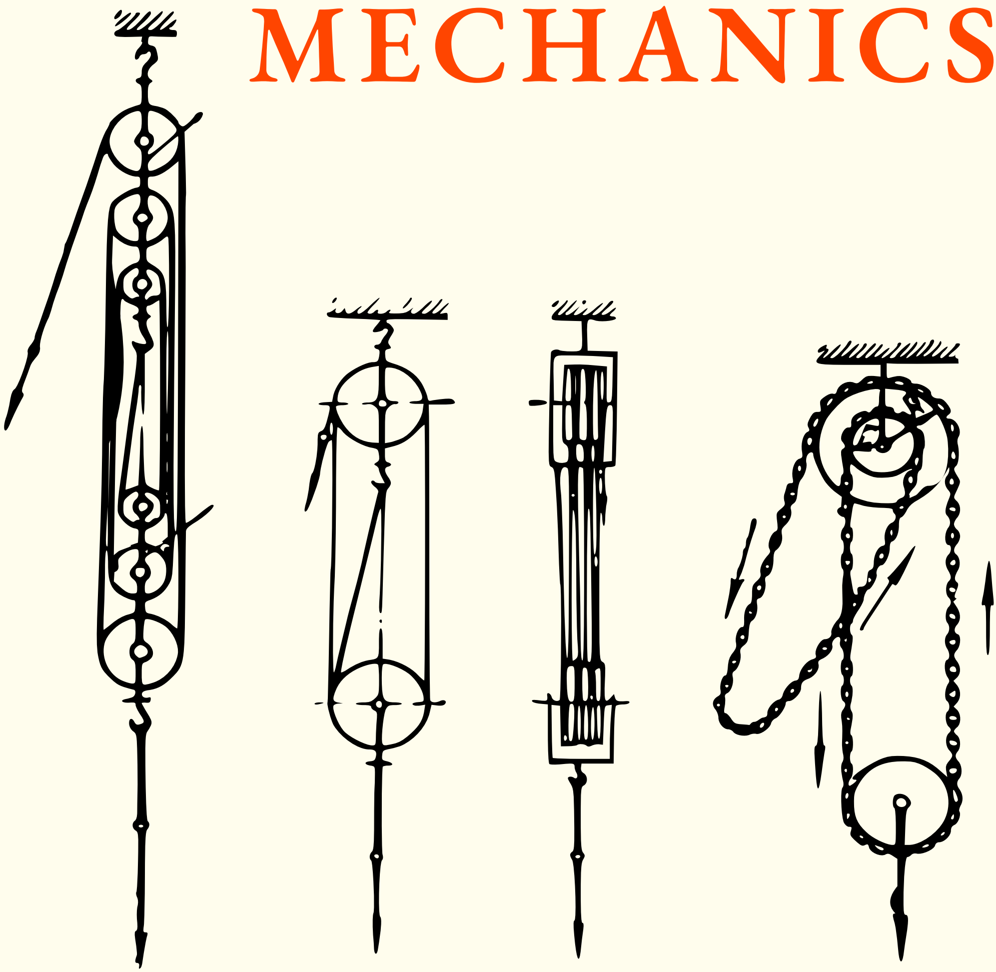
L. Levinson

FUNDAMENTALS OF ENGINEERING MECHANICS



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FUNDAMENTALS OF ENGINEERING MECHANICS

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PART ONE

THEORETICAL MECHANICS

INTRODUCTION

1. Mechanical Motion

There are a great many forms of motion. An incalculable number of bodies are in motion on the earth, which in its turn is rotating about its axis and also travelling around the sun, while the sun itself and all its planets are in movement relative to the stars, which in their turn are also moving through space. But in all these instances we have to do with only one form of motion - the motion of bodies themselves. Science has established that heat, light, electricity, and chemical and many other phenomena are also forms of motion. Furthermore, life itself in all its manifestations is a form of motion.

The perpetual movement of matter causes all the natural phenomena about us.

Motion occurs in space and in time, therefore space and time are inseparable from matter in motion. When a body changes its position in respect to other bodies, we say it is in motion. This relative change in position of a body is called *mechanical motion.*

The science dealing with the laws of mechanical motion is called *mechanics.*

2. The State of Rest as a Relative Phenomenon

The state of rest is a concept we continually meet with in mechanics. For example, when a railway bridge is built firmly and rigidly to make it immovable, we infer that it is in a state of rest only relative to the earth. For actually the bridge is in motion together with the earth as the latter rotates about its axis, travels around the sun, and moves with the whole solar system.

There is no such thing as an absolute state of rest. In mechanics when we speak of an immovable body we have in mind its *relative* state of rest, that is, its immovability in respect to some other body, usually the earth. And when we say that the headstock of a lathe is fixed we infer that it is rigidly fastened to the frame which we assume to be immovable. A body assumed to be immovable is called *a basic system.*

3. Fundamental Elements of Mechanics

The motion of a body occurs in space, therefore *space* is one of the fundamental elements in mechanics. *Time* is likewise such an element in mechanics.

Every point of a moving body describes a path of definite form relative to a basic system; this path is called a *trajectory*. A trajectory may be either of straight or curved lines, in accordance with which the motion of a point is then described either as *rectilinear* or *curvilinear*. A moving point traverses a definite *distance*, the length of which as covered in a definite interval of time will depend upon the *speed* of the moving point. If the point travels equal distances in equal intervals of time, its speed will be constant and its motion is then said to be *uniform*. In other cases the motion is said to be *non-uniform*, or *variable*.

If speed changes at an equal rate in equal intervals of time, the motion is said to be either *uniformly accelerated* or *uniformly retarded*. A change in speed is called *acceleration*.

In investigating mechanical motion of bodies and their state of rest (as a particular case of motion), another quantity is met with which determines the action of one body upon another; that quantity is called *force*.

All these elements will be dealt with in detail along with others pertaining to mechanics, as we proceed.

4. Basic Units of Measure Used in Mechanics

In order to express quantities in figures, definite basic units of measure are required. Such units are:

the metre, written m, for measuring length and distance;
the kilogramme, written kg, for measuring force;
the second, written sec, for measuring time.

Units representing other quantities in mechanics are derived from the above units. Speed is represented as a fraction formed by dividing distance by time, thus:

$\frac{\text{unit of length}}{\text{unit of time}}$, or $\frac{\text{m}}{\text{sec}}$, or m $\times$ sec^{-1}; acceleration is represented by the magnitude $\frac{\text{unit of velocity}}{\text{unit of time}}$, i. e., $\frac{\text{m}}{\text{sec} \times \text{sec}}$ or $\frac{\text{m}}{\text{sec}^2}$ or m $\times$ sec^{-2}; and so forth with other magnitudes.

It is sometimes found more convenient to derive certain units directly from m, kg, and sec. Low speeds, for instance, are expressed as $\frac{\text{mm}}{\text{sec}}$; the speed of a train or a plane as $\frac{\text{km}}{\text{hr}}$; the speed at which a lathe cuts metal as $\frac{\text{m}}{\text{min}}$; great forces are expressed in tons (ton), etc.

In solving problems the units of measure must be brought into proper correlation and thenceforth strictly followed to obtain correct results.

Illustrative Problem 1. How much greater or smaller is the unit of velocity $\frac{km}{hr}$ than the unit of velocity $\frac{m}{min}$?

Solution: to solve this problem, the units must first be reduced to a common denomination: since 1 km = 1,000 m and 1 hr = 60 min, it follows that

$$1\ \frac{km}{hr} = \frac{1{,}000\ m}{60\ min};$$

therefore

$$1\ \frac{km}{hr} : 1\ \frac{m}{min} = \frac{1{,}000\ m}{60\ min} : 1\ \frac{m}{min} = \frac{1{,}000}{60} = 16\frac{2}{3}.$$

Hence $1\ \frac{km}{hr}$ is $16\frac{2}{3}$ times greater than $1\ \frac{m}{min}$.

5. A Material Point and a Solid Body

Bodies whose motion is dealt with in theoretical mechanics are assumed as consisting of a very great number of infinitely small particles. The size of each particle is imagined so small as to approach a geometric point. Each such particle is known as a *material point.*

Hence, *any body is regarded as being the sum, or a system, of material points.*

In studying motion in mechanics, the body concerned is frequently represented by a single material point. The motion of a ship, for instance, may be designated as the motion of just such a material point. A moving ball attached to a long string may also be considered a material point.

Accordingly, the concept *material point* may signify either a very small particle of a body, or a whole body considered as a point.

A material point is a body whose dimensions are so small as to be negligible with respect to other geometric values involved in the given problem.

Bodies dealt with in theoretical mechanics are assumed to be ***absolutely rigid*** and unchangeable in size and shape under the influence of another body.

6. The Science of Mechanics

Mechanics deals with a variety of problems, but notwithstanding this variety they fall under one of the following classifications:

1. Determining the trajectory described by the points of a moving body, the position of any one of the points in its trajec-

tory, its speed and acceleration, etc., in short, the solution of problems concerned with the movement of a body as a whole or of any of its individual points independent of the force applied.

This branch of mechanics is called *kinematics*. Kinematics deals with the relationship between the geometric elements

M. Lomonosov

of motion and time, irrespective of the forces acting on the body in motion.

2. Determining the nature of motion of a body as related to the forces acting on the body, or, conversely, determining the forces causing the motion. This type of problem is dealt with in the branch of mechanics called *kinetics*.

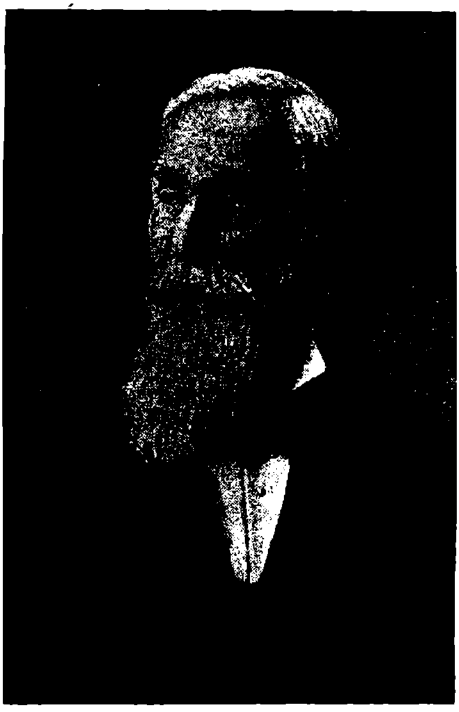
Mechanics also treats of terrestrial bodies in a state of rest, that is, a state of equilibrium. Here we seek the conditions under which forces acting on a body are brought into equilibrium, for knowing these conditions, engineers can ensure rigidity and strength to the structures they are building.

That part of kinetics dealing with equilibrium of forces and the consequent state of rest of a body is known as *statics*, while the investigation of motion of bodies under the action of forces applied to them constitutes another branch of kinetics called *dynamics*.

Such are the sciences embraced by mechanics, and their fundamentals are taken up in the first part of this book in the following order: statics, kinematics, and dynamics.

7. Chief Stages in the Historical Development of Mechanics

It took thousands of years for man to find scientific explanations for mechanical phenomena. The first known attempts of the kind were conducted during the 4th century B. C. Implements and mechanical devices of the time were extremely simple,



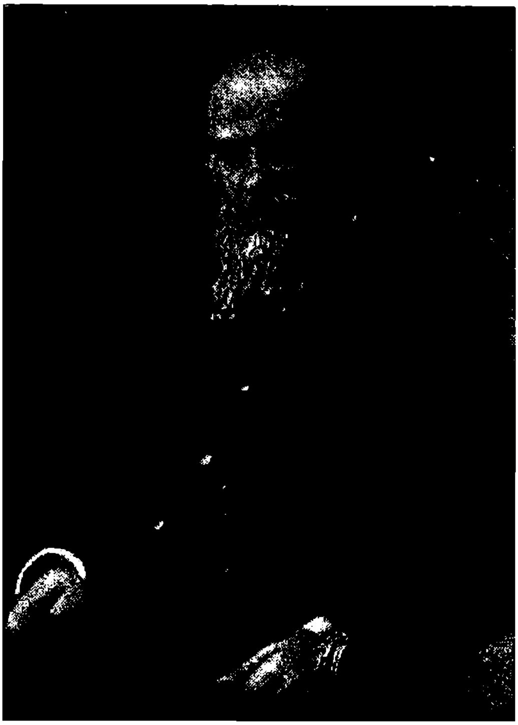
P. Chebyshev.

knowledge of mechanics was correspondingly limited and the devices known—the lever, pulley, windlass, etc. —were studied for the most part from the standpoint of statics to attain an understanding of equilibrium of forces.

Some of the most important work in the field of statics was done by Archimedes (287-212 B. C.), who carried on research on the laws of the lever, centre of gravity, and other phenomena.

After Archimedes, there was little advance in mechanics until the 15th century A. D., when it began to develop intensively, spurred on by the transition from primitive handicraft to improved methods of production. During this period Leonardo da Vinci (1452-1519) made several discoveries in the field of mechanics, while Stevinus (1518-1620) further developed many of Archimedes' principles of statics and investigated the mechanical properties of the inclined plane.

In the 17th century mechanics was further enriched by Galileo Galilei (1564-1642). Galileo's work in this sphere was carried forward by Isaac Newton (1642-1727), who improved the

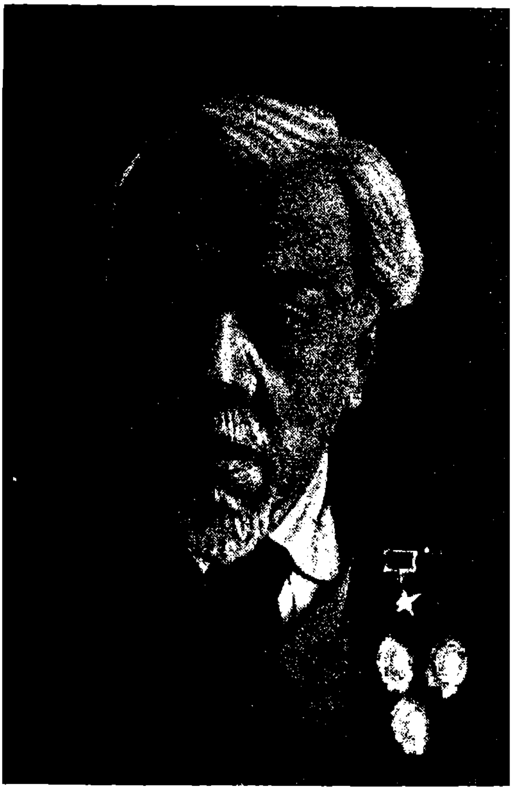


N. Zhukovsky

formulation of some of Galileo's laws and developed mechanics to the level of a science. The mechanics of Galileo and Newton, now known as *classical mechanics*, formed the foundation for the subsequent intensive growth of that science.

The 18th century saw the advancement of a new science called analytical mechanics, whose founder was the Russian mathematician and mechanic, Academician L. Euler (1707-1783).

An outstanding name of the eighteenth century was that of the Russian scholar M. Lomonosov (1711-1765), eminent for his discoveries in various spheres of science, including mechanics; another major contribution was his discovery of the law of the conservation of matter and energy.



S. Chaplygin

Another amongst the first Russian scientists notable for their work in mechanics, was Academician S. Kotelnikov who in 1774 published a book on equilibrium and motion of bodies.

Beginning with the 19th century mechanics made rapid strides and its principles were applied with ever greater frequency to practical problems. The Russian scientist P. Chebyshev (1821-1894) carried out extensive research on, and created the foundation of, a branch of mechanics called the "Theory of Mechanisms and Machines". Russian scientists also contributed enormously to the knowledge of the mechanics of liquids and gases. Amongst them N. Zhukovsky (1847-1921) holds a leading position. Known as the "Father of Russian Aviation", he is the founder of the Russian theoretical school in that field, his works forming the basis of the general science of aerodynamics and aviation as a whole. Academician S. Chaplygin (1869-1942), one of Zhukovsky's outstanding pupils, solved a number of important problems in contemporary super-speed aviation and other pressing questions of mechanics of great theoretical and practical significance.

STATICS

CHAPTER I

FUNDAMENTALS OF FORCE, AND AN INTRODUCTION TO STATICS

8. Forces

Some examples of mechanical phenomena are: a stone falling to the ground, a tramcar passing from a rectilinear to a curvilinear stretch of track due to pressure on the sides of the wheels by the rails, the deformation of the spring and consequent lowering of the pan of weighing scales when an object is placed thereon. In all of the above instances a change of motion, or, as it is called, of mechanical position of a body, is brought about by the action of another body upon the one in question. In the first and third of the above instances that other body is the earth, while in the second instance it is the rails.

In mechanics, action exerted by one body upon another is called a *force*.

It must be noted that these are instances of the interaction of two bodies (the earth and a stone, rails and wheels, the body being weighed and a spring). When body A exerts a force upon body B, body B exerts a force of equal magnitude upon A but in the opposite direction.

9. Statics

As has already been said, statics deals with the equilibrium of forces. In order to find if a system of forces is in equilibrium, or what conditions are required to maintain a given equilibrium, it is necessary first to effect either a composition of the given forces, that is, to replace all the forces by a simple system of forces or by a single force that will exert the same action, or to resolve the forces into their components. Hence the laws of composition and resolution of forces are of primary importance in statics.

Among all forces acting on a body there is always one which is manifest by an attraction towards the centre of the earth; that force is *weight*, or gravity.

Another constant force is *friction*; let us say we want to shift an object along the ground. But we know from experience that it is not always possible to do this and that the action will depend upon a number of conditions, one of the most important being the resistance of the surface of the ground, i.e., the force of friction. All other conditions being equal, friction will vary directly with the weight of the object; the heavier the body, the greater will be the friction.

The force of gravity and the force of friction are very important factors in solving a great variety of problems; therefore they will also be investigated in this section of the book.

10. Elements Which Determine a Force

The action of one body upon another, known in mechanics as the application of a force, may be exerted in various directions. Hence, *direction is the first element of a force.* For example, the force of gravity which affects everything around us acts towards the centre of the earth, i.e., vertically.

However, the direction of a force is not sufficient to determine the action it will exert upon a body. For obviously the greater the force, the greater the action in the given direction. Therefore *the second element required to determine a force is its magnitude.*

To express the magnitude of a force it must be measured by some definite force taken as a unit. The most convenient way to measure the magnitude of a force in mechanics is to compare it with the force of gravity to which all bodies on earth are subject. For that reason the kilogramme (kg), which is the weight of one cubic decimetre of water, has been accepted as the unit of weight for measuring the magnitude of any force in mechanics. The instrument used for this purpose is called a *dynamometer.*

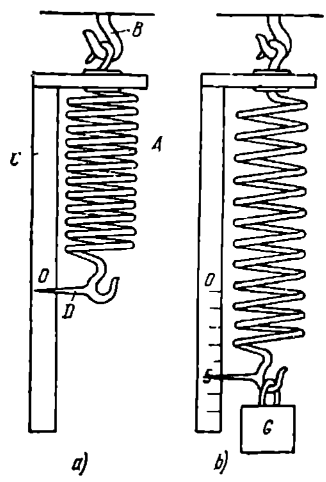


Fig. 1

The simplest type of dynamometer is illustrated in Fig. 1. A spiral spring *A*, with a pointer *D* attached to a hook on the lower end, is suspended from a stationary hook *B*. A strip *C* is fastened rigidly to the upper end of the spring. A mark *O* is made to indicate the position of the pointer *D* when at rest (Fig. 1*a*). When a load, let us say a 5 kg weight, is suspended from the hook on the lower end of the spring (Fig. 1*b*), it will stretch the spring so that the pointer will finally come to rest at a new position which is then marked 5. Thus the force of gravity acts on the load and causes the spring to stretch to a definite extent. Now if the load is removed and we subsequently stretch the spring again by hand till the pointer reaches the same figure 5, we may say that we are exerting a force of 5 kg upon the spring. And if we hang a 2.5 kg weight on the spring, we would see that the pointer comes to rest halfway between O and 5. Therefore if we divide the distance between O and 5 into five equal parts, we may then measure forces to within one-kilogramme divisions; or by dividing the same distance into ten equal parts, we could measure forces to within half-kilogramme divisions. Dynamometers constructed on the principle of the deformation of a spring are called *spring dynamometers*.

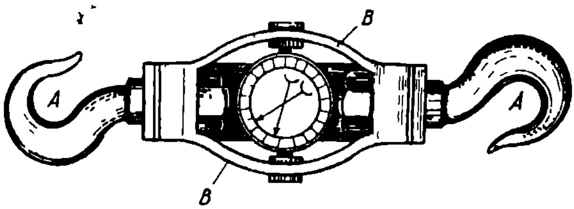


Fig. 2

Fig. 2 shows another type of spring dynamometer used to measure forces of large magnitude, from 2 to 5 tons. When pulling forces are applied to hooks *A* and *A*, plates *B* and *B* will be brought closer to each other and displace one of the pointers on the dial. By hitching such a dynamometer between a locomotive and a train of cars, the pulling power of the locomotive can be measured.

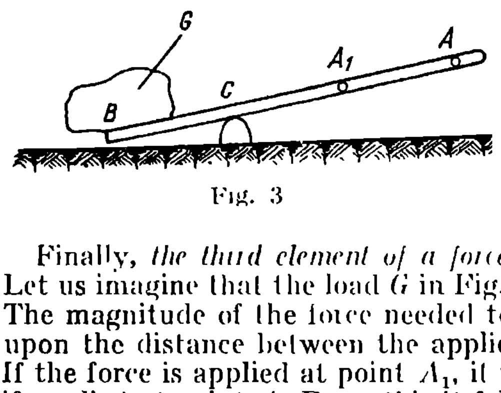


Fig. 3

Finally, *the third element of a force* is its *point of application.* Let us imagine that the load *G* in Fig. 3 is to be raised by a lever. The magnitude of the force needed to raise the load will depend upon the distance between the applied force and the fulcrum *C*. If the force is applied at point A_1, it will have to be greater than if applied at point *A*. From this it follows that in order to determine the action that a given force will have on a given body, its point of application must be known. In practice, the force will not act at a geometric point but will bear over a certain area. But for convenience in computation it is assumed as applied at one definite point of contact.

Wherefore, *the elements required to determine a force are direction, magnitude, and point of application.*

11. Graphic Method of Representing Forces

The action of forces is much easier to understand when they are represented graphically. Let us take a simple example.

Assume it necessary to represent the pressure exerted on a horizontal plate by a stationary ball whose weight $G = 2.25$ kg and which is lying on the plate's surface at a distance of a mm from its front edge and b mm from its left edge.

The application of the force G occurs at the point of contact between the flat plate and the ball. By drawing a straight line KL at a distance of a from the front edge of the plate (Fig. 4) and another straight line MN at a distance of b from the left edge, we find the point of application A at their intersection. At this point a force equal to the weight of the ball is acting on the plate. Since the force of gravity acts directly downwards, we draw a vertical line BC through point A along the path of the force. Now all that remains is to mark off on the latter line the force to be represented, to do which scale must be selected, such as 10 mm to one kg. We then mark off a section of 22.5 mm on the line BC from point A downwards to D, and draw an arrow at D indicating that the direction of the force is downwards.

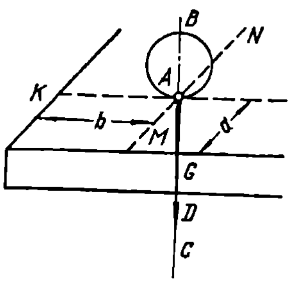


Fig. 4

The line upon which the force has been laid out (in this case line BC) is called the *line of action of the force*. This is a term we shall subsequently make frequent use of.

Force, as we see, is a quantity possessing *direction*. Other quantities possessing direction are also applied in mechanics (velocity, acceleration, etc.) and are all called *vector quantities* or *vectors*, as distinguished from quantities which have no direction (as, for instance, area, volume, etc.) and which are called *scalar quantities*.

A vector is delineated as part of a straight line whose length is based on a scale equal to the value of the given vector, while its direction is taken as the direction of this value.

To show the direction of the vector, an arrow is drawn on the segment of the line where the vector is laid out. A vector occurring in the text is designated by the same letters as indicated on the length of its delineation, except that a vinculum is drawn above these letters; for instance, the vector represented by the length AD in Fig. 4 is written in the text as $\overline{AD}$.

A vector may also be designated by only one letter instead of two in the text, but printed in bold type. For example, if the vector $\overline{AD}$ has a value of G, it is designated as **G**.

In the above example the letter A marks the beginning of the vector while the letter D — its end. To denote a vector the letter which marks its beginning is always written first and then follows the letter marking its end.

If the vector **G** were to act from points D to A, then it would be written as DA.

12. A System of Forces and Its Resultant

Fig. 5 represents a body with forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ applied at points A, B, and C. The aggregate of forces acting upon a body is called a *system of forces.*

If we find one force **R** exerting the same action upon a body as the whole system of indicated forces, then that force can be used to replace the system of forces.

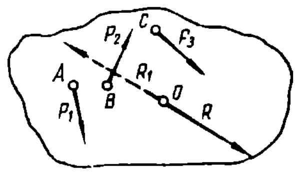


Fig. 5

A force which exerts the same action as a given system of forces is called the *resultant of forces.*

Forces whose concurrent action can be replaced by a resultant of forces are called the *components* of the force.

The resultant of forces is found through the composition of forces.

13. Two Equal Forces Acting in Opposite Directions Along a Straight Line Connecting Their Points of Application, Are in Equilibrium

Assume two men holding the ends of a pole and pulling in opposite directions. If the pole shifts toward one of the men, we will say he is pulling it with greater force than the other; if the pole does not shift, we will say the two men are pulling with equal force. In Fig. 6, if forces $\mathbf{P}_1$ and $\mathbf{P}_2$ applied at points A and B are equal in magnitude and acting in opposite directions along a straight line connecting their points of application, ***they will be in equilibrium*** and will cause no change in the mechanical state of the body. In like manner forces $\mathbf{P}_1$ and $\mathbf{P}_2$ may be applied to one point (Fig. 7).

Fig. 6

From this it follows that *if we add two more equal forces acting in opposite directions along a straight line to a system of forces already acting upon a body, the mechanical state of that body will not be changed.*

14. The Point of Application May Be Altered Along the Line of Action of a Force

Let us assume that a force **P** is applied to a body at point A (Fig. 7) and that at another point, say B, we apply two forces $\mathbf{P}_1$ and $\mathbf{P}_2$, each equal in magnitude to **P** and acting along its line of action but in opposite directions. As previously explained, this will not change the mechanical state of the body, but as a result we will have a system of three forces **P**, $\mathbf{P}_1$, and $\mathbf{P}_2$ acting along one line. But forces **P** and $\mathbf{P}_2$ are equal to each other

Fig. 7

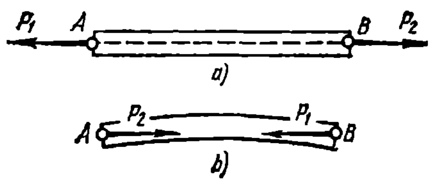


Fig. 8

and in mutual equilibrium and consequently the mechanical state of the body to which the three forces are applied depends on force $\mathbf{P}_1$ alone, which is equal in magnitude to P and acting in the same direction. In other words, we have altered the point of application of force **P** to point B.

This property of forces is frequently used in solving problems of mechanics.*

15. Equilibrant of a Force

Let us return to Sec. 12 (Fig. 5), where it was shown that the forces $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ may be replaced by their resultant **R**.

Now let us apply force $\mathbf{R}_1$, equal in magnitude, opposite in direction, and collinear with **R**, to point O. On the basis of what has been statedin Sec. 13, **R** and $\mathbf{R}_1$ are equivalent and consequently the forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ are equivalent to $\mathbf{R}_1$ which means that a body acted upon by this system of forces will be in equilibrium. A force $\mathbf{R}_1$ which is equal, opposite, and collinear with the resultant **R** of a system of forces acting on a body is called the *equilibrant* of that system. This line of reasoning

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* A clarification is required here. Let us assume that two forces $\mathbf{P}_1$ and $\mathbf{P}_2$, equal in magnitude and acting in opposite directions along the same line of action, are applied to a thin bar at points A and B (Fig. 8*a*). Force $\mathbf{P}_1$ may be moved to point B, and force $\mathbf{P}_2$ to point A (Fig. 8*b*), but it is apparent that in the first instance the bar would tend to *stretch*, and in the second would tend to *contract* and take the shape shown in Fig. 8*b*. Therefore from the physical standpoint the situation of the point of application of a force along its line of action is not a matter of indifference. Hence, as already stated in Sec. 5, fundamental deductions in theoretical mechanics are based on the assumption that a body on which forces are acting is absolutely rigid and unchangeable.

may be carried further. Let us assume that we have a system consisting of $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{R}_1$. Then their resultant would be equivalent in magnitude to force $\mathbf{P}_1$, and would be opposite and collinear with it, while $\mathbf{P}_1$ would be the equilibrant. This means that *in a system of forces in equilibrium, any one of the forces is the equilibrant of all the other forces.*

Oral Exercise

What is the difference between the resultant and the equilibrant of forces?

16. Composition of Collinear Forces

• Given two collinear forces $\mathbf{P}_1$ and $\mathbf{P}_2$ acting on a body in the same direction at points A and B (Fig. 9a). The problem is to compose the forces, i.e., to find their resultant.

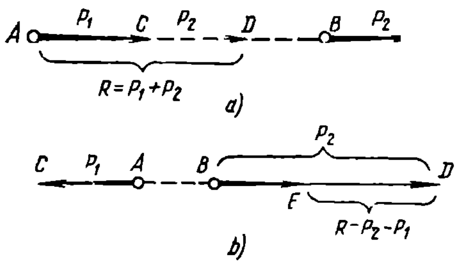


Fig. 9

We know by experiment that the action of two such forces on the mechanical state of a body will be the same as the action of another force equal in magnitude to the sum of the two forces and acting in the same direction. We can find this sum graphically by altering the point of application of force $\mathbf{P}_2$ to the end C of the vector $\mathbf{P}_1$, as shown in the figure. Then the resultant can be represented by the vector $\overline{AD}$. The same result may be obtained if the end of the vector $\mathbf{P}_1$ is transferred to point B. From all of which the resultant

$$R = P_1 + P_2.$$

Or let us take the case of two opposite forces acting along one line (Fig. 9b). Here it is necessary to compose the forces $\mathbf{P}_1$ designated by $\overline{AC}$, and $\mathbf{P}_2$ designated by $\overline{BD}$, into their resultant, the second force being greater than the first ($P_2 > P_1$). Let us assume force $\mathbf{P}_2$ to be the sum of two forces acting in the same direction—the force designated by the vector $\overline{BE}$, which is equal, opposite, and collinear with $\mathbf{P}_1$, and a second designated by $\overline{ED}$. Forces $\overline{AC}$ and $\overline{BE}$ are in equilibrium since they are equal, opposite, and collinear. As a result the two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ are reduced to one force, i.e., to the resultant $\overline{ED}$, which is equal in magnitude to the difference between them and acts in the direction of the greater force:

$$\mathbf{R} = P_2 - P_1.$$

If instead of the resultant of two forces we must find the resultant of more than two forces with a common line of action, then each direction of forces is first summed up and the resultant of the two forces determined as described above.

If it be assumed that the forces acting in one direction are positive and those acting in the other are negative* then their algebraic sum will be the resultant.

Wherefore *the resultant of two or more collinear forces is the algebraic sum of their components.*

Illustrative Problem 2. Find the resultant of the following five collinear forces: $P_1 = 400$ kg, $P_2 = -200$ kg, $P_3 = -350$ kg, $P_4 = 100$ kg, and $P_5 = -175$ kg.

Solution: the resultant $R = P_1 + P_2 + P_3 + P_4 + P_5 = 400 - 200 - 350 + 100 - 175 = -225$ kg, acting in the direction opposite to $\mathbf{P}_1$ and $\mathbf{P}_4$.

Illustrative Problem 3. A man weighing 82 kg is standing on floor scales and pulling vertically on a rope hanging from the ceiling. What force is he exerting on the rope if the scales are registering 45 kg?

Solution: the figure registered on the scales shows the force with which the man is pressing down upon them. This force, which is the resultant of the weight of the man and the force exerted by means of the rope (in opposition to the weight of the man), we shall designate as P.

If we take the forces acting vertically downward as positive and force P as negative (the man is pulling himself upwards), then we have the following equation:

$$82 - P = 45, \quad \text{from which} \quad P = 37 \text{ kg}.$$

17. Constraints and the Reactions of Constraints

Various kinds of motion occur when bodies are acted upon by forces. Most frequently we meet with the motion of a body whose free choice of position in space is restricted by other bodies. Instances of this type of motion are: the movement of a body on the earth's surface, the revolution of a shaft in a bearing, the movement of the carriage of a lathe along the guides of its bedway, etc.

This kind of motion of a body is called *restricted motion.* The conditions restricting the motion of a body are called *constraints.*

The action on a body by other bodies exercising constraint is measured by a force called the *reaction of the constraint.*

In Fig. 10 the ball rolling down the inclined plane under the force of gravity is subject to the reaction of that plane, designated by force **N** perpendicular to the plane KM.

Or, take a beam lying freely on two supports (Fig. 11). The weight of the beam exerts pressure on both supports, while

* The choice of which forces to designate as negative and which as positive varies with each case and has no influence on the final result.

the supports in their turn exert opposing forces $\mathbf{Q}_1$ and $\mathbf{Q}_2$ on the beam and are called *the reactions at the supports.*

It is important to stress the following: *the force exerted by a body on a support is applied to a point on the support, whereas the reactive force of the support is applied at a point on the body, with the result that the two forces have different points of application.*

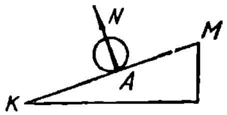


Fig. 10

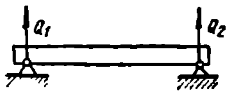


Fig. 11

So is it in the case of the ball on the inclined plane when the pressure of the ball is exerted at a point on the plane and the reactive force is applied to a point on the ball. The two points are contiguous at point A. In the same way the points of application of pressure of a beam are on the supports, whereas the points of application of the reactions are on the beam.

Oral Exercise

A ball weighing 2 kg is resting on a horizontal plate. What is the direction of the reaction of the plate, where is its point of application, and what is its magnitude?

18. Questions for Review

1. What do we call a force in mechanics and how does it manifest itself?
2. Is there any difference between the *line of action of a force* and the *direction of a force*?
3. In what sequence should the elements designating a force be delineated on a drawing?
4. What quantity is a vector?
5. What is the difference between the resultant of a system of forces and equilibrant?
6. What is meant by the expression *the algebraic sum of forces*?
7. What is the difference between the force that a body exerts on a support and the reaction at the support?
8. What is the direction of the reactions exerted on the wheels of a locomotive at rest?

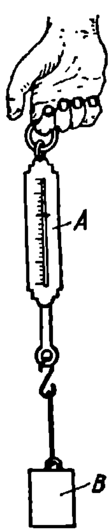


Fig. 12

19. Exercises

1. Load B, weighing 3.5 kg (Fig. 12) and attached by a cord to dynamometer A, is lowered to a table in such a way that the cord remains taut. The pointer on the dynamometer registers 1.5 kg. To what is the reactive force of the table applied, what is its direction, and what is its magnitude?

2. A man is standing on floor scales and pulling on a rope suspended from a dynamometer fastened to the ceiling. The dynamometer registers 15 kg while the scales on which the man is standing show 65 kg. Find 1) the weight of the man and 2) the magnitude and directions of the reactions exerted on the man by the platform of the scales and by the rope.

CHAPTER II

COPLANAR SYSTEMS OF CONCURRENT FORCES

20. Finding the Resultant of Two Forces Acting at an Angle

So far we have investigated systems of forces having a single line of action and whose resultant is collinear with these forces.

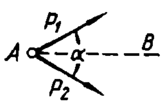


Fig. 13

Now we shall take up the question of finding the resultant of two forces whose lines of action intersect at a point and form an angle.

The simplest system of forces acting at an angle is obtained when the forces are of equal magnitude as shown in Fig. 13, where forces P_1 and P_2 are equal and are acting at an angle α. Since there is no reason for the resultant to be directed closer to one force than to the other, it should bisect the angle. If the

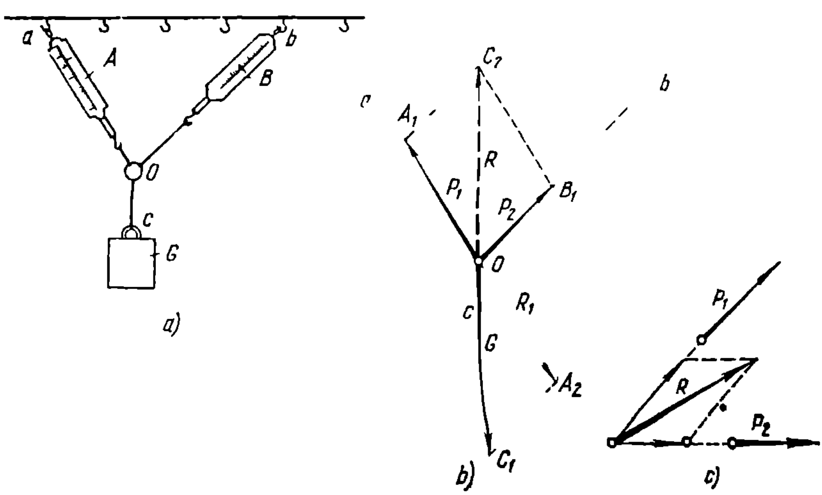


Fig. 14

forces were of different magnitudes, the resultant would remain in the same plane and make a different division of the angle formed by the lines of action of the forces.

Let us make the following experiment.

Suspend two spring dynamometers A and B from hooks in a stationary horizontal bar as shown in Fig. 14a. Tie their hooks to ring O with cords of any length. Then hang load **G** from the ring by a third cord. The cords will become taut, the load will swing and the pointers of the dynamometers will shift back and forth. Finally the dynamometers and the weight will stop fluctuating and the whole system will reach a state of equilibrium.

Now let us see what forces are acting on the ring O. The spring of dynamometer A is acting on it from the left, and the spring of dynamometer B from the right. The weight **G** of load c is acting on it directly downwards. And since the system is in a state of rest, the three forces are in equilibrium.

Now let us take a sheet of cardboard, place it behind this system of forces, and with a pencil mark points a and b the points from which dynamometers A and B are hung--and the centre of the ring O, and also draw a straight line Oc designating the position of the cord from which the load is hung.

Now we can delineate on the cardboard all the forces acting on the ring and meeting at its centre. First we draw lines Oa, Ob and Oc to represent the lines of action of the forces acting on the ring (Fig. 14b), the weight of the load is known and the magnitude of the two other forces is taken from the dynamometers. By choosing a suitable scale we can lay off corresponding lengths from point O along the three lines and add arrows showing the directions along which the forces are acting. As a result there will be three forces designated on the cardboard: $\mathbf{P}_1$, $\mathbf{P}_2$ and **G**, expressed by the vectors OA_1, OB_1, and OC_1.

These three forces are in equilibrium. Assuming that force **G** is the equilibrant of forces $\mathbf{P}_1$ and $\mathbf{P}_2$, we then delineate vector OC_2, which represents force **R**, equal in magnitude to force **G** and acting in the opposite direction. In accordance with Sec. 12, the force **R** is the resultant of $\mathbf{P}_1$ and $\mathbf{P}_2$.

