Lezioni sulle equazioni a derivate parziali," Gheroni, Torino, 1954.
13. Rekach V. G., "An Aid to Solving Problems in Applied Elasticity," Vysshaya Shkola, Moscow, 1973 (in Russian).
14. Courant R., Hilbert D., "Methoden der mathematischen Physik," 2 verb. Aufl., Bd. I, II, Springer-Verlag, Berlin, 1931. Transl.—

"Methods of Mathematical Physics," Interscience Publ., New York-London, Vol. 1, 1953; Vol. 2, 1962.

15. Smirnov M. M., "Problems in Equations of Mathematical Physics," 2nd ed., Gostekhizdat, Moscow, 1954 (in Russian).
16. Trefftz E., *Mathematische Elastizitätstheorie*, in "Handbuch der Physik, Bd. VI, Mechanik der elastischen Körper," Springer-Verlag, Berlin, 1928, S. 92.
17. Galerkin B. G., *On the Analysis of Stresses and Strains in an Elastic Isotropic Body*, Doklady Akad. Nauk SSSR. Ser. A, No. 14, 353-358 (1930) (in Russian).
18. Galerkin B. G., *On the General Solution of the Elasticity Problem in Three Dimensions Using Stress and Displacement Functions*, Doklady Akad. Nauk SSSR. Ser. A, No. 10, 281-286 (1931) (in Russian).
19. Papkovich P. F., *Expression for the General Integral of the Basic Equations of the Theory of Elasticity in Terms of Harmonic Functions*, Izv. Akad. Nauk SSSR. Otd. Matem. i Estestv. Nauk, No. 10, 1425-1435 (1932) (in Russian).
20. Grodskii G. D., *Integration of the Equations of Equilibrium for an Elastic Body of Revolution with an Axially Symmetric Distribution of Body and Surface Forces*, Izv. Akad. Nauk SSSR. Ser. Matem. i Estestv. Nauk, No. 10, 1537-1550 (1934) (in Russian).
21. Neuber H., *Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie*, Z. angew. Math. und Mech., 14(4), 203-212 (1934).
22. Neuber H., "Kerbspannungslehre. Grundlagen für genaue Spannungsrechnung," Springer-Verlag, Berlin, 1937.
23. Galin L. A., *Three-dimensional Contact Problems of the Theory of Elasticity for Punches of Circular Shape in Plan*, Prikl. Matem. i Mekh., 10 (4), 425-448 (1946) (in Russian).
24. Galerkin B. G., *Elastic Rectangular and Triangular Simply Supported Thick Plates Subjected to Bending*, Doklady Akad. Nauk SSSR. Ser. A, No. 10, 273-280 (1931) (in Russian).
25. Mindlin R. D., *Force at a Point in the Interior of a Semi-infinite Solid*, Physics, 7(5), 195 (1936); see also in "Proc. First Midwestern Conf. Solid Mech., Univ. of Illinois," Urbana, 1953, p. 111.
26. Abramyan B. L., *On the Problem of the Axially Symmetric Deformation of a Circular Cylinder*, Doklady Akad. Nauk ArmSSR, 19(1), 3-12 (1954) (in Russian).
27. Kolosov G. V., "Application of a Complex Variable to the Theory of Elasticity," Objed. Nauchno-tekhn. Izd., Moscow-Leningrad, 1935 (in Russian).
28. Muskhelishvili N. I., "Some Basic Problems of the Mathematical Theory of Elasticity; Fundamental Equations, Plane Theory of Elasticity, Torsion and Bending," 5th ed., Nauka, Moscow, 1966 (in Russian). Transl.— P. Noordhoff, Groningen, 1953. (Translation of the 3rd ed., Izd. Akad. Nauk SSSR, Moscow-Leningrad, 1949.)
29. Vlasov V. V., *Initial Function Method in the Plane Problem of Elasticity*, Izv. Vysshikh Uchebn. Zavedenii. Stroitel'stvo i Arkhitektura, No. 2, 97-111 (1958) (in Russian).