If we draw a straight line extending from A_1 (the end of the vector of force $\mathbf{P}_1$) parallel to the vector of force $\mathbf{P}_2$, and another line from B_1 parallel to the vector of force $\mathbf{P}_1$ (the end of the vector of force $\mathbf{P}_2$) we will find that these two lines intersect at point C_2, that is, at the end of the vector of the resultant **R**. Hence, by thus constructing a parallelogram of the vectors $\overline{OA}_1$, and $\overline{OB}_1$ of the forces $\mathbf{P}_1$ and $\mathbf{P}_2$, we obtain their resultant **R** in magnitude and direction, designated by the diagonal OC_2 of the parallelogram.

Such a parallelogram is called a *parallelogram of forces.*

Wherefore the principle for the composition of two forces acting at an angle may be stated as follows: ***the resultant of two forces with a common point of application and acting at an angle*** *is equal in magnitude and direction to the diagonal of a parallelogram constructed with the two forces as adjacent sides.*

In the above instance the component forces present a common point of application (the centre of the ring shown in Fig. 14*a*). But if the forces are exerted at different points, they may be shown as applied to a point where these lines intersect and a parallelogram with their vectors forming adjacent sides may be constructed as shown in Fig. 14*c*.

The resultant obtained in this way is called *the geometric (or vectorial) sum of component forces.*

This principle of the parallelogram is also employed in the composition of other vector quantities acting at an angle.

It will be recalled from geometry that any side of a triangle is less than the sum of the other two sides and larger than their difference. By applying this to the resultant in Fig. 14*b* we obtain

$$P_1 + P_2 > R > P_1 - P_2.$$

When the angle under which forces are acting is changed, their resultant will also change: if the angle decreases, the resultant will increase and vice versa; with an angle of 0°, the two components will have both the same line of action and of direction and their resultant will be $P_1 + P_2$, whereas at an angle of 180° their resultant will be $P_1 - P_2$. With these extreme angles between component forces, their geometric sum becomes their algebraic sum. This means that the principle used in the composition of two collinear forces is part of the principle of the parallelogram of forces.

Illustrative Problem 4. Two forces $\mathbf{P}_1$ and $\mathbf{P}_2$ of equal magnitude are acting at an angle of 120° Find their resultant (Fig. 15).

Solution: the parallelogram constructed on the vectors of these forces is a rhombus, for which reason the diagonal OC bisects angles AOB and ACB. Therefore $\angle AOC = \angle COB = 60° = \angle ACO = \angle OCB$.

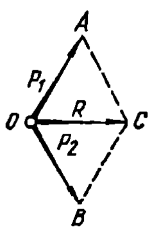


Fig. 15

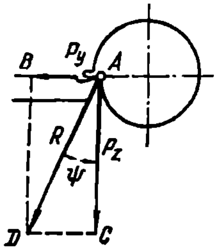


Fig. 16

It follows that the resultant forms an angle of 60° with each of the component forces. Furthermore it is easy to see that the triangles OAC and OBC are equilateral, which means that $OC = OA = OB$. Hence the resultant in this case is equal to each of the components.

Illustrative Problem 5. A groove is being cut in a workpiece machined on a lathe (Fig. 16). It has been determined with a dynamometer that radially acting force $P_y = AB = 55$ kg, and vertically acting force $P_z = AC = 93$ kg. Find the magnitude and the direction of the resultant **R** $= AD$.

Solution: since the components are perpendicular to one another, the parallelogram constructed on their vectors will be a rectangle and its diagonal can be determined by the Pythagorean Theorem:

$$R = \sqrt{55^2 + 93^2} = 108 \text{ kg}.$$

From the triangle ADC we obtain $DC = AC \tan \varphi$, therefore

$$\tan \varphi = \frac{DC}{AC} = \frac{55}{93} = 0.592 \text{ and the angle } \varphi = 30°36'$$

21. Resolving a Force into Two Components Applied at One Point and Acting at an Angle

The reverse of the composition of forces is called the *resolution*, of a force into its components. *To resolve a force into two components signifies finding two forces whose combined action will be the same as the given force*, i.e., *finding two forces whose resultant will be equal to the given force.*

It can be easily found that such a problem may have an infinite number of solutions. Let us assume it necessary to resolve the force $\mathbf{Q} = \overline{OA}$ into two components (Fig. 17). By drawing two lines, OK and OL, through the point of application O, and lines AC and AB parallel to them, from point A, we obtain a parallelogram $OBAC$ from which we see that forces OB and OC are components of force **Q**. But other lines of action could likewise be taken for the components—say OM and ON. Then we would obtain the parallelogram $ODAE$, from which it follows that the force **Q** may be the resultant of two other forces OD and OE.

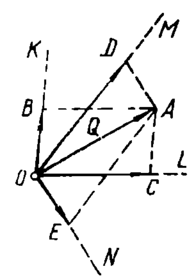


Fig. 17

Therefore the problem as it is stated is indeterminate and additional conditions must be included to obtain a single solution for each specific case, as shown in the following examples.

1. Resolve force **Q** (Fig. 18*a*) into two components $\mathbf{P}_1$ and $\mathbf{P}_2$, whose lines of action MN and ST intersect with the action line of **Q** at point O.

By altering the point of application of force **Q** to point O we obtain vector $\overline{OC} = \mathbf{Q}$. Then we construct a parallelogram $OACB$ on that vector by delineating two lines from point C, that is, CB parallel to MN and CA parallel to ST. The resulting vectors $\overline{OA}$ and $\overline{OB}$ will designate the sought component forces $\mathbf{P}_1$ and $\mathbf{P}_2$.

2. Resolve force **Q** (Fig. 18*b*) into two components of which one, $\mathbf{P}_1$, is defined in magnitude and direction.

We extend the lines of action of the two given forces to their point of intersection O and construct vectors $\overline{OA} = \mathbf{P}_1$ and

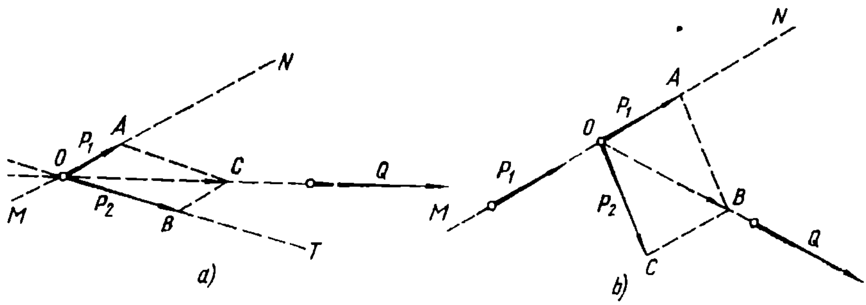


Fig. 18

$\overline{OB} = \mathbf{Q}$. We then connect B and A and delineate $OC \parallel AB$ and $BC \parallel MN$, thus obtaining the desired second component $\mathbf{P} = OC$.

Oral Exercise

Can a given force be resolved into components, each of which acts at a right angle to it?

Illustrative Problem 6. A vertical load P of 1,200 kg is supported by a triangular bracket ABC. Find the forces acting on AB and BC if $AB = 600$ mm and $AC = 800$ mm (Fig. 19).

Solution: the forces acting on the indicated elements are components of force **P** and directed along these elements. By constructing the parallelogram $BEDF$ on vector $\overline{BD}$, which represents force **P**, we obtain both components $\mathbf{P}_1$ and $\mathbf{P}_2$. The first is directed from point B towards point E, that is, from A along the support AB and tends to stretch the latter. The second component $\overline{BF}$ is directed towards point C, that is, towards the point which fixes the element BC and therefore tends to compress that element.

By measuring the two components by the same scale as used in designating force **P**, we can find their magnitudes.

These magnitudes can also be found by calculation, as follows.

Since triangles ABC and BED are similar, then

$$\frac{BE}{BD} = \frac{P_1}{P} = \frac{AB}{AC} = \frac{600}{800} = \frac{3}{4},$$

hence

$$P_1 = \frac{3}{4}P = \frac{3}{4} \times 1{,}200 = 900 \text{ kg}.$$

Furthermore,

$$\frac{ED}{BD} = \frac{P_2}{P} = \frac{BC}{AC} = \frac{\sqrt{600^2 + 800^2}}{800} = \frac{1{,}000}{800} = \frac{5}{4}$$

and

$$P_2 = \frac{5}{4} P = \frac{5}{4} \times 1{,}200 = 1{,}500 \text{ kg}.$$

Illustrative Problem 7. Slide-block K (Fig. 20) which is subject to the action of force **P**, is moving along a straight horizontal bar AB at an even speed. What reaction does the bar exert on the slide-block?

Solution: the reaction we seek is perpendicular to the bar, i.e., it is directed from it vertically upwards. We resolve force **P** into two components $\bar{CE}$ and $\overline{CF}$, of which one is perpendicular, and the other parallel to AB. The parallelogram of forces $CEDF$ so formed is a rectangle, of which side $\overline{CE}$ represents the downward pressure of the slide-block on the bar, and side $\overline{CF}$ represents the force acting in the same direction as the movement of the slide block*.

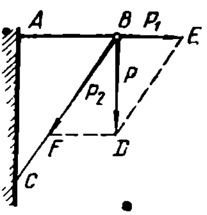


Fig. 19

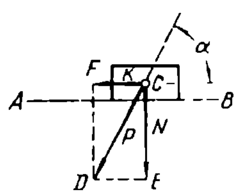


Fig. 20

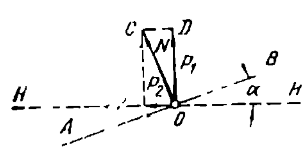


Fig. 21.

Assuming that force $P = 80$ kg and the angle α which it forms with the bar is equal to 60°, then $\angle DCE = 90° - \alpha = 30°$.

From the right triangle DCE we derive

$$CE = CD \cos 30° = 80 \times 0.866 = 69.3 \text{ kg},$$

hence the reaction sought is equal to 69.3 kg and acting opposite to force **N**.

Illustrative Problem 8. In Fig. 21, line AB represents the axis of the cross section of the wing of an airplane travelling horizontally along the line HH_1. The pressure of the air **N** perpendicular to the wing is designated by the vector $\overline{OC}$. Find the lifting power that the air exerts on the plane.

Solution: By resolving **N** into two components, we obtain vertical $\mathbf{P}_1 = \overline{OD}$ and horizontal $\mathbf{P}_2 = \overline{OE}$. The first gives the magnitude of the vertical pressure of the air and is equal to the lifting power of the plane.

22. The Composition of Several Forces Lying in One Plane and Intersecting at One Point

The principle of the parallelogram can also be employed to solve problems involving more than two component forces. By moving any two of the forces along their lines of action so as to have a common point of application and constructing a parallelogram of forces, their resultant $\mathbf{R}_1$ can be found. Then the next force is moved to the common point of action and

* This force is in equilibrium due to the force of friction induced on the surface of contact between the slide-block and the rod.

combined with resultant **R** to obtain $\mathbf{R}_2$. Thus we continue with the rest of the forces until finally by combining the last component with the resultant of all the other forces we obtain the resultant of the whole system. The order in which we combine the forces will have no influence on the final result, but of course the partial resultants will differ.

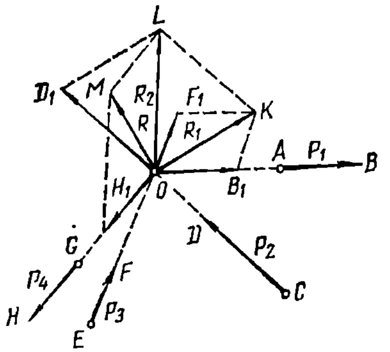


Fig. 22

Fig. 22 shows a system of four meeting forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ whose lines of intersecting action are at point O. By transferring the points of application $\mathbf{P}_1$ and $\mathbf{P}_3$ to this same point and constructing a parallelogram OB_1KF_1, we obtain our first partial resultant $\mathbf{R}_1 = OK_1$. This we combine with force $\mathbf{P}_2$, which we have also transferred to O, and obtain the second partial resultant $\mathbf{R}_2 = OL$. Finally, by constructing parallelogram $OLMH_1$ on vectors OL and OH_1 of forces $\mathbf{R}_2$ and $\mathbf{P}_4$, we obtain the resultant $\mathbf{R} = OM$ for the whole system of forces.

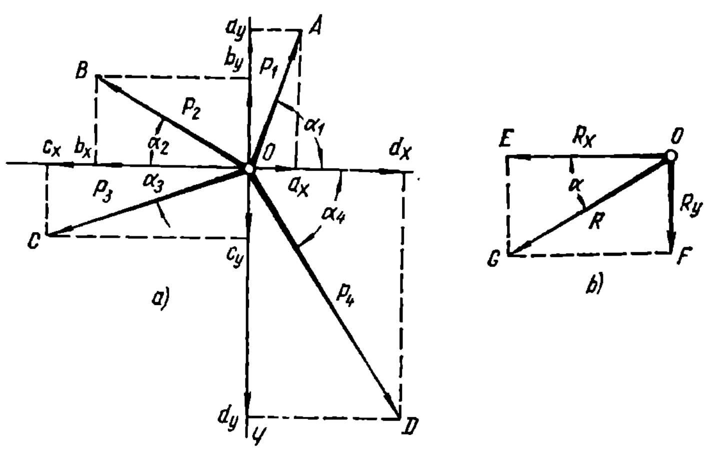


Fig. 23

The forces need not necessarily be combined in this order. We could first have combined $\mathbf{P}_1$ and $\mathbf{P}_2$ and then combined $\mathbf{P}_4$ with their resultant, etc., and the final result would have been the same.

There are other methods of combining concurrent forces. Let us take four forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ (Fig. 23).

First delineate two perpendicular coordinates xx and yy through a common point of application O (see Fig. 23a). Then we resolve each of the given forces into two components whose lines of action coincide with axes xx and yy. By connecting the perpendiculars Aa_x and Aa_y from point A (the end of the vector of force $\mathbf{P}_1$) with axes xx and yy, we resolve $\mathbf{P}_1$ into two components Oa_x and Oa_y. The process is then repeated with the other given forces. As a result we obtain four components Oa_x, Od_x, Ob_x, and Oc_x acting along axis xx, and the same number of components Ob_y, Oa_y, Oc_y, and Od_y acting along axis yy. In short, we replace the given forces by these eight forces.

Now we work out the algebraic sum of the forces acting along each of the axes xx and yy. For the forces acting along axis xx we obtain the resultant $R_x = Ob_x + Oc_x - Oa_x - Od_x$, designated by vector OE in Fig. 23b. In exactly the same way we find the resultant of forces acting along axis yy, i.e., $R_y = Oc_y + Od_y - Ob_y - Oa_y$ as designated in Fig. 23b by vector OF. In this way we reduce all the given forces to two, acting at right angles to each other. By constructing the parallelogram $OEGF$, we obtain the desired resultant $\mathbf{R}$ designated by vector OG.

The magnitude and direction of this resultant can be found by calculation without resorting to delineation.

Firstly, designate the angles formed with axis xx by forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, etc., as α_1, α_2, α_3, etc. From triangle AOa_x we obtain $Oa_x = OA \cos\alpha_1 = P_1 \cos\alpha_1$; in the same way we find that $Od_x = P_4 \cos\alpha_4$, $Ob_x = P_2 \cos\alpha_2$, and $Oc_x = P_3 \cos\alpha_3$.

From the same triangles we find each vertical component of the given forces: $Oc_y = OC \sin\alpha_3 = P_3 \sin\alpha_3$, $Od_y = P_4 \sin\alpha_4$, $Ob_y = P_2 \sin\alpha_2$, and $Oa_y = P_1 \sin\alpha_1$. Hence,

$$R_x = P_2 \cos\alpha_2 + P_3 \cos\alpha_3 - P_1 \cos\alpha_1 - P_4 \cos\alpha_4;$$

and $$R_y = P_3 \sin\alpha_3 + P_4 \sin\alpha_4 - P_2 \sin\alpha_2 - P_1 \sin\alpha_1.$$

The magnitude of resultant $\mathbf{R}$ is determined from triangle OEG as the hypotenuse of a right triangle:

$$R = \sqrt{R_x^2 + R_y^2}.$$

Angle α formed by this resultant with axis xx is determined from the same triangle: $R_y = R_x \tan\alpha$, from which $\tan\alpha = \frac{R_y}{R_x}$.

In general, the resultant of a system of n forces meeting at one point is determined by the following equations:

$$\left.\begin{aligned} R_x &= P_1\cos\alpha_1 + P_2\cos\alpha_2 + P_3\cos\alpha_3 + \ldots + P_n\cos\alpha_n \\ R_y &= P_1\sin\alpha_1 + P_2\sin\alpha_2 + P_3\sin\alpha_3 + \ldots + P_n\sin\alpha_n \end{aligned}\right\} \quad (1)$$

$$R = \sqrt{R_x^2 + R_y^2}, \quad (2)$$

$$\tan\alpha = \frac{R_y}{R_x}. \quad (3)$$

Wherefore, *the resultant of a system of coplanar concurrent forces is equal to the geometric sum of two forces, each of which is the algebraic sum of the components of the system along two axes intersecting at right angles.*

The tangent of the angle formed by a resultant and a horizontal component is equal to the ratio of the algebraic sum of the vertical components to the algebraic sum of the horizontal components.

Oral Exercises

1. If $R_x = 0$, and $R_y \neq 0$, what is the magnitude and direction of the resultant?
2. If $R_x = R_y$, what will be the direction of the resultant?

Illustrative Problem 9. Given a system of concurrent forces in which $P_1 = 20$ kg, $P_2 = 25$ kg, $P_3 = 30$ kg, and $P_4 = 40$ kg (Fig. 23). The angles these forces form with the horizontal axis are $\alpha_1 = 70°$, $\alpha_2 = 30°$, $\alpha_3 = 20°$ and $\alpha_4 = 60°$, from which the resultant of these forces must be found.

Solution: R_x and R_y are calculated through Equations (1):

$$R_x = -20 \cos 70° + 25 \cos 30° + 30 \cos 20° - 40 \cos 60° =$$
$$= -20 \times 0.342 + 25 \times 0.866 + 30 \times 0.940 - 40 \times 0.5 = 23 \text{ kg.}$$
$$R_y = -20 \sin 70° - 25 \sin 30° + 30 \sin 20° + 40 \sin 60° =$$
$$= -20 \times 0.94 - 25 \times 0.5 + 30 \times 0.342 + 40 \times 0.866 = 13.6 \text{ kg.}$$

The magnitude of the resultant, according to Equation (2),

$$R = \sqrt{23^2 + 13.6^2} = 26.7 \text{ kg.}$$

and the angle α which it forms with the horizontal axis

$$\tan \alpha = \frac{13.6}{23} = 0.591, \text{ whence we obtain } \alpha = 30°36'.$$

23. Equilibrium of a System of Coplanar Forces Intersecting at One Point

As we have already learnt above, a system of forces is in equilibrium if each of its forces is the equilibrant of all the others, that is, if each of its forces is equivalent to, and opposing, the resultant of the other forces. In combining forces according to the method shown in Fig. 22 we obtain a resultant $\overline{OL}$ equal in magnitude and opposite in direction to force $\mathbf{P}_4$ and the general resultant is zero, i.e., the system of forces would be in equilibrium.

For a system of concurrent forces to be in equilibrium, their resultant must be zero.

Let us see what conditions must be satisfied for this to hold true.

Let us return to Fig. 23. As we have already seen, resultant **R** can be defined as the geometric sum of the forces $\mathbf{R}_x$ and $\mathbf{R}_y$, each of which is the algebraic sum of the components of the system obtained by resolving its forces along axes xx and yy.

Therefore if the system is to be in equilibrium the condition $\sqrt{R_x^2 + R_y^2} = 0$ must be satisfied. Since R_x^2 and R_y^2 are always positive quantities, this will be the case only if $\mathbf{R}_x$ and $\mathbf{R}_y$ are each equal to zero.

Hence by employing Equations (1) we obtain the following conditions for the coplanar system of convergent forces to be in equilibrium:

$$R_x = P_1 \cos\alpha_1 + P_2 \cos\alpha_2 + P_3 \cos\alpha_3 + \ldots + P_n \cos\alpha_n = 0 \quad (4)$$

$$R_y = P_1 \sin\alpha_1 + P_2 \sin\alpha_2 + P_3 \sin\alpha_3 + \ldots + P_n \sin\alpha_n = 0. \quad (5)$$

Wherefore *in order that a coplanar system of convergent forces be in equilibrium, it is necessary and sufficient that each algebraic sum of the components of those forces along two perpendicular axes be equal to zero.*

Illustrative Problem 10. Given a coplanar system of convergent forces (Fig. 24*a*) in which $P_1 = 23$ kg, $P_2 = 27.5$ kg, $P_3 = 21.3$ kg, $P_4 = 30$ kg, and $P_5 = 30$ kg. The angles formed by their lines of action with the horizontal axis *xx* passing through the point of intersection of the forces are respectively $\alpha_1 = 26°$, $\alpha_2 = 68°$, $\alpha_3 = 15°$, $\alpha_4 = 59°$ and $\alpha_5 = 31°$. Determine whether the system is in equilibrium.

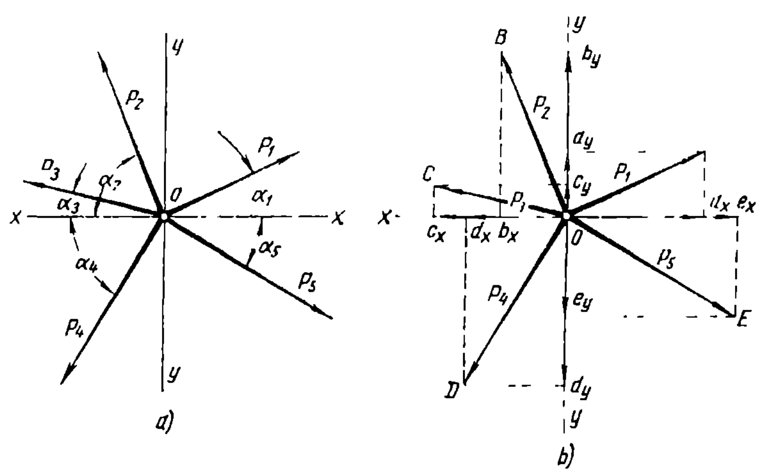


Fig. 24

Solution: by applying Equations (4) and (5) we obtain

$R_x = P_1 \cos 26° - P_2 \cos 68° - P_3 \cos 15° - P_4 \cos 59° +$
$+ P_5 \cos 31° = 23 \times 0.899 - 27.5 \times 0.375 - 21.3 \times 0.966 -$
$- 30 \times 0.515 + 30 \times 0.857 = 20.68 - 10.31 - 20.58 - 15.45 +$
$+ 25.71 = 46.39 - 46.34 = 0.05$ kg ≈ 0.
$R_y = P_1 \sin 26° + P_2 \sin 68° + P_3 \sin 15° - P_4 \sin 59° - P_5 \sin 31°$
$= 23 \times 0.438 + 27.5 \times 0.927 + 21.3 \times 0.259 - 30 \times 0.857 -$
$- 30 \times 0.515 = 10.07 + 25.49 + 5.52 - 25.71 - 15.45 = 41.16 -$
$- 41.00 = 0.08$ kg ≈ 0.

Therefore the system is in equilibrium*.

In Fig. 24*b* this problem is solved graphically (the scale used is 1 kg = 1 mm). As may be seen from the drawing, $R_x = 0$ and $R_y = 0$.

24. Lines of Action of Three Non-Parallel Balanced Coplanar Forces that Intersect at One Point

Fig. 25 represents a body under the action of three balanced forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. Since any one of these forces is the equilibrant of the other two, it must be equal and opposite to the resultant of the other two forces. If we transfer forces $\mathbf{P}_1$ and $\mathbf{P}_2$ to their point of intersection O and find their resultant $\overline{OC}$, we may see that force $\mathbf{P}_3$ must have the same line of action as resultant $\mathbf{R}$; in other words, it must pass through the same point O at which $\mathbf{P}_1$ and $\mathbf{P}_2$ intersect. Wherefore, *if three non-parallel forces lying in one plane are in equilibrium, their lines of action will intersect at one point.*

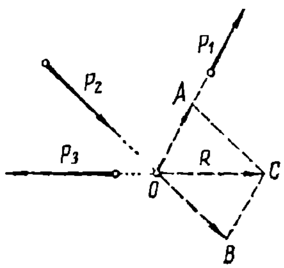


Fig. 25

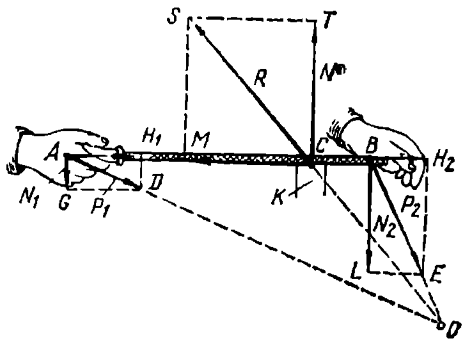


Fig. 26

Illustrative Problem 11. A man is filing a workpiece *K* held in a vise (Fig. 26). In order that the file move evenly, his hands exert forces $\mathbf{P}_1$ and $\mathbf{P}_2$ at each of its ends (points *A* and *B*) and thus overcome resistance $\mathbf{R}$ of the workpiece. In short, forces $\mathbf{P}_1$ and $\mathbf{P}_2$ compensate force $\mathbf{R}$, and all three forces meet at point *O*.

25. Questions for Review

1. What system of forces is a concurrent system?
2. Is there only one solution to a problem in which a force is to be resolved into two components if either the magnitude or the direction of one of the components is given?
3. What is the answer to Question 2 if both the magnitude and direction of one of the components are given?
4. What is the answer to the same question if the magnitude of one of the components and the direction of the other are given?

---

* Since our calculations have been approximate, we may neglect the quantities 0.05 and 0.08 kg, inasmuch as they are of no importance as compared with the forces given.

5. After the successive composition of forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, respectively (Sec. 22, Fig. 22) by means of the principle of the parallelogram, it was found that the system was in equilibrium. What is the position of the line of action of resultant $\mathbf{R}$, and what is its magnitude?

6. The system of forces in Fig. 25 is in equilibrium. By constructing a parallelogram of forces $\mathbf{P}_2$ and $\mathbf{P}_3$, prove that force $\mathbf{P}_1$ is their equilibrant.

7. Will a system of forces be in equilibrium if only one of the Equations (4) and (5) is satisfied?

26. Exercises

3. Show five forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, and $\mathbf{P}_5$ with a common point of application and whose lines of action form angles of 30°, 120°, 270°, 300°, and 330°, respectively, with the horizontal axis (lay out the angles counterclockwise). The magnitudes of the forces are $P_1 = 150$ kg, $P_2 = 200$ kg, $P_3 = 120$ kg, $P_4 = 180$ kg, and $P_5 = 80$ kg.

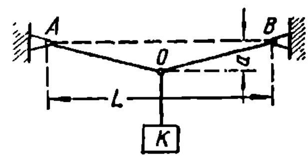


Fig. 27

4. Find the resultant of two forces $\mathbf{P}_1$ and $\mathbf{P}_2$, when each is 100 kg and their lines of action intersect at right angles.

5. Fig. 27 represents a cable fastened at points A and B with load K of weight $\mathbf{G}$ suspended from it in the middle. Calculate the force $\mathbf{P}$ exerted on each half of the cable if $L = 5$ m, $a = 600$ mm, and the weight G of load $K = 100$ kg.

6. The workpiece A in Fig. 28 is being machined lengthwise on a lathe by cutter B. A perpendicular force $\overline{OC}$ of 127 kg designated by $\mathbf{N}$ is acting on the cutting edge. Find forces $\mathbf{P}_x$ and $\mathbf{P}_y$ acting on the cutter in the direction of the axis of the workpiece

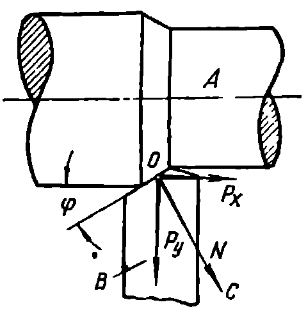


Fig. 28

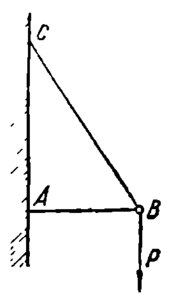


Fig. 29

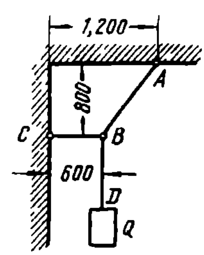


Fig. 30

and also perpendicularly to it if angle φ between the cutting edge and the workpiece (called the main angle in plan) equals 35°.

7. Find the forces acting on supports AB and BC of the triangular bracket in Fig. 29 if $AB = 800$ mm, $AC = 1{,}200$ mm, and $P = 900$ kg.

8. Load Q weighing 200 kg (Fig. 30) is suspended from joint B between two bars fastened on hinges A and C. Find the magnitude and direction of the forces acting on the two bars.

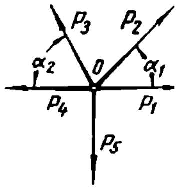


Fig. 31.

9. In Fig. 31 five forces are applied at point O: $P_1 = 20$ kg, $P_2 = 40$ kg, $P_3 = 30$ kg, $P_4 = 33.3$ kg, and $P_5 = 53.3$ kg. Angles $\alpha_1 = 45°$ and $\alpha_2 = 60°$. The lines of action of forces $\mathbf{P}_1$ and $\mathbf{P}_4$ coincide and are horizontal and force $\mathbf{P}_5$ is directed vertically downward. Find the resultant.

CHAPTER III

COPLANAR PARALLEL FORCES, AND THE MOMENT OF A FORCE

27. Composition of Parallel Forces Acting in One Direction

The principle of the parallelogram obviously cannot be applied to the composition of parallel forces. To arrive at a principle for the composition of two parallel forces we must replace them by two intersecting forces having the same action as the given forces.

Let us assume we are to find the resultant of the two parallel forces $\mathbf{P}_1$ and $\mathbf{P}_2$ in Fig. 32.

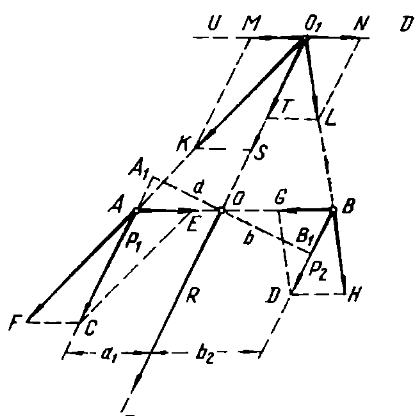


Fig. 32

We connect the points of application A and B of the two forces with line AB and resolve $\mathbf{P}_1$ into two arbitrary components $\overline{AE}$ and $\overline{AF}$, of which the first is directed along line AB. We then resolve force $\mathbf{P}_2$ in such a way as to make its component $\overline{BG}$ also act along line AB and have the same magnitude as force $\overline{AE}$. Then its second component $\overline{BH}$ will be fully determinate. Since the components $\overline{AE}$ and $\overline{BG}$ have been constructed equal in magnitude and acting in opposite directions, they will therefore balance each other, it then follows that forces $\mathbf{P}_1$ and $\mathbf{P}_2$ can be replaced by forces $\overline{AF}$ and $\overline{BH}$ which are acting at an angle.

We transfer the points of application of these two forces to their intersection at O_1, that is, we lay out $\overline{O_1K} = \overline{AF}$ and $\overline{O_1L} = \overline{BH}$. Then we resolve each of these forces in two directions: UD parallel to AB, and O_1Z parallel to the lines of action of the given forces $\mathbf{P}_1$ and $\mathbf{P}_2$. As a result we obtain four forces $\overline{O_1M}$, $\overline{O_1N}$, $\overline{O_1S}$, $\overline{O_1T}$.

It is apparent that triangles O_1MK and EAC are equal to each other because $O_1K = AF$ and the adjacent angles are equal. From this it follows that $MK = AC$, and since $MK = O_1S$, then $O_1S = AC$, i.e., $\overline{O_1S} = \mathbf{P}_1$. And since the triangles are equal, it also follows that $\overline{O_1M} = \overline{AE}$.

In the same way we can prove that $\overline{O_1T} = \mathbf{P}_2$ and $\overline{O_1N} = \overline{BG}$. Therefore, since $\overline{O_1M}$ and $\overline{O_1N}$ are equal and opposite, they are in equilibrium. Accordingly, the resultant of forces $\overline{O_1K} = \overline{AF}$ and $\overline{O_1L} = \overline{BH}$ can be expressed by the algebraic sum of the forces $\overline{O_1S}$ and $\overline{O_1T}$, and since these two forces are equal in magnitude, respectively, to the given forces $\mathbf{P}_1$ and $\mathbf{P}_2$, the resultant of forces $\overline{AF}$ and $\overline{BD}$ is expressed by $P_1 + P_2$. However, the action of forces $\overline{AF}$ and $\overline{BD}$ is the same as the action of forces $\mathbf{P}_1$ and $\mathbf{P}_2$. Hence we conclude that the resultant of forces $\mathbf{P}_1$ and $\mathbf{P}_2$ is equal to their sum and we have therefore determined its magnitude. Now it remains for us to find the position of point O at which the line of action of resultant O_1Z and line AB intersect. Since triangle O_1AO and O_1KS are similar, it follows that $\frac{OA}{KS} = \frac{O_1O}{O_1S}$; or if we consider that O_1S represents the magnitude of force $\mathbf{P}_1$, we then obtain $\frac{OA}{KS} = \frac{O_1O}{P_1}$.

In the same way we obtain the proportion

$$\frac{OB}{TL} = \frac{O_1O}{P_2}.$$

By dividing the first proportion by the second and bearing in mind that $KS = TL$, we obtain

$$\frac{OA}{OB} = \frac{P_2}{P_1}.$$

If we designate a_1 as the length of OA adjacent to the point of application of $\mathbf{P}_1$, and b_1 as the length of OB, the proportion will become

$$\frac{a_1}{b_1} = \frac{P_2}{P_1}.$$

We then delineate line A_1B_1 through point O perpendicular to the lines of action of the given forces and designate a as segment OA_1 and segment OB_1 as b. Since triangles OAA_1 and OBB_1 are similar, we obtain $\frac{a}{b} = \frac{a_1}{b_1}$; hence, $\frac{a}{b} = \frac{P_2}{P_1}$, whence it follows that

$$P_1a = P_2b.$$

Wherefore, *the resultant of two parallel forces having the same direction is equal to their sum, is parallel to them, and acts in the same direction,* i.e.,

$$R = P_1 + P_2. \qquad (6)$$

The point of application of the resultant divides the line that connects the points of application of the given forces into a ratio inversely proportional to their magnitudes.

In other words, the line of action of the resultant passes between the lines of action of the component forces at distances inversely proportional to their magnitude, i.e.,

$$\frac{a_1}{b_1} = \frac{P_2}{P_1}, \qquad (7)$$

$$\frac{a}{b} = \frac{P_2}{P_1}. \qquad (8)$$

By multiplying the means and the extremes of Eqs (7) and (8) we obtain

$$P_1 a_1 = P_2 b_1,$$

$$P_1 a = P_2 b. \qquad (8a)$$

For some problems Equation (8) may be presented in a more convenient form by expressing it as a derivative proportion: $\frac{a+b}{b} = \frac{P_2 + P_1}{P_1}$.

By exchanging the means, we obtain $\frac{a+b}{P_2+P_1} = \frac{b}{P_1}$.

An exchange of the means in Equation (8) results in

$$\frac{a}{P_2} = \frac{b}{P_1}.$$

Consequently $\frac{a+b}{P_2+P_1} = \frac{b}{P_1} = \frac{a}{P_2}$, and since $P_1 + P_2$ is equal to the resultant **R**, then $\frac{a+b}{R} = \frac{b}{P_1} = \frac{a}{P_2}$.

Oral Exercises

1. Does the derivation of Eqs (6) and (8) depend on the magnitudes of forces $\overline{AE}$ and BG (Fig. 32)?

2. What will be the position of point O, through which the line of action of the resultant passes (Fig. 32), if $\mathbf{P}_1$ and $\mathbf{P}_2$ are equal in magnitude?

28. Composition of Parallel Forces Acting in Opposite Directions

Assume it necessary to find the resultant of two parallel forces $\mathbf{P}_1$ and $\mathbf{P}_2$ (Fig. 32*a*) acting in opposite directions when $P_1 > P_2$. Solution: we replace force $\mathbf{P}_1$ by two forces—force

$\mathbf{P}_2'$ applied at point B and equal in magnitude to force $\mathbf{P}_2$ but acting in the opposite direction, and force $\mathbf{R}$ which is equal in magnitude to the difference $P_1 - P_2$ and applied at the point determined by the ratio $\frac{OA}{AB} = \frac{P_2}{R}$. Under these conditions we may regard force $\mathbf{P}_1$ as the resultant of the two parallel forces $P_1 - P_2$ and $\mathbf{P}_2'$. However, force $\mathbf{P}_2'$ being the equilibrant of force $\mathbf{P}_2$, it follows that the given system of forces has been reduced to the one force $R = P_1 - P_2$. Hence this is the desired resultant. In short, the magnitude of the resultant has been found as

$$R = P_1 - P_2. \qquad (6a)$$

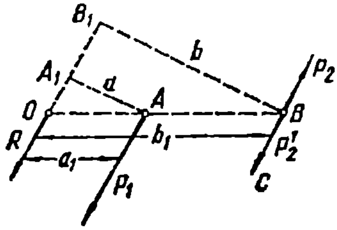


Fig. 32a

Having chosen the point of application of force $\mathbf{R}$ so that the distances OA and AB satisfy the condition

$$\frac{OA}{AB} = \frac{P_2'}{R},$$

we convert this condition into a derivative proportion and obtain

$$\frac{OA}{AB + OA} = \frac{P_2'}{R + P_2'}.$$

By taking a_1 to represent OA, and b_1 to represent OB and bearing in mind that force $\mathbf{P}_2'$ is equal in magnitude to force $\mathbf{P}_2$ and that $\mathbf{R}$ is equal to the difference $P_1 - P_2$, we obtain

$$\frac{a_1}{b_1} = \frac{P_2}{P_1 - P_2 + P_2} \quad \text{or} \quad \frac{a_1}{b_1} = \frac{P_2}{P_1}.$$

By extending the perpendiculars $AA_1 = a$ and $BB_1 = b$ from points A and B to the line of action of resultant $\mathbf{R}$, we obtain similar triangles AA_1O and BB_1O, from which it follows that

$$\frac{a_1}{b_1} = \frac{a}{b},$$

and Equation (7) may then be expressed as

$$\frac{a}{b} = \frac{P_2}{P_1},$$

from which

$$P_1a = P_2b.$$

Wherefore, *the resultant of two parallel forces acting in opposite directions is equal to their difference, is parallel to them, and acts in the direction of the greater force.*

The line of action of the resultant lies beyond the larger force at distances from the component forces equal to the inverse proportion of these latter forces.

If it is necessary to find the resultant of a system of parallel forces of which some are acting in one direction and the rest in the opposite direction, the simplest method is to first combine those acting in one direction and then those in the other direction, and finally combine their two resultants.

Oral Exercise

As the magnitude of P_2 approaches that of $\mathbf{P}_1$, what will be the position of point O (Fig. 32*a*) through which the resultant **R** passes, provided the points of application of $\mathbf{P}_2$ and $\mathbf{P}_1$ remain the same?

29. Resolution of a Force into Parallel Components

The resolution of a force into two parallel components is just as indeterminate a problem as the resolution of a force into components directed at an angle; additional conditions must be given in each individual case, such as the points of application (or lines of action) of both components, or the point of application (or line of action) and magnitude of one of the components, or the point of application of one of the components and the ratio of their magnitudes, etc. The problem will then become determinate and can be solved by applying the equations used in the two foregoing sections.

Let us assume it necessary to resolve force **P** (Fig. 33) into two components $\mathbf{P}_1$ and $\mathbf{P}_2$ acting in one direction, with the magnitude of force $\mathbf{P}_1$ given. The distance between the action lines of forces **P** and $\mathbf{P}_1$ is equal to a. The magnitude of force $\mathbf{P}_2$ is found from Equation (6):

$$P_2 = P - P_1.$$

The distance x between its line of action and force **P** is found through Equation (8), according to which $P_1a = P_2x$, whence

$$x = \frac{P_1}{P_2} a.$$

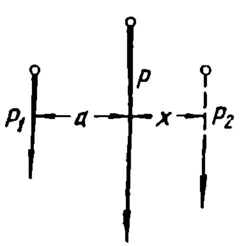


Fig. 33

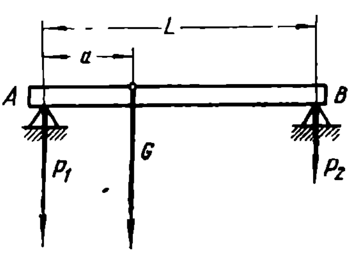


Fig. 34

Illustrative Problem 12. A load G weighing 1,200 kg is travelling along a beam resting on two supports A and B (Fig. 34). Find the bearing loads $\mathbf{P}_1$ and $\mathbf{P}_2$ exerted by the load on each support if it comes to rest at a distance $a = 1{,}500$ mm from the left support. The length of beam $L = 4{,}500$ mm.

Solution: as the load moves, the bearing loads on the supports change and when it is at the extreme left, its entire weight will rest on support A. As it moves from left to right the bearing load on the left support decreases while that on the right increases and at the extreme right the entire weight of the load will be borne by support B.

Therefore force $\mathbf{G}$ is the resultant of the components $\mathbf{P}_1$ and $\mathbf{P}_2$ acting in one direction, and so by applying Eq. (8*a*), we find that, in the given position of the load,

$$P = \frac{Ga}{L} = \frac{1{,}200 \times 1{,}500}{1{,}500} = 400 \text{ kg},$$

and

$$P_1 = 1{,}200 - 400 = 800 \text{ kg}.$$

The reactions at the supports are equal to $\mathbf{P}_1$ and $\mathbf{P}_2$ in magnitude and act in the opposite direction.

30. The Centre of Coplanar Parallel Forces

Given a system of coplanar parallel forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ (Fig. 35). By combining forces $\mathbf{P}_1$ and $\mathbf{P}_2$ applied at points A and B, we find the first partial resultant $\mathbf{R}_1$ applied at O_1. After connecting point O_1 and the point of application C of force $\mathbf{P}_3$, we combine this force with $\mathbf{R}_1$ and thereby obtain the second partial resultant $\mathbf{R}_2$. By going through the same process with this force and the last component $\mathbf{P}_4$, we obtain the resultant $\mathbf{R}$ of the entire system, with the point of application at O.

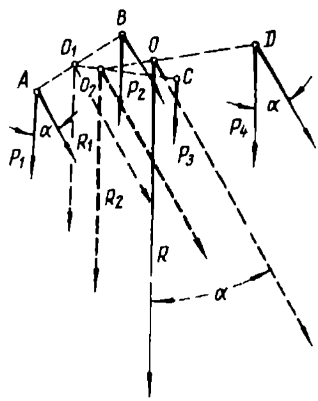


Fig. 35

Now let us assume that all the given forces are rotated about their points of application through a freely-chosen angle α in their plane, as indicated by the dotted lines. Since $\mathbf{P}_1$ and $\mathbf{P}_2$ have not changed in magnitude and their points of application remain the same, the point of application O_1 of resultant $\mathbf{R}_1$ also remains the same. The point of application C of force $\mathbf{P}_3$ likewise remains unchanged, from which it follows that the point O_2 of application of their resultant $\mathbf{R}_2$ is also unchanged. By carrying this line of reasoning to its conclusion we see that the point of application O of resultant $\mathbf{R}$ of all the forces also remains the same.

If all the forces in the system are rotated through the same angle, their resultant will also rotate through the same angle and its magnitude and point of application will remain unaltered.

The point of application of the resultant O is called the *centre of parallel forces.*

Wherefore, *the centre of parallel forces remains unchanged no matter what their direction if they retain their magnitude and points of application.*

31. Moment of a Force in Respect to a Point

We know from experience that it is easiest to get a work-piece gripped in a vise if we apply pressure on the handle as far as possible from the axis of rotation O of the screw (Fig. 36). A greater force will have to be applied at point A_1 than at A to produce an equally tight grip. Hence the rotating action of a force with respect to the axis of rotation O depends not only on the magnitude and direction of the force, but also on the distance of the line of action of force **P** from that axis.

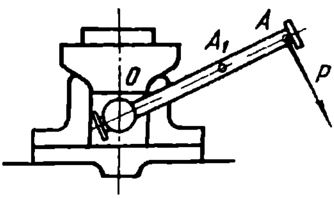


Fig. 36

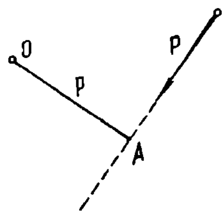


Fig. 37

The quantity used in mechanics to measure the rotating action of a force is called the *moment of a force.*

Assume a force **P** to be applied to a body (Fig. 37). From any freely-chosen point O we extend a line OA perpendicular to the action line of the force. The product of the magnitude of the force and the length of the perpendicular is called the moment of the force **P** with respect to point O. By denoting M as the magnitude of the moment, we then obtain

$$M = Pp, \tag{9}$$

in which P represents the magnitude of the force, and p is the length of the perpendicular connecting point O with the line of action of the force.

The point O with respect to which the moment of force has been taken is called the *moment centre*, and the distance p from the moment centre to the action line of the force is called the *arm of the force with respect to the said point.* The product of a force and its arm is called the *moment of the force with respect to a given point.*

Since force is measured in kilogrammes and the arm in units of length (m, cm), the moment of a force is expressed in kilogramme-metres (kg-m) or kilogramme-centimetres (kg-cm).

From what has been said it is apparent that the larger the magnitude of a force and the larger its arm, the greater is its tendency to produce rotation with respect to a given moment centre. If the moment centre coincides with the line of action of the force, the moment of the force will be zero since its arm will amount to zero.

Let us assume that force $\mathbf{P}_1$ and arm p_1 produce the moment M_1, and the second force $\mathbf{P}_2$ and arm p_2 produce the moment M_2 with respect to the same point. Then $M_1 = P_1p_1$ and $M_2 = P_2p_2$. Furthermore, let us assume that the two moments are equal, i.e., that $P_1p_1 = P_2p_2$. From this it follows that

$$\frac{P_1}{P_2} = \frac{p_2}{p_1}.$$

Wherefore, *when moments of forces with respect to one and the same point are equal, the magnitudes of the forces will be inversely proportional to their arms.*

In order fully to determine the action of a force on a body, it is necessary to take into consideration not only its magnitude, but also the direction in which it tends to produce rotation. Thus in Fig. 37 the mutual positions of the force and the moment centre indicate that they tend to produce clockwise rotation. If force **P** were acting in the opposite direction, or if point O were on the other side of the action of the force, the moment would tend to produce counterclockwise rotation.

Hereafter we shall call a moment *positive* if it tends to produce clockwise rotation, and *negative* if it tends to produce counterclockwise rotation.

Oral Exercises

1. Will a moment of force with respect to a given moment centre change if the point of the force is altered along its line of action?

2. On what line are points situated with respect to which the moments of the force are zero?

3. Can forces of different magnitude produce equal moments with respect to one and the same centre? Under what condition?

Illustrative Problem 13. A workpiece is being machined on a lathe by means of a cutter A (Fig. 38). The distance l from the cutting edge to the base of the tool is 60 mm, the vertical component P of the pressure exerted on the cutting edge by the workpiece is 900 kg. Find the moment of force **P** with respect to point B where the cutter is fastened in its support.

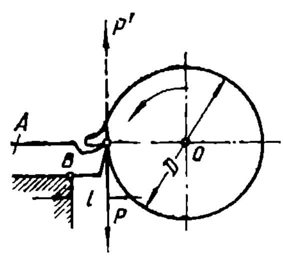


Fig. 38

Solution: the moment tending to rotate the cutter (the "bending moment") is found from the equation $M = Pl = 900 \times 60 = 54{,}000$ kg-mm $= 5{,}400$ kg-cm. It can be seen that as the length of the moment arm p increases (the distance from the cutting edge to the base of the tool), the bending moment will also increase.

Illustrative Problem 14. In the preceding example, **P** is the force the workpiece exerts vertically upon the cutter and causes the reaction **P**′ of the cutter on the workpiece. This reaction is equal and opposite to the force **P**. Find the moment of the force **P**′ with respect to point O on the axis of the workpiece if its diameter $D = 80$ mm.

Solution: in this case the arm of the sought moment is equal to $\frac{D}{2}$ and the moment $M = \frac{P'D}{2} = 900 \times 40 = 36{,}000$ kg-mm $= 3{,}600$ kg-cm.

Under the action of this moment the workpiece (together with the spindle with which it is tightly joined) will tend to twist. This is called torque.

32. Moment of a Resultant

Assume we have two parallel forces $\mathbf{P}_1$ and $\mathbf{P}_2$ acting in the same direction and that their resultant **R** has been found (Fig. 39). Let us freely choose a point C as the moment centre. The moment of the resultant **R** with respect to this point will be

$$M_R = Rc = (P_1 + P_2)c. \qquad \text{(a)}$$

Let us express the moments of the component forces in respect to the same point O:

$$M_{P_1} = -P_1(a - c) = -P_1 a + P_1 c;$$

$$M_{P_2} = P_2(b + c) = P_2 b + P_2 c.$$

By adding the members of the right and left parts of the two equations we obtain

$$M_{P_1} + M_{P_2} = P_2 b - P_1 a + (P_1 + P_2)c.$$

By applying Equation (8) we derive

$$P_2 b - P_1 a = 0.$$

Therefore the sum of the moments of the component forces is equal to $(P_1 + P_2)c$. But as may be seen from (a) above, the moment of the resultant is also equal to $(P_1 + P_2)c$, the moments having been taken together with their algebraic signs.

It is easy to see that we would have arrived at the same solution no matter where we had taken the moment centres in the same plane (say at C_1, C_2, C_3, etc.).

If there had been more than two parallel forces, then by combining them in succession and applying the same principle as above for each partial resultant, we would finally find that the moment of the resultant is also equal to the algebraic sum of the moments of all its components.

We have therefore proved an important relationship for moments of forces in general, and in the particular case for parallel forces acting in one direction. In more detailed courses on theoretical mechanics this relationship is proved to hold true for any distribution of forces in a plane.

Wherefore, *the moment of the resultant of a coplanar system of forces is equal to the algebraic sum of the moments of the component forces with respect to one and the same moment centre.*

This may be expressed for an n number of forces by the following equation:

$$M_R = M_{P_1} + M_{P_2} + M_{P_3} + \ldots + M_{P_n}. \qquad (10)$$

In using this equation it must be borne in mind that the algebraic sign of the moment of each force must be retained.

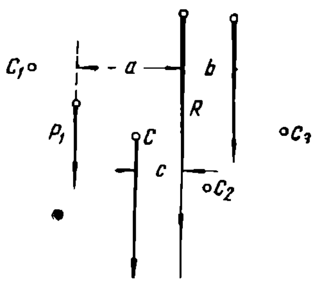


Fig. 39

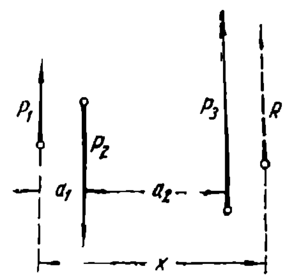


Fig. 40

Illustrative Problem 15. Find the resultant of the parallel forces $P_1 = 30$ kg, $P_2 = 53$ kg, and $P_3 = 70$ kg, by using Equation of moments (10), the distances between their lines of force being $a_1 = 80$ mm and $a_2 = 250$ mm (Fig. 40).

Solution: the magnitude of resultant $R = P_1 + P_3 - P_2 = 30 + 70 - 53 = 47$ kg and, consequently, is directed upwards.

Let us take the moment centre on the line of action of force $\mathbf{P}_1$ and let x denote the unknown arm of the resultant. Then the equation of moments is $-Rx = P_2a_1 - P_3(a_1 + a_2)$, from which, after substituting figures for the letters they represent, we find $x = 401$ mm.

33. The Couple

Let us now return to the composition of two parallel forces acting in opposite directions. As has already been shown in Sec 28, the resultant of two such forces is equal to their difference. Now let us assume that the two components are equal in magnitude as illustrated in Fig. 41. In this case, according to Equation (6a) the resultant is zero. However, a body under the action of two such forces would not be in equilibrium. We know that such forces would tend to produce rotation of the body. A good illustration of such a system of forces is shown in Fig. 42 — two hands turning a reamer.

A system of two equal parallel forces acting in opposite directions is called a *couple.*

A couple possesses no resultant.

A couple is characterised by the magnitude of the forces constituting it and by the distance between their lines of action. The distance between the action lines of the forces forming a couple is called the *arm of the couple.*

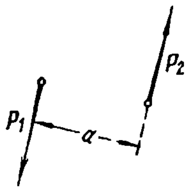


Fig. 41

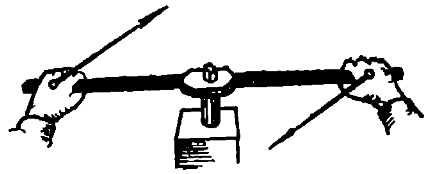


Fig. 42

The action of a couple on a body to which it is applied is directly proportional to the magnitude of the forces composing it and to the length of its arm. This action is measured by the *moment of the couple* and is the product of the magnitude of one of the forces and the arm of the couple. Therefore if we denote the arm of the couple in Fig. 41 as a and its moment as M, we obtain the moment of the couple as the expression $M = P_1a = P_2a$ or, in general, the equation

$$M = Pa. \tag{11}$$

The moment of a couple, just as the moment of a force, is measured in kg-m, kg-cm, etc. In order to determine the action of a couple, it is necessary to know not only the magnitude of its moment, but also the direction in which it tends to rotate the body. Just as with the moment of a force, we shall consider the moment of a couple to be positive if it tends to produce clockwise rotation, and negative if counterclockwise.

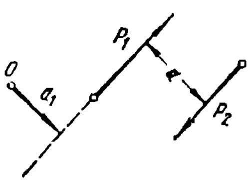


Fig. 43

Fig. 43 represents a couple $\mathbf{P}_1$ and $\mathbf{P}_2$ with an arm a and where the moment of the couple $M = P_1a = P_2a$. The moments of forces comprising the couple with respect to an arbitrary point O lying in the plane of the couple are expressed as

$$M_{P_1} = -P_1a_1$$

and

$$M_{P_2} = P_2(a_1 + a).$$

Combining these two moments, we obtain

$$M_{P_1} + M_{P_2} = -P_1a_1 + P_2(a_1 + a) = -P_1a_1 + P_2a_1 + P_2a.$$

Since forces $\mathbf{P}_1$ and $\mathbf{P}_2$ are equal in magnitude, then

$$P_2a_1 - P_1a_1 = 0$$

and

$$M_{P_1} + M_{P_2} = P_2a.$$

Thus, the algebraic sum of the moments of forces $\mathbf{P}_1$ and $\mathbf{P}_2$ with respect to point O is equal to the moment of the couple.

Wherefore, *the moment of a couple is equal in magnitude and possesses the same sign as the algebraic sum of the moments of the forces comprising it with respect to any point lying in the plane of the couple.*

A couple can be balanced only by another couple which is equal in moment and opposite in sign. It cannot be balanced by one force.

Oral Exercises

1. Given two couples, the arm of one of which is one-fifth the length of the arm of the other. What would be the ratio between the forces comprising the couples if the moments of the couples were equal?

2. The arm of one couple is m times less than that of a second couple. The magnitude of the forces comprising the first couple is n times greater than that of the second. What is the ratio of the moments of the two couples?

34. Equilibrium of a Coplanar System of Parallel Forces

Let us see what conditions a system of parallel forces in a plane must satisfy for a body to which they are applied to be in equilibrium.

Assume a system of parallel forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, and $\mathbf{P}_5$ as in Fig. 44*a*. As has already been shown, the magnitude of the resultant of forces $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_5$ is equal to the sum of these forces. By combining them in succession we obtain the line of action of resultant $\mathbf{R}_1$. Then we combine forces $\mathbf{P}_1$ and $\mathbf{P}_4$ acting in the opposite direction and obtain the other partial resultant $\mathbf{R}_2$. Let us assume that $\mathbf{R}_2$ is equal and opposite in direction to the first resultant $\mathbf{R}_1$. The system is therefore reduced to two equal and opposite forces and its resultant is zero, which means that it is in equilibrium. If the sum of the forces acting in one direction would not be equal to the sum of the forces acting in the other, or, in other words, if the algebraic sum of all the forces would not be zero, the system would not be in equilibrium.

In Fig. 44*b* another system of forces is represented. Proceeding as before, we obtain resultant $\mathbf{R}_1$ of forces $\mathbf{P}_1$, $\mathbf{P}_3$, and $\mathbf{P}_4$, and then the second partial resultant $\mathbf{R}_2$ of forces $\mathbf{P}_2$, $\mathbf{P}_5$, and $\mathbf{P}_6$.

These two resultants are also equal in this case, but their lines of action differ and they form a couple.

We therefore see that if a system of parallel forces is to be in equilibrium, it is not enough for the algebraic sum of forces to be equal to zero; another condition that must be fulfilled is that the system is not reduced to a couple, i. e., that the moment of the couple equals zero.

How can it be determined whether this second condition is satisfied?

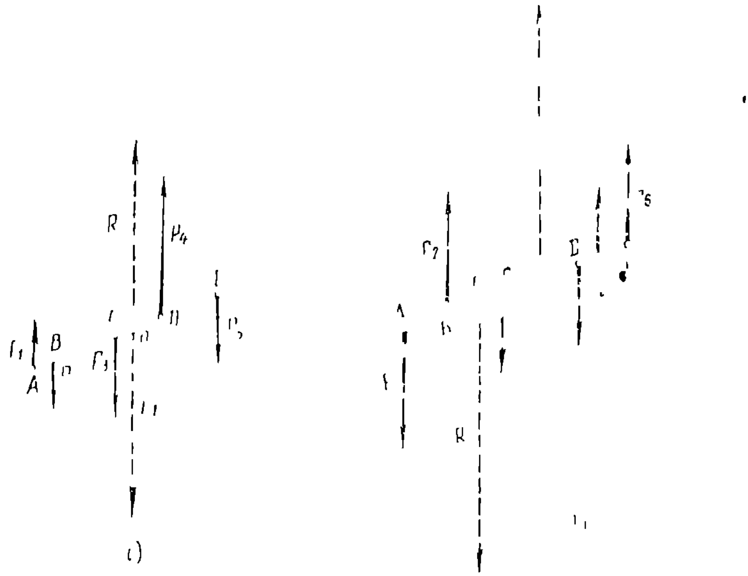


Fig. 44

It will be recalled from the preceding section that the moment of a couple is equal to the algebraic sum of the moments of the forces comprising the couple with respect to any point lying in its plane. This means that the obtained moment of the given couple will be equal to the algebraic sum of the moments of forces $\mathbf{R}_1$ and $\mathbf{R}_2$. But $\mathbf{R}_1$ is the resultant of the group of forces $\mathbf{P}_1$, $\mathbf{P}_3$ and $\mathbf{P}_4$, while $\mathbf{R}_2$ is the resultant of the second group of $\mathbf{P}_2$, $\mathbf{P}_5$, and $\mathbf{P}_6$, and it has been proved in Sec. 32 that the moment of the resultant is equal to the algebraic sum of the moments of the component forces. Hence the moment of the resultant $\mathbf{R}_1$ is equal to the algebraic sum of the moments of the forces $\mathbf{P}_1$, $\mathbf{P}_3$, and $\mathbf{P}_4$, and the moment of the resultant $\mathbf{R}_2$ is equal to the sum of the moments of the forces $\mathbf{P}_2$, $\mathbf{P}_5$, and $\mathbf{P}_6$.

We then come to the conclusion that the moment of the couple to which the given system may be reduced will be zero if the algebraic sum of the moments of all the forces P_1, P_2, P_3, P_4, P_5, and P_6 is zero.

Hence a system of an n-amount of coplanar parallel forces will be in equilibrium under the following conditions:

1) the algebraic sum of all the forces must be zero, i. e.,

$$P_1 + P_2 + P_3 + \ldots + P_n = 0; \qquad (12)$$

2) the algebraic sum of the moments of all the forces with respect to any point in the plane of the system must be zero, i. e.,

$$M_{P_1} + M_{P_2} + M_{P_3} + \ldots + M_{P_n} = 0. \qquad (13)$$

Both of these conditions must be satisfied *simultaneously* for a system of parallel forces to be in equilibrium.

Assume the lever AB, with its fulcrum at O (Fig. 45a), to be acted upon by forces $\mathbf{Q}_1$ and $\mathbf{Q}_2$ at its left side and by force $\mathbf{P}$ at its right. For the lever to be in equilibrium the conditions expressed in Eqs (12) and (13) must be satisfied. The forces $\mathbf{P}$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ and the reaction of the fulcrum $\mathbf{R}$ are acting on the lever; then if the forces acting downwards are regarded as positive, Eq. (12) becomes

$$P + Q_1 + Q_2 - R = 0,$$

from which $R = P + Q_1 + Q_2$.

We have thus determined the reaction of the fulcrum. Now let us determine the magnitude of force $\mathbf{P}$ by applying the second condition required to obtain equilibrium. By taking point O as the moment centre, Eq. (13) becomes

$$Pb - Q_1a_1 - Q_2a_2 = 0,$$

from which $P = \frac{Q_1a_1 + Q_2a_2}{b}$.

If all the forces were applied on one side of the fulcrum (Fig. 45b), we would have

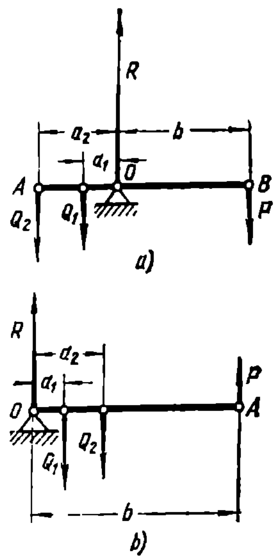


Fig. 45

$$Q_1 + Q_2 - P - R = 0, \text{ from which } P + R = Q_1 + Q_2.$$

To determine force $\mathbf{P}$ in this case, we will write the equation of moments with respect to point O as follows:

$$Q_1a_1 + Q_2a_2 - Pb = 0 \text{ from which } P = \frac{Q_1a_1 + Q_2a_2}{b}$$

and the reaction at the fulcrum $R = Q_1 + Q_2 - P = Q_1 + Q_2 - \frac{Q_1a_1 + Q_2a_2}{b}$.

Now let us take another example.

Fig. 46 represents a beam lying horizontally on supports A and B with one end extending beyond its support as an overhang. Forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are acting downwards on the beam. It is required to find the reactions at the supports $\mathbf{R}_A$ and $\mathbf{R}_B$.

From Eq. (12) we obtain one equation with two unknowns:

$$R_A + R_B = P_1 + P_2 + P_3 + P_4.$$

First we will determine reaction $\mathbf{R}_B$ by taking the algebraic sum of the moments with respect to point A:

$$P_1a + P_2(a + b) + P_3(a + b + c) - R_B(a + b + c + d) + P_4(a + b + c + d + e) = 0.$$

After finding $\mathbf{R}_B$ we insert it in the first equation and find $\mathbf{R}_A$.

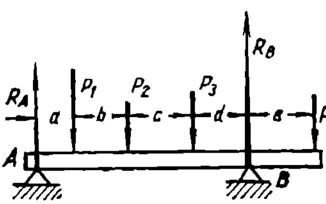


Fig. 46

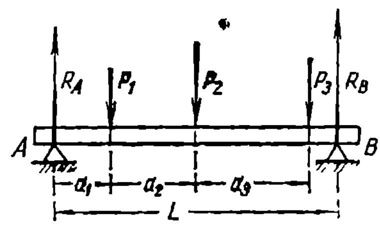


Fig. 47

Illustrative Problem 16. The beam lying on supports A and B in Fig. 47 is acted on by forces $P_1 = 200$ kg, $P_2 = 300$ kg and $P_3 = 250$ kg. Find the reactions at the supports $\mathbf{R}_A$ and $\mathbf{R}_B$ if $a_1 = 1$ m, $a_2 = 1.5$ m and $a_3 = 2$ m, and the distance L between the supports = 5 m. The weight of one linear m of the beam is 20 kg and its length 5.5 m.

Solution: the beam is in equilibrium under the action of forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, of its own weight acting directly downwards, and of the reactions at the supports. First we must determine reactions $\mathbf{R}_A$ and $\mathbf{R}_B$ caused by forces $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$.

Since the reactions act upwards, Eq. (12) becomes

$$P_1 + P_2 + P_3 - R'_A - R'_B = 0, \text{ i.e., } R'_A + R'_B = 200 + 300 + 250 = 750 \text{ kg}.$$

We obtain the second equation by reducing to zero the algebraic sum of the moments with respect to any point in the plane of the forces. For the sake of simplicity we will take point A as the moment centre, with respect to which the moment of reaction $\mathbf{R}_A$ is zero:

$$P_1a_1 + P_2(a_1 + a_2) + P_3(a_1 + a_2 + a_3) - R'_BL = 0;$$

substituting numerical values, we obtain

$$200 \times 1 + 300 \times 2.5 + 250 \times 4.5 - R'_B \times 5 = 0,$$

from which

$$R'_B = 415 \text{ kg and } R'_A = 750 - 415 = 335 \text{ kg}.$$

If we had taken the moment centre on the line of action of force **P**, the equation of moments would have become

$$R'_A \times 1 + 300 \times 1.5 + 250 \times 3.5 - 4R'_B = 0,$$

from which

$$4R'_B - R'_A = 1{,}325.$$

In solving this equation in combination with the equation $R'_A + R'_B = 750$, we would have obtained the same result.

The weight of the beam itself is $5.5 \times 20 = 110$ kg and is distributed equally between the two supports. Hence the full reaction $\mathbf{R}_A$ at support A is $335 + \frac{110}{2} = 390$ kg and the reaction $\mathbf{R}_B$ at support B is $415 + 55 = 470$ kg.

35. The Moment of a Force in Respect to an Axis

Fig. 48*a* represents a vertical shaft OO_1 capable of revolving in its bearings and having at its upper end an elbow OA forming a right angle AOO_1 with the shaft. At point A force **P** is applied which, just as the axis of the elbow, lies in the plane MN and is perpendicular to the axis of shaft OO_1. Under the action of this force the shaft begins to revolve.

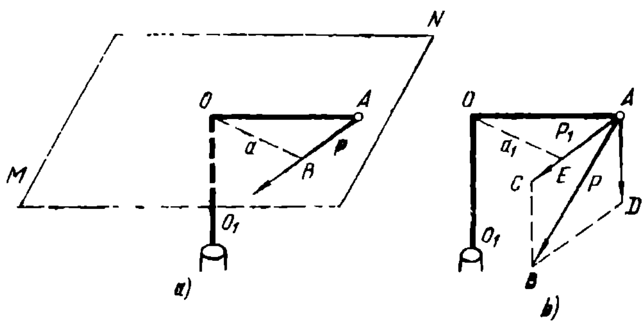


Fig. 48

Now let us assume that force **P** does not lie in the plane MN but acts at an angle with it (Fig. 48*b*). Let us resolve this force into two components: $\overline{AC}$ in the plane of rotation of elbow OA, and $\overline{AD}$ lying parallel to the axis of the shaft OO_1.

It is obvious that force $\overline{AC}$ will tend to make the shaft rotate, whereas force $\overline{AD}$ will tend to push the shaft downward in line with its axis (to avoid this the lower end of the shaft is constrained by a thrust bearing). It is evident that in the first case (Fig. 48*a*) it is the entire force **P** that rotates the shaft, while in the second case it is only one of its components. In the first case the tendency of the force to produce rotation is measured by its moment with respect to point O, which is equal to Pa and where a represents the arm (the length of the perpendicular OB extending from rotation centre O to the action line of the force). In the second case the tendency of force **P** to produce rotation is measured by the moment of force $\mathbf{P}_1$ with respect to the same point O, and is equal to P_1a_1 in which a_1 represents the arm of the force with respect to point O. Component $\mathbf{P}_1$ of force **P** is a projection of force **P** on plane MN, perpendicular to the axis OO_1. The moment of force $\mathbf{P}_1$, equal to P_1a_1, will be the moment of force **P** with respect to the axis.

It will be found that if a force lies in a plane perpendicular to an axis, its moment with respect to the axis will be equal to its moment in relation to the point where the axis intersects the plane. From Fig. 48*b* it is apparent that the moment of force **P** decreases as the angle CAB increases. For that reason it is of greater advantage to apply a force so that it will act in a plane perpendicular to the axis of rotation of a body.

36. Questions for Review

1. What is a moment of force with respect to a point?
2. Is it possible to select a point in relation to which the moment of force will be zero? Is there only one such point?
3. What is the relation between the moments of the resultant of a system of parallel forces and the moments of its components?
4. What conditions must a system of parallel forces satisfy if it is to be in equilibrium?

37. Exercises

10. Given two parallel forces $\mathbf{P}_1$ and $\mathbf{P}_2$ acting in one direction (Fig. 49). The distance l between their lines of action is 120 mm. Find the line of action and the magnitude of the resultant if $P_1 = 48$ kg and $P_2 = 144$ kg.

11. Solve Problem 10 with $\mathbf{P}_1$ and $\mathbf{P}_2$ acting in opposite directions.

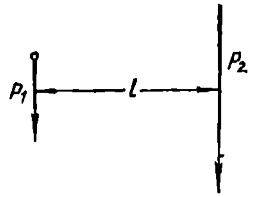


Fig. 49

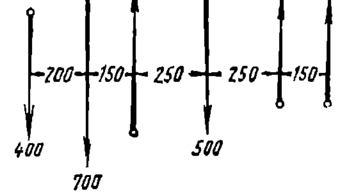


Fig. 50

12. Find the line of action and the magnitude of the resultant of the system of parallel forces shown in Fig. 50 (forces are denoted in kg, and lengths in mm).

13. Find the resultant of the system of parallel forces represented in Fig. 51 if $P_1 = 100$ kg, $P_2 = 900$ kg, $P_3 = 800$ kg, $P_4 = 300$ kg, and $a = 300$ mm, $b = 600$ mm, and $c = 200$ mm.

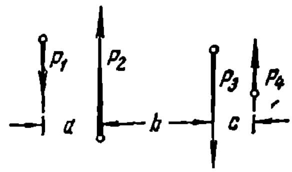


Fig. 51

14. A beam lying freely on two supports A and B (Fig. 52) is under the vertical action of forces $P_1 = 300$ kg, $P_2 = 300$ kg, $P_3 = 150$ kg, and $P_4 = 240$ kg. Find the reactions of the supports caused by these forces if $a_1 = 1.8$ m, $a_2 = 0.9$ m, $a_3 = 0.9$ m, $l = 4.5$ m and $L = 6$ m.

15. The fulcrum of lever AB in Fig. 53 is at point O and is under the action of forces $P_1 = 120$ kg and $P_2 = 60$ kg. The distances between the lines of action of these forces and the fulcrum are $a_1 = 360$ mm and $a_2 = 375$ mm. What must be the magnitude of a force $\mathbf{P}_3$ applied at end B of the lever to keep the lever in equilibrium if its length $l = 960$ mm? What will be the reaction R of the fulcrum (assuming the lever itself to be weightless)?

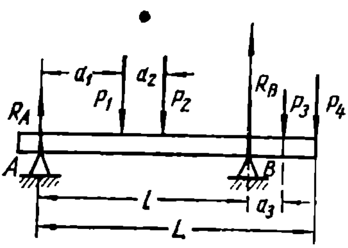


Fig. 52

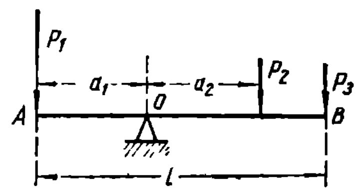


Fig. 53

CHAPTER IV

CENTRE OF GRAVITY, AND STABILITY OF BODIES

38. Centre of Gravity, and Centre of Parallel Forces

The force of the earth's attraction (gravity) acts on all the particles of a body. Gravitational forces always act on the particles of a body in a line directed towards the centre of the earth and therefore converge. But the angle of deviation from the parallel is extremely small, amounting to only one second along a meridian on the earth's surface for two points situated 31 m apart. And since bodies with which engineering mechanics is concerned are infinitesimal as compared with the radius of the earth, the forces of gravity acting on their particles are considered parallel.

When we combine all the elementary gravitational forces acting upon all the particles of a body, we obtain their resultant. This resultant of the forces of gravity acting on all the particles of a body is called the *weight of the body*. The point of application of this resultant is called the *centre of gravity of the body*.

It will be seen that the centre of gravity is also the centre of parallel forces, and as already explained, holds true no matter what the direction in which the forces act if only they remain parallel. From this it follows that the centre of gravity of a body remains unchanged irrespective of the position of the body with regard to the earth's surface.

39. Centre of Gravity of Certain Bodies of Simple Form

In many engineering calculations where the weight of bodies must be taken into account, it is necessary to know the exact position of the centre of gravity. In some cases it is very easy to find the centre of gravity.

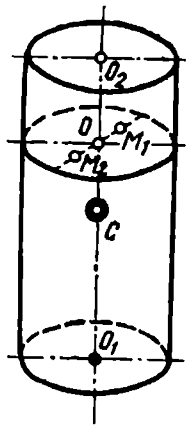
Let us investigate several instances where bodies are of simple geometric form.

1. *The centre of gravity of a sphere coincides with its geometric centre.* The truth of this statement is apparent from the fact that the resultant of all the elementary gravitational forces acting on the particles along one diameter passes through its centre, which is also the centre of the sphere.

2. *The centre of gravity of a right circular cylinder* (Fig. 54). Let us make a cut through any arbitrary point O perpendicular to axis O_1O_2. Taking a particle M_1 in this section we then choose another particle M_2 on the same diameter and at an equal distance from the centre O of the section. It follows that the resultant of the elementary gravitational forces acting on these two particles passes through the centre of the section. Following the same procedure with respect to any point of the cylinder, we come to the conclusion that *the centre of gravity of the whole cylinder lies on its axis O_1O_2 and at half its altitude at point C.*

3. *The centre of gravity of a right regular prism* (Fig. 55). By reasoning as in the case of the above right circular cylinder, we reach a similar conclusion, i. e., that *the centre of gravity of a right regular prism lies on its axis and at half its altitude.*

But there is one important factor to be borne in mind. It is evident, from what has been said, that we assume the elementary gravity forces acting on the particles as being equal in magnitude. This presupposes that the body is uniform throughout. Such a body is known as *homogeneous**. If this condi-



● Fig. 54

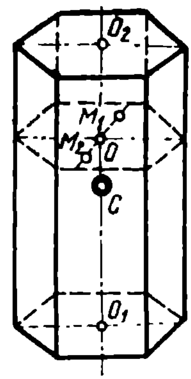


Fig. 55

tion is not satisfied, the process of finding the centre of gravity becomes complex, as may be seen from the following example.

Illustrative Problem 17. Fig. 56 represents a cylindrical shaft with a length $L = 1{,}000$ mm and made of two materials of different specific gravity. Along its length $l = \frac{L}{2} = 500$ mm it is made of aluminium with a specific gravity $\gamma_1 = 2.6$ g/cu cm, while the remainder of its length DB is made of steel with a specific gravity $\gamma_2 = 7.85$ g/cu cm. Find the centre of gravity of the shaft.

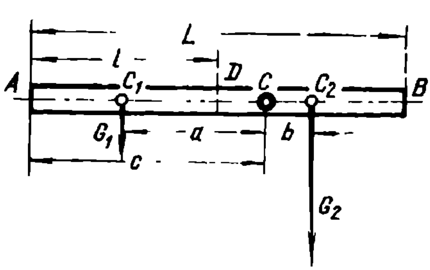


Fig. 56

Solution: if the shaft were homogeneous, its centre of gravity would be on its axis and halfway along its length, i.e., within section D at a distance $l = 500$ mm from its end. But in the case in hand it will be necessary to determine the weight of each component of the shaft before finding its true centre of gravity.

By first denoting the area of the cross-section of the shaft as F, the weight of its aluminium part AD will then be expressed as $G_1 = Fl\gamma_1 = 2.6\,Fl$ g, and the weight of its steel part DB as $G_2 = Fl\gamma_2 = 7.85\,Fl$ g. The point of application of the first force G_1 is C_1 in the middle of AD, and that of the second force G_2 at point C_2 in the middle of DB. The distance between points C_1 and C_2 is 500 mm. In order to find the overall centre of gravity C of the shaft we must find the point of application

* Henceforth it shall be assumed that a body is homogeneous unless the contrary is stipulated.

of the resultant of the two already obtained parallel components, as follows:

$\frac{a}{b} = \frac{7.85\ Fl}{2.6\ Fl} = 3$, from which $a = 3b$; and since $a + b = 500$ mm, then $a = 375$ mm and $b = 125$ mm.

Hence the sought centre of gravity C of the shaft lies at a distance of $c = \frac{l}{2} + a = 250 + 375 = 625$ mm from its left end.

40. Centre of Gravity of Plane Figures

Fig. 57 represents a homogeneous disc of uniform thickness, i. e., a cylinder of small height as compared to its diameter. It is apparent from what has already been said that the centre of gravity of the disc lies in the centre of its middle section MN dividing its thickness in half. Therefore instead of the whole disc we may deal with its middle section, where we may assume all the material of the disc to be concentrated. Hence we may regard the centre of gravity of this disc as the centre of gravity of the material area of a circle. In exactly the same way we may regard the centre of gravity of a triangular plate ABD (Fig. 58) as the centre of gravity of its middle section, i. e., as the centre of gravity of the area of a triangle; and so forth with other plane figures.

Now let us consider methods of finding the centre of gravity of a number of plane figures.

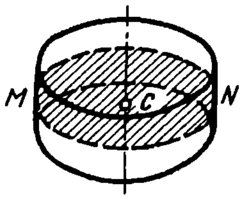


Fig. 57

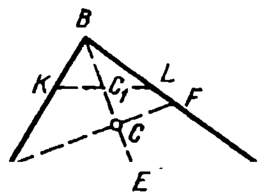


Fig. 58

1. *The centre of gravity of the area of a circle* lies in its geometric centre.

2. *The centre of gravity of the area of a triangle* lies at the intersection of its medians.