30. Papkovich P. F., *On One Form of Solution of the Plane Problem of Elasticity for a Rectangular Strip*, Doklady Akad. Nauk SSSR, Novaya Seriya, 27(4), 335-339 (1940) (in Russian).
31. Fadle J., *Die Selbstspannungs-Eigenwertfunktionen der quadratischen Scheibe*, Ing.-Arch., 11(2), 125-149 (1940).
32. Kitover K. A., *On the Use of Special Systems of Biharmonic Functions for the Solution of Some Problems of Elasticity*, Prikl. Matem. i Mekh., 16(6), 739-748 (1952) (in Russian).
33. Varvak P. M., "Development and Application of the Grid Method to the Analysis of Plates," Izd. Akad. Nauk UkrSSR, Kiev, Part I, 1949; Part II, 1952 (in Russian).
34. Zhemochkin B. N., "Theory of Elasticity," Stroivoenmorizdat, Moscow, 1948 (in Russian).
35. Galerkin B. G., *On the Analysis of Stresses in Dams and Retaining Walls of Trapezoidal Section*, Sbornik Leningr. Inst. Inzh. Putei Soobshcheniya, No. 99, 147-170 (1929) (in Russian).
36. Filonenko-Borodich M. M., "Theory of Elasticity," 4th ed., Fizmatgiz, Moscow, 1959 (in Russian). Transl. — Foreign Languages Publishing House, Moscow, 1963.
37. Savin G. N., "Stress Concentration Round Holes," Gostekhizdat, Moscow-Leningrad, 1951 (in Russian).
38. Vlasov V. Z., Leont'ev N. N., "Beams, Plates, and Shells on an Elastic Foundation," Fizmatgiz, Moscow, 1960 (in Russian).
39. Filon L. N. G., *On an Approximate Solution for the Bending of a Beam of Rectangular Cross-section Under Any System of Load, with Special Reference to Points of Concentrated or Discontinuous Loading*, Phil. Trans. Roy. Soc. London, Ser. A, 201 (334), 63-155 (1903).
40. Ribière C., *Sur la flexion des pièces épaisses*, Compt. rend. Acad. sci., Paris. Ser. A, 126, 402-404 (1898); Sur la résistance des massifs épais, ibid., 126, 1190-1192 (1898).
41. Vlasov V. V., "Initial Function Method in Problems of the Theory of Elasticity and Structural Mechanics," Stroiizdat, Moscow, 1975 (in Russian).
42. Prokopov V. K., *On a Plane Problem of Elasticity for a Rectangular Region*, Prikl. Matem. i Mekh., 16(1), 45-56 (1952) (in Russian).
43. Varvak P. M., *On the Analysis of Deep Beams*, Sbornik Trudov. Kievsk. Stroit. Inst., Vyp. 3, 143-178 (1936) (in Russian).
44. Golovin Kh. S., "An Elastostatic Problem," Izv. Tekhnol. Inst. St. Petersburg, 3(1881) (in Russian).
45. Kats A. M., "Theory of Elasticity", Gostekhizdat, Moscow, 1956 (in Russian).
46. Rekach V. G., *Some Symmetrical Supporting Conditions for Bodies on an Absolutely Rigid and Smooth Foundation in the Plane Problem of Elasticity*, Nauchn. Doklady Vysshei Shkoly. Stroitel'stvo, No. 1, 5-12 (1959) (in Russian).
47. Romanovskii P. I., "Fourier Series. Field Theory. Analytic and Special Functions. Laplace Transformation," 3rd ed., Fizmatgiz, Moscow, 1961 (in Russian).

48. Vainberg D. V., *Local Stresses in a Flat Circular Ring Due to Two Concentrated Forces*, Prikl. Matom. i Mekh., 13(2), 151-158 (1949) (in Russian).
49. Rekach V. G., *Application of Boundary Value Homogeneous Solutions to the Analysis of Triangular Dams*, Trudy Univ. Druzhby Narodov Im. Patrice Lumumba, 9(2), 84-108 (1965) (in Russian).
50. Borisov N. E., *Application of Homogeneous Boundary Value Solutions to the Analysis of a Wedge on an Absolutely Rigid Foundation*, Izv. Vysshikh Uchebn. Zavedenii. Stroitel'stvo i Arkhitektura, No. 6, 41-48 (1968) (in Russian).
51. Borisov N. E., *Application of Homogeneous Boundary Value Solutions to the Analysis of a Wedge on an Elastic Foundation*, Sbornik Nauchn. Rabot Penzenskogo Inzhenerno-stroit. Inst., Vyp. 5, 216-232 (1970) (in Russian).
52. Solyanik-Krassa K. V., "Torsion of Shafts of Variable Section," Gostekhizdat, Moscow-Leningrad, 1949 (in Russian).
53. Gray A., Mathews G. B., "A Treatise on Bessel Functions and Their Applications to Physics," 2nd ed., Macmillan Co., London, 1931.
54. Novozhilov V. V., "Theory of Elasticity," Sudpromgiz, Leningrad, 1958 (in Russian). Transl.— Pergamon Press, New York, 1961.
55. Melan E., *Ein Beitrag zur Torsion von Rotationskörpern*, Technische Blätter, Prag, No. 48, 417-419; No. 49/50, 427-429 (1920).
56. Melan E., Parkus H., "Wärmespannungen infolge Stationärer Temperaturfelder," Springer-Verlag, Wien, 1953.
57. Lykov A. V., "Theory of Heat Conduction," Vysshaya Shkola, Moscow, 1967 (in Russian).
58. Boley B. A., Weiner J. H., "Theory of Thermal Stresses," Wiley, New York-London, 1960.
59. Maslov G. N., *Thermal Stresses and Strains in Concrete Blocks Based on the Theory of Elasticity*, Izv. Nauchnoissled. Inst. Gidrotekhniki, 13, 18-67 (1934) (in Russian); see also Maizel V. M., "Thermoelastic Problem," Izd. Akad. Nauk UkrSSR, Kiev, 1951 (in Russian).
60. Bezukhov N. I., "Fundamentals of the Theory of Elasticity, Plasticity, and Creep," Vysshaya Shkola, Moscow, 1961 (in Russian).
61. Korenev B. G., "Some Problems of Elasticity and Heat Conduction Solvable in Terms of Bessel Functions," Fizmatgiz, Moscow, 1960 (in Russian).
62. Shtaerman I. Ya., "Contact Problem of the Theory of Elasticity," Gostekhizdat, Moscow-Leningrad, 1949 (in Russian).
63. Galin L. A., "Contact Problems of the Theory of Elasticity", Gostekhizdat, Moscow, 1953 (in Russian).
64. Egorov K. E., *Deformation of the Base of a Circular Rigid Foundation Under an Eccentric Load*, Sbornik Trudov. Nauchnoissled. Inst. Osnovanii i Fundamentov, No. 11, 119-138 (1948) (in Russian).
65. Gorbunov-Posadov M. I., "Analysis of Structures on an Elastic Foundation," Stroiizdat, Moscow, 1953 (in Russian).
66. Florin V. A., "Analysis of Foundations of Hydraulic Structures," Stroiizdat, Moscow, 1948 (in Russian).