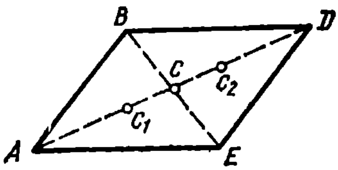
Given triangle ABD (Fig. 58). We delineate median BE connecting vertex B with the midpoint E of its base AD. Then we delineate segment KL at any arbitrary place parallel with base AD. Since the triangle BKL is similar to BAD, then $KC_1 = C_1L$. Hence the resultant of all elementary gravity forces acting on all the particles lying along segment KL is at point C_1, the intersection of median BE with that segment.

Following the same procedure with respect to any other linear segment parallel to base AD, we see that the centre of gravity of the triangle lies on median BE.

Now let us delineate another median AF to the side BD. Using the same method as with median BE, we find that the centre of gravity also must lie on this median. From this we conclude that *the centre of gravity of the area of a triangle lies at the intersection of its medians.*

In geometry it is proved that the point of intersection of the medians of a triangle divides them in a ratio of 1 : 2, i. e., $CE = 1/2BC$ and $CF = 1/2AC$. From this it follows that the centre of gravity C lies at a distance $CE = 1/3BE$ or $CF = 1/3AF$, i.e., at a distance of one-third the length of a median from the side to which it has been delineated.

3. *The centre of gravity of the area of a parallelogram* (Fig. 59).



59

Delineate diagonals AD and BE. The diagonals of a parallelogram are divided at their midpoints by their point of intersection. Hence segment AC of diagonal AD is a median of triangle ABE, and segment CD of the same diagonal is a median of triangle BDE. For this reason the centres of gravity C_1 and C_2 of these two triangles lie on the diagonal AD, and the centre of gravity of the whole parallelogram lies on this same diagonal. In the same way we can prove that the centre of gravity lies on the second diagonal BE.

Wherefore, *the centre of gravity of the area of a parallelogram lies at the point of intersection of its diagonals.*

Obviously this deduction also refers to the rhombus, the rectangle, and the square, since all these are forms of the parallelogram.

4. Knowing how to find the centre of gravity of the area of a triangle and of parallelograms of all types, we can find the centre of gravity of any figure that can be divided into such elements.

Let us assume we want to find *the centre of gravity of the area of a freely chosen quadrangle* (Fig. 60).

We first divide the quadrangle into two triangles ABD and ADE by the delineation of diagonal AD. We then delineate medians to the midpoint of side AD, mark the centres of gravity C_1 and C_2 of the areas of the two triangles and connect them by means of segment C_1C_2. Next we divide the quadrangle with a second diagonal BE, forming triangles ABE and BDE. By repeating the above process we also obtain segment C_3C_4. The desired centre of gravity is found at the intersection of this segment and segment C_1C_2.

Illustrative Problem 18. Find the centre of gravity of the try square shown in Fig. 61.

Solution: the centre of gravity of the plank *I* of the square lies at the intersection of diagonals AE and BD, and the centre of gravity of leg *II* lies at the intersection of diagonals DL and KM. The centre of gravity C

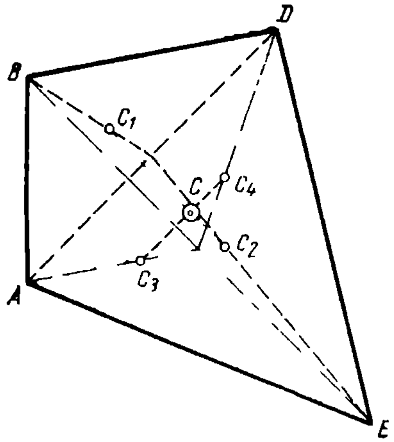


Fig 60

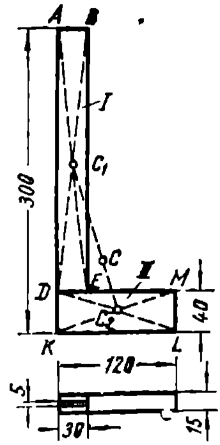


Fig. 61

of a whole square lies somewhere on line C_1C. In order to find point C, we must divide line C_1C, so as to obtain a ratio inversely proportional to the weights of the two sides of the square, or, since the square is homogeneous, inversely proportional to their volumes. The volume of plank $I = (300 - 40) \times 30 \times 5 = 39,000$ cu mm, while the volume of leg $II = 120 \times 40 \times 15 = 72,000$ cu mm. By dividing the segment C_1C_2 in such a way as to satisfy the condition

$$\frac{C_1C}{CC_2} = \frac{72}{39},$$ we obtain the centre of gravity C.

41. Practical Method of Determining the Centre of Gravity of a Plate

Let us assume it necessary to find the centre of gravity of the flat plate of irregular outline as shown in Fig. 62. We suspend it from its corner A by the cord KA and when it comes to rest it will be in a state of equilibrium. The weight of the plate will be equal to the reaction from the cord at point A. These two forces have a common line of action which coincides with the vertical line AD and on which, therefore, lies the centre of gravity. We then delineate this vertical line AD on the plate and then suspend the plate from some other point, let us say

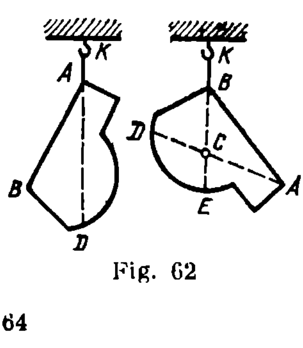


Fig. 62

corner B, and following the same procedure we delineate the vertical line BE on the plate. Since the centre of gravity must lie on both BE and AD, we conclude that it must be at point C of their intersection.

42. The Stability of a Body Having a Point or an Axis as Support

Make this experiment: take some pointed object, let us say a centre-punch, which is symmetrical in relation to its longitudinal axis, and stand it vertically on its sharp end upon a horizontal surface MN (Fig. 63a). In this position the weight G of the punch, applied at its centre of gravity C, will be equal to the reaction at the horizontal plane. But we know that if we thus stand the punch vertically, the moment we release our hold it will begin to fall. This is explained by the fact that when the axis of the punch leaves its vertical position, a moment of force caused by the weight G is induced which tends to rotate the punch about its point of support A (Fig. 63b).

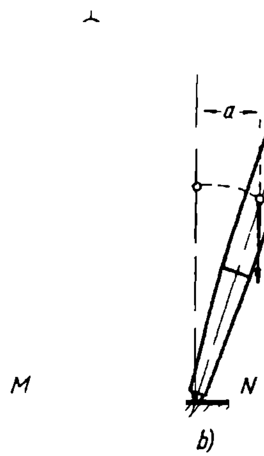


Fig. 63

This position of a body, in which the slightest force is sufficient to upset its equilibrium, is known as the *state of unstable equilibrium*.

The characteristic of this state of unstable equilibrium is that when the body leaves this position its centre of gravity is lowered.

Let us investigate another example. The ball represented in Fig. 64 is made of two materials of different specific gravity, the specific gravity of the material of segment K being the

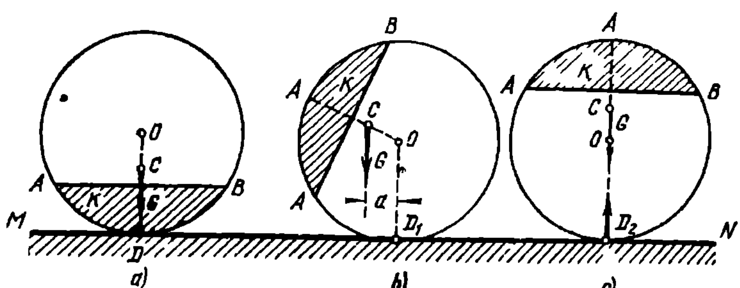


Fig. 64

greater. The centre of gravity of the ball will therefore not be at its centre O but at some other point C lying on the radius OD which is perpendicular to the separation plane AB (Fig. 64*a*). In the position shown in Fig. 64*a* the weight **G** of the whole ball is equalised by the reaction from the point of support applied to the ball at point D.

If we turn the ball so that it takes the position shown in Fig. 64*b*, we will see that its weight **G** induces a moment equal to Ga in respect to the point of support D_1, which will act in such a direction that the centre of gravity will be lowered when we remove our hand, therefore the ball will be induced to turn back

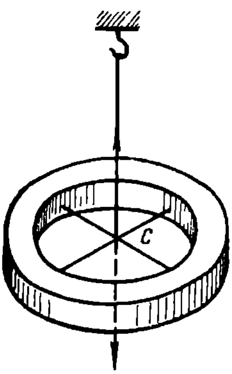


Fig. 65

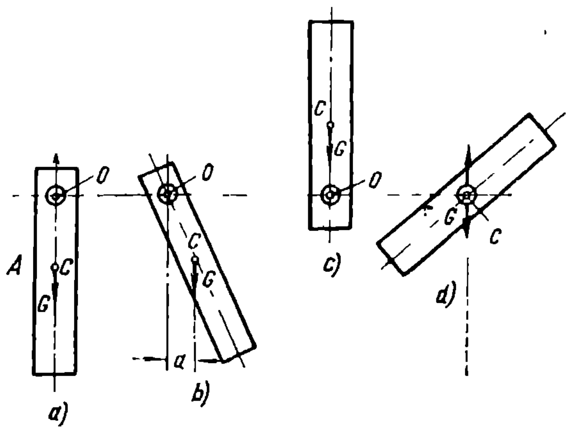


Fig. 66

until it reaches its original position (Fig. 64*a*)* in which it will be again in a state of equilibrium.

A position to which a body returns after the force which has disturbed its equilibrium has ceased to act, is called *a state of stable equilibrium*.

It is characteristic of this state of stable equilibrium that its centre of gravity is raised under the influence of the force disturbing its equilibrium.

If this ball be placed in the position shown in Fig. 64*c*, it will be in the position of unstable equilibrium similar to that of the centre-punch shown in Fig. 63*a*.

Finally, if a body is given support at its centre of gravity, its weight will be equalised by the reaction from the support no matter what position it is in. For example, the ring (Fig. 65) suspended at the point of intersection of two cords in its middle plane will remain in a state of equilibrium in any placed posi-

* Actually the ball will assume this position only after rolling back and forth several times.

tion because its centre of gravity will remain always unchanged. In the same way a homogeneous ball in any position will remain in a state of equilibrium when placed on a horizontal plane.

A position in which a body remains in equilibrium, no matter what its position with respect to a support, is called *a state of indifferent equilibrium.*

It is characteristic of indifferent equilibrium that the centre of gravity remains at the same height no matter what the position of the body.

All the above classes of equilibrium refer to a body supported at one point. Now let us examine a case when a body is supported on a fixed axis around which it can freely rotate. Assume that the plank A in Fig. 66*a* is fastened to a shaft freely supported in bearings*. If we move the plank so that its position becomes as shown in Fig. 66*b*, its centre of gravity will have been displaced higher. If left to itself, under the action of the moment of its weight Ga, the plank will rotate back, and after swinging back and forth a few times will take up its original position (Fig. 66*a*) which is therefore a *stable* position. If we arrange the plank in the position shown in Fig. 66*c*, a slight force is all that will be needed to start it rotating and its centre of gravity will drop until finally the plank takes a stable position. Therefore its original position was one of *instability* (Fig. 66*c*).

Finally, if the plank were held on the shaft in such a way that its centre of gravity coincided with the axis of the shaft (Fig. 66*d*), it would always be in a state of indifferent equilibrium no matter what its position.

As we shall see later, it is often necessary for machine parts revolving about a fixed axis to be arranged in a state of indifferent equilibrium. This process is known as balancing.

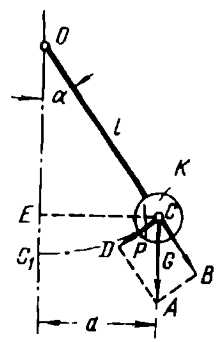


Fig. 67

Illustrative Problem 19. Fig. 67 shows a light rod suspended on axis O and holding a disc K whose weight is $G = 5$ kg. This pendulum is pulled to the position shown in the figure and then released. Find the magnitude of the force acting on it at the instant it begins to swing to a position of stable equilibrium. The centre of the disc is at a distance $a = 200$ mm from the vertical, and $OC = l = 340$ mm.

Solution: we will neglect the weight of the rod and consider that the centre of gravity of the pendulum coincides with the centre of disc C. We then resolve the force of the weight G into components $\overline{CB}$, acting along the rod, and $\overline{CD}$ perpendicular to it. As is apparent from the drawing, component $\overline{CB}$ cannot induce the pendulum to swing, but the second component $\overline{CD}$, which is tangent to the arc described by the

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* The bearings are not shown in the drawing.

radius OC, will induce the displacement of the centre of gravity of the disc in the direction of C_1, the position of stable equilibrium.

Since $\angle CAD = \angle ACB = \angle EOC$, therefore $\Delta CAD \sim \Delta CEO$, hence $CD : CE = CA : OC$. Accordingly, the component we are seeking

$$P = \frac{CA \cdot CE}{OC} = \frac{Ga}{l} = \frac{5 \times 200}{340} = 2.94 \text{ kg}.$$

If the angle of inclination α had been given instead of distance a, we would have found magnitude $a = EC$ from the right triangle OEC, whose leg $EC = OC \sin \alpha$.

This problem can be solved more simply by applying the deduction made in Sec. 31: the moment of force **G** with respect to axis O is equal to Ga, and the moment of component $\overline{CB}$ is zero (its line of action intersects axis O, and its arm is zero). Whence $Ga = Pl$, from which $P = \frac{Ga}{l}$, the result we have already obtained.

43. The Stability of a Body on a Horizontal Surface

Fig. 68 represents a body K with its base supported on a horizontal surface MN. If we rotate it about edge E, its centre of gravity C will rise and describe the arc CC_1. If we take our hand away, the body will rotate in reverse about the same edge E and return to its original position $ABDE$ which is accordingly a position of *stability*. In this position the weight of the body is equalised by the reaction from the surface. This will be the case till we place the body in position $A_1B_1D_1E$ indicated by the

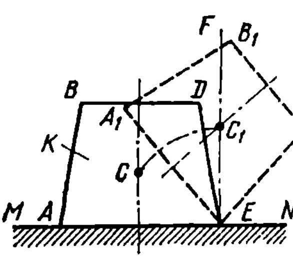

Fig. 68

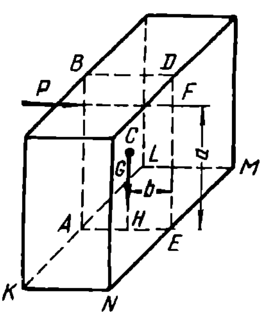

Fig. 69

dotted line and in which its centre of gravity is on the vertical plane passing through edge E. If we take our hand off the body while it is in this position it will begin to rotate either to the right or to the left and the centre of gravity will drop until it reaches the lowest point possible. Wherefore position $A_1B_1D_1E$ is one of *unstable equilibrium*.

Let us investigate under what conditions a body will maintain a position of stable equilibrium: assume that a parallelepiped of weight G is standing on its base $KLMN$ on a horizontal surface (Fig. 69). Assume we apply a force **P** to the body with the line of action lying in the middle plane $ABDE$. In respect to edge NM this force will induce the moment Pa, in which $a = EF$ and is the arm of the force **P**. The tendency of this moment to tilt over the parallelepiped about the edge NM is counteracted by the moment of force of its weight **G** which has the same edge NM for its moment centre. The arm of this moment $b = EH$ and is found by constructing a perpendicular to the line of action of the force of gravity from point E. The condition that must be satisfied for the parallelepiped to maintain its equilibrium is that the algebraic sum of these two moments with respect to point E be equal to zero:

$$Pa - Gb = 0.$$

The moment of force **P** is the *tilting moment*, while the moment of force **G** is the *stability moment*. If $Pa > Gb$, the block will rotate round edge NM, but if $Pa < Gb$ it will maintain its stable position on the surface.

If $Pa < Gb$, then $P < \frac{Gb}{a}$, from which we see that the greater the moment of stability and the shorter the arm of force **P** with respect to axis NM, the more stable the body will be.

In calculating the stability of cranes, dams, retaining walls, smokestacks, etc., there must always be a definite reserve of stability which is expressed by the ratio

$$k = \frac{M_G}{M_P},$$

in which M_G is the moment of stability, and M_P the tilting moment. This ratio is called the *coefficient of stability*. It is apparent from what has been said that this coefficient must always be greater than 1.

However, from the above it must not be thought that the weight of a body always contributes to its stability. Fig. 70*a* represents a body $ABDE$ which will overturn about the edge E under the action of its own weight **G** which induces a tilting moment Ga. In order to keep the body in the position shown, a force must be applied which will induce a moment equal in magnitude and acting in the opposite direction. It is seen that the body will fall over because the line of action of the force of gravity intersects the supporting surface beyond the base of the body.

In Fig. 70*b* the body is similar in height to that of Fig. 70*a* but is stable because the action line of the force of gravity passes through the supporting area within the base of the body. Whereas the body in Fig. 70*c* has the same area of support as that in Fig. 70*a* but is also stable because its centre of gravity has been lowered.

Wherefore, *a body on a horizontal surface is in a position of stable equilibrium if the resultant of all the forces acting on it, including its own weight, intersects the area of support within the configuration of the base.*

The greater the area of its base and the lower its centre of gravity, the more stable the body.

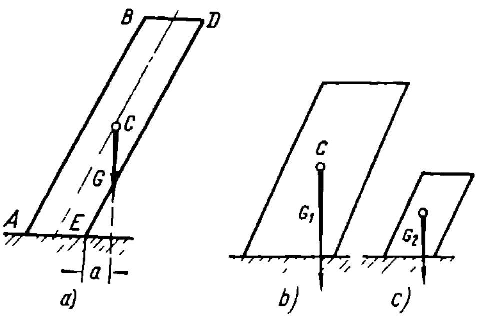

Fig. 70

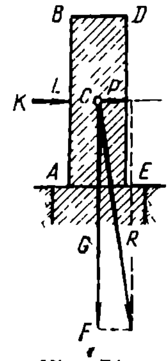

Fig. 71

Illustrative Problem 20. The weight **G** of a wall *ABDE* which is rectangular in cross section (Fig. 71) is expressed by vector *CF* and the greatest wind pressure by vector *KL*, both vectors being drawn to the same scale. Check the stability of the wall.

Solution: we displace the force of wind pressure along its line of action to the centre of gravity *C* and then construct a parallelogram of forces (in this case a rectangle) on the vectors of forces **P** and **G**. Since the action line of the resultant **R** intersects the supporting area *AE* within the configuration of the base, the wall will maintain its condition of stable equilibrium.

14. Questions for Review

1. Will the centres of gravity of two homogeneous bodies, both of similar shape and dimensions but made of materials possessing different specific gravities, be in the same position?
2. Will the centres of gravity of two cylinders of similar dimensions, one homogeneous and the other made of horizontal layers of materials possessing different specific gravities, be in the same position?
3. A rectangular frame *ABCD* has two sides *AD* and *BC* made of one material, and the other two sides *AB* and *CD* of a material of different specific gravity. Will this frame have the same centre of gravity as a frame made entirely of one material?
4. Will the ring in Fig. 65 retain its condition of indifferent equilibrium if the point of intersection of the cords by which it is hung does not lie in its middle plane?
5. Will the metal strip in Fig. 66*d* be in a condition of indifferent equilibrium if the geometric axis of the shaft to which it is fixed does not pass through the midpoint of its width?

45. Exercises

16. Find the centre of gravity of the area of a triangle ABC with sides $AB = 120$ mm, $BC = 90$ mm, and $AC = 150$ mm.

17. The triangle in Ex. 16, made in the form of a frame, is of homogeneous wire of uniform cross-section. Find its centre of gravity.

Hint to solution. Draw vectors at the centres of gravity of the sides, proportionate to their lengths, then find the centre of these parallel forces.

18. Find the centre of gravity of a trapezoidal plate $ABCD$ (Fig. 72) whose dimensions $a = 60$ mm, $b = 20$ mm, and $c = 10$ mm*.

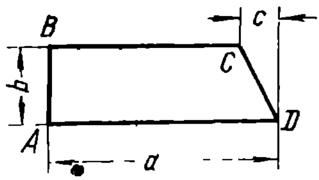

Fig. 72

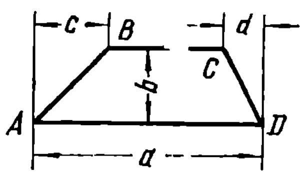

Fig. 73

19. Solve similarly for Fig. 73, but with dimensions $a = 60$ mm, $b = 20$ mm, $c = 20$ mm, $d = 10$ mm.

20. Solve similarly for Fig. 74, but with dimensions $a = 60$ mm, $b = 20$ mm, $c = 20$ mm, $d = 10$ mm.

21. Fig. 75 shows a disc with two bosses of equal size on either side. Find its centre of gravity.

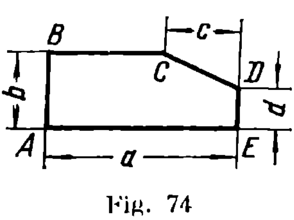

Fig. 74

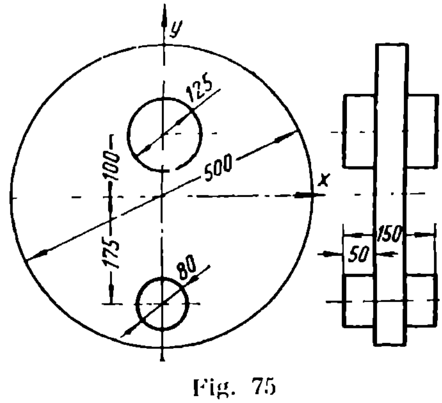

Fig. 75

22. The cast-iron disc A in Fig. 76 has a boss whose centre of gravity is at a distance $a = 290$ mm from the axis of the disc. Find the weight of the load K fastened to the disc at a distance $b = 420$ mm from the same axis and on the same diameter in

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* Exercises 18 to 20 are to be solved by the method given in Sec. 40, item 4.

order to keep the disc in a state of indifferent equilibrium in respect to its axis; the dimensions of the boss $d = 80$ mm and $c = 100$ mm, and its weight $\gamma = 7.25$ g/cu cm.

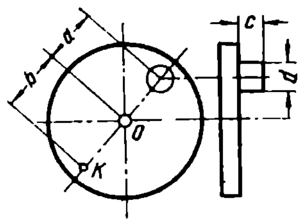


Fig. 76

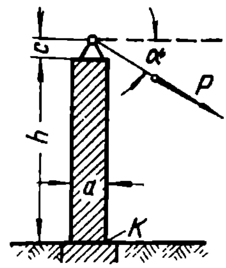


Fig. 77

23. Fig. 77 represents a pillar with a bar fastened to its top, forming an angle $\alpha = 30°$ with the horizontal and subjected to the action of force $P = 200$ kg. The pillar is square in cross-section with one side $a = 0.5$ m, its height $h = 4$ m, and $c = 200$ mm. Find the tilting moment of the pillar with respect to edge K and also its coefficient of stability if 1 cu m of the pillar weighs 2,200 kg.

Hint to solution. Resolve force **P** into vertical and horizontal components.

CHAPTER V

FRICTION

46. Harmful Frictional Resistance

We know from experience that the amount of energy required to pull a load across a surface depends on the character of the surface: it is much easier to pull a loaded sledge over packed snow than over bare earth, or a cart over an asphalt road than over a cobbled road, etc. For whenever an object moves in respect to another against which it is pressed with a certain force, it gives rise to a force opposing the motion. This force is called *friction.*

Hence *the resistance to the motion of two bodies in contact with one another is determined by friction.*

Let us assume that a workpiece is being machined longitudinally on a lathe. If there were no friction between the carriage and the bedways, the force transmitted to the carriage by the feed mechanism would be expended on the cutting process alone. However, part of this force is exerted in overcoming friction, which means that more power must be expended by the motor. Accordingly, friction is called *detrimental resistance.*

When a body moves and encounters the resistance of a surrounding medium like air or liquid, this kind of resistance can also be considered detrimental; and the faster the body moves the greater will be the resistance. There are also other forms of detrimental resistance. Whereupon it is very important to know what measures can be taken to counteract resistance, and in particularly friction.

However, it must be noted that although friction is accepted as detrimental, it is frequently a necessity, as we shall see further.

47. Sliding and Rolling Friction

There are several types of friction. Let us illustrate.

Imagine a point on the carriage of a lathe located on the surface where it is in contact with the bedway. As the carriage moves, this point will coincide with a countless number of points on the bedway lying on a straight line along which the carriage moves. This kind of movement is called *sliding* and the friction arising from it on the contiguous surfaces is called *sliding friction*.

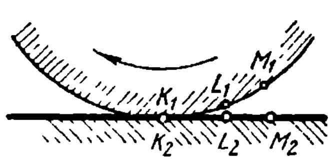


Fig. 78

The movement of a wheel on a rail (Fig. 78) is an entirely different matter. Assume that at a certain moment point K_1 on the wheel will come in contact with point K_2 on the rail. After an interval, two other points will come into contact, let us say L_1 and L_2, then points M_1 and M_2, and so on. If the segments of the arcs K_1L_1, L_1M_1, etc. are equal to corresponding segments K_2L_2, L_2M_2, etc., then this kind of movement is called *rolling*. Characteristic of rolling is that each point on one of the contiguous bodies comes into contact with a definite point on the other body, and the resistance that thus arises is known as *rolling friction*.

If the arc segments K_1L_1, L_1M_1, etc., are not equal to segments K_2L_2, L_2M_2, etc., we would then have a combination of rolling and sliding and the friction produced will also be of both kinds.

Sliding friction is sometimes called friction of the first type, while rolling friction is known as friction of the second type.

We thus see that sliding and rolling are two entirely different kinds of movement, for which reason in each case the resistance is likewise different.

Oral Exercises

1. Name the kind of friction produced in each of the following instances:

a) a shaft revolving in the bushings of a bearing;

b) the spindle of a lathe revolving in roller or ball bearings;

c) the rotation of a workpiece against the dead centre of a lathe.

2. What kind of friction is developed between the wheels and the ground when the wheels turn without moving a car?

48. Basic Laws of Sliding Friction and the Coefficient of Sliding Friction

Friction is a complex physical phenomenon and the amount of it produced in each case depends on a number of factors. Let us examine several of the factors which apply to sliding friction.

Make the following simple experiment. Place a known weight on a small square plate lying on a horizontal surface (Fig. 79). Attach a spring dynamometer to the plate by a cord and put the whole in motion by pulling the dynamometer. It will require

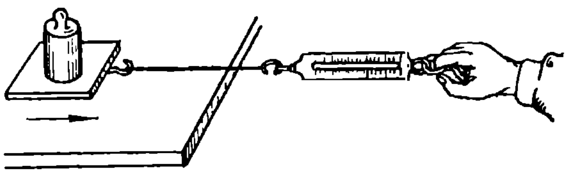


Fig. 79

a definite force to make the plate move at an even speed; the dynamometer will indicate this force which will be equal and opposite to the force of resistance to the motion, that is, to the force of sliding friction. It will also be seen that at the instant just before the plate begins to move, the dynamometer will indicate a greater force than when the plate subsequently begins to move smoothly. Friction is caused by the pressure of the plate on the supporting surface, i. e., by the weight of the load and the plate acting perpendicularly to the supporting surface and called *normal pressure*.

The following laws of sliding friction have been established experimentally:

1. *Total friction is proportional to normal pressure.* Experiments show that the force of friction **F** increases or decreases in exactly the same proportion as the sum weight **Q** of the plate and the load. This means that the force of friction comprises a certain part of normal pressure and can be expressed by the equation

$$\frac{F}{Q} = f, \text{ or } F = fQ. \qquad (11)$$

The factor f represents *the coefficient of sliding friction*, or *the coefficient of friction of the first type*. Whereupon it may be said that *the force of sliding friction is equal to normal pressure multiplied by the coefficient of sliding friction.*

Since forces **Q** and **F** are expressed in the same units, *the coefficient of sliding friction is an abstract quantity.*

2. Let us repeat the experiment but with a larger plate. If we choose the load so that the weight of the plate and the load is the same as before, we shall see no change in the force necessary to move the plate. This means that the force of friction is the same as in the first experiment.

Wherefore, *the force of friction does not depend on the area of contact.*

This can be expressed differently. If we represent the area of contact in the first experiment by S_1 cm², and in the second experiment by S_2 cm², then the force q acting on 1 cm² and called *specific pressure*, can be expressed in the first case by $q_1 = \frac{Q}{S_1}$, and in the second case by $q_2 = \frac{Q}{S_2}$.

Wherefore, *the force of sliding friction does not depend on specific pressure.*

3. Continuing our experiments with the plate, we find that the amount of friction will change if either the plate or the horizontal supporting surface are of different materials. For example, if we use a planed supporting surface in one case and a polished surface in the second, it is obvious that in the latter case there will be less friction. Furthermore, there will be less friction between lubricated surfaces than between dry ones.

Wherefore, *if normal pressure is unchanged, total friction will depend on the material of the contacting bodies, the finish of their surfaces, and the nature and amount of lubrication.*

4. Finally, *total friction does not depend on sliding velocity, although the force necessary at the start of sliding is greater than when momentum (retained motion) has been achieved*, as has already been stated at the beginning. For which reason a differentiation is made between *static and kinetic friction.*

Approximate coefficient values of sliding friction for different materials under various conditions are given in Supplement I.

Oral Exercises

1. Knowing only normal pressure, is it possible to establish the amount of friction that can be developed?

2. What must be known in order to find the amount of friction that can be developed?

Illustrative Problem 21. What force will be necessary to slide a wooden box weighing 1,200 kg over horizontal pine boards if the coefficient of friction $f = 0.30$?

Solution: using Eq. (11) we obtain

$$F = 0.3 \times 1{,}200 = 360 \text{ kg}.$$

The force required can be no smaller than this, but it will take a somewhat greater effort to start the box moving.

Illustrative Problem 22. To a solid cast-iron block is applied a force $P = 2$ kg along the same line of movement which causes it to slide at a constant speed on horizontal guides; weight of block $G = 20$ kg. What is the coefficient of friction?

Solution: using Eq. (14) we obtain

$$f = \frac{F}{Q} = \frac{P}{G} = \frac{2}{20} = 0.1.$$

Illustrative Problem 23. A cast-iron block with a weight $G = 12$ kg is moving at constant speed along a horizontal cast-iron surface under the action of force $P = 23$ kg (Fig. 80). Find the coefficient of friction if the force **P** forms an angle $\alpha = 14°$ with the vertical axis.

Solution: the force of friction is the result of the action of normal pressure and which is the sum of the weight of the block **G** and the vertical component of force **P**. First we must find this component. From ΔABC we obtain $Q = P \cos \alpha$.

Hence the full normal pressure $Q_1 = Q + G = P \cos \alpha + G$. It follows that the force of friction $F = Q_1 f = (P \cos \alpha + G) f$. When speed is constant, the motive force $T = P \sin \alpha$ and is equal to the force of friction, i. e.,

$$(P \cos \alpha + G)\, f = P \sin \alpha,$$

from which

$$f = \frac{P \sin \alpha}{P \cos \alpha + G} = \frac{23 \sin 14°}{23 \cos 14° + 12} = \frac{23 \times 0.242}{23 \times 0.97 + 12} = 0.16.$$

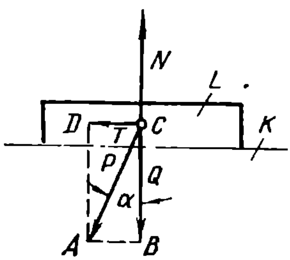


Fig. 80

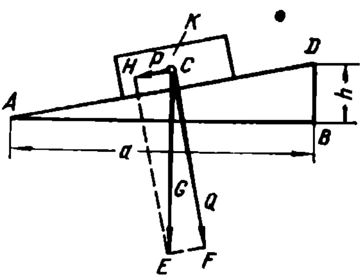


Fig. 81

Illustrative Problem 24. Fig. 81 represents a block K sliding at constant speed down an inclined plane ABD under its own weight **G**. Find the coefficient of friction when $a = 400$ mm and $h = 100$ mm.

Solution: resolve the force **G** into two components: **Q** perpendicular to the inclined surface AD, and **P** parallel to AD. The force of friction $F = fQ$ and is equalised by the component **P**. We must determine this component.

From the similarity of triangles EHC and ABD we evolve $\frac{P}{h} = \frac{Q}{a}$, from which $P = Q \frac{h}{a}$.

Since this component is equal to the force of friction, we obtain

$$Q \frac{h}{a} = fQ.$$

from which the coefficient of friction $f = \frac{h}{a} = \frac{100}{400} = 0.25.$

49. Dry and Fluid Friction

The force of friction depends on the condition of contacting surfaces. If the surfaces are dry they will come into direct contact with each other as shown in Fig. 82; no matter how smooth the surfaces seem to be, they will always retain irregularities whose magnitude will depend upon polishing precision. Under the action of force Q these irregularities will undergo deformation, the protrusions of one surface squeezing into the hollows of the other. This interlocking of contact surfaces will give rise to cohesion and resist the relative motion of both surfaces. Such resistance is called *dry friction*.

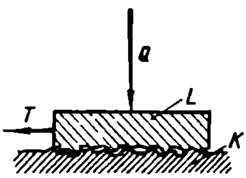


Fig. 82

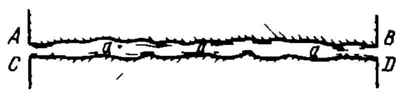


Fig. 83

Now let us assume there is a layer of lubricant between the contiguous surfaces as shown in Fig. 83. If the layer *a* is thick enough, it will completely separate the surfaces *AB* and *CD* and their irregularities will not come into contact with one another; instead of resistance between surface irregularities, there will be interaction between the particles of the lubricant. This kind of friction is called *fluid friction*. It is easy to understand that in this case there will be less resistance to relative movement than in the case of dry friction. It is also obvious that there will be less heat produced and less wear of contacting machine parts. That is why lubricating directions for machines must be strictly observed.

As shown by experiment, the thickness of the lubricating layer ranges from 0.005 mm to 0.05 mm.

Phenomena connected with fluid friction between machine parts were first thoroughly investigated towards the end of the past century by the outstanding Russian scientist N. Petrov, the author of the Hydrodynamic Theory of Friction now used in calculations concerning lubrication of major contacting parts of machinery.

Such calculations for determining the force of friction must take into account the mutual speed of contacting surfaces, normal specific pressure, and the thickness of the lubricant as well as its viscosity (which latter characterises the adhesion between particles).

Friction is sometimes intermediate between the dry and fluid kind; this occurs when the lubricating layer does not completely cover the irregularities of contiguous surfaces, in which case it is called either *semi-dry* or *semi-fluid* friction, depending on which it more closely approximates.

50. Coefficient of Rolling Friction

There is one feature that distinguishes rolling from sliding: since theoretically a cylinder comes into contact with a flat surface along a straight line, and a ball and a flat surface touch at one point, great pressure developes at these places on both bodies and deforms them there. The schematic diagram in Fig. 81 shows how a cylinder is flattened along arc *ab* as it rolls over a straight surface, pressing into the supporting plane and developing a ridge in front which resists the rolling of the cylinder. There are also other factors that cause resistance to rolling, one of which involves irregularities on both contacting surfaces (the larger the irregularities, the greater the resistance).

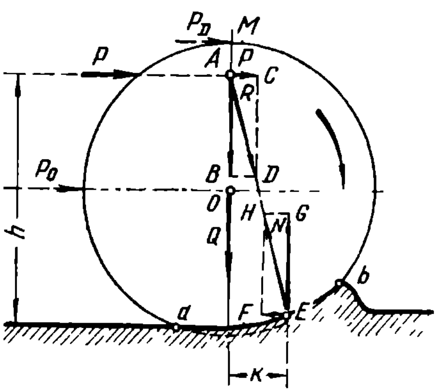


Fig. 81

Now let us see how to determine the amount of resistance developed to rolling. The roller represented in Fig. 81 is under the action of load **Q** (which includes its own weight), and also of force **P** acting horizontally at a height h above the supporting surface. By transferring both these forces to point A the point where their lines of action intersect we construct our parallelogram of forces and obtain the resultant **R** represented by vector $\bar{AD}$. If the roller is to be in equilibrium, some other force must be applied to equalise the resultant **R**. Such a force is the reaction **N** of the supporting surface acting normally to the contiguous surfaces (that is, perpendicular to their tangent) at point E. For forces **R** and **N** to be in equilibrium they must be equal in magnitude and opposite in direction. Hence forces **R** and **N** are equal in magnitude.

We resolve force **N** into two components $\bar{EF}$ and $\bar{EG}$, acting horizontally and vertically, respectively. It is evident that triangles EHG and ACD are congruent. Therefore $\bar{EF}$ is equal in magnitude to force **P**, and $\bar{EG}$ to force **Q**. We thus obtain two couples, **P** and $\bar{EF}$, and **Q** and $\bar{EG}$. These couples must be in equilibrium, and their moments must be equal and have opposite signs. The moment of the first couple is Ph and is positive. The moment of the second couple is Qk (where k represents the distance between the point of application of reaction N and the vertical plane passing through the axis of the roller) and is negative. Since the equation of these moments is $Ph = Qk$, we find the magnitude of force **P** needed to overcome the resistance to the motion of the roller as follows:

$$P = k\frac{Q}{h}. \qquad (15)$$

The magnitude of arm k of the couple will depend, above all, on the hardness of the materials of which the two contiguous bodies are made and also on the condition of their surfaces. Accordingly, the magnitude of k is taken as the *coefficient of rolling friction*. As distinguished from the coefficient of sliding friction, *it is a denominate quantity expressed in linear units* (cm, mm). It goes without saying that k and h must both be given in the same units.

If force **P** is applied at the level of centre O, then in Eq. (15) h will be equal to the radius R of the roller and

$$P_0 = k\frac{Q}{R}. \qquad (16)$$

But if force **P** is applied at point M at the height h, which latter is equal to diameter D, then

$$P_D = k\frac{Q}{D}. \qquad (17)$$

From what has been said it is evident that the harder the contiguous bodies and the more polished their surfaces, the smaller will be the coefficient of rolling friction.

Coefficients of rolling friction for a few materials are given in Supplement II.

In order to find the force necessary to move a wheeled vehicle, it is necessary to take into account the sliding friction developed between the wheels and their axles in addition to the rolling friction developed between the wheels and the road (or rails). In solving problems of this kind a formula is used expressing the relationship between the tractive effort **P** and normal pressure **N** acting on the axle, which also makes allowance for both rolling and sliding frictions:

$$P = fN. \qquad (18)$$

The coefficient f is called the *general coefficient of friction.*

Oral Exercises

1. What is the chief difference between the coefficient of sliding friction and the coefficient of rolling friction?

2. Is it more advantageous in rolling to apply the motive force **P** nearer to the supporting surface or farther from it?

Illustrative Problem 25. A wooden drum together with its contents weighs 1.2 tons. What force **P** must be applied to it at the height of its axis to keep it rolling at a constant speed over a horizontal wooden floor if the diameter of the drum $D = 1.5$ m.

Solution: by expressing the weight of the drum in kilogrammes and its radius in centimetres and applying a coefficient of friction of 0.08 cm, we find that $P = 0.08 \times \frac{1,200}{75} = 1.3$ kg.

If the same load in a wooden box is pulled over a wooden floor and the coefficient of sliding friction for wood upon wood $f \approx 0.5$, then the force needed would be

$$P = 1,200 \times 0.5 = 600 \text{ kg}.$$

Illustrative Problem 26. It is well known that the dimensions of bodies alter with changes in temperature. This factor must be taken into account, among other things, in planning steel bridges. Since a bridge must have two supports (or "chairs"), one of them must be made movable. Fig. 85 represents schematically such a movable chair: between the lower immovable shoe A and the upper shoe B, attached to the bridge girder, cylindrical rollers are inserted.

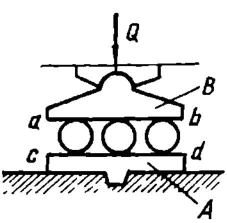


Fig. 85

Assuming the force Q transmitted by the bridge to the support to be 200 tons and the diameter of the rollers d to be 150 mm, and that all elements of the support are made of steel, find the force of resistance P developed by the support when the bridge lengthens in the summer and contracts in winter.

Solution: in the given case the rollers are moving along two surfaces *ab* and *cd*. Since both shoes are of the same material, the coefficient of friction is the same for both, and the sought force of resistance P is equal to $F_1 + F_2$, with $\mathbf{F}_1$ representing rolling friction on surface *ab*, and $\mathbf{F}_2$ that on surface *cd*. Using Eq. (17) we obtain

$$F_1 + F_2 = k\frac{Q}{d} + k\frac{Q + 3G}{d} = k\frac{2Q + 3G}{d},$$

where G represents the weight of one roller.

The weight of the rollers are neglected since they are insignificant as compared with force **Q**; by taking $k = 0.006$ cm, we obtain

$$P = k\frac{2Q}{d} = 0.006 \times \frac{2 \times 200,000}{15} = 160 \text{ kg}$$

acting along the length of the bridge.

51. Function of Friction in Nature and in Engineering

As we have already said, resistance caused by friction is considered undesirable only in a comparative sense. For without friction it would be impossible to walk even on a level surface or for locomotives to move on rails. Nor would any object stay put on an inclined surface nor nails hold boards together, etc.

In engineering, friction plays a double function. On the one hand it is detrimental because it creates added resistance to the motion of machine parts; to overcome this resistance it is necessary to expend additional energy which could otherwise be used for the work of the machine. On the other hand friction plays a positive role, for without friction, nuts and bolts would be useless, belts would not transmit rotational motion, etc.

Therefore, we must reduce friction between moving machine parts to a minimum and increase friction to a maximum in other parts where it is desirable.

52. Questions for Review

1. Blocks B and C are lying on the horizontal surface A (Fig. 86). The force of friction between B and A is represented by $\mathbf{F}_1$, and between B and C by $\mathbf{F}_2$. A force $\mathbf{P}$ is acting on block C. State how the two blocks will move in the following three cases:

a) when force $\mathbf{P}$ is less than $\mathbf{F}_2$ but more than $\mathbf{F}_1$;

b) when force $\mathbf{P}$ is less than $\mathbf{F}_1$ but more than $\mathbf{F}_2$;

c) when force $\mathbf{P}$ is less than either $\mathbf{F}_1$ or $\mathbf{F}_2$.

2. In Fig. 84, will force $\mathbf{P}$ slide the roller instead of rolling it? What would be necessary to slide the roller?

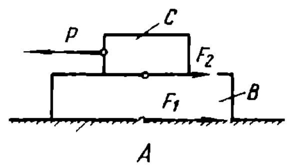


Fig. 86

53. Exercises

24. To maintain the constant speed of a 120 kg load over a horizontal surface, it requires a 15 kg force applied in the direction of the moving load. What is the coefficient of friction?

25. If there were no rollers between shoes A and B in Illustrative Problem 26 (Fig. 85), how much greater would force $\mathbf{P}$ be, considering that the coefficient of dry friction of steel upon steel $f = 0.15$?

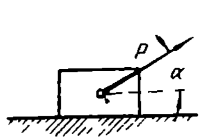


Fig. 87

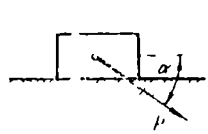


Fig. 88

26. A force $\mathbf{P}$ is applied to a block of weight $G = 20$ kg In one case force $\mathbf{P}$ acts upwards at an angle $\alpha = 35°$ to the horizontal (Fig. 87), and in the other downwards (Fig. 88). What must be the value of $\mathbf{P}$ in both cases to keep the block moving at a constant speed if the coefficient of friction $f = 0.25$?

27. A steel sliding block of weight $G = 10$ kg is rising at a constant speed between cast-iron guides (Fig. 89) under the

action of force **P** which forms an angle α — 30° with the vertical axis. Find the magnitude of force **P** if the guides are lubricated (f = 0.08).

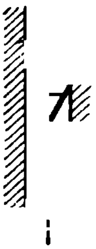


Fig. 89

28. What would be the solution to Exercise 27 if the steel block were sliding downward at a constant speed?

29. A load on a steel plate is being moved over a wooden surface with the aid of steel rollers whose diameter 100 mm (Fig. 90). Find the force **P** required if the combined weight G of the load and the steel plate equals 300 kg, the coefficient of friction between the plate and the rollers k 0.005 cm and the coefficient of friction between the rollers and the wooden surface k_1 0.25 cm (the weight of the rollers is to be neglected).

30. What must be the angle α of the inclined plane in

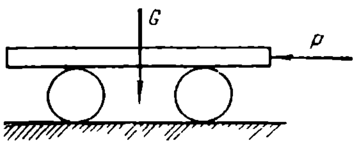


Fig. 90

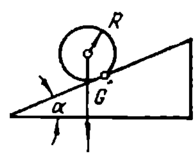


Fig. 91

Fig. 91 so that the cylinder, whose radius is **R**, will roll down at a constant speed under the action of its own weight if the coefficient of rolling friction equals k?

KINEMATICS

CHAPTER VI

THE TRAJECTORY OF A PARTICLE DISPLACEMENT AND TIME

54. Fundamentals of Kinematics

Assume it necessary to set a lathe for the longitudinal machining of a shaft. This must be done so that correct cutting speed and feed are assured with a given thickness of the chip, i.e., so that the right number of revolutions are transmitted to the shaft and the cutter advances the required distance during each revolution. This operation is accomplished by setting the devices that actuate the spindle and the carriage (both driven by the motor).

In doing all this no calculations are made concerning the forces acting on the various parts of the lathe. In other words, the problem is solved through *kinematics*, that branch of mechanics which treats of motion independent of the forces causing it. For kinematics deals with space and time as inseparable from motion.

In order to determine the position of a body in space it must first be known how to determine the position of any one of its points at a given moment of time. Therefore in order to study the motion of a body as a whole, it is first necessary to establish the kinematic relationship between the elements of movement of one of its particles. For this purpose kinematics is subdivided into *kinematics of a particle* and *kinematics of a body*. We shall see, however, that it is sufficient in many cases to know only the motion of one particle in order to solve problems concerning the motion of a body as a whole.

55. Trajectories and Their Influence on Principal Types of Motion

A moving point occupies different positions in space at different moments of time. A continuous path described by a point in motion is called the *trajectory* of the point. The form of trajectory is one of the factors serving to classify its motion.

If the trajectory is a path confined to a plane, it is classified as *coplanar*. The path described by a point on the rim of a wheel rolling along a straight track, or by a point on the cutter of a lathe, are each examples of a coplanar trajectory. If the path does not fall into one plane, it is called *spatial*. An example of such a path is a point on a nut being screwed onto a bolt, or of a point on the cutting edge of a drill. If the path is a straight line it is called *rectilinear* as distinguished from *curvilinear* (when it describes a curve). Curvilinear motion may be of different kinds according to the shape of the curve described by the particle: it is circular if the path is a circle or a segment of a circle; or it may be elliptical, helical, etc.

Oral Exercises

1. Name the kind of motion for a point on each of the following items:

a) the revolving spindle of a lathe;
b) the cutter of a lathe during longitudinal feed;
c) the cutter of a lathe when working with a template;
d) a drill clamped to the tailstock of a lathe while it is drilling.

2. Give examples of other kinds of motion.

Illustrative Problem 27. Assume that a straight line *On*, tangential to a circle, rolls on the circumference of the circle without sliding. Plot the curve traced by point *O* on the line (Fig. 92).

Solution: assume firstly that point *O* is in contact with the circle; after an interval of time some other point *a* on the line will come in contact with point *1* on the circle, then points *b* and *2* will coincide, etc. From this it follows that the segment of the line *Oa* is equal to arc *O1*, the line segment *ab* is equal to arc *12*, and so on.

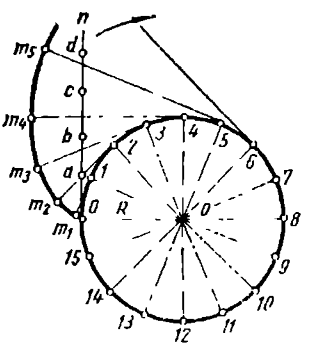


Fig. 92

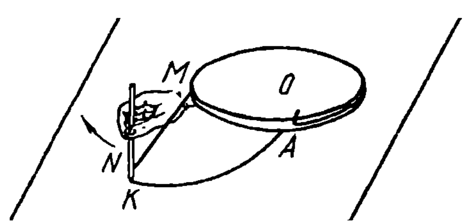


Fig. 93

We then plot arcs *O1*, *12*, *23*, etc., on the circle. Since a tangent is perpendicular to the radius of a circle at the point of contact, we delineate perpendiculars to the radii at points *1*, *2*, *3*, etc., and then plot $1m_1$, $2m_2$, $3m_3$, etc., equal to arcs *O1*, *O1* + *12*, *O1* + *12* + *23*, etc., thus obtaining points m_1, m_2, m_3, etc., lying on the path of point *O*. It will be found more convenient to divide the circumference into several equal segments and then lay out the required number of segment lengths on the respective tangents.

Since only the chord of an arc can be measured with a compass, the greater the number of segments into which we divide the circumference the more precise will be the curve we construct. The curve thus obtained is called the *involute of a circle*, or *a developed curve*.

Accordingly, if a straight line can roll without slipping round the edge of a disc in the way we have already described, its points will describe trajectories in the form of involutes of a circle.

An involute of a circle can be constructed in another way: take a flat disc O (Fig. 93) and fasten to it one end of a thin string, to the other end of which a sharp pencil is fastened. Then wind the string round the disc and place it on a sheet of paper with the pencil at point A. Holding the disc firmly in place on the paper, begin to draw a line with the pencil while unwinding the string, keeping it taut all the time. The curve obtained will be an involute. In the position shown in Fig. 93 the pencil has drawn the segment AK of the involute, the length MK of the string being equal to the length of the arc MA.

Involute curves are widely used in machine engineering, particularly in designing gear wheels, where the profiles of the teeth are in most cases obtained through such curves.

56. Determining the Distance Traversed by a Point According to Its Positions on the Trajectory

A trajectory alone is not sufficient to completely define the position of a particle. We must also know its displacement during a given interval of time and also its direction; that is, we are interested in its current location on the trajectory.

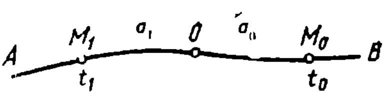


Fig. 94

Assume the curve AB (Fig. 94) to be the path described by particle M. We shall calculate the displacement of the particle at different moments starting from any fixed reference point O, called the *origin*.

Let us assume that at moment t_0 the moving particle was at point M_0, a distance of a_0 from the origin O, and at moment t_1 was displaced from right to left and is at point M_1, a distance of a_1 from the origin but in the opposite direction. Furthermore, let us assume that the particle again changes its direction and moves from left to right and at moment t_2 is at point O. It follows that during the entire time interval the particle M traversed a distance equal to the sum of the arcs

$$a_0 + a_1 + a_1 = a_0 + 2a_1.$$

Since a particle may occupy positions of equal distance on either side of the origin, its displacements must be identified by algebraic signs; if a displacement to the right of the origin O is considered positive, then one to the left will be negative.

Illustrative Problem 28. Point M is moving along a rectilinear path at such a speed that its displacement s from the origin at all moments of time satisfies the equation $s = 25 + 7t - t^2$, in which s is the distance from the origin expressed in centimetres and t is the time in seconds. Find the positions of the point on its path at moments of time $t_0 = 0$ sec, $t_1 = 1$ sec, $t_2 = 2$ sec, $t_3 = 3$ sec, $t_4 = 4$ sec, and $t_5 = 5$ sec (Fig. 95).

Solution: assume point O to be the origin. To find the initial displacement from the origin when t_0 = zero, we substitute zero for t in the equation, and find that $s_0 = 25$ cm from point O. This means that at the first moment, M was at position M_0 or 25 cm to the right of origin O. Substituting 1, 2, 3, 4 and 5 sec for t in the equation, we obtain the respective displacements $s_1 = 28$ cm, $s_2 = 23$ cm, $s_3 = 10$ cm, $s_4 = -11$ cm,

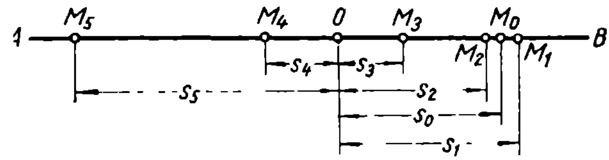


Fig. 95

and $s_5 = -40$ cm. Then let us plot positions M_1, M_2, M_3, M_4 and M_5 of the moving point at those moments. Positive displacements are laid out to the right of the origin and negative ones to the left. The distance traversed by the point in five seconds becomes in this case $M_0M_1 + M_1M_5 = s_1 - s_0 + s_1 + s_5 = 28 - 25 + 28 + 40 = 71$ cm. In Fig. 95 displacements are laid out at a scale of 1 : 10.

57. Plotting a Trajectory According to Given Coordinates

It has just been demonstrated that in order to find the position of a moving particle at any moment it is necessary to lay off its displacement on the trajectory from the origin.

The next question is, what information is needed to plot the trajectory itself?

Assume line AB (Fig. 96) to represent a coplanar trajectory. Delineate axes Ox and Oy perpendicular to each other. At a certain moment of time t_0 the moving particle will be at the initial position A, then at moment t_1 it will be at position M_1, at moment t_2 at position M_2, etc. Now from A, M_1, M_2, etc., plot the perpendiculars Aa_0, M_1a_1, M_2a_2, etc., to axis Ox, and perpendiculars Ab_0, M_1b_1, M_2b_2, etc., to axis Oy. It will be found that the lengths of these perpendiculars determine the position of the moving point at a definite moment.

Therefore by using two axes perpendicular to each other, we are able to plot the trajectory if we know the length of the perpendiculars. Each segment of these perpendiculars, giving the distance of the particle from the axes Ox and Oy, is called a *coordinate*, and the axes themselves are *coordinate axes*. Each segment Oa_0, Oa_1, Oa_2, etc., which indicates the distance of the particle from the axis Oy, is called an *abscissa*, while the axis Ox is known as the *axis of the abscissae*.

Linear segments Ob_0, Ob_1, Ob_2, etc., indicating the distance of the particle from axis Ox are called *ordinates*, and axis Oy is called the *axis of the ordinates*. In short, by delineating the abscissa and the ordinate of the moving particle for a given moment and constructing perpendiculars, we find the position of the particle at that moment at the intersection of the perpendiculars. Then by drawing a smooth line through the points thus acquired, we obtain the path of the moving particle at the chosen scale.

Illustrative Problem 29. A particle is moving along a trajectory determined by coordinates from the equations

$$x = 2t^2 \text{ and } y = 5 + 3t,$$

in which the coordinates x and y are given in centimetres, and the time t in seconds. Plot the trajectory for the first five seconds.

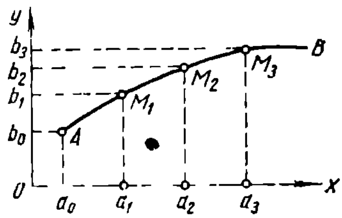


Fig. 96

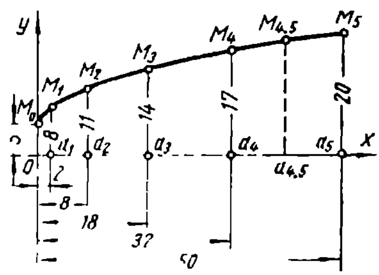


Fig. 97

Solution: first we delineate the coordinate axes Ox and Oy (Fig. 97) and then calculate the coordinates for the initial moment $t = 0$ and for the moments at the end of the first, second, third, etc., seconds. Substituting 0 for t in the equations given, we find that $x = 0$ and $y = 5$ cm. Using a scale of 1 : 10, we delineate from point O the segment $OM_0 = 5$ mm on axis Oy. Then substituting one second for t in the equations, we obtain $x_1 = 2$ cm and $y_1 = 8$ cm. Accordingly, by using the scale chosen, we lay out abscissa $Oa_1 = 2$ mm, and on the perpendicular delineated to a_1 we lay out the ordinate $a_1M_1 = 8$ mm. As a result we obtain the second point M_1 on the trajectory. Repeating this process for all five seconds we obtain six points on the trajectory. By joining all these points by a smooth curve, we obtain the sought trajectory*.

It must be noted that when the trajectory is known, the position of a particle at any moment during the interval $t = 0$ to $t = 5$ sec can be found. Thus, if we want to determine the position of a particle at $t =$ 4.5 sec, we can calculate the abscissa $x = 2 \cdot 4.5^2 = 40.5$ cm, lay it out to scale ($Oa_{4.5} = 40.5$ mm), and then delineate the perpendicular at point $a_{4.5}$. Hence, point $M_{4.5}$ is the required position of the moving particle.

58. The Displacement-Time Graph

It is frequently convenient to represent the displacement of a moving particle from its origin in relationship to time by means of a rectangular system of coordinates.

* If the points in any part of the trajectory are found to be too far apart to draw a smooth curve, it will been necessary to take some intermediate value, such as $t = 2.5$ or 3.5 seconds, etc.

Assume that the path of particle M is represented by curve AB at a definite scale (Fig. 98a). Points M_1, M_2, M_3, etc., will denote the positions of particle M at moments t_1, t_2, t_3, etc. The initial position is M_0 and the origin is point O.

By employing a rectangular system of coordinates Ot and Os at a suitable scale (Fig. 98b), the axis of the abscissae Ot

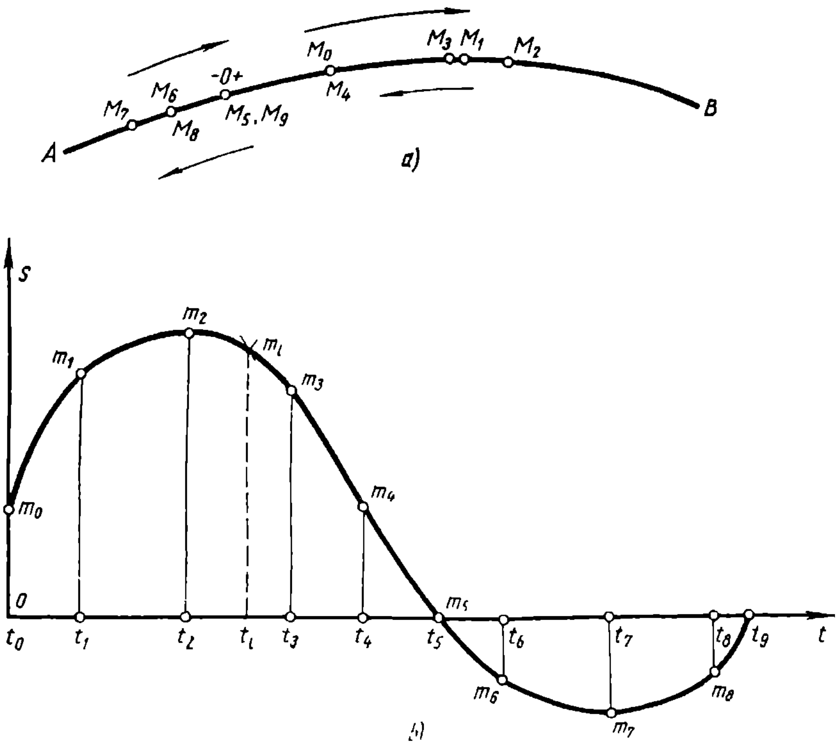


Fig. 98

will represent the time of displacement, while the axis of the ordinates Os will represent the distance of displacement of particle M from the origin O. After laying out the intervals of time denoted by t_1, t_2, etc., we chart perpendiculars to them, representing the displacement of the moving particle from the origin O at the corresponding moment of time. Displacements to the right of point O (Fig. 98a) will be regarded as positive, and those to the left as negative. Positive values are plotted above axis Ot (Fig. 98b) and negative ones below. By joining the points found in this way (m_0, m_1, m_2, etc.) by a smooth line, we obtain a curve which instantly shows the displacement of the moving particle from origin O (Fig. 98a) at any moment of time from $t = t_0 = 0$, to $t = t_9$.

The curve acquired in this way is called a *displacement curve* (Fig. 98*b*) and shows graphically the displacement of the moving particle from a fixed point of reference. It illustrates that at the initial moment when $t = 0$, displacement is represented by the ordinate Om_0 and, according to the chosen scale, is equal to arc OM_0 in Fig. 98*a*; displacement then increases at moment t_2 where it is equal to the ordinate t_2m_2. Then it diminishes to zero at moment t_5 (in Fig. 98*a* point M_5 coincides with O, that is, particle M passes through point O as it moves from right to left), and subsequently the particle, continuing to move in the same direction, passes into the area of negative displacement and at moment t_7 reaches its greatest distance t_7m_7 from the origin, equal to the length of arc OM_7 as shown in Fig. 98*a*. At this moment the point changes its direction and approaches the origin and at moment t_9 aligns with point O.

Thus we see that ordinates corresponding to positive displacements lie above the axis of the abscissae, while ordinates corresponding to negative displacements are below.

The distance of the particle from the origin can be determined for any instant of time on the displacement curve. For example, at the moment of time t_i, it is expressed by ordinate t_im_i.

The displacement curve also makes it possible to determine the increment of displacement of the particle during any interval of time. Thereby it is also known as the *curve of the trajectory* or the *displacement-time graph*.

59. Questions for Review

1. Name the kind of trajectory (coplanar or spatial) described by a point on the following items: a) the chuck of a lathe, b) the chuck of a drilling machine, c) the pulley of a machine tool, d) a die stock when cutting threads by hand.

2. Name the kind of trajectory (rectilinear or curvilinear) described by a point on the following items: a) the facing tool on a lathe, b) the ram of a shaping machine, c) the load screw of a lathe, and the half-nut in the apron.

3. What is the displacement from the fixed reference point of a moving point for a moment of time when the displacement curve intersects the abscissae axis?

60. Exercises

31. Draw the involute of a circle, 10 mm in diameter, generated by a straight line rolling once around the circle's circumference.

32. A particle is moving in a rectilinear trajectory in such a way that its displacement s from the fixed reference point satisfies the equation $s = 20 + 7t - 3t^2$, in which s is expressed in centimetres and t in seconds. Using a suitable scale, plot the path of the particle at moments $t_1 = 1$ sec, $t_2 = 2$ sec, $t_3 = 3$ sec, $t_4 = 4$ sec, and $t_5 = 5$ sec.

33. The trajectory of a moving particle is determined by the coordinates $x = 10t$ and $y = 10 + 9t$, in which t is the time in seconds. Plot the trajectory.

34. Describe the motion of the particle represented by the displacement curves in Figs 99 and 100, stating a) whether the particle is moving incessantly or whether at some interval of

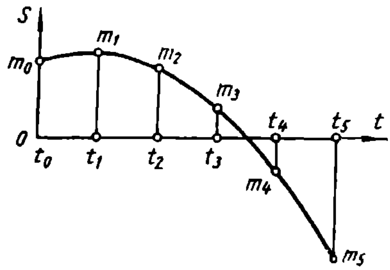


Fig. 99

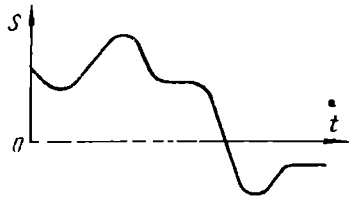


Fig. 100

time it is motionless with respect to the origin, b) at what interval of time it approaches the origin in the area of positive displacement, c) at what interval of time it approaches the origin in the area of negative displacement, d) whether or not the particle passes through the origin and at what moment, e) at what moment of time the particle is furthest from the origin

CHAPTER VII

RECTILINEAR MOTION OF A PARTICLE

61. Uniform Motion

The simplest kind of motion of a particle is when its trajectory is a straight line, in which instance the particle is said to have rectilinear motion. But as we have already noted, a knowledge of the shape of its trajectory is not sufficient to fully define the motion of a point. It is also necessary to know its displacement from its origin, i.e., from its fixed point of reference.

Assume that a particle in traversing a rectilinear trajectory AB (Fig. 101) is at the initial moment at M_0 — a distance of $OM_0 = s_0$ from the origin O. As it moves to the right it comes to point M_1 at moment t_1, a distance of $OM_1 = s_1$, and at moment t_2 at point M_2 a distance of $OM_2 = s_2$, from the origin. Accordingly, during the interval of time $t_1 - t_0$ the particle covers a distance $s_1 - s_0$, and during the interval of time $t_2 - t_1$ a distance $s_2 - s_1$. Dividing the distances traversed by the corresponding intervals of time we obtain $\frac{s_1 - s_0}{t_1 - t_0}$ and $\frac{s_2 - s_1}{t_2 - t_1}$.

Let us assume that the above ratios are equal:

$$t_1 \quad t_0 \; - \; \overline{t_2} - t_1 .$$

This would mean that the distances covered by the particle are equal during equal intervals of time. When this is true, the motion is said to be *uniform*, and the length of the path traversed by the particle increases as many times as the corresponding intervals of time. In brief, we may say that *when a particle possesses uniform motion the distance it traverses is directly proportional to the time expended.*

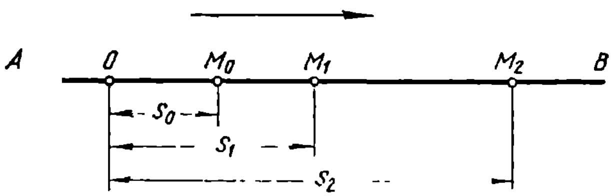


Fig. 101.

If $t_2 - t_1 = t_1 - t_0$, then $s_2 - s_1 = s_1 - s_0$, that is, *when a particle possessed uniform motion, the distances it traverses during equal intervals of time are equal to each other.*

62. Velocity and Displacement When Motion Is Uniform

Let s designate the general displacement of a particle possessing uniform motion for a given interval of time t. Accordingly, the greater the distance s traversed and the less time expended during this displacement, the faster will be the motion or velocity. Then if we designate velocity as v, we obtain

$$v = \frac{s}{t}, \tag{19}$$

that is, *velocity of uniform motion is expressed by a quotient obtained when the distance traversed by a particle is divided by the time expended.*

If at the initial moment the particle is at a distance s_0 from the origin, and at the end of the interval of time t is at a distance s from the origin, then its velocity will be expressed as

$$v = \frac{s - s_0}{t}. \tag{20}$$

From this equation we obtain

$$s = s_0 + vt, \tag{21}$$

in which s_0 represents the displacement of the particle from the origin at the initial moment. If the position of the particle at the initial moment is taken as the origin, the displacement s_0 will be zero and the distance traversed will be

$$s = vt. \tag{22}$$

Wherefore, *the distance traversed by a particle engaged in uniform motion is equal to its velocity multiplied by the time in which the distance is covered.*

Since distance is measured in units of length, therefore velocity is expressed as $\frac{\text{unit of length}}{\text{unit of time}}$.

If the metre is taken as the unit of length and the second as the unit of time, velocity is expressed as $\frac{m}{sec}$; if length is in kilometres and time is in hours, velocity will be $\frac{km}{hr}$, etc. Velocity may be converted from one unit into another, as for example:

$$1\ \frac{km}{hr} = \frac{1{,}000\ m}{60\ min} = \frac{1{,}000\ m^{*}}{60 \cdot 60\ sec}, \text{ etc.}$$

Velocity is determined not only by its numerical value but also by its *direction*. Therefore *velocity is a vector quantity***. In the case of rectilinear motion, velocity is directed along the trajectory in the direction of motion.

Illustrative Problem 30. A 1,000 mm shaft is being machined on a lathe. If the spindle executes 800 revolutions per minute and the feed is 0.2 mm per revolution, how long will it take the cutter to pass down the entire length of the shaft?

Solution: first the velocity of the cutter must be found. At 800 rpm the cutter moves at the rate of $0.2 \times 800 = 160$ mm per min, that is, its velocity is

$$v = 160\ \frac{mm}{min}.$$

To execute the whole operation, the cutter must move along the length of the bedway for a distance $s = 1{,}000$ mm. Accordingly, the required time $t = \frac{s}{v} = \frac{1{,}000}{160} = 6.25\ \text{min} = 6\ \text{min}\ 15\ \text{sec}$.

63. The Graph Illustrating Displacement and Velocity for Uniform Motion

Let us consider how to plot a graph expressing the relationship between displacement and time for uniform motion.

Delineate a rectangular system of coordinates with the time axis Ot and the displacement axis Os (Fig. 102*a*). Lay out on the

* We know from algebra that $\frac{a}{b} = ab^{-1}$, therefore velocity may sometimes be expressed as $m \times sec^{-1}$, $m \times min^{-1}$, etc.

** Vectors of velocity are designated just as vectors of force (Sec. 11).

ordinate axis the segment OA representing at a definite scale the displacement of the moving particle at the initial moment from the fixed point of reference. Then by applying Eq. (21), calculate the displacement s of the particle from the origin at moments t_1, t_2, t_3, etc., and construct a displacement-time graph as was shown before. We will thus find that the line passing through points A, m_1, m_2, etc., is straight. From this it follows that to construct the line AB, it is sufficient to lay out the linear segment OA representing the displacement s_0 of the particle at the initial moment, and the ordinate of one other moment. By thus connecting the two points with the line AB we obtain in graphic form the relationship given in Eq. (21).

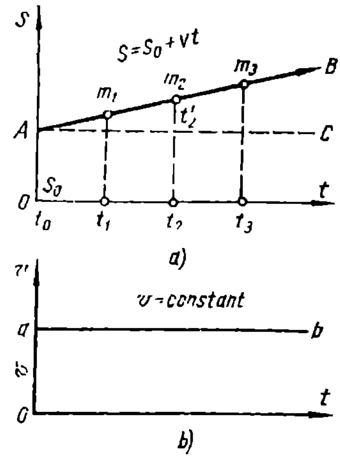


Fig. 102

With such a diagram it is possible to determine for any given moment the displacement of the moving particle from the origin and the distance it has covered. For instance, its displacement at moment t_2 is represented by the ordinate t_2m_2, and the distance covered in the interval of time $t_2 - t_0$ is shown by segment $t_2'm_2$.

Now let us take another rectangular system of coordinates (Fig. 102*b*) where the axis Ot represents time as before, and the ordinate axis Ov shows velocity, all at an appropriate scale (ordinate Oa).

Since the velocity is uniform, it can be illustrated by a straight line ab from point a parallel with Ot.

These graphs illustrate an instance when the particle is moving in the same direction as its initial displacement s_0, as laid out from the origin, and when the movement is positive. In this case the displacement of the particle has increased from the origin. But if motion were in the opposite direction, its velocity would

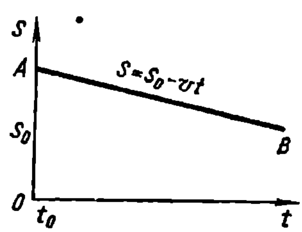


Fig. 103

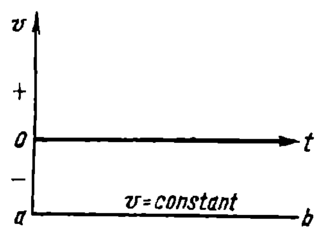


Fig. 104

be negative and Eq. (21) would take the form

$$s = s_0 - vt. \tag{23}$$

Accordingly, the displacement of the moving particle from the origin would diminish with time (Fig. 103) and its velocity while remaining constant would become negative; hence the linear segment representing it would be constructed below axis Ot (Fig. 104).

Since the displacement-time relationship is expressed by a straight line, *uniform motion obeys the principle of the straight line.*

Oral Exercises

Displacement-time graphs for two particles having uniform motion are plotted at similar scales both for time and displacement. The line AB for one particle forms a greater angle with horizontal line AC (Fig. 102*a*) than for the other. What can be said about the velocities of these two particles?

Illustrative Problem 31. A workpiece 2,800 mm long is being machined on a planer with a cutting speed of $v_p = 21$ m/min and a speed on the return stroke of $v_x = 30$ m/min. Construct the displacement-time and velocity-time graphs.

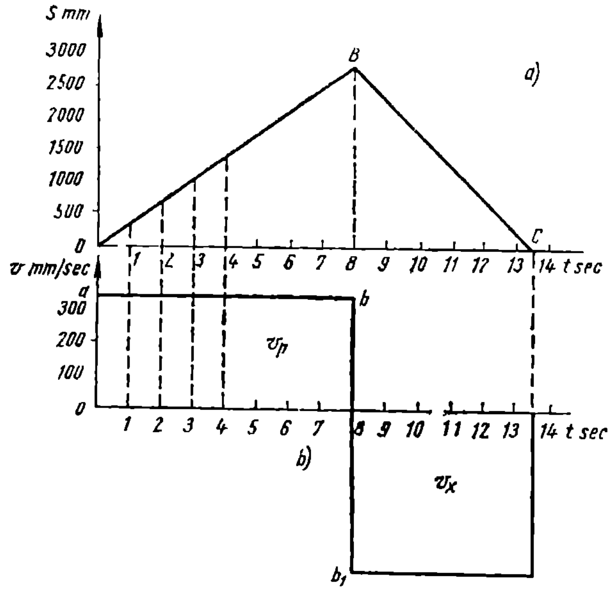


Fig. 105

Solution: at the velocities indicated, the time required for the cutting stroke

$$t_{cs} = \frac{2.8}{21} = \frac{2}{15} \text{ min} = \frac{2}{15} \times 60 \text{ sec} = 8 \text{ sec},$$

while the time required for the return stroke

$$t_{rs} = \frac{2.8}{30} = \frac{7}{75} \text{ min} = \frac{7}{75} \times 60 \text{ sec} = 5.6 \text{ sec}.$$

We delineate axes Ot, Os and Ot, and Ov (Fig. 105). On the Ot axes we lay out the time at a scale of 5 mm = 1 sec; on axis Os we lay out displacement at a scale of 1 : 100; and on axis Ov we take a scale of 1 mm = 20 mm sec for the velocity.

At the end of the eighth second, displacement of any particle in the workpiece is 2,800 mm from the origin (Fig. 105*a*, point *B*). After this the workpiece moves in the opposite direction, and in 13.6 sec is at its initial position at point C.

Velocity v = 21 m/min = 350 mm/sec and is constant till the end of the eighth second (point *b* in Fig. 105*b*), after which it changes in sign (the planer's table begins to move in the opposite direction).

64. Variable (or Non-Uniform) Motion, and Averages of Velocity and Acceleration

When a particle covers different distances in equal intervals of time, it is said to have *variable*, or *non-uniform*, motion.

Let s_1 represent the displacement of a particle from the origin at the moment t_1, and s_2 show its displacement at moment t_2. Then the distance covered during the interval of time $t_2 - t_1$ will be equal to $s_2 - s_1$. By dividing this distance by the corresponding time interval, we obtain a velocity v_{av} called *average velocity* for the given interval of time:

$$v_{av} = \frac{s_2 - s_1}{t_2 - t_1}. \tag{24}$$

Actually in the given example the particle does not travel at a constant velocity during the entire time interval. Average velocity v_{av} is merely the speed at which the particle would traverse the same distance $(s_2 - s_1)$ in the same interval of time $(t_2 - t_1)$ if it moved at a uniform speed. Therefore average velocity does not give the actual velocities at which the particle moves at various moments of time. Nevertheless, in engineering it is often necessary to know average velocity.

Variable motion differs from average velocity in that it refers to *a very small interval of time*; hence the actual velocity of a particle having variable motion is *instantaneous* for the given moment. But if from a given moment of time t the motion should become uniform, the instantaneous velocity at that given moment would be equal to its succeeding uniform motion.

From this it is apparent that the smaller the interval of time in Eq. (24), the closer will the average velocity be to instantaneous velocity.

Since the velocity of a particle possessing variable motion is not constant, it is continually receiving a certain acceleration which may be either positive or negative. In the first instance velocity will increase, while in the second it will decrease.

If at moment t_1 the velocity is v_1, and at moment t_2 it is v_2, *the difference in velocity $v_2 - v_1$ divided by the interval of time $t_2 - t_1$ will be equal to the average acceleration a_{av} for that interval of time*

$$a_{av} = \frac{v_2 - v_1}{t_2 - t_1} \qquad (25)$$

Just as with velocity, the smaller the interval of time $t_2 - t_1$, the closer will be the average acceleration to instantaneous acceleration

Similar to velocity, acceleration is a vector quantity And if the sign of acceleration is the same as that of velocity, it will have the same direction as the motion If, on the contrary, its sign differs, then its direction will be opposite to the motion

As we see from Eq (25), acceleration is expressed by

$$\frac{\text{unit of length}}{\text{unit of time}} : \text{unit of time} = \frac{\text{unit of length}}{(\text{unit of time})^2}.$$

Thus, if velocity is expressed as m/sec, the measuring unit of acceleration will be m/sec² = m · sec⁻² (to be read *metres per second per second*)

Illustrative Problem 32. The ram of a shaping machine, moving non-uniformly, completes a cutting stroke of 400 mm in 1.25 sec By dividing 1.25 sec into 8 equal intervals it was found that during the first interval the cutter moved a distance of $s_1 = 22$ mm, in the second interval it moved $s_2 - s_1 = 71 - 22 = 19$ mm, in the third interval it moved $s_3 - s_2 = 134 - 71 = 63$ mm, in the fourth interval the movement was $s_4 - s_3 = 200 - 134 = 66$ mm Find the average velocity of the ram for the entire 1.25 sec and then for each of the four equal intervals of the given time

Solution The average velocity for the entire 1.25 sec will be

$$v_{av} = \frac{400}{1.25} = 320 \text{ mm/sec} = 19.2 \text{ m/min}$$

For the first interval of time $t_1 - t_0$

$$v_{av_1} = \frac{22 \times 8}{1.25} \approx 141 \text{ mm/sec} = 8.45 \text{ m/min}$$

For the second interval of time $t_2 - t_1$

$$v_{av_2} = \frac{19 \times 8}{1.25} \approx 314 \text{ mm/sec} = 18.82 \text{ m/min}$$

For the third interval of time $t_3 - t_2$

$$v_{av_3} = \frac{63 \times 8}{1.25} = 403 \text{ mm/sec} = 24.19 \text{ m/min}$$

For the fourth interval of time $t_4 - t_3$

$$v_{av_4} = \frac{66 \times 8}{1.25} \approx 422 \text{ mm/sec} = 25.34 \text{ m/min}.$$

Thus we see that the average velocities for separate intervals of time greatly differ not only from each other, but also from the average velocity for the entire stroke of the ram.

65. Uniformly-Variable Motion. Velocity and Acceleration

The simplest form of variable motion is that which is *uniformly variable*, i.e., when the change in velocity is equal for like intervals of time. To express this in another way it may be said that *variable motion, in respect to which acceleration is constant, is uniformly accelerated.*

Let us see how the velocity of a uniformly-accelerated particle is determined for a given moment.