67. Leonov M. Ya., "*The General Problem of the Pressure of a Circular Punch on an Elastic Half-space, Prikl. Matem. i Mekh.*, 17(1), 87-98 (1953) (in Russian).
68. Martynenko M. D., *Fundamental Boundary Value Problems of Potential Theory for Regions with Slits*, Ukr. Matem. Zh., 15(4), 431-437 (1963) (in Russian).
69. Rekach V. G., *A Contact Problem of the Theory of Elasticity for a Flat Bar of Circular Shape*, Izv. Vysshikh Uchebn. Zavedenii. Stroitel'stvo i Arkhitektura, No. 2, 76-80 (1958) (in Russian).
70. Galin L. A., *Mixed Problems of the Theory of Elasticity with Frictional Forces for a Half-plane*, Doklady Akad. Nauk SSSR. Novaya Seriya, 39(3), 88-93 (1943) (in Russian).
71. Sherman D. I., *Plane Problem of the Theory of Elasticity with Mixed Boundary Conditions*, Trudy Seismol. Inst., Akad. Nauk SSSR, No. 88, 1-32 (1938) (in Russian).
72. Rayleigh J. W., *On Waves Propagated Along the Plane Surface of an Elastic Solid*, Proc. London Math. Soc., 17, 4-11 (1887).
73. Miller G. F., Pursey H., *On the Partition of Energy Between Elastic Waves in a Semi-infinite Solid*, Proc. Roy. Soc. London. Ser. A, 233 (1192), 55-69 (1955).
74. Dolling H.-J., *Die Abschirmung von Erschütterungen durch Bodenschlitze*, Bautechnik, No. 5, 151-158; No. 6, 193-204 (1970).
75. Barkan D. D., "Dynamics of Bases and Foundations," Stroivoenmorizdat, Moscow, 1948 (in Russian).
76. Zabylin M. I., Igol'nikov V. V., *Investigation of the Parameters of Elastic Waves Propagating in Soils from Industrial Sources*, Izv. Vysshikh Uchebn. Zavedenii. Stroitel'stvo i Arkhitektura, No. 11, 30-37 (1971) (in Russian).
77. Love A. E. H., "Some Problems of Geodynamics," Cambridge Univ. Press, 1911. Reprint, Dover Publ., New York, 1967.
78. Radok J. R. M., *On the Solution of Problems of Dynamic Plane Elasticity*, Quart. Appl. Math., 14(3), 289-298 (1956).
79. Sneddon I. N., Berry D. S., *The Classical Theory of Elasticity*, in "Encyclopedia of Physics, Vol. VI, Elasticity and Plasticity" (S. Flügge, ed.), Springer-Verlag, Berlin, 1958, pp. 1-126.
80. Sneddon I. N., "Fourier Transforms," McGraw-Hill Book Co., New York, 1951.
81. Petrashen G. I., *On a Rational Method of Solving Elastodynamic Problems in the Case of Laminate-isotropic Regions with Plane-parallel Interfaces*, Uchenye Zapiski Leningr. Gos. Univ., Ser. Matem. Nauk, No. 208, 5-57 (1956) (in Russian).
82. Smirnov V. I., Sobolev S. A., *Sur une méthode nouvelle dans le problème plan des vibrations élastiques*, Trudy Seismol. Inst., Akad. Nauk SSSR, No. 20, 37 pp. (1932) (in French).
83. Smirnov V. I., Sobolev S. A., *Sur l'application de la méthode nouvelle a l'étude des vibrations élastiques dans l'espace à symétrie axiale*, Trudy Seismol. Inst., Akad. Nauk SSSR, No. 29, 49 pp. (1933) (in French).
84. Erugin N. P., *On Functionally Invariant Solutions*, Uchenye Zapiski Leningr. Gos. Univ., Ser. Matem. Nauk, No. 15, 101-134 (1948) (in Russian).