Let the velocity of the moving particle at the initial moment be v_0. If the acceleration is a, then the increase in velocity during the interval of time t will be at. Hence, the velocity at the end of the interval will be

$$v_t = v_0 + at. \quad (26)$$

If the initial velocity of the particle $v_0 = 0$, the final velocity will be

$$v_t = at. \quad (27)$$

But it must be borne in mind that acceleration may be either positive or negative. If it is positive, it will have the same direction as the motion, and the motion is then known as *constant acceleration*. If it is negative, its direction will be opposite to the motion and the motion is then said to have *constant deceleration*. In the latter case, acceleration is written with a negative sign in Eq. (26).

Oral Exercises

1. How does the velocity of a moving point that possesses uniformly-variable motion change if acceleration is positive?
2. How does it change if acceleration is negative?

Illustrative Problem 33. A train travelling at a velocity of 45 km/hr began going downgrade and increased its velocity to 54 km/hr in 1.5 min. Find its acceleration.

Solution: applying Eq. (26), initial velocity $v_0 = 45$ km/hr = = 12.5 m/sec and the interval of time $t = 1.5$ min = 90 sec; velocity at the end of this interval will be

$$v_t = v_{90} = 54 \text{ km/hr} = 15 \text{ m/sec}.$$

Substituting for numerical values, we obtain

$$15 = 12.5 + a \times 90, \text{ from which } a = \frac{15 - 12.5}{90} = 0.028 \text{ m/sec}^2.$$

66. Displacement When Motion Is Uniformly Accelerated

Having found how to determine velocity at any given moment for a moving particle possessing constant acceleration, let us now find its displacement. We shall begin by expressing Eq. (26)

graphically to show the relationship between velocity, acceleration, and time.

We shall use the rectangular system of coordinates Ot and Ov (Fig. 106), with time as the axis of abscissae and velocity as the axis of ordinates. We have already seen (Fig. 102*b*) that when motion is uniform (which means that velocity is constant) the velocity-time graph is a straight line parallel to the time axis Ot. When motion acquires constant acceleration, this line will be sloping and form an angle with axis Ot.

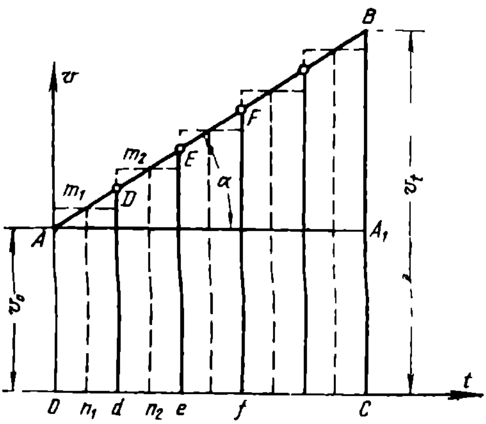


Fig. 106

At the initial moment when $t = 0$, the velocity of the particle will be equal to v_0. Therefore we delineate the linear segment OA on axis Ov, thus representing to scale the magnitude of the velocity at that moment. When motion has acquired constant acceleration, the increase in velocity will be proportional in time. Hence, after calculating the velocity for a certain moment of time, we construct the perpendicular at the corresponding point on the abscissae axis and on it we lay out the velocity to scale. Then we plot a straight line through point A and the point obtained, thus constructing a velocity-time graph which expresses the principle for changes in velocity.

In order to find displacement s of a moving particle during a given time interval t, we divide this time interval into several equal parts (Od dc $ef = \ldots$). Then by adding the initial velocity and the final velocity for each of these parts and dividing the sums by 2, we find their average velocity. In this way we calculate that during the time interval Od there is uniform motion with velocity expressed by the ordinate $n_1m_1 = \frac{OA + dD}{2}$; during the time interval de velocity is expressed by the ordinate $n_2m_2 = \frac{dD + eE}{2}$, and so forth.

Then we delineate a straight line through point m_1 parallel to axis Ot. In the resulting rectangle the base Od expresses the interval of time in which the motion takes place, while its altitude n_1m_1 shows the velocity. Accordingly, the area of the rectangle, measured at a corresponding scale, will give the displacement of the particle moving uniformly during the interval of time

Od. This area is equal to the area of the trapezoid $OADd$ because n_1m_1 is its middle line.

Therefore the displacement of the particle during the time interval Od is represented by the area of the trapezoid $OADd$. In the same way we can prove that its displacement during interval de is represented by the area of the trapezoid $dDEe$, etc.

Hence the path traversed by a particle possessing constant acceleration during the time interval as shown by the linear segment OC, is given at a corresponding scale by the area of the trapezoid $OABC$ bounded by the ordinates equalling the initial and final velocities, the velocity curve (when the motion has constant acceleration, by the line AB), and the time axis.

On this basis we may say that displacement

$$s = \frac{v_0 + v_t}{2} t;$$

and if we replace v_t by $v_0 + at$, we obtain

$$s = v_0 t + \frac{at^2}{2}. \qquad (28)$$

From Fig. 106 we see that component $v_0 t$ is expressed by the area of the rectangle OAA_1C, and the second component $\frac{at^2}{2}$ by the area of triangle ABA_1, inasmuch as A_1B represents the increase in velocity at, while AA_1 is the time t.

Therefore, *the displacement of a particle possessing constant acceleration is equal to the product of the initial velocity and time, plus half the product of the acceleration and the square of time.*

Sometimes in determining displacement it is more convenient to use a different equation derived from Eq. (28) as follows.

From Eq. (26) we evolve

$$t = \frac{v_t - v_0}{a}.$$

If we substitute this value for t in Eq. (28), then

$$s = v_0 \frac{v_t - v_0}{a} + \frac{a}{2} \times \frac{(v_t - v_0)^2}{a^2},$$

from which

$$s = \frac{v_t^2 - v_0^2}{2a}. \qquad (29)$$

Accordingly, *the displacement of a point is equal to half the difference of the squares of the final and initial velocities divided by the acceleration.*

It should be understood from the above that the value of acceleration must be inserted into these equations with the correct sign: if the motion possesses constant acceleration it will have a plus sign; but if it is constant deceleration, then the sign will be minus.

If at the initial moment of the interval of time from which reference is taken the particle's speed is zero, then $v_0 = 0$ should be used in Eqs (28) and (29), in which case

$$s = \frac{at^2}{2} \tag{30}$$

and

$$s = \frac{v_t}{2a}. \tag{31}$$

If the particle moves with constant deceleration and stops at the end of t seconds, then $v_t = 0$ in Eqs (26) and (29).

The same units of measure must be used on both sides in all equations. Let us take Eq. (28) as an example. If the left side is expressed in metres, the first member of the right side will be in $\frac{m}{sec} \times sec = m$, and the second member $\frac{m}{sec^2} \times sec^2 = m$. We thus give all the members of the equation the same units of measure.

Oral Exercises

1. What will be the direction of line AB in Fig. 106 when motion possesses constant deceleration?
2. Are all the members of Eq. (29) in the same units of measure?

Illustrative Problem 34. While travelling at a speed of 45 km/hr a train began going downgrade at a constant acceleration and covered the entire 2,500 m of downgrade in two minutes. What was the train's acceleration on the downgrade and at what speed was it travelling when it reached level track.

Solution: the train's initial speed $v_0 = 45$ km/hr $= 12.5$ m/sec. By employing Eq. (28) we obtain

$$2{,}500 = 12.5 \times 120 + \frac{a \times 120^2}{2}, \text{ from which } a = 0.139 \text{ m/sec}^2.$$

Hence the train's acceleration $a = 0.139$ m/sec² and when it reached level trackage it was travelling at a speed of

$$v_{1\,0} = 12.5 + 0.139 \times 120 = 29.18 \text{ m/sec} = 105.1 \text{ km/hr}.$$

Illustrative Problem 35. A train was travelling at a speed of 72 km/hr when the brakes were applied. It then travelled with constant deceleration for three minutes before it came to a dead stop. How far did the train travel from the time the brakes were applied till it came to a dead stop?

Solution: employing Eq. (26) in which the final speed $v_t = 0$, we determine the acceleration a: $v_0 = 72$ km/hr $= 20$ m/sec, and $t = 180$ sec, whence we derive $0 = 20 + a \times 180$, from which $a = \frac{1}{9}$ m/sec².

Now Eq. (28) can be used to find the distance the train travelled after braking:

$$s = 20 \times 180 - \frac{180^2}{9 \times 2} = 1{,}800 \text{ m} = 1.8 \text{ km}.$$

Illustrative Problem 36. A train was travelling at a speed of 54 km/hr when its brakes were applied, from which time it travelled 900 m with constant deceleration before it came to a dead stop. How long did it take the train to stop after the brakes were applied?

Solution: we find acceleration from Eq. (29):

$$a = \frac{v_t^2 - v_0^2}{2s}.$$

If $v_t = 0$, $v_0 = 54$ km/hr $= 15$ m/sec, and $s = 900$ m, we obtain $a = -0.125$ m/sec². By using Eq. (26) in which $v_t = 0$, $v_0 = 15$ m/sec, and $a = -0.125$ m/sec², we obtain $t = 120$ sec $= 2$ min.

67. Vertical Motion Under the Force of Gravity

The vertical motion of a body* under the force of gravity is an example of rectilinear motion with constant acceleration. When a body is thrown upwards with a certain initial velocity its motion will be evenly retarded, i.e., its velocity will gradually diminish; and when it has reached a certain height the body will pause for an instant and then begin falling with constant acceleration. Acceleration due to gravity is always the same —9.81 m/sec² and is designated by the letter g.

In order to apply equations (26-31) deduced for uniformly-variable motion, the acceleration of gravity g is used instead of acceleration a, and with the appropriate sign as a prefix.

A body projected vertically upwards with an initial velocity v_0 will acquire *constant deceleration* inasmuch as the force of gravity acts in the opposite direction, in which case g must be used with a *minus* sign and Eq. (26) will be

$$v_t = v_0 - gt \tag{32}$$

The height h which a body thrown upwards will reach from the initial moment, is found through Eq. (28) as follows:

$$h = v_0 t - \frac{gt^2}{2}, \tag{33}$$

while Eq. (29) gives

$$h = \frac{v_t^2 - v_0^2}{-2g},$$

or

$$h = \frac{v_0^2 - v_t^2}{2g}. \tag{34}$$

When the body reaches its highest point, its velocity v_t becomes zero and accordingly Eq. (32) becomes

$$v_0 = gt,$$

* The motion of a body may be regarded as the motion of its centre of gravity and the body considered a material point.

from which

$$t = \qquad (35)$$

Wherefore, *the time consumed for a body to rise to its highest point is equal to its initial velocity divided by the acceleration of the force of gravity.* In this case Eq. (34) becomes

$$h = \frac{v_0^2}{2g}. \qquad (36)$$

from which

$$v_0^2 = 2gh,$$

or

$$v_0 = \sqrt{2gh}. \qquad (37)$$

Wherefore, *initial velocity is equal to the square root of twice the product of the height multiplied by the acceleration of gravity.*

When a body is falling freely, its movement coincides with the direction of gravity acceleration, for which reason it then possesses *constant acceleration,* and gravity acceleration g must therefore be used with a *plus* sign.

If the initial velocity of a falling body is zero, then $v_0 = 0$, and Eqs (27), (30) and (31) respectively become

$$v_t = gt, \qquad (38)$$

$$h = \frac{gt^2}{2} \qquad (39)$$

$$h = \frac{v_t^2}{2g} \qquad (40)$$

From Eq. (40) we obtain

$$v_t^2 = 2gh,$$

or

$$v_t = \sqrt{2gh}. \qquad (41)$$

Wherefore, *the velocity of a body at the end of its fall is equal to the square root of twice the product of gravity acceleration multiplied by the height of the fall.*

A comparison of Eqs (37) and (41) will show that $v_t = v_0$. Wherefore, *the final velocity of a falling body is the same as its initial velocity but opposite in direction.*

Fig. 107*a* shows the displacement-time curve of a freely falling body with an initial velocity $v_0 = 0$; the time axis Ot is divided into equal segments each of which represents 0.5 sec, while each division of the displacement axis Os represents one metre. Using Eq. (39) and taking succeeding numerical values of t as 0.5 sec, 1 sec, etc., and g as 9.81 m/sec², we will find corresponding displacement of a body from its initial position, i.e., 1.226 m in 0.5 sec, 4.905 m in 1.0 sec, 11.036 m in 1.5 sec, and 19.62 m in 2.0 sec, etc. By constructing the ordinates for these moments of time, we then obtain a number of points to connect with a smooth line OA* which is accordingly the displacement-time curve. If, for example, it be necessary to find how far the body fell in 1.75 sec after the initial moment, we find the point on the axis of abscissae that represents the moment and construct a perpendicular to it to find its displacement.

Fig. 107*b* is a velocity-time graph. As is apparent from Eq. (38), velocity changes in direct proportion to time, i.e., the relationship between velocity and time is expressed by a straight line. Let us then employ Eq. (38) to find the velocity at some given moment, for instance, at the end of the first second $v_1 = 9.81 \times 1 = 9.81$ m/sec and plot a velocity-time graph to a scale. Since the velocity at the initial moment is zero, we delineate OB from the origin through the point obtained. This is the velocity-time curve.

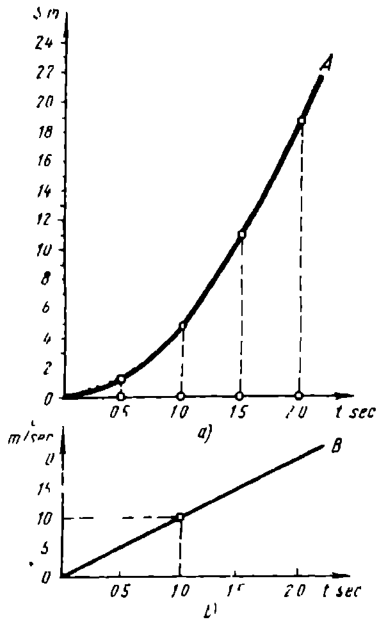


Fig. 107

Illustrative Problem 37. From what height would a body fall if it takes ten seconds to reach the ground, and what is its velocity at the final moment?

Solution: from Eq. (39)

$$h = \frac{9.81 \times 10^2}{2} = 490.5 \text{ m}$$

And from Eq. (38)

$$v_t = 9.81 \times 10 = 98.1 \text{ m/sec}.$$

68. Questions for Review

1. What is the difference between non-uniform motion and uniform motion?
2. State the law governing the displacement of a uniformly-moving particle from the origin.
3. State the law governing the change in velocity of a particle possessing uniformly-variable motion when its initial velocity is zero.
4. When is acceleration considered positive and when negative?
5. What kind of motion has a body when projected upwards?
6. What kind of motion has a freely falling body?

69. Exercises

35. A workpiece is being machined on a planer whose cutting stroke is 1,500 mm. It takes the machine nine seconds to complete a cutting and return stroke. Find the velocity v_{cs} of the cutting stroke and velocity v_{rs} of the return stroke if the latter is twice the former.

36. One minute after leaving the station a train had travelled 450 m with constant acceleration. Find its acceleration a and velocity v.

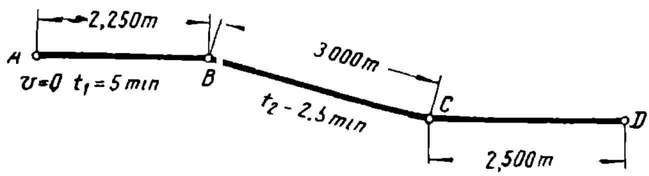


Fig. 108

37. A train is travelling from A to D along the stretch of track represented in Fig. 108. Its initial speed at A is zero. It takes the train 5 minutes to cover the level stretch of track AB which is 2,250 m in length, and 2.5 min to cover the downgrade BC which is 3,000 m in length. On reaching C on the level stretch, the brakes are applied and the train stops 2,500 m beyond, at D. Find the deceleration on stretch CD, the time it takes the train to get from A to D, and its average speed for the whole distance.

38. What height will a stone reach, and how much time will its entire flight take (upward and downward) if it is hurled vertically upward with an initial velocity $v_0 = 39.24$ m/sec?

39. Draw the displacement-time and velocity-time graphs for a body hurled vertically upward with an initial velocity $v_0 = 19.62$ m/sec.

CHAPTER VIII

THE COMPOSITION OF SIMPLE MOTIONS OF A PARTICLE

70. Compound Motion, and Absolute and Relative Motion

Let us assume that an overhead crane (Fig. 109) is transporting a load along a factory shop. The crane travels the length of the shop in the direction of the arrow A. At the same time the crane's crab, to which the load is hung by means of the hook K, is moving athwart the overhead crane in the direction shown by arrow B.

It is seen that the motion of the load is the sum of two motions at right angles to each other: the motion of the overhead crane with respect to the earth, and the motion of the crane's crab with respect to the overhead crane. Hence the motion of the load is compound and its nature depends upon the motion of the crane and its crab, i.e., upon component motions. The motion of the overhead crane in respect to the earth is called *absolute* motion, while that of its crab in respect to the crane is known as *relative* motion.

If the crane moves a distance KA in respect to the earth and the crab's hook simultaneously moves a distance KB in relation

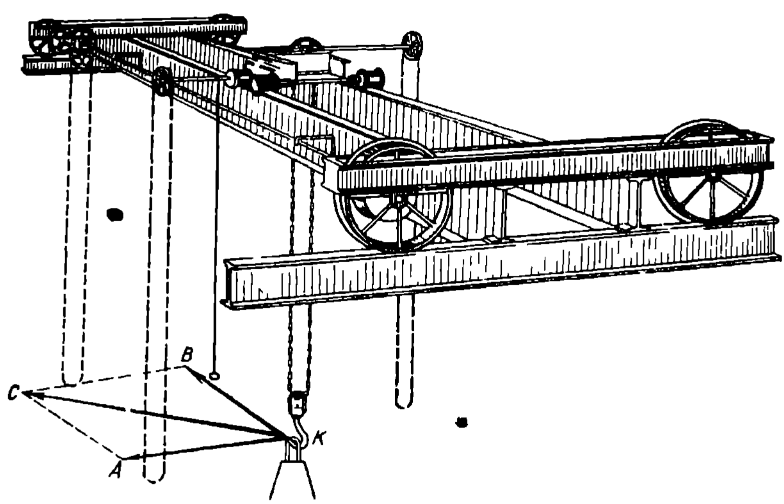


Fig. 109

to the crane, it may be said that the displacement due to absolute motion is equal to KA, while the displacement due to relative motion is equal to KB.

Since all bodies are actually always in motion, then all kinds of motion dealt with in mechanics are relative and in each individual case we arbitrarily assume one or another body to be motionless. In most instances the motion of a body is measured in relation to the earth and we call the motion of that body absolute. Thus in the cited example the movement of the load in respect to the overhead crane is relative motion, while the movements of the crane itself and the load relative to the shop is absolute motion.

In this example the motion of the load is conditioned by both absolute and relative motion and it is such compound motion that we most often have to deal with in machines. However, mechanics is also concerned with the motion of bodies that are not connected with each other. For instance, let us assume a train leaves a station. Subsequently, after a sufficient lapse of time, another train will be sent out after it along the same track so that the two trains will at no time approach each other closer than safety permits. In solving such a problem the thing that interests us above all is the relative speed of both trains and the distance between them.

71. The Composition of Uniform Collinear Motions

The simplest case of compound motion is that of two collinear components having either the same or opposite directions.

Fig. 110 represents two bodies *1* and *2*, in contact along plane AB. At the initial moment, point M_2 on body *2* is in contact with point M_1 on body *1*. Let us assume that the two bodies are moving at the same time in such a way that at moment t point M_1 has moved from left to right for a distance s_1 in respect to an immovable surface, and point M_2 has moved a distance s_2 from right to left in respect to point M_1. In other words, the displacement of point M_2 due to absolute motion from left to right is designated by s_1, and displacement due to relative motion from right to left is indicated by s_2. What is the resultant displacement of point M_2?

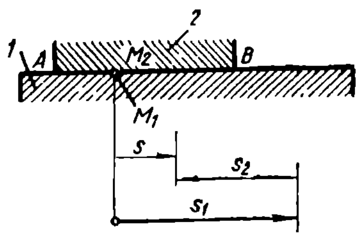


Fig. 110

To answer this question we reason in the following way: assume that point M_2 was not displaced in respect to point M_1, in which case its absolute motion would also be equal to s_1 and would be acting from left to right. But since point M_2 was actually displaced in respect to M_1 from right to left for a distance s_2, then its displacement in respect to the immovable plane, that is, its resultant displacement from left to right, becomes

$$s = s_1 - s_2.$$

Obviously if both displacements had been from left to right, the resultant displacement of point M_2 would also have been from left to right:

$$s = s_1 + s_2.$$

By considering displacement from left to right as positive and displacement from right to left as negative, and assuming that both displacements had been from right to left, we would compute as follows:

$$-s = -s_1 + (-s_2) = -(s_1 + s_2).$$

By resorting to the same reasoning in dealing with any number of component motions, we would find that *in compound rectilinear motion the absolute displacement of a point is equal to the algebraic sum of the component displacements.* This can be expressed by the following equation:

$$s = s_1 + s_2 + s_3 + \dots + s_n, \tag{42}$$

in which each component displacement must be prefixed with its proper sign.

Assume that all component displacements have uniform motion and occur within a certain interval of time t. We shall denote their velocities as $v_1, v_2, v_3 \dots v_n$. Whereupon $s_1 = v_1 t$, $s_2 = v_2 t$, $s_3 = v_3 t \dots$, $s_n = v_n t$. By substituting these values for the displacements in Eq. (42) we obtain

$$s = v_1 t + v_2 t + v_3 t + \dots + v_n t = (v_1 + v_2 + v_3 + \dots + v_n) t,$$

from which $\frac{s}{t} = v_1 + v_2 + v_3 + \dots + v_n$.

However, $\frac{s}{t} = v$ which is the velocity of the compound motion and also uniform. Accordingly,

$$v = v_1 + v_2 + v_3 + \dots + v_n. \tag{43}$$

Wherefore, *if the components of compound motion are collinear and uniform, the velocity of the compound motion is equal to the algebraic sum of the velocities of the components.*

Oral Exercises

1. If a particle possesses two kinds of motion, can its absolute displacement be zero at any moment, and under what conditions?

2. At a certain moment, point M, on body *2* in Fig. 110 is in contact with point M_1 of body *1*, after which point M_1 moves from left to right for a distance s_1, and point M_2 moves from right to left for a distance s_2 in respect to point M_1 during the same interval of time. Find the absolute displacement of M and its direction in each of the following four cases: a) when $s_1 > s_2$, b) when $s_1 < s_2$, c) when $s_1 = s_2$, d) when $s_1 = 0$.

Illustrative Problem 38. Town B is situated 22.5 km down the river from town A. A boat makes the trip from A to B in 1.5 hr, and from B to A in 2.5 hr. Assuming the motion of the boat to be uniform, find the velocity of the current $\mathbf{v}_1$, and the velocity of the boat $\mathbf{v}_2$ with respect to the water.

Solution: velocity $\mathbf{v}_2$ represents the velocity of the boat in relation to the water, irrespective of whether the water is flowing or standing still. Therefore in moving with the current, the boat moves with an absolute velocity, in respect to the bank, of $v_1 + v_2$. In moving against the current the absolute velocity of the boat is $v_2 - v_1$. Hence we have two equations:

$$(v_1 + v_2) \times 1.5 = 22.5 \text{ and } (v_2 - v_1) \times 2.5 = 22.5.$$

By solving these equations we obtain $v_1 = 3$ km/hr and $v_2 = 12$ km/hr.

72. The Composition of Rectilinear Uniform Motions Which Are at an Angle to One Another

Now let us learn how to combine rectilinear uniform motions when they are directed at an angle to one another.

Assume that we have set the longitudinal feed of a lathe so as to give the carriage an axial displacement of AB (Fig. 111), and as it moves we actuate the cutter with a constant crosswise movement by turning the handle of the cross feed. Thus all points on the cutter receive two motions—the absolute longitudinal motion of the carriage and the relative crosswise motion of the cross feed. Let us investigate the motion of apex A of the cutter. Assume that during a certain interval of time the apex and the carriage are displaced to position M_1, while in the relative motion of the cross feed the apex is displaced to position N_1. Let us assume that these two displacements are successive: apex A is first longitudinally displaced for a distance AM_1 along the axis and then it moves a distance $M_1A_1 = AN_1$ crosswise. As a result of these two displacements, apex A reaches point A_1.

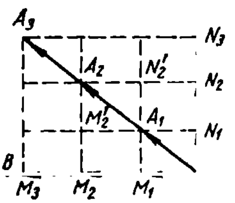


Fig. 111

Thus position A_1, which has been taken up by apex A of the cutter, becomes the vertex of the parallelogram $AM_1A_1N_1$ (in this case a rectangle).

Similarly we find that during the next interval of time the point of the cutter is displaced to point A_2 which is the vertex of the parallelogram $A_1M_2'A_2N_2'$, and so forth with subsequent displacements.

We shall prove that the displacement of the cutter's apex from position A to position A_2 is rectilinear, i.e., that the diagonals AA_1 and A_1A_2 lie on the same straight line. Assume that displacements AM_1, AN_1 and M_1M_2, N_1N_2 occur in equal intervals of time. Then $M_1M_2 = AM_1$ and $N_1N_2 = A_1N_2'$. Since $A_1M_2' = M_1M_2$ and $M_2'A_2 = N_1N_2$, therefore the triangles $A_1M_2'A_2$ and AM_1A_1 are congruent and $\angle A_2A_1M_2' = \angle A_1AM_1$, that is, the linear segments A_1A_2 and AA_1 lie on the same straight line. It also follows from the similarity of the same two triangles that these two linear segments are equal to each other, which means that point A in its compound motion receives equal displacements in equal intervals of time; in short, it is clear that the compound motion is as uniform as its components.

By dividing the displacements by the time which they consumed, we obtain the velocity of each one. Hence, if AM_1 represents the velocity of the absolute motion and AN_1 the velocity of the relative motion, then the diagonal AA_1 will indicate the direction and magnitude of the velocity of the resultant motion.

Wherefore, *the resultant motion of a point having two rectilinear uniform motions is rectilinear and uniform.*

The resultant displacement of a point is equal in magnitude and direction to the diagonal of a parallelogram constructed on the basis of component displacements.

The resultant velocity is equal to the diagonal of a parallelogram constructed on the basis of component velocities.

It can be proved that if the components of a motion have an initial velocity of zero and are uniformly accelerated and rectilinear, the compound motion will also be uniformly accelerated and rectilinear.

Illustrative Problem 39. What should be the ratio between the velocities of the longitudinal feed $\mathbf{v}_1$ of a lathe and the cross feed $\mathbf{v}_2$ in order to cut the truncated cone $ABCE$ shown in Fig. 112*a* if $D = 80$ mm, $d = 60$ mm, and $l = 100$ mm?

Solution: the velocity of longitudinal displacement of the cutter added to the velocity of its crosswise displacement will give the velocity of the compound motion towards the cone, i. e., will be actuated along the diagonal of the parallelogram $A_1F_1E_1E$ constructed on the bases of component velocities A_1F_1 and A_1E_1 (Fig. 112*b*).

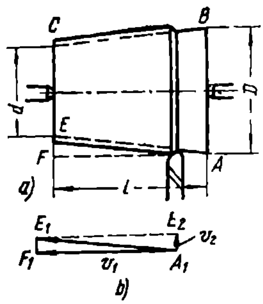


Fig. 112

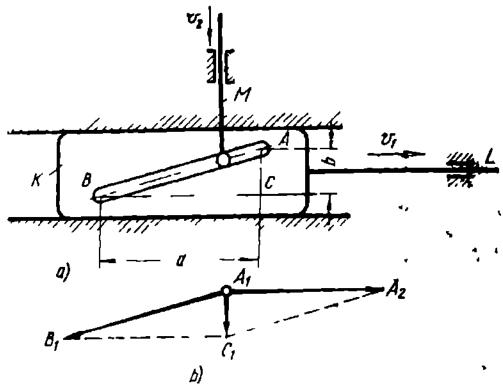


Fig. 113

From the similarity of triangles $A_1F_1E_1$ and AFE it follows that $\frac{A_1F_1}{AF} = \frac{E_1F_1}{EF}$, from which, after substituting the numerical values $AF = 100$ mm and $EF = \frac{D - d}{2} = \frac{80 - 60}{2} = 10$ mm, we find that

$$\frac{v_1}{100} = \frac{v_2}{10}, \text{ from which } \frac{v_1}{v_2} = \frac{100}{10} = 10.$$

Hence the ratio of longitudinal feed to cross feed should be

Illustrative Problem 40. The plunger K under the action of rod L in Fig. 113*a* is in reciprocating motion between fixed guides at a velocity $v_1 = 60$ mm/sec. There is a roller in the groove AB of the plunger to which is fastened a sliding follower M that slips up and down between immovable guides. Find velocity $\mathbf{v}_2$ of the follower if the groove AB forms an angle ABC with the line of motion of the plunger and if $BC = a = 120$ mm and $AC = b = 30$ mm.

Solution: the resultant motion of the follower M may be regarded as a compound motion: the absolute motion of the block moving from left to right during the given moment, and the relative motion of the roller in the groove of the plunger. We therefore construct a parallelogram of velocities on the bases of the velocities of the motion components (Fig. 113*b*). By taking any arbitrary point A_1 and choosing a scale, we lay out vector A_1A_2 representing the velocity $\mathbf{v}_1$ of the plunger and from the same point A_1 we delineate a straight line parallel to the velocity of the follower M to point C_1 where it intersects with line A_2C_1 which is parallel to the axis of the groove AB, and then complete the parallelogram $A_1A_2C_1B_1$. It is evident that the component A_1B_1, which transmits the velocity to the centre of the roller in respect to the plunger, is directed from right to left, as it should be: for if the plunger were moving from left to right, the motion of the roller in respect to the plunger would be in the opposite direction. By measuring the diagonal A_1C_1 of the parallelogram and multiplying its length by the chosen velocity scale, we obtain the velocity of the follower $\mathbf{v}_2$.

This velocity may also be found by calculation, as follows. From the similarity of triangles ABC and $A_1B_1C_1$ we may calculate

$$\frac{A_1C_1}{AC} = \frac{B_1C_1}{BC} \quad \text{or} \quad \frac{v_2}{b} = \frac{v_1}{a},$$

from which

$$v_2 = v_1 \frac{b}{a} = 60 \times \frac{30}{120} = 15 \text{ mm/sec.}$$

73. Resolving a Velocity into Its Components

In mechanics it is frequently found necessary to carry out the reverse of the composition of velocities when it is required to resolve a velocity into two components. In its general form this problem is as indeterminate as the resolution of forces, but in each specific case it is solved in conjunction with additional data (direction of component velocities, magnitude and direction of one of these, etc.), as may be seen from the following example.

Illustrative Problem 41. Drops of rain strike the windows of a railway carriage travelling at a velocity $\mathbf{v}_1$ and leave streaks that form an

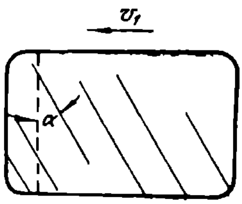


Fig. 114

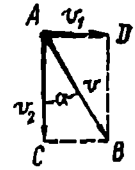


Fig. 115

angle of 30° with the vertical (Fig. 114). Find velocity v_2 of the raindrops in respect to the earth.

Solution: In respect to the window, each drop is moving vertically downwards with a velocity v and horizontally with a velocity v_1' but in the direction opposite to that of the movement of the train. Hence we can construct the parallelogram $ACBD$ (Fig. 115), we lay out vector $\overline{AD}$ representing velocity v_1', delineate a straight line at an angle $\alpha = 30°$ to the vertical, and then plot a vertical line down from point D. These two lines intersect at point B. Then we finish the parallelogram by delineating side AC which represents the velocity of the raindrop v_2 at the same scale as vector AD. By calculation we then find that $v_2 = v_1 \cot \alpha$.

74. Questions for Review

1. The carriage of a lathe is moving from right to left with a certain velocity. The cross feed is set parallel to the axis of the lathe and is moving from left to right with the same velocity. What is the resultant velocity of the cutter?

2. What would be the answer to Question 1 if the cross feed were set at an angle to the axis of the lathe?

3. The belt of an escalator moves upward with a velocity v_1 and a man is walking down the escalator with a velocity v. What is the resultant velocity with which the man moves in the following three cases: a) when $v > v_1$, b) when $v < v_1$, and c) when $v = v_1$?

75. Exercises

40. A steamer, whose speed is 10 km/hr is plying up a river that has a current of 4 km/hr. What is the resultant velocity of the steamer, and what would it be if it were plying through still water?

41. A steamer plying downstream covers 30 km in two hours. In still water the steamer's speed is 12 km/hr. How far could it have travelled upstream in the same two hours?

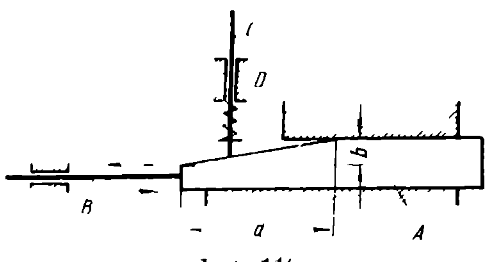


Fig. 116

42. The plunger A in Fig. 116 moves between fixed guides in reciprocating motion under the action of rod B. The end of the follower C is sliding in fixed guides and is pressed to the inclined surface of the plunger by a spring. Find the speed v at which the follower moves when the speed of the plunger is 600 mm/min and if $a = 300$ mm and $b = 50$ mm. Also find speed v_1 with which the end of the follower moves on the inclined surface of the plunger.

CHAPTER IX

CURVILINEAR MOTION OF A PARTICLE

76. Uniform and Non-Uniform Curvilinear Motion of a Particle

Thus far we have been treating rectilinear motion. Now let us examine a more complex kind of motion when the trajectory traversed by a particle is a curved line in one plane.

Fig. 117 represents such a trajectory. At the moment of time t_1 the moving particle is at point M_1, and at the moment of time t_2 it is at point M_2. Therefore, during the interval between t_1 and t_2, the particle has traversed a path as represented by the curved line M_1M_2. If the motion is such that the particle traverses equal distances in equal intervals of time (however small such intervals may be) the motion will be uniform. Otherwise the motion will be non-uniform, or variable. The major difference between curvilinear and rectilinear motion is that in the former the path traversed by a moving particle is composed of curved segments instead of straight ones.

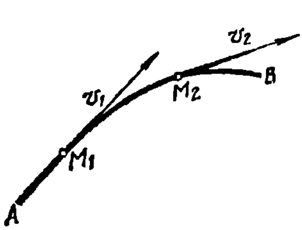


Fig. 117

77. The Velocity of a Particle Possessing Curvilinear Motion

The *rate* of velocity of a particle possessing curvilinear motion is determined in the same way as for one of rectilinear motion, except that it will be a quotient derived by dividing the trajectory's *curved-line segments* by corresponding intervals of time. Thus, when the motion of the particle displaced from point M_1 to point M_2 (Fig. 117) is uniform, its velocity is expressed as a quotient obtained by dividing the length of the arc M_1M_2 by the time taken by the particle to traverse that distance. If the motion were non-uniform, this quotient would represent average velocity. And the shorter the arc M_1M_2, the closer that average velocity will be to the actual (instantaneous) velocity of the particle.

Now let us learn how to determine the *direction* of velocity of a particle having curvilinear motion.

When a particle has rectilinear motion its direction remains constant, whereas with curvilinear motion its direction continually changes according to the curvature of its trajectory. From this we conclude that the direction of its velocity also changes.

How then is the direction of velocity determined?

Let us assume that at the moment the moving particle is at position M_1 (Fig. 117), the constraint causing it to diverge from a rectilinear path were removed. Obviously from that point the particle would move in a straight line; to be exact, it would be a straight line tangent to its trajectory at point M_1. From this it follows that its velocity too will be directed along that tangent in the direction of the motion of the particle and can be represented by vector $\mathbf{v}_1$ at a definite scale. In the same way the velocity of the particle at point M_2 can be represented by vector $\mathbf{v}_2$ in the direction of the tangent to its trajectory at that point. Wherefore, *the direction of velocity of a particle possessing curvilinear motion is tangent to its trajectory at the point corresponding with the given moment of time and is the same as the direction of its motion.*

By way of illustration, let us imagine we are swinging a stone, tied to a cord, in a horizontal circle. At a certain critical speed the cord breaks and the motion of the stone changes from curvilinear to rectilinear, directed at a tangent to its curved trajectory and with the velocity it had the instant just before the string broke.

78. Acceleration of a Particle Possessing Curvilinear Motion

Assume a particle to be traversing the curved trajectory AB in Fig. 118*a*. At one moment it is at point M_1 and at the succeeding short interval of time Δt* it is at point M_2. Let velocity $\mathbf{v}_1$ of

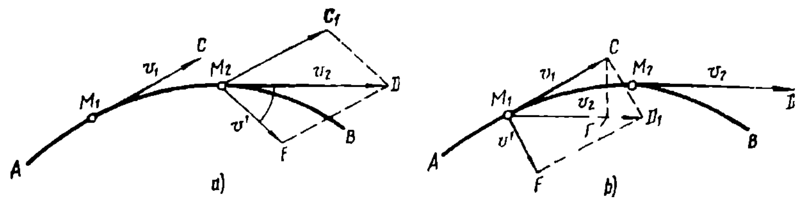


Fig. 118

the particle at point M_1 be expressed by the vector $\overline{M_1C}$ and at point M_2 by vector $\overline{M_2D}$. Now let us determine the change in velocity during the interval of time Δt, proceeding as follows (Fig. 118*b*). Delineate vector $\overline{M_1D_1}$ from point M_1, equal to vector $\overline{M_2D}$ of velocity $\mathbf{v}_2$, that is, equal in length, parallel to, and having the same direction. Then resolve velocity $\mathbf{v}_2$ into two components (according to the principle of the parallelogram) one of which, $\mathbf{v}_1$, will have a known magnitude and

* The sign Δ, the Greek letter "delta", is usually used to designate small quantities.

direction. In the parallelogram M_1CD_1E, the side M_1E will represent velocity **v**′ which expresses the change in velocity of the moving particle in the interval of time Δt during which the particle moved from point M_1 to point M_2. Then by dividing velocity **v**′ by the time Δt, we obtain the average acceleration a_{av}:

$$a_{av} = \frac{v'}{\Delta t} \tag{44}$$

The shorter the interval of time Δt, the closer will be the average acceleration to the acceleration of the particle at the instant it is at point M_1 in its trajectory.

Thus we see that *acceleration of a particle having curvilinear motion, unlike its velocity, is not directed along the tangent to the trajectory but forms an angle with it lying inside the curvature of the trajectory.*

79. Tangential and Normal Acceleration

We have learnt that acceleration along a curved trajectory defines the change in velocity both in magnitude and direction, for which reason it is known as *total acceleration*. We shall see later that in solving problems concerning curvilinear motion, it will be found necessary to consider, separately, acceleration due to changes in the *magnitude of velocity* and that due to changes in the *direction of velocity* caused by the curvature of the trajectory.

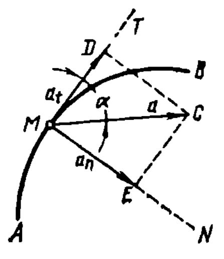


Fig. 119

Fig. 118*b* illustrates both such kinds of acceleration. On the velocity vector $\overline{M_1D_1}$ we lay out segment M_1F equal in magnitude to vector $v_1 = \overline{M_1C}$. It will be found that segment FD_1 expresses the change in the magnitude of velocity of the particle, whereas segment CF expresses the change in direction of the velocity.

Assume that acceleration a of a particle at position M (Fig. 119) is expressed by vector $\overline{MC}$. Just as in velocity, we resolve this acceleration into two components by the principle of the parallelogram, one along the tangent to trajectory MT at point M, and the second in the direction of MN perpendicular to the tangent. As a result we obtain the rectangle $MDCE$ in which $\overline{MD}$ expresses acceleration $\mathbf{a}_t$ while the vector $\overline{ME}$ shows acceleration $\mathbf{a}_n$.

Wherefore, *acceleration having the same direction as the tangent along which velocity is directed expresses a change in magnitude of velocity and is called tangential acceleration* $\mathbf{a}_t$, *whereas acceleration* *directed perpendicular to the tangent represents a change in direction of velocity and is called normal acceleration* $\mathbf{a}_n$*

From this it follows that if the tangential acceleration of a particle is in the same direction as its velocity, the particle possesses positive acceleration, if it is in the opposite direction, it possesses negative acceleration and the motion of the particle is retarded, and if it is zero, then the motion is uniform

Accordingly, possible cases of motion of a particle in a plane may be tabulated as follows:

Acceleration	Change in Velocity	Motion
1 Both kinds of acceleration, i.e., $\mathbf{a}_t$ and $\mathbf{a}_n$	both in magnitude and direction	curvilinear, non uniform
2 Acceleration $\mathbf{a}_n$ only	in direction	curvilinear, uniform
3 Acceleration $\mathbf{a}_t$ only	in magnitude	rectilinear, non-uniform

If there is no acceleration of either form, motion is rectilinear and uniform.

There is a simple relationship between total acceleration and its components. From the right triangle MCD (Fig 119) it follows that $CD = MD \tan \alpha$, in which α is the angle formed by total acceleration and the tangent Hence

$$a_n = a_t \tan \alpha \tag{45}$$

Since the leg of the triangle is equal to the hypotenuse multiplied by the sine of the opposite angle or the cosine of the adjacent angle, we obtain

$$a_n = a \sin \alpha \tag{46}$$

$$a_t = a \cos \alpha \tag{47}$$

Finally, according to the Pythagorean Theorem,

$$a = \sqrt{a_t^2 + a_n^2} \tag{48}$$

* "Normal acceleration" is so called because a line perpendicular to a tangent at the point of contact is called a "normal".

80. Normal Acceleration of a Particle Possessing Uniform Circular Motion

Let us investigate a specific case of curvilinear motion when the trajectory of a particle moving with constant velocity is in the form of a circle with radius R (Fig. 120). In this case there is only normal acceleration $\mathbf{a}_n$, since tangential acceleration $\mathbf{a}_t$ is zero (case 2 in the tabular representation given above).

Proceeding as in Sec. 78, we obtain the component $\mathbf{v}'$ of velocity $\mathbf{v}_2$ expressing a change in velocity in the time interval Δt during which the particle traverses the arc M_1M_2. Since the particle is travelling with uniform velocity, vectors $\overline{M_1C}$ and $\overline{M_2D_1}$ are equal in *magnitude* and, as distinguished from the general case previously presented, the vector M_1E represents a change in the *direction* of velocity.

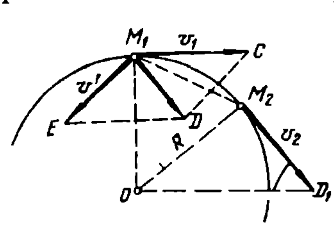


Fig. 120

Thus we see that, under these conditions, M_1DE forms an isosceles triangle, since $M_1D = M_1C$. In the same way M_1OM_2 is also an isosceles triangle because OM_1 and OM_2 are radii of the same circle. Furthermore, these two triangles are similar since $\angle M_1OM_2 = \angle M_1DE$ (their sides being mutually perpendicular) and therefore the remaining angles of one triangle are equal to the angles of the other triangle.

From this it follows that

$$\frac{M_1E}{M_1M_2} = \frac{M_1D}{OM}$$

from which

$$\overline{M_1E} = \frac{M_1D}{OM_1} M_1M_2. \qquad \text{(a)}$$

The velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the particle at points M_1 and M_2, expressed by the vectors M_1C and $\overline{M_2D_1}$, are equal in magnitude. By designating this magnitude as v, we obtain $M_2D_1 = v_2 = v$. By also taking into account that $OM_2 = R$ and by substituting these values in Eq. (a), we obtain

$$M_1E = \frac{v}{R} M_1M_2.$$

By dividing both sides of the equation by the time Δt during which the particle moved from M_1 to M_2, we obtain

$$\frac{M_1E}{\Delta t} = \frac{v}{R} \times \frac{M_1M_2}{\Delta t} \qquad \text{(b)}$$

The left side of the above equation expresses the average acceleration for the given interval of time. As this interval decreases, average acceleration will approach normal acceleration a_n, in which case the chord M_1M_2 may be assumed to be equal to the corresponding arc and the quotient $\frac{M_1M_2}{\Delta t}$ will represent velocity v. A substitution of these values in Eq. (b) offers the equation in its final form:

$$a_n = \frac{v^2}{R} \tag{49}$$

In this way we have obtained the following important relationship: *normal acceleration of a particle moving in a circle is equal to its velocity squared, divided by the radius of the circle.*

Now let us see what units are used to express this acceleration. The numerator in Eq. (49) is expressed in $\left(\frac{\text{unit of length}}{\text{unit of time}}\right)^2 = \frac{(\text{unit of length})^2}{(\text{unit of time})^2}$, hence the measuring unit of a_n will be

$$\frac{(\text{unit of length})^2}{(\text{unit of time})^2} : (\text{unit of length}) \qquad \frac{\text{unit of length}}{(\text{unit of time})^2}$$

i.e., the same measuring units as used for acceleration of rectilinear motion (Sec. 61).

This acceleration is directed towards the centre of the circle in which the particle is travelling (for which reason it is sometimes called centripetal).

81. Total Acceleration of a Particle Moving in a Circle

The above case is of a particle moving in a circle with constant velocity. But if motion is non-uniform, then aside from normal acceleration as determined by Eq. (49), the particle will also have tangential acceleration coinciding with the tangent in either direction. If the magnitude of this acceleration is constant, motion will be uniformly accelerated and displacement of the particle for any interval of time will be found through the formulae for rectilinear motion as deduced in Sec. 66 and will be equal to the length of the arc traversed.

In uniform circular motion, total acceleration is the same as for normal acceleration. In non-uniform curvilinear motion, total acceleration is determined by Eq. (48) as the square root of the sum of the squares of tangential and normal acceleration, while the angle they form with the tangent is evolved either by Eq. (46) or (47).

Illustrative Problem 42. A particle is travelling in a circle whose radius $R = 1$ m. It possesses a constant tangential acceleration of 0.2 m/sec². At the initial moment its velocity is $v_0 = 0$. Find the velocity and acceleration of the particle at $t = 3$ sec after the beginning of its motion, and determine the distance the particle covers in that interval of time.

Solution: by employing Eq. (27) we find velocity v_3 at the end of the third second:

$$v_3 = a_t t = 0.2 \times 3 = 0.6 \text{ m/sec}.$$

Normal acceleration, according to Eq. (49), is

$$a_n = \frac{v_3^2}{R} = \frac{0.36}{1} = 0.36 \text{ m/sec}^2.$$

Total acceleration at the end of the third second is found by Eq. (48):

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{0.04 + 0.129} = 0.412 \text{ m/sec}^2,$$

and the tangent of the angle it forms with the contacting tangent is obtained by Eq. (45):

$$\tan \alpha = \frac{a_n}{a_t} = \frac{0.36}{0.2} = 1.8,$$

from which $\alpha = 61°$.

The distance covered by the particle in three seconds is found through Eq. (30):

$$s = \frac{a_t t^2}{2} = \frac{0.2 \times 9}{2} = 0.9 \text{ m}.$$

82. Questions for Review

1. What is the direction of velocity, in respect to its trajectory, of a particle having curvilinear motion?
2. What is individually expressed by tangential and normal acceleration and what is their direction?
3. Is it possible for a particle with curvilinear motion not to have tangential acceleration? Is it possible for it not to have normal acceleration?
4. What is uniform motion that possesses acceleration?

83. Exercises

43. A particle with an initial velocity of zero moves for 5 sec with constant acceleration in a circle whose radius is 2 m and covers a distance of 3 m. Find its velocity and its total accelerations at the end of the fifth second.

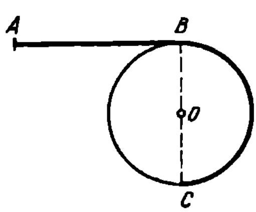


Fig. 121

44. The particle in Fig. 121 abandons position A with an initial velocity of zero and, moving with a constant acceleration is at position B in three seconds, 0.45 m from position A, after which it travels with a constant velocity in a circle whose radius is 0.5 m. Find its velocity v and its acceleration at the opposite point C (AB is tangent to the circle).

CHAPTER X

SIMPLE MOTIONS OF A HARD BODY

84. The Difference Between the Motion of a Hard Body and That of a Particle

Thus far we have studied the motion of a particle. Now we shall examine the simplest motions of a hard body which we have already classified as an unchangeable system of material particles.

When a body is in motion its various particles traverse different trajectories with diverse velocities and accelerations. By way of illustration let us take the slider-crank mechanism shown in Fig. 122.

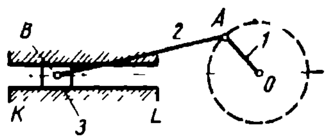


Fig. 122

Crank 1 is fastened rigidly to *shaft O* and turns with it. It is hinged, by means of crankpin A, to one end of *connecting rod 2*, the other end of which is hinged by means of pin B to *slider 3*, moving in fixed guides KL. As the crank turns, its particles all describe circles of different radii and consequently move with diverse velocities, whereas the particles of the slider describe identical rectilinear trajectories and with an identical velocity. The connecting rod moves in its own way and quite differently from either the crank or the slide; its right end in the centre of the crankpin A describes a circle whereas its left end in the centre of pin B moves in a straight line. The trajectories executed by the rest of its particles are curves of various shapes

In this chapter we shall learn how to solve problems concerning the simpler kinds of motion of a hard body, assuming in all cases that the body possesses *plane motion*, which means that *all its particles describe trajectories parallel to one and the same fixed plane.* All the elements of the mechanism just examined possess such motion, since the particles of these elements continuously trace paths lying in planes parallel to one and the same vertical plane.

85. Linear Translation

We shall begin by examining the simplest case of the motion of a hard body.

Imagine a train moving on straight rails; all points on the train, with the exception of the axles, wheels, and other elements whose motion is relative in respect to the bodies of the cars and the locomotive, are tracing identical trajectories; these trajectories are parallel to the rails and consequently parallel to each other. This is also true of all the particles in the slider 3 of the slider-crank mechanism in Fig. 122 inasmuch as the guides KL are straight. The same may be said of all the particles in the mobile jaw of a parallel vise and other mechanisms of the same nature.

Now let us take up a more complicated example. The plate B in Fig. 123 can travel either to the right or to the left on the flat horizontally fixed guide A, as shown by arrows *1*. The plate C to which rod D (ending with roller E) is rigidly fixed can slide back and forth on guides on the surface of plate B in a direction perpendicular to the lower guide A, as shown by arrows *2*. The roller E attached to C travels in a curved groove GH in plate F which is part of A.

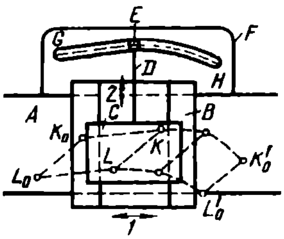


Fig. 123

Assume plate B to be moving along guide A; obviously the motion of plate C, due to the curved guide GH, will be relative to plate B and be compounded with the motion of plate B itself in the direction of arrows *1*. As a result of these two motions the trajectories traversed by all points on plate C or rod D will be identical and parallel to guide GH. For instance, a freely-selected point K will trace the trajectory K_0K_0', and point L will move along the path L_0L_0', etc. Thus, plate C and all particles connected with it trace identical and parallel paths*.

If we select any line on the plate C, for example KL joining points K and L, or any other line joining two points on the plate, they will remain parallel to themselves when the plate moves. The same may be said of any line joining two points on the train mentioned above, or on the carriage of a lathe, or the jaws of a vise, etc.

Wherefore, *when a hard body moves in such a way that any line joining any two of its points moves parallel to itself, the body is said to have motion of translation.*

In any motion of translation of a rigid body each point of the body will possess the same motion, that is, the same displacement, velocity, and acceleration at any instant.

On the basis of all this we come to the following important conclusion: the relationships we have already deduced for moving points can be used to solve problems concerning motion of translation.

If the trajectory of any point of a body describing motion of translation is a straight line, the movement of the whole body is said to have *rectilinear translation*. If, on the other hand, the trajectories are curves, then the motion is called *curvilinear*

* This kind of motion is made wide use of, such as on lathes which work with a template, or for the machining of bodies of rotation having a curvilinear profile or conical surfaces.

translation. Such is the motion of plate C in the example above.

A specific case of curvilinear translation is *circular translation;* here all points describe circles of an equal radius. This is illustrated in Fig. 124. Crank A is fastened rigidly to shaft O; plate B is hinged to the other end of the crank, its centre of gravity being lower than the axis of the hinge O_1, and occupies a vertical position under its own weight. When the crank moves about axis O, plate B will move in such a way that any line KL joining two of its points will move parallel to itself, and points K, L, etc., will describe circles of an equal radius. Hence the motion of plate B is circular translation.

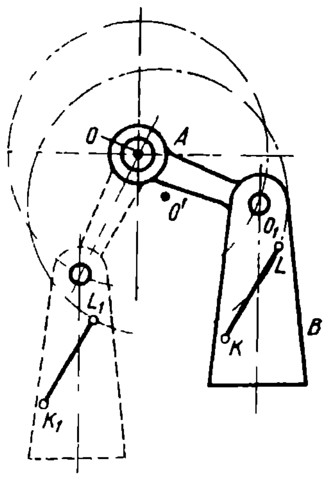


Fig. 124

Oral Exercises

1. What is the motion of the ram of a shaping machine, or the table of a planing machine?
2. What is the motion of the cutter described in Sec. 72 (Fig. 111)?

86. Rotation of a Body Around a Fixed Axis, and Angular Displacement

Now let us study the rotary motion of a body when the axis of rotation occupies a fixed position.

Assume body A in Fig. 125 to be rotating about axis O which is perpendicular to the plane of the drawing. Also assume that point K of the body occupies position K_0 at a certain moment. As the body rotates, this point will describe a circle with a radius of OK_0 equal to the length of a perpendicular drawn from the point to the axis of rotation and called the *rotational radius*.

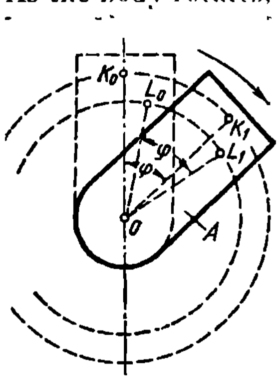


Fig. 125

Now let us delineate a plane through point K and the rotational axis. This plane will move with the body A. Assume that this plane occupies position OK_0 at the initial moment, and, after a certain interval of time, moves to position OK_1 and forms a certain two-facet angle $\varphi = \angle K_0OK_1$. This angle is formed by the initial and final positions of the rotational radius. In the same way a plane passing through any point L and the axis of rotation will, in the same interval of time, form the same angle φ as it moves from the initial position OL_0 to the final position OL_1. Therefore the angle formed by the swing of a body about its rotational radius is the same for any point of the body for the same interval of time. This angle serves to measure the rotation of the body as a whole and is called its *angular displacement during a given interval of time.*

87. Angular Velocity and Angular Acceleration

If a rotating body forms equal angles with the rotational axis in equal intervals of time, its rotation will be uniform; otherwise it will be non-uniform, or variable.

Assume that the angular displacement of a body is equal to φ_1 at the end of a time interval t_1, and φ_2 at the end of a time interval t_2, both being measured from the same initial position. Then its angular displacement for the interval of time $t_2 - t_1$ will be equal to $\varphi_2 - \varphi_1$. We find its *average angular velocity* for this interval of time by dividing the angular displacement by time as follows:

$$\omega_{av} = \frac{\varphi_2 - \varphi_1}{t_2 - t_1}. \tag{50}$$

Here it is not amiss to repeat what was said in Sec. 64 concerning the average velocity of a point having non-uniform motion: the smaller the interval of time $t_2 - t_1$, the closer the average angular velocity to the instantaneous velocity at the time moment t_1. Accordingly, the angular velocity of a point having non-uniform rotation is not constant. Let the angular velocity of a given point be ω_1 at the instant of time t_1, and ω_2 for the instant of time t_2. It then follows that the change in angular velocity during the interval of time $t_2 - t_1$ is $\omega_2 - \omega_1$. The ratio between the change in angular velocity and the interval of time in which it took place is called *average angular acceleration* and is expressed as

$$\varepsilon_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1}. \tag{51}$$

If acceleration possesses the same sign as angular velocity, the body will have positive acceleration; otherwise its rotation will be retarded.

Since angular displacement is measured in angular units, the measuring unit for angular velocity will be

$\frac{\text{unit of angular measure}}{\text{unit of time}}$ and for angular acceleration will be

$$\frac{\text{unit of angular measure}}{\text{unit of time}} : \text{unit of time} = \frac{\text{unit of angular measure}}{(\text{unit of time})^2}.$$

88. Linear Velocity of the Points of a Rotating Body

We have learnt that all points of a rotating body describe trajectories in the form of a circle. Geometry shows that the greater the radius and central angle, the greater will be the length of an arc. Since all rotational radii of a rotating body turn through the same angle, the length of the trajectories traversed by points situated at different distances from the axis of rotation will vary and be *proportional to the rotational radii.* For instance, the length of the arc K_0K_1 described by point K in Fig. 125 is as proportional to the length of the arc L_0L_1 described by point L as the rotational radius OK is to the rotational radius OL. Thus the various points of a rotating body receive different displacements in equal intervals of time. From this it follows that the velocities with which the points are displaced will also depend on the length of their rotational axes. Wherefore, *the velocities of the points of a rotating body are also proportional to their rotational radii.*

The velocity with which a point on a rotating body moves is called its *linear velocity* and is expressed as $\frac{\text{unit of length}}{\text{unit of time}}$.

Accordingly, the angular velocity of a body is a measure of the rotation of the whole body as well as all its points and is the same for all rotational axes. Whereas the linear velocity of points situated at different distances from the rotational axis will differ. From this it is further concluded that their acceleration will also differ.

89. Uniform Rotation of a Body Around a Fixed Axis

If the angular displacement of a body is the same for equal intervals of time, it is said to have uniform rotation. It is evident in this case that angular velocity will be constant.

Assume that a body rotates uniformly for an interval of time t. Then its angular velocity will be

$$\omega = \frac{\varphi}{t}. \tag{52}$$

The unit of measure used to express velocity will depend on the numerator and denominator of the right half of this formula: if angular displacement is expressed in degrees and time in seconds, then angular velocity will be expressed in degrees per second $\left(\frac{\text{deg}}{\text{sec}}\right)$. If time is in minutes, then it will be $\frac{\text{deg}}{\text{min}}$, etc. In this book the angle φ will henceforth be expressed in degrees.

Eq. 52 is sometimes expressed in a different form. The *radian* (a unit frequently used for angular measurement) is the central angle whose arc is equal to the radius of a circle. By designating x as the value of the radian in degrees. and R as the radius of the arc corresponding to it, we obtain $\frac{2\pi R}{360} x = R$, from which $x = \frac{180°}{\pi} = 57°17'44''$.

Whereupon the angular displacement expressed in radians would be $\frac{\varphi°}{x°} = \frac{\pi}{180°} \varphi°$, and the angular velocity

$$\omega = \frac{\pi}{180°} \times \frac{\varphi°}{t} \times \frac{1}{\text{sec}} .$$

In engineering, uniform rotation is almost always expressed in number of revolutions per minute and designated as n (rpm of the rotor of an electric motor, of the spindle of a lathe, etc.), in which case angular velocity is expressed as follows: when a body makes one revolution per minute, it turns through 360° in one minute; if it makes n revolutions per minute, it turns through $360n$ degrees and in one second it turns through $\frac{360n}{60} = 6n$ degrees. Hence if a shaft revolves at the rate of n revolutions per minute, it means that its angular velocity

$$\omega = 6n \frac{\text{deg}}{\text{sec}} = 360n \frac{\text{deg}}{\text{min}} \qquad (53)$$

Let us examine the motion of separate points of a uniformly rotating body. Fig. 126 represents a sheave which executes n rpm about its geometric axis O. Let us take point K on the outer rim of the sheave, the diameter of which we will denote as D. When the sheave executes one revolution, the point K will describe a circle of diameter D; this means that its trajectory will be equal to πD, in which π is the ratio of the circumference to the diameter of a circle. By executing n rpm, the trajectory traversed by the point will equal πDn and in one second would be $\frac{\pi Dn}{60}$. Since the diameter of the sheave is given in millimetres, the linear velocity of point K will be

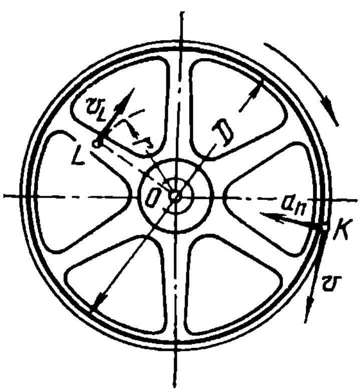


Fig. 126

$$v = \frac{\pi Dn}{60 \times 1{,}000} \text{ m/sec.} \qquad (54)$$

All points on the outer rim of the sheave (those farthest from the axis of rotation) will have the same linear velocity, known as the *peripheral velocity of the sheave.*

Now let us take a point L lying at a distance r from the axis (but not on the rim).

We obtain the linear velocity of the point L by following the same line of reasoning as with point K:

$$v_L = \frac{2\pi r n}{60 \times 1{,}000} \text{ m/sec.}$$

By dividing v by v_L we obtain

$$\frac{v}{v_L} = \frac{D}{2r}.$$

Wherefore, *the ratio of linear velocities of points on a rotating body is equal to the ratio of their diameters, or, which is the same thing, of the radii of the circles they describe.*

Eq. (54) expresses the peripheral velocity of a body (or the linear velocity of its points) depending on the diameter and number of revolutions per minute. If it is necessary to find the number of revolutions when the diameter and peripheral velocity are known, the equation becomes

$$n = \frac{60 \times 1{,}000 v}{\pi D} \text{ rpm.} \tag{55}$$

When peripheral velocity and the number of revolutions per minute are known, the diameter in millimetres is found by the following equation:

$$D = \frac{60 \times 1{,}000 v}{\pi n} \text{ mm.} \tag{56}$$

Velocity in Eqs (55) and (56) is given in m/sec.

Oral Exercises

1. Two points, one twice the distance from the axis as the other, lie on the same radius of a rotating body. What is the ratio of velocities of the two points?
2. What is the ratio of their normal acceleration?

Illustrative Problem 43. A sheave with a diameter $D = 2{,}000$ mm fixed rigidly to a shaft whose diameter $d = 125$ mm, is rotating uniformly at a rate $n = 240$ rpm. Find the peripheral velocities v_1 and v of the sheave and the shaft, respectively, and the normal acceleration of a point on the rim of the sheave.

Solution: applying Eq. (54), we find the peripheral velocity of the sheave as follows:

$$v_1 = \frac{\pi D n}{60 \times 1{,}000} = \frac{3.14 \times 2{,}000 \times 240}{60 \times 1{,}000} = 25.12 \text{ m/sec.}$$

Peripheral velocity of the shaft is either found in the same way, or solved on the basis that the linear velocities are proportional to the diameters: $\frac{v_2}{v_1} = \frac{d}{D}$, from which we obtain $v_2 = v_1 \frac{d}{D} = 25.12 \times \frac{125}{2{,}000} = 1.57$ m/sec.

The normal acceleration of a point lying on the rim of the sheave is calculated by using Eq. (49), in which case the diameter must be expressed in metres because velocity is given in m/sec:

$$a_n = \frac{v_1^2}{D/2} = \frac{2v^2}{D} = \frac{2 \times 25.12^2}{2} = 631 \text{ m/sec}^2.$$

Illustrative Problem 44. How many revolutions per minute must be transmitted to a high-speed steel drill of 14 mm in diameter in order to bore into soft cast iron at the rate of 50 m/min (the cutting speed for drilling is equal to the peripheral velocity of the drill).

Solution: by applying Eq. (55) we obtain

$$n = \frac{1{,}000v}{\pi D} = \frac{1{,}000 \times 50}{3.14 \times 14} = 1{,}137 \text{ rpm}.$$

Illustrative Problem 45. What diameter must a sheave be given if it is to attain 1,500 rpm and have a peripheral velocity of 22 m/sec?

Solution: Eq. (56) gives us

$$D = \frac{60 \times 1{,}000v}{\pi n} = \frac{60 \times 1{,}000 \times 22}{3.14 \times 1{,}500} = 280 \text{ mm}.$$

90. Diagrams Showing the Relationship Between Peripheral Velocity, Diameter, and Number of Revolutions

In spite of the comprehensiveness of the foregoing equations, their use involves tedious calculations which must be often executed in the workshop (as, for instance, in determining the number of revolutions to be imparted to the spindle of a lathe for a given cutting speed). Therefore in solving practical problems it is more convenient to use diagrams which make it possible to find desired magnitudes quickly and with sufficient accuracy.

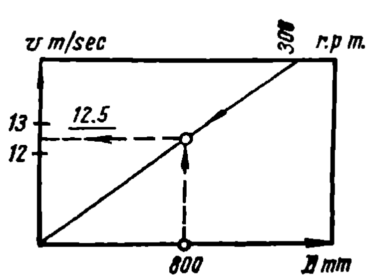


Fig. 127

Diagrams which plot the relationship between peripheral velocity, diameter, and number of revolutions are known as *nomographs*. With their help peripheral velocity may be found if the other two magnitudes are known. For example, if the diameter $D = 800$ mm and the number of revolutions $n = 300$ rpm, peripheral velocity is found by inspection to be $v = 12.5$ m/sec (Fig. 127).

In practice, two types of nomographs are widely used—radial and logarithmic. The plotting of nomographs and their use in practical calculations is explained in special courses on production technology.

91. Uniformly-Accelerated Rotation of a Body Around a Fixed Axis

When a change in angular velocity of a rotating body is equal for equal intervals of time, the body is said to possess *uniform acceleration*. If angular velocity is on the increase, the body is said to have uniform positive acceleration; if it is on the decrease, the body is said to possess uniform negative acceleration, or uniform deceleration.

By comparing the rotation of a body with the rectilinear motion of a material point, we find that angular displacement in the former is analogous to rectilinear displacement in the latter. In a similar manner angular velocity and angular acceleration, which are characteristic of rotation, correspond to the velocity and acceleration of a body possessing rectilinear motion. Therefore the equations giving the relationship between angular displacement, angular velocity, and angular acceleration can be deduced in the same way as accomplished for displacement, velocity, and acceleration of a particle of uniform rectilinear motion (Secs 65 and 66). Such an operation will yield the following formulae:

Angular velocity at moment t

$$\omega_t = \omega_0 + \varepsilon t, \tag{57}$$

in which ω_0 is initial angular velocity, and ε is angular acceleration, which is constant when rotation is uniformly accelerated. If initial angular velocity $\omega_0 = 0$, then

$$\omega_t = \varepsilon t. \tag{58}$$

Analogous to Eq. (28), we obtain angular displacement for time t:

$$\varphi = \omega_0 t + \frac{\varepsilon t^2}{2}, \tag{59}$$

and in accordance with Eq. (29)

$$\varphi = \frac{\omega_t^2 - \omega_0^2}{2\varepsilon}. \tag{60}$$

Finally, if initial angular velocity is zero,

$$\varphi = \frac{\varepsilon t^2}{2} \tag{61}$$

and

$$\varphi = \frac{\omega_t^2}{2\varepsilon}. \tag{62}$$

Illustrative Problem 46. A sheave begins to rotate with uniform acceleration at 12.5 revolutions for the first 5 seconds. What are its angular and peripheral velocities at the end of that time if its diameter $D = 2{,}000$ mm?

Solution: we first find angular displacement, bearing in mind that one revolution corresponds to a turn of 360°:

$$\varphi = 360 \times 12.5 = 4{,}500°.$$

Since the initial angular velocity is zero, we use Eq. (61) to find angular acceleration:

$$\varepsilon = \frac{2\varphi}{t^2} = \frac{9{,}000}{5^2} = 360 \text{ deg/sec}^2.$$

Angular velocity at the end of the fifth second is found through Eq. (58):

$$\omega_5 = \varepsilon t = 360 \times 5 = 1{,}800 \text{ deg/sec}.$$

This angular velocity corresponds to $\frac{1{,}800}{360} = 5$ rev/sec. Hence the peripheral velocity at that moment

$$v_5 = \frac{\pi 2{,}000 \times 5}{1{,}000} = 31.4 \text{ m/sec}.$$

Illustrative Problem 47. A sheave with a diameter of 1,200 mm rotates at the rate of $n = 400$ rpm. When power is cut off, it continues to rotate with uniform deceleration, coming to a stop in 2 min 30 sec. Determine the number of revolutions it executed after power was cut off, and the tangential acceleration of a point on its rim during the same interval before stopping.

Solution: the angular velocity of the sheave at the moment of transition from uniform motion to uniform deceleration is found by using Eq. (53):

$$\omega_0 = 6n = 6 \times 400 = 2{,}400 \text{ deg/sec}.$$

To find the angular deceleration we use Eq. (58), but instead of the final angular velocity, we apply the initial velocity:

$$\varepsilon = \frac{\omega_0}{t} = \frac{2{,}400}{150} = 16 \text{ deg/sec}^2.$$

Now we can find the angular displacement through Eq. (61):

$$\varphi = \frac{16 \times 150^2}{2} = 180{,}000 \text{ degrees}.$$

Inasmuch as one revolution equals 360°, the sheave has made $\frac{180{,}000}{360} = 500$ revolutions.

To find tangential acceleration on the rim of the sheave, we calculate the length of the arc corresponding to angular acceleration $\varepsilon = 16$ deg/sec². The diameter of the sheave D being 1,200 mm = 1.2 m, then the length of the arc corresponding to a central angle of 16° will be

$$\frac{\pi D 16}{360} = \frac{\pi 1.2 \times 16}{360} \text{ m} = \frac{4\pi}{75} \text{ m}.$$

Therefore the tangential acceleration on the rim is

$$\frac{4 \times 3.14}{75} \approx 0.17 \text{ m/sec}^2.$$

92. Questions for Review

1. What is motion of translation? Name the different kinds of translation.

2. If a railway carriage passes from a rectilinear to a curvilinear track, can its motion still be called translation?

3. If the same railway carriage passes from a rectilinear horizontal track to a rectilinear inclined track, can its motion still be called translation?

4. What kind of motion does the foot-rest of a bicycle pedal have?

5. Does the magnitude of angular velocity depend upon the magnitude of the rotational radius?

6. Does the magnitude of linear velocity depend upon the magnitude of the rotational radius?

7. Two cylinders of different diameters are rotating about their geometric axes. What ratio should there be between the number of revolutions they attain per unit of time so that their peripheral velocities remain the same?

8. Two sheaves of different diameters execute the same number of revolutions per minute. What can be said about their angular and peripheral velocities?

93. Exercises

45. If a sheave has a diameter of 160 mm and its motion is uniform, what must be its rpm to achieve a peripheral velocity of 24 m/sec?

46. A sheave is turning at the rate of 1,500 rpm and with a peripheral velocity of 22 m/sec. What is its diameter?

47. A steel workpiece with a diameter of 60 mm is being machined on a lathe with a high-speed steel cutter. What is the cutting speed (peripheral velocity) if the workpiece attains 1,140 rpm?

48. A brass workpiece 50 mm in diameter is being machined by a high-speed steel cutter at the rate of 430 m/min. Calculate the rpm of the workpiece.

49. A sheave with a diameter of 1,100 mm had at one moment $t = 0$ a peripheral velocity of 9 m/sec and 12 m/sec following an elapse of 2.5 minutes. Assuming the rotation of the sheave to be uniformly accelerated, find the angular and tangential accelerations on the rim, and also the angular and peripheral velocities following an elapse of 1.5 min after the initial moment $t = 0$.

50. A flywheel 1,500 mm in diameter attained 60 revolutions in the first 45 seconds after starting. Assuming its motion to be uniformly accelerated, find its angular and tangential accelerations and its angular and peripheral velocities at $t = 60$ sec.

51. A flywheel turns at a speed of 210 rpm. When power is cut off it continues to rotate but with uniform deceleration and stops after an elapse of 4 min 24 sec. Find the angular acceleration of the flywheel and the number of revolutions it executes after power was cut off.

DYNAMICS

CHAPTER XI

FUNDAMENTALS OF DYNAMICS

94. Definition of Dynamics

In the preceding section on kinematics we studied the motion of a hard body and its various points. But there can be no change in motion or, as it is called, the mechanical state of a body, unless another body (a force) is acting upon it. Therefore in order to obtain a complete picture of the motion of a body, we must know the relation between its motion and the forces acting upon it. This problem is dealt with in that section of mechanics known as dynamics. It may accordingly be said that *dynamics deals with the motion of a body in connection with the forces acting upon it.*

There are two overarching problems to be taken up in dynamics: 1) determining the forces that cause the motion of a body on the basis of the kinematics of that motion; 2) determining the motion that a body achieves under the action of forces exerted on it.

95. The First Law of Mechanics (Newton's First Law)

We know by experiment that a body at rest cannot change this state unless another body acts upon it and that it will continue in such a state for an indefinite time. We also know that if a body possesses uniform rectilinear motion, it requires the action of another body to change this motion.

Let us assume we have pushed a ball lying on the floor. As we watch we see that it has acquired rectilinear motion. The harder the material of which the ball is made and the smoother its surface as well as the surface of the floor, the longer it will continue to move in a straight line and the less change will there be in its velocity. If the ball were in a vacuum it would continue its motion still longer. Thus we see that the floor and surrounding air influence the ball, cause its motion to be non-uniform, and impart a negative acceleration to it.

From this we deduce that if it were left to itself and were free of the influence of other bodies, the ball would have acquired uniform rectilinear motion with a velocity constant in magnitude and direction.

The property of a body to maintain its momentum (or also its state of rest) is called *inertia.*

We have thus reached a conclusion expressing the substance of the Law of Inertia, or Newton's First Law: *a body will remain in a state of rest. or of uniform rectilinear motion, until some other body forces it to change that state.*

It is important to bear in mind that the action of one body upon another need not necessarily occur through direct (visible) contact. For instance, a body projected horizontally will not exhibit rectilinear motion; it will achieve curvilinear motion due to the earth's invisible attraction.

96. The Basic Equation of Dynamics (Newton's Second Law)

Let us make the following experiment. The plunger B in the guides A (Fig. 128) can be forced to the right by the spring D. We pull the plunger to the left and fasten it in place by gripping its handle C with the screw K. We then place two balls, E and F, against the plunger. Both balls have the same diameter but their materials are of different specific gravity and are, therefore, of different weights. We then release the handle C and the plunger jerks suddenly to the right, simultaneously pushing the two balls in the same direction. We observe that they both acquire rectilinear motion, but displacement for each in the same interval of time is different: the lighter ball travels faster and outstrips the heavier one. If the balls had had the same specific gravity, they would have moved with equal velocity and been stopped by the resistance to their motion, at an equal distance from the initial position.

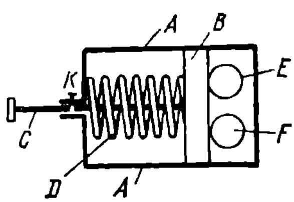


Fig. 128

If we repeat this experiment but with the spring squeezed tighter (the spring pushed further to the left than in the first experiment), we shall see both balls move with greater velocity than before; nevertheless the velocity of the heavier ball will steadily become less than that of the lighter one.

From these experiments we deduce the following: at the initial moment both balls are in a state of rest. Under the action of the spring which imparts equal forces to both balls via the plunger, they are put in motion but each with its own velocity. In other words, under the action of equal forces, the two bodies received unequal accelerations, the heavier receiving lesser acceleration. Furthermore, by comparing the second experiment with the first, we see that a greater force nevertheless imparts more acceleration to one and the same body.

From this it is apparent that there is some kind of relationship between a force and the acceleration it imparts to a body. Let us make another experiment to determine this relationship.

To the car A (Fig. 129) standing on straight and horizontal rails we fasten one end of a dynamometer B, the other end of which we fasten to a cord C which we pass over pulley D and tie to a weight G_1. Then we allow the car to move under the pull of the cord caused by load G_1 and make a note of the magnitude

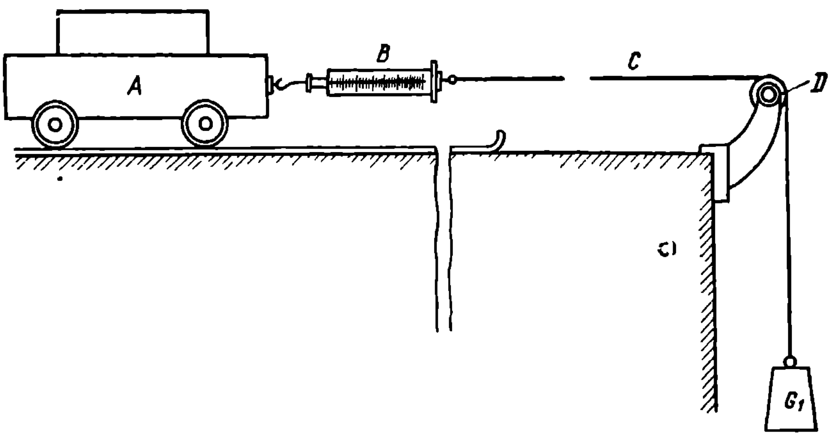


Fig 129

of the force $\mathbf{P}_1$ indicated on the dynamometer. By studying the motion of the car (e g , measuring the distances it travels in equal intervals of time) we find that it acquires uniform acceleration. We then find the magnitude of its acceleration a_1 by means of the distance it travels in a definite interval of time.

Then replacing load G_1 by load G_2, we repeat the experiment and find that under the action of the second force $\mathbf{P}_2$ as indicated by the dynamometer, the car receives an acceleration of a_2. If the car is constructed so as to offer very little friction in its movement along the rails, we shall find as a result of a number of similar experiments that the ratio between forces $\mathbf{P}_1$ and $\mathbf{P}_2$ differs very little from the ratio of accelerations the forces impart to the car. We thereby establish that the magnitudes of the forces are directly proportional to the magnitudes of accelerations which they impart:

$$\frac{P_1}{P_2} = \frac{a_1}{a_2},$$

or, after replacing the middle members,

$$\frac{P_1}{a_1} = \frac{P_2}{a_2} = \text{a constant quantity.}$$

No matter how many times we repeat this experiment but with different loads, we shall see that the ratio of a force to the acceleration it imparts to one and the same body is always the same.

Wherefore, *the ratio of a force to the acceleration it imparts is a constant quantity for every body.* If we denote this quantity by the letter m, we obtain

$$\frac{P}{a} = m$$

or,

$$P = ma. \tag{63}$$

From this equation it follows that the greater the magnitude of m, the greater the force required to impart one and the same acceleration to a body. The quantity m is called the *measure of mass* of a body, or, to put it simply, the *mass* of a body.

Since according to the Law of Inertia a body tends to either remain at rest or retain its uniform rectilinear motion, it is understood that when acceleration is imparted to the body, it will resist that acceleration; and the greater its mass, the greater its resistance. Whence the mass of a body is considered to be a measure of its inertia.

Eq. (63) which expresses Newton's Second Law, is the *basic equation of dynamics* and can be formulated as follows: *force is equal to mass multiplied by acceleration.* Moreover, acceleration attains the same direction as the force imparting it.

Oral Exercises

1. If the magnitude of a force acting on a body is increased n times, how will it effect the acceleration of the body?

2. The mass of particle A is n times greater than the mass of particle B, and the acceleration imparted to A is also n times greater than that imparted to B. How much greater is the force imparted to A than to B?

97. Law of the Independent Action of Forces

Assume a particle to be moving with an acceleration $\mathbf{a}_1$ under the action of force $\mathbf{P}_1$ (Fig. 130) and that at a certain moment another force $\mathbf{P}_2$ begins to act on the particle. If the particle were under the action of force $\mathbf{P}_2$ alone, it would receive an acceleration $\mathbf{a}_2 = \frac{P_2}{m}$, in which m is the mass of the particle.

However, we know that under the action of the two forces $\mathbf{P}_1$ and $\mathbf{P}_2$, the particle will move with an acceleration $\mathbf{a}$ represented by the vector $\overline{OC}$, which is also the diagonal of the parallelogram $OACB$, constructed on both accelerations $\mathbf{a}_1 = \overline{OA}$ and $\mathbf{a}_2 = \overline{OB}$ as two of its sides. In other words we may say that the acceleration of a particle is equal to the geometric sum of the two accelerations.

By multiplying the two accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ by the mass of the particle, we evolve the forces $\mathbf{P}_1$ and $\mathbf{P}_2$. Therefore we may regard the parallelogram $OACB$ as being constructed (to scale) on the vectors of forces $\mathbf{P}_1$ and $\mathbf{P}_2$, and vector OC as representing (to the same scale) the resultant of the two component forces $\mathbf{P}_1$ and $\mathbf{P}_2$.

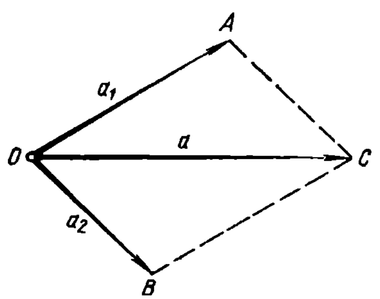


Fig. 130

From this we arrive at the following deduction: *if a moving particle is under the action of several forces at once, the acceleration the particle receives is equal to the geometric sum of the accelerations produced separately by each of the forces acting on it.*

Let us assume that a particle moving under its own momentum (its motion is uniformly rectilinear) begins to be acted upon at a certain moment by a force $\mathbf{P}$ having a constant direction. As a result, the particle will receive a given acceleration in the direction of this latter force. If the particle had been at rest when acted upon by the force, it would have received a definite velocity in the direction of the force. But since it was also under the action of its own momentum, its velocity will be the sum of the velocity produced by its momentum and that produced by force $\mathbf{P}$ (assuming the latter had been applied to the particle as if it were at rest).

This may be formulated as follows: *the action that a force will have upon a particle does not depend upon whether the particle is at rest or in motion, or whether one or several forces are acting upon it.*

From this it also follows that if a particle is moving under its own momentum and a system of forces in equilibrium is applied to it, its motion will continue to be uniform and rectilinear.

This principle of mechanics is called *the law of the independent action of forces*, or *the law of the joint action of forces.*

98. Propositions Deduced From the Laws of Mechanics

The following set of propositions, confirmed by experiment, emerges from the laws of mechanics that we have investigated.

1. Assume a particle having rectilinear motion and being under the action of a force. According to Newton's Second Law, its motion will have acceleration. *If the force should be removed, the particle will continue to move under its own inertia (momentum) with a uniform rectilinear motion and its velocity will be that attained at the time the force was removed.* Such would be the motion of a train, travelling on straight horizontal rails, after steam is cut off and if there were no resistance to its motion. The smaller the resistance, the longer will the train move under its own inertia and the more nearly uniform will its motion be.

2. Assume that a particle has curvilinear motion. As is apparent from Newton's first two laws, such motion can only occur under the action of a force. *If the force is removed, the particle will continue to move, but in a straight line tangent to its path at the moment the force has been removed.* An example of this is a stone tied to the end of a cord and being whirled around by a hand holding the other end of the cord. If the cord breaks, the stone will fly off in a direction tangent to the circle described by its centre of gravity under the constraining action of the cord. A particle torn off a rotating grindstone will acquire the same motion.

3. Now let us consider the motion of a train on straight and horizontal rails. In order for the train to maintain uniform motion, the locomotive must develop a definite tractive force to overcome the harmful resistance which is opposite in direction to the motion of the train. If the tractive force is greater than this resistance, the surplus will impart positive acceleration to the train and make it move faster. But if the resistance is greater than the tractive force, the surplus resistance will impart a negative acceleration to the train, i.e., an acceleration opposite in direction to the motion of the train. This would cause the movement of the train to be retarded.

From what has just been said, the following important deduction can be made: since on the one hand the train possesses uniform rectilinear motion under the action of the tractive force of the locomotive, and on the other hand of the force of resistance, and since both these forces are exactly equal in magnitude, the forces are in equilibrium.

Wherefore, *if a particle under the action of forces possesses uniform rectilinear motion, the forces will be in equilibrium, have no influence on its motion, and the particle will move under its own inertia*; and conversely, *if the forces applied to a particle* *are in equilibrium, it will either have uniform rectilinear motion or will remain in a state of rest.*

This is one of the most important principles of engineering mechanics. It simplifies all problems concerning rectilinear and uniform motion since it makes it possible to solve them through the principles of statics.

The following table schematically presents all the above deductions.

Kind of Force Required to Move a Material Point from Its State of Rest	The Resulting Motion of the Particle
1. A force constant in magnitude and direction	Uniformly accelerated, rectilinear
2. A force variable in magnitude and constant in direction	Rectilinear
3. A force which imparts non-uniform curvilinear motion but which ceases at a given moment	Uniformly rectilinear motion along the tangent to its trajectory
4. A force which imparts non-uniform rectilinear motion but which ceases at a given moment	Uniformly rectilinear motion from the moment the force ceases

Oral Exercises

1. A particle moving under its own inertia (momentum) comes under the action of a constant force having a direction opposite to the motion of the particle. What effect will the force have on the motion of the particle?

2. A particle moving under its own inertia (momentum) comes under the action of two forces equal in magnitude and opposite in direction. What effect will they have on the motion of the particle?

99. Units of Measure in Engineering and Physics

Eq. (63) expresses the relationship between three quantities — force, mass, and acceleration. Acceleration is expressed in $\frac{\text{unit of length}}{(\text{unit of time})^2}$. Since the basic unit of length used in engineering is the metre (m) and of time it is the second (sec), hence acceleration is expressed in $\frac{\text{m}}{\text{sec}^2} = \text{m} \times \text{sec}^{-2}$ (read metres per second per second).

As for other quantities in the said equations, we may choose either force or mass as the basic unit, and express one in terms of the other. If we take force as the basic unit and in the form of the kilogramme just as we did in statics, then let us see how mass will be expressed.

We have already established the following relationship: $m = \frac{P}{a}$.

By substituting 1 kg for the force P and 1 m/sec² for acceleration a, we may express the unit of mass through these units as

$$kg : \frac{m}{sec^2} = \frac{kg \times sec^2}{m} = kg \times m^{-1} \times sec^2.$$

This system of units (kg, m, sec) in which the unit of mass is expressed through these very units, has been adopted in engineering and is called the *engineering system of units*. This is the system we shall henceforth use.

Now let us try taking the unit of mass as our basic unit. If we use the gramme as this unit, Eq. (63) will give the following relationship for the unit of force:

$$\text{unit of force} = (\text{unit of mass}) \times (\text{unit of acceleration}) = g \times \frac{\text{unit of length}}{(\text{unit of time})^2}.$$

In the system of units as used in physics, the unit of length is the centimetre and the unit of time is the second, according to which the unit of force, called the *dyne*, is expressed as

$$g \times \frac{cm}{sec^2} = g \times cm \times sec^{-2}.$$

This system is called the *physical* or *absolute system of units*, or is simplified by the technical sobriquet CGS (centimetre, gramme, second).

100. Relationship Between Mass and Weight of a Body

Let us assume that a body is falling freely in a vacuum where it meets with no resistance. As we know, a body falls because of the force of gravity, or in other words, of its weight. And since this force, acting upon it, is constant both in magnitude and direction, the body falls with a constant acceleration.

Hence it is obvious that the Basic Equation of Dynamics (63) is also applicable to this case, when the active force is gravity. But instead of the force P in Eq. (63) we substitute the weight of the body G, and instead of acceleration a, we apply the acceleration due to the force of gravity g. Whereupon the equation becomes

$$G = mg. \qquad (64)$$

Since the mass of a body is constant but its acceleration may be a diverse quantity, the weight G of one and the same body may possess different numerical values along different latitudes of the earth, a fact proved by weighing a body by means of a spring balance. From this we see that there is an appreciable difference between the mass of a body and its weight. All bodies have mass, and mechanics deals with the mass of all bodies as unchangeable. But the weight of a body is determined by the gravity of the earth and varies along different parts of the earth's surface, depending on the magnitude of gravitational acceleration.

Illustrative Problem 48. A body having an initial velocity of 10 m/sec moves 200 m in 5 sec when a force of 20 kg is applied to it. What is its weight (the acceleration due to gravity is taken 9.81 m/sec²)?

Solution: since the body possesses uniformly accelerated motion, we apply Eq. (28):

$$S = v_0 t + \frac{at^2}{2},$$

whence, by substituting corresponding numerical values, we obtain $a = 12$ m/sec², and the mass of the body

$$m = \frac{P}{a} = \frac{20}{12} = \frac{5}{3} \text{ kg} \times \text{m}^{-1}\,\text{sec}^2.$$

Therefore the weight of the body

$$G = m \times g = \frac{5}{3} \times 9.81 = 16.35 \text{ kg}.$$

Illustrative Problem 49. A body of weight $G = 29.43$ kg is moving under its own inertia at a velocity $v_0 = 10$ m/sec. At a certain moment a force $P = 2$ kg is applied to it acting in the opposite direction to its motion. Find the velocity of the body three seconds after the force P is applied.

Solution: since the force is acting in the opposite direction to the motion of the body, the acceleration imparted to it is negative and the motion of the body is uniformly retarded. From Eq. (26) we obtain

$$v_3 = v_0 - at = 10 - 3a.$$

Acceleration $a = \frac{P}{m}$, mass $m = \frac{G}{g} = \frac{29.43}{9.81} = 3$ kg × m⁻¹ sec²,

hence acceleration $a = \frac{2}{3}$ m/sec² and the sought velocity

$$v_3 = 10 - 3 \times \frac{2}{3} = 8 \text{ m/sec}.$$

101. Law of Action and Reaction (Newton's Third Law)

If body A receives a certain acceleration under the action of a force, it means that another body B is exerting this force on body A. Body B may act on A either in direct contact or at a distance, the latter as in the case of the force of gravity.

Let us make the following experiment.

We place trucks A and B on rails (Fig. 131), and connect them by a spring C, the cars being at such a distance from each other that the spring is somewhat taut. When we release both trucks simultaneously they move towards each other. We measure the distance each has covered and calculate the acceleration of each truck. We denote the acceleration of truck A as a_1 and of truck B as a_2. After calculating the masses of the trucks and comparing them with their corresponding acceleration, we find (provided the experiment has been carried out with sufficient precision) that the equation $m_1a_1 = m_2a_2$ holds true to a sufficient extent.

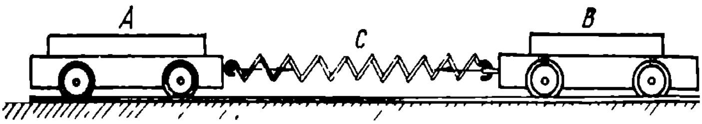


Fig. 131

However, according to Newton's Second Law, the product of mass and acceleration is equal to the force imparting acceleration. Therefore, we find that a force has acted on truck A from left to right and a force of the same magnitude has acted on truck B from right to left (in the opposite direction).

This result confirms the Third Law of Mechanics *(Newton's Third Law)*, which when stated briefly, is that *action and reaction are equal.*

The interaction of two bodies is the result of two forces which are equal and opposite. Hence, forces act in pairs when they are applied to two interacting bodies.

We have already seen this law applied with respect to the equilibrium of bodies in statics, when we learnt that the pressure of a body on its support gives rise to an equal and opposite reaction.

102. Questions for Review

1. Explain the Law of Inertia.
2. A force acting on body A is n times greater than a force acting on body B; the mass of body B is n times greater than the mass of body A. What is the ratio between the accelerations imparted to the two bodies?
3. How is the unit of mass expressed in engineering and in absolute systems of units?
4. Under what conditions does a particle, under the action of a system of forces, acquire uniform rectilinear motion?

103. Exercises

52. Find the mass of a body having a weight of 1,963 kg.

53. The tractive force of a locomotive, after allowing for all resistances to its motion, is 12,000 kg and it imparts an acceleration to the train of $a = 0.1$ m/sec². What is the weight of the train and what will be its velocity following an elapse of 45 sec after it begins to move?

54. What tractive force (including that needed to overcome resistance) is necessary to give a train, weighing 2,000 tons, an acceleration of 0.05 m/sec if the resistance to its movement amounts to 0.005 of the weight of the train?

55. Three minutes after starting, a train weighing 1,200 tons is travelling at a speed of 40 km/hr on a straight and horizontal track. What is the tractive force of its locomotive (considering it constant) if the resistance to its motion is 0.005 of the weight of the train?

56. How long will it take to stop a tramcar travelling on a horizontal track at a speed of 35 km/hr and how far will it travel after the brakes have been applied if all the resistances to the tram's motion, including that created by the brakes, amount to 200 kg per ton of weight of the tram?

CHAPTER XII

INTRODUCTION TO DYNAMICS OF A MATERIAL POINT

104. Dynamics of a Material Point

When we were investigating kinematics we found that if a hard body is rotating about a fixed axis, its various points are displaced in circular trajectories of different radii, velocities, and accelerations. But when a body possesses motion of translation, the elements comprising this motion are exactly the same for all points on the body. Hence, in considering motion of translation of a body under the action of applied forces, we may ignore its dimensions and take a point (usually its centre of gravity) which represents the place of concentration of the entire mass of the body. As already explained at the beginning of this book, such a point, which is made to represent the body as a whole, is known as a material point.

However, the use of a material point is not restricted to motion of translation alone. It is also useful in more complicated types of motion; let us assume a ball is rolling on a surface. As the ball rolls, its centre describes a simple curved or straight trajectory, whereas its other points describe various complicated curved trajectories. If, in solving the problem, we are interested only in the motion of the centre of the ball, we may consider the ball as a material point situated at its centre and containing its whole mass. Accordingly, henceforth when speaking of the motion of a body, we shall assume the body to be a material point whose mass is equal to the mass of the whole body.

105. The Action of the Force of Gravity on the Motion of a Vertically-Projected Body

Assume a body to be thrown vertically upwards. If it were not attracted by the earth. it would retain the velocity imparted to it at the initial moment and move under its own momentum at a constant rectilinear velocity. But the body is acted upon by the force of gravity whose magnitude is determined by the acceleration g, which it imparts to the body, and by the mass of the body.

Therefore the velocity of the body at any moment t during its flight upwards is equal to the difference between the constant velocity v_0 with which it would have been displaced under its own momentum, and velocity gt which it acquires at the same moment from the force of gravity. From this we derive Eq. (32), already stated in the section on kinematics:

$$v_t \qquad gt.$$

When velocity gt, as imparted to the body by the force of gravity, becomes equal in magnitude to the velocity of its motion due to inertia (momentum), the velocity of the body will become zero. At that moment the body will reach its highest point and then begin to fall under the action of gravity alone (if the resistance of the air is not taken into consideration). The body will acquire uniformly-accelerated motion and its velocity at any moment t will be equal to gt.

This is the explanation of the kinematic relationships already mentioned in Sec. 66.

106. The Motion of a Body Thrown Upwards at an Angle to the Horizon

Now let us consider a more complicated case of bodily motion under the influence of the force of gravity: a body is thrown upwards not vertically but at an angle to the horizon (Fig. 132).

Assume that a body M is thrown from an initial position M_0 in the direction of N and with a velocity $\mathbf{v}_0$, with its trajectory forming an angle α to the horizon. If the force of gravity did not act on the body, it would be displaced with uniform rectilinear motion in the direction of M_0N with a constant velocity $\mathbf{v}_0$. But the force of gravity causes the body to diverge steadily from a straight line so that at a given moment it will fall back to earth at a spot M and at a distance of $L = M_0M$ from its initial position.

We resolve the velocity v_0, as represented to scale by vector $\overline{M_0A}$, into two components: v_{0hor} in a horizontal direction and v_{0vert} in a vertical direction. Whatever may be the shape of the trajectory described by the body*, its horizontal component of motion will be uniform because no force is acting upon it in that direction. In the vertical component, however, the velocity of the body will not be uniform and at any moment of time can be expressed by the difference between initial velocity v_{0vert} and velocity gt as determined by acceleration due to

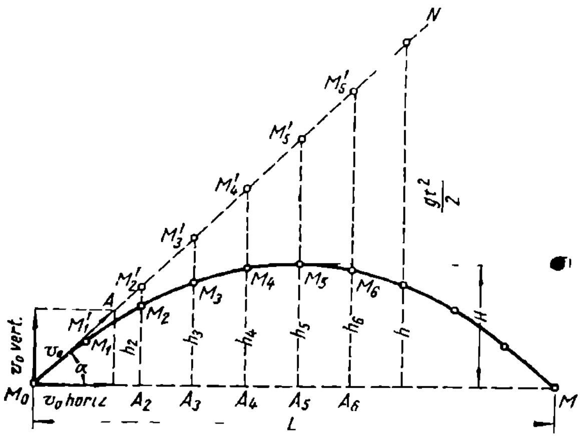


Fig. 132

gravity. The velocity v_{vert} with which the body is displaced vertically is found by means of Eq. (32), which in this case becomes

$$v_{vert} = v_{0vert} - gt.$$

Thus we see that the vertical motion of the body becomes uniformly decelerated and at the moment when gt equals v_{0vert}, the vertical component of the body's velocity will become zero, at which moment the body will reach its highest point. After this the body will begin to move with positive acceleration because of the increase in the vertical component, and at a given moment the body will strike the ground.

Now what is the trajectory traced by the centre of gravity of the body?

We know from experience that it will be curvilinear, and we may plot its path by the following method; let T represent the time elapsed while the body is in the air. We divide that time into several equal intervals t_1, t_2, t_3, etc., and indicate points

* It must be remembered that we are assuming this body to be a material point.

M_1', M_2', M_3', etc., on line M_0N to represent the hypothetical positions of the body at moments t_1, t_2, t_3, etc., had its motion been due only to inertia. And since such a supposed motion would have been uniform, $M_0M_1' = v_0t_1$, $M_0M_2' = v_0t_2$, $M_0M_3' = v_0t_3$, etc., and the height that the body attained would have been represented by the linear segments $M_1'A_1$*, $M_2'A_2$, $M_3'A_3$, etc. However, under the action of gravity the body becomes displaced downwards with uniformly accelerated motion, covering the distance $M_1'M_1 = \frac{gt_1^2}{2}$ during the interval t_1, distance $M_2'M_2 = \frac{gT_2^2}{2}$ during the interval $T_2 = 2t_1$, and distance $M_3'M_3 = \frac{gT_3^2}{2}$ during the interval $T_3 = 3t_1$, etc. Thus the segments $M_1'M_1$, $M_2'M_2$, $M_3'M_3$, etc., are related to each other just as $\frac{gt_1^2}{2} : \frac{gT_2^2}{2} : \frac{gT_3^2}{2}$, etc. But $T_2 = 2t_1$, $T_3 = 3t_1$, etc. Consequently, $M_1'M_1 : M_2'M_2 : M_3'M_3$, etc., $= t_1^2 : 4t_1^2 : 9t_1^2$, etc., $= 1 : 4 : 9$, etc. Hence, by plotting segments $M_1'M_1$, $M_2'M_2 = 4M_1'M_1$, $M_3'M_3 = 9M_1'M_1$ vertically downward from points M_1', M_2', M_3', etc., we obtain a number of points lying on the trajectory of the moving body. In more detailed treatments on this subject it is shown that the path traced by this body marks a curve called a *parabola*.

Now let us find the lapse of time T that is consumed by the body to move from position M_0 to position M, and determine the maximum height it attains at H and the length of its flight L.

As already explained, the motion upwards is uniformly retarded and at the highest point the vertical component of velocity is zero. Accordingly, $v_{tvert} = v_{0vert} - gT' = 0$, from which $T' = \frac{v_{0vert}}{g}$, in which T' is the lapse of time it takes the body to reach its highest point. Since $v_{0vert} = v_0 \sin\alpha$, the time the body takes to move from the initial to the highest point is

$$T' = \frac{v_0 \sin\alpha}{g}.$$

It can be easily shown that the time the body consumes in moving from its highest point to point M will be the same. Hence the time the body takes to traverse the whole path from M_0 to M will be

$$T = \frac{2v_0 \sin\alpha}{g}. \tag{65}$$

The next step is to determine the height H attained by the body. When we use Eq. (29), v_t must be taken as zero because at the highest point the final vertical velocity is zero. The initial velocity in this case is $v_{0vert} = v_0 \sin\alpha$, and acceleration a is the same as acceleration g caused by the force of gravity. All this offers us the following expression

* Segment $M_1'A_1$ has not been delineated in Fig. 132 to avoid complicating the drawing.

in order to determine the highest point the body reaches:

$$H = \frac{v_0^2 \sin^2 \alpha}{2g}. \quad (66)$$

And finally, we find the distance L at which the body is displaced. Since its horizontal component of motion is uniform, we may write

$$L = v_{0hor} T. \quad (a)$$

Velocity v_{0hor} is found as the leg adjacent to the acute angle α of a right triangle having a given hypotenuse v_0:

$$v_{0hor} = v_0 \cos \alpha.$$

By substituting this value for v_{0hor} and the value found above for the time of the flight T, we obtain

$$L = v_0 \cos \alpha \frac{2v_0 \sin \alpha}{g} = \frac{v_0^2}{g} 2 \sin \alpha \cos \alpha.$$

From trigonometry we know that the expression $2 \sin \alpha \cos \alpha$ is equal to the sine of a double angle α, i.e., $2 \sin \alpha \cos \alpha = \sin 2\alpha$; from which we finally obtain

$$L = \frac{v_0^2}{g} \sin 2\alpha. \quad (67)$$

Since 1 is the greatest possible value for a sine when an angle is 90°, hence a body will cover its greatest distance when its angle of projection $2\alpha = 90°$, or $\alpha = 45°$.

The vertical motion of a body is a specific case of the kind of motion we have been examining. Indeed, when motion is along a vertical line the angle $\alpha = 90°$ and $2\alpha = 180°$, $\sin 180° = 0$, whereupon $L = 0$, i.e., the body returns to its initial position after its fall. Moreover, when $\alpha = 90°$, then $\sin 90° = 1$ and $H = \frac{v_0^2}{2g}$, which is the relationship already obtained in Sec. 66.

Illustrative Problem 50. A gun fires its projectile at an angle $\alpha = 30°$ to the horizon and with a muzzle velocity $v_0 = 500$ m/sec. Calculate the distance and time of flight of the projectile if the flight had been through a vacuum.

Solution: the time of flight is determined by Eq. (65) as follows:

$$T = \frac{2 \times 500 \sin 30°}{9.8} \approx 51 \text{ sec.}$$

While the distance of the flight, according to Eq. (67) will be

$$L = \frac{v_0^2}{g} \sin 2\alpha = \frac{500^2}{9.81} \sin 60° \approx 22.07 \text{ km.}$$

107. Tangential and Normal Forces When a Particle Moves in a Circular Trajectory

Let us assume that the rod K, to the end of which the ball C is attached, is rotating in a horizontal plane about the axis O (Fig. 133). If the centre of the ball moves about the circle with non-uniform motion, the change in magnitude of velocity will be expressed by tangential acceleration $\mathbf{a}_t$, while the change in direction will be expressed by the normal acceleration a_n

a_n defined by Eq. (49):

$$a_n = \frac{v^2}{R}$$

We multiply each of these accelerations by the mass m of the ball. The product ma_t of the mass of the ball, multiplied by the tangential acceleration a_t, gives the magnitude of the force $\mathbf{T} = \overline{CD}$ and which is directed along the tangent to the circle followed by the centre of the ball. This is called the *tangential force*. The factum $ma_n = \frac{mv^2}{R} = N = \overline{CB}$ expresses the *centripetal* or *normal force* and represents the magnitude of the force directed towards the centre.

Accordingly, the tangential force

$$T = ma_t \tag{68}$$

and the normal force

$$N = \frac{mv^2}{R} \tag{69}$$

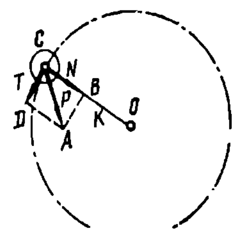


Fig. 133

These two forces are components of the force $\mathbf{P} = \overline{CA}$ and are represented by the diagonal of the rectangle $ABCD$ equal to

$$P = \sqrt{\left(\frac{mv^2}{R}\right)^2 + (ma_t)^2} = m\sqrt{\frac{v^4}{R^2} + a_t^2}. \tag{70}$$

If motion is uniform, $a_t = 0$, whereupon the tangential force $T = 0$.

108. Inertial Forces

Let us assume that body B begins to act with a certain force upon body A when the latter is in a state of rest. We have already learnt that this action will impart acceleration to body A. However, according to the Law of Inertia, body A will tend to remain in a state of rest and thus display a certain resistance to a change in its state of rest. This resistance takes the form of a force exerted on body B by body A. In other words, we may say that the action of body B on body A gives rise to a reaction on the part of the latter which, according to the Third Law of Mechanics, is opposite in direction and equal in magnitude to that action.

The reactions experienced by one body from a second body to which it, and it alone, is imparting velocity, is called the *force of inertia*.

From this the following important deduction is made. If body A receives acceleration under the action of body B, the force causing that acceleration is applied to body A; this force of inertia is equal and opposite and applied to body B. Th[illegible] *these two forces are applied to different interacting bodies,* f[illegible] which reason they differ from two equal and opposite forces applied to one and the same body as heretofore discussed in the chapter Statics.

It must be finally emphasised that there can be no force of inertia if there is no force imparting acceleration to a body. Hence, the two forces act simultaneously.

109. Inertial Forces in Rectilinear Motion of a Particle

Let us assume that the slider K in Fig. 134 is moving within straight guides under the action of the connecting rod L. We will apply the equation $P = ma$, where **P** represents the force exerted by the rod on the slider, and **a** is the acceleration of the motion of the slider. Force **P** gives rise to reaction $\overline{OA}_1$ of the slider and which is applied to the rod. According to the Third Law of Mechanics, this reaction is equal and opposite to force **P**, from which it follows that its direction is opposite to that of the acceleration of the slider. By designating this reaction as $\mathbf{P}_i$, we obtain $P_i = - P = -ma$. This then will be the force of inertia developed by the slider and applied to the rod.

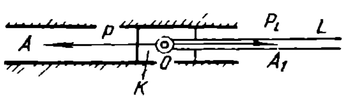


Fig. 131

Assume that a locomotive and its tender are moving along a straight and horizontal track when, at a certain moment, the tractive force of the locomotive increases and imparts corresponding acceleration **a** to the tender. The additional force exerted upon the tender by the locomotive is expressed by $P = ma$, in which m is the mass of the tender (which latter is considered as a material point, ignoring the rotation of the wheels and axles). It follows that from the moment the said tractive force increases, the tender will begin to exert on the locomotive a force of inertia $P_i = -ma$, direction of which is opposite to force **P**.

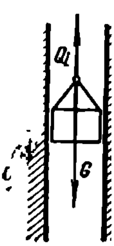


Fig. 135

Illustrative Problem 51. The mine cage of weight $G = 300$ kg represented in Fig. 135 descends into a shaft with an acceleration of 2 m/sec². What is the pull of the cable where it is fastened to the cage?

Solution: a downward-pulling force equal to the weight of the cage $G = 300$ kg is acting on the cable. As it descends with an acceleration $a = 2$ m/sec², the cage develops a force of inertia $Q_i = ma = \frac{300 \times 2}{g} =$ $= 61.2$ kg which is transferred to the cable and directed upwards.

Accordingly, the cable will be drawn taut by a force

$$R = G - Q_i = 300 - 61.2 = 238.8 \text{ kg}.$$

If the cage had ascended with the same acceleration as when descending, the force exerted on the cable would be

$$R = G + Q_i = 361.2 \text{ kg}.$$

110. Inertial Forces Acting Upon a Particle Moving in a Circular Trajectory

We have already learnt that in general the force applied to a particle possessing circular motion can be resolved into two components—one, the force **T** tangent to the curve and the other, the force **N** normal to the curve. The tangential force **T** imparts to the moving particle an acceleration which determines its change in magnitude of velocity, while the normal force **N** changes the direction of velocity.

Hence, the following two inertial forces will be acting simultaneously on one body that imparts to another body tangential and normal accelerations: *tangential inertial force* $T_i = -T = -ma_t$, and *inertial force* $\mathbf{N}_i$ which latter is equal and opposite to the normal force.

Let us investigate the second of the two inertial force $\mathbf{N}_i$ —in greater detail.

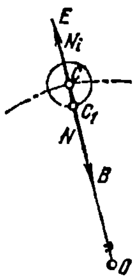


Fig. 136

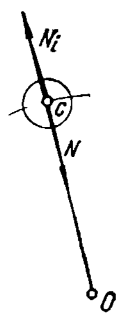


Fig. 137

Assuming that the centre of ball C in Fig. 136 is moving in a circle, propelled by a light rod which is rotating uniformly about axis O. Since in this instance tangential acceleration is zero, the ball is being acted upon by the normal force **N** alone which compels it to move in a circle. This force is applied to the ball at point C_1 and is directed towards the centre O of the circle.

Simultaneously with force **N**, an inertial force $\mathbf{N}_i$, equal and opposite to it, is acting upon the ball and represents the resistance of the ball to a change in direction of velocity that it would have had due to inertia. This force which is applied to the *rod* at point C_1 and directed from the centre along radius OC, is called *centrifugal force*.

Since we regard the ball in this case as a material point situated at its centre of gravity C where the entire mass of the ball is hypothetically concentrated, both forces **N** and $\mathbf{N}_i$ may be considered as being transmitted along their lines of action to point C, as indicated in Fig. 137. It must be remembered, however, that these forces are applied to different bodies and for that reason cannot attain equilibrium.

The force which, due to inertia, a particle possessing uniform circular motion exerts upon a constraining body, is centrifugal force.

In accordance with Eq. (69), the magnitude of centrifugal force is determined by the equation

$$N_i = \frac{mv^2}{R}, \tag{71}$$

whereas the magnitude of tangential force of inertia

$$T_i = ma_t. \tag{72}$$

By applying Eq. (51), we can give the following form to Eq. (71):

$$N_i = \frac{m}{R}\left(\frac{\pi R n}{30}\right)^2 = \frac{\pi^2 n^2 m R}{900} \text{ kg}, \tag{73}$$

in which m — the mass of the particle;

n — the number of revolutions per minute;

R — the distance, in metres, of the particle from the axis of rotation.

Finally, if the mass of the particle is expressed in terms of weight G, the equation takes another form:

$$N_i = \frac{G}{9.81} \times \frac{\pi^2 n^2 R}{900} = 0.00112\ GRn^2. \tag{74}$$

Oral Exercises

1. Under what conditions will the tangential force of inertia of a moving particle have constant magnitude?
2. Answer Question 1 in respect to centrifugal force.

Illustrative Problem 52. A round workpiece 60 mm in diameter, ready for machining, is fixed between the centres of a lathe. The cutting speed has been set at 425 m/min. What will be the magnitude of the centrifugal force as set up by the rotation of the workpiece if its centre of gravity is shifted 1.5 mm from the axis of rotation* and its weight $G = 1.6$ kg?

Solution: in order to employ Eq. (74) it will be necessary to find the rpm that must be imparted to the workpiece. By applying Eq. (55) we obtain

$$n = \frac{1{,}000v}{\pi D} = \frac{1{,}000 \times 425}{3.14 \times 60} = 2{,}257 \text{ rpm}$$

Let us take this figure in round numbers, i.e., 2,250 rpm of the spindle.

* Such a deviation from the centre is called *eccentricity*.

We now calculate the magnitude of centrifugal force by means of Eq. (74), in which eccentricity R must be taken in metres, i.e., R = 1.5 mm = 0.0015 m:

$$N_1 = 0.00112 \times 1.6 \times 0.0015 \times 2{,}250^2 = 13.6 \text{ kg}.$$

We see that here centrifugal force will be $\frac{13.6}{1.6} = 8.5$ times the weight of the workpiece. This will harm the centres of the lathe and increase wear on the bearings of the spindle.

Illustrative Problem 53. A train that had been running along a straight track reached a curve. While the train had been travelling along the straight track, the weight of each car was balanced by the reactions of the rails and both rails were carrying equal loads. But when the train reached the curve a centripetal force **N** arose, which forced the centre of gravity of the cars to begin moving in a curved line; simultaneously a centrifugal force began to act from the car wheels towards the rails and applied to the outer rail where it comes into contact with the flanges of the wheels.

Let us assume that the centre of curvature lies to the right, then the centrifugal force $\mathbf{N}_1$ will act on the left—the outer rail (Fig. 138a). This force tends to wrench the rails loose and also retards the motion of the train by causing increased friction between the wheels and rails. It may even result in the train jumping the track. How can all this be avoided?

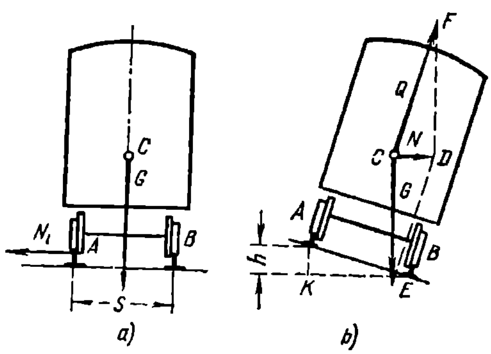


Fig. 138

Solution: To overcome the bad effect of this centrifugal force, the roadbed is banked in order to raise the outer rail above the inner one (Fig. 138b) and attain a difference in their heights h. This height must be so chosen that the reaction **Q** of the rails against the car is perpendicular to the cross-section of the roadbed, thus eliminating any possible lateral force that may be exerted on the rails. To achieve this, the normal force **N** must equal the resultants of the force of gravity **G** and the reaction **Q**. By regarding the car as a material point situated at its centre of gravity C, we derive the parallelogram $CEDF$ in which the diagonal $\overline{CD}$, which is horizontal (it corresponds with the rotational radius), represents the normal force $N = \frac{mv^2}{R}$, where m is the mass of the car and R — the radius of curvature. Sides $\overline{CE}$ and $\overline{CF}$ represent, respectively, the weight of the car $G = mg$ and the reaction of the rails **Q**.

We can now calculate the magnitude of h the difference between the height of the rails. From the similarity of triangles CDF and ABK we obtain $\frac{CF}{s} = \frac{CD}{h}$, whence $h = \frac{CDs}{CF}$ (s is the width of the track as indicated in Fig. 138a).

Since $\overline{CD}$ is a scale representation of centripetal force $N = \frac{mv^2}{R}$, and $\overline{CF}$ that of the reaction **Q** which may be determined as the hypotenuse of the right triangle CDF, we obtain

$$h = \frac{mv^2s}{RQ} = \frac{mv^2s}{R} \times \frac{1}{\sqrt{N^2 + G^2}} = \frac{mv^2s}{R} \times \frac{1}{\sqrt{\left(\frac{mv^2}{R}\right)^2 + (mg)^2}}$$

which, after its proper algebraic transformation, becomes

$$h = \frac{v^2s}{Rg\sqrt{1 + \frac{v^4}{g^2R^2}}}.$$

The values v and R as occurring in practice are such as to make $\frac{v^4}{g^2R^2}$ a negligible quantity. For example, when $v = 60$ km/hr $= 16.6$ m/sec and $R = 300$ m, then $\frac{v^4}{g^2R^2} = 0.0089$. Hence it can be ignored, thus greatly simplifying the equation, which becomes

$$h = \frac{v^2s}{Rg}$$

It is to be seen from this equation that the greater the speed of the train and the smaller the radius of curvature, the greater must be the height of the outer rail above the inner one.

In planning railways, h is determined by both the average speed that a train is expected to attain on the given curve, and the radius of curvature.

111. Forces of Inertia as Applied in Engineering

The forces of inertia play a very important part in modern engineering with its high speeds and accelerations. It is difficult to imagine a machine without some rotating part, and since rpm attain magnitudes of tens of thousands, centrifugal force is a factor of particular significance. From Illustrative Problem 52 we have already seen that centrifugal force may reach several times the weight of a given body.

Assume that the centre of gravity of a rotating body of weight **G** considered as a material point, is situated at a distance of ϱ from the rotational axis. According to Eq. (74), if $n = 20{,}000$ rpm, then the centrifugal force N_t will be equal to 4[illegible]3,000 $G\varrho$. If the weight of the body G is 1 kg and eccentricity ϱ is as small as 0.5 mm = 0.0005 m, then centrifugal force N_t will be equal to 224 kg. We thus see that this force is 224 times greater than the weight of the body itself. This will cause much wear on bearings and shaft journals and also cause knocking, all of which may result in a breakdown. Hence great precision must be given to centerings of rapidly rotating machine parts to ensure that the centre of gravity lies exactly on rotational axes. To do this, the parts are either counterweighted or surplus material is removed. This is known as *balancing a part*. For instance, if the centre of gravity of the sheave in Fig. 139 is found not be on its geometric axis O but at a distance of OC from it at some point C, the centre of gravity can be shifted so as to make it align with the axis. To do this, it is necessary either to attach an added load at point A diametrically opposite C, or to reduce the weight of the sheave at point B on the same diameter by boring a hole of required dimensions in the rim. There are special machines called "centering machines" used to balance parts.

Fig. 140 represents a bearing A ready for machining and fixed to a faceplate with an angle bar B. Although the centre of gravity of the faceplate coincides with the axis of the spindle,

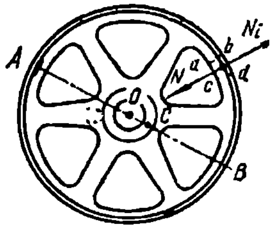