85. Sinitsyn A. P., "Structural Analysis for Thermal Shock," Stroiizdat, Moscow, 1971 (in Russian).
86. Danilovskaya V. I., *Thermal Stresses in an Elastic Half-space Produced by a Sudden Heating of Its Boundary*, Prikl. Matem. i Mekh., 14(3), 316-318 (1950) (in Russian); see also On a Dynamic Problem of Thermoelasticity, Prikl. Matem. i Mekh., 16(3), 341-344 (1952) (in Russian).
87. Chree C., *Long Rotating Circular Cylinders*, Proc. Cambridge Phil. Soc. 7, 283-305 (1892).
88. Chree C., *The Equations of an Isotropic Elastic Solid in Polar and Cylindrical Co-ordinates*, Their Solution and Application, Trans. Cambridge Phil. Soc., 14, 250-369 (1889).
89. Pochhammer L., *Ueber die Fortpflanzungsgeschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotropen Kreiscylinder*, J. reine und angew. Math., 81, 324-336 (1876).
90. Sneddon I. N., *The Stress Produced by a Pulse of Pressure Moving Along the Surface of a Semi-infinite Solid*, Rend. Circ. mat. Palermo, Ser. II, 1(1), 57-62 (1952).
91. Borodachev N. M., *Vibrations of an Elastic Half-plane*, Trudy Kuibyshev. Inzhenerno-stroit. Inst., Vyp. 5, 291-300 (1958) (in Russian).

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Elements of Hereditary Solid Mechanics

Yu. RABOTNOV, Mem. USSR Acad. Sci.

Boltzmann-Volterra hereditary mechanics describes such processes when the state of a mechanical system depends on the entire history of the actions it has been subjected to. The considerable development of this theory in the last few decades was determined by a multitude of its technical applications connected with the studying of the creep of metals, plastics, concrete, rock and other bodies. The book sets out the formal fundamentals of the theory, its applications to the description of the behaviour of real materials, and some methods for solving problems of the linear hereditary theory of elasticity and the nonlinear theory of creep. Special attention is given in the appendices to the use of weakly singular operators.

The book will be of interest for quite a broad circle of readers—engineers, scientists, students and post-graduates.

The Theory of Elasticity

Yu. AMENZADE, Corr. Mem. Azerb. SSR Acad. Sci.

This book contains relevant data from tensor analysis (the exposition of the fundamentals of the theory of elasticity is given at contemporary advanced level and in modern form), the plane problems of theory of elasticity are considered with the help of method of functions of complex variables and method of integral transforms. The book deals with the theory of rotation and bending of prismatic bodies. Hertz contact problem and certain six-symmetric problems. It also contains theory of propagation of elastic waves in an infinite medium and surface waves of Rayleigh, etc. Examples of theory of bending of thin sheets are given. The textbook excels in clarity and originality and is illustrated by numerous examples. The textbook is intended for university students.

This book is designed to be used as an aid to solving elasticity problems in college and university courses in engineering.
The book covers all subjects of the mathematical theory of elasticity. It contains material which forms the basis for structural analysis and design. Numerous problems illustrate the text and somewhat complete it. Along with classical problems, they include cases of practical significance.
The author does not emphasize any particular procedure of solution, but instead considerable emphasis is placed on the solution of problems by the use of various methods. Most of the problems are worked out and those which are left as an exercise to the student are provided with answers or references to the original works.

Professor Vladimir Germanovich Rekach, D.Sc., is the Head of the Department of Strength of Materials at the Patrice Lumumba Peoples' Friendship University in Moscow.
His main scientific interests are structural design, analysis of curved bars and vibration problems. The title of his doctoral thesis was "The Analysis of Spherical Shells". He is the author of 28 articles and 8 books (3 as coauthor).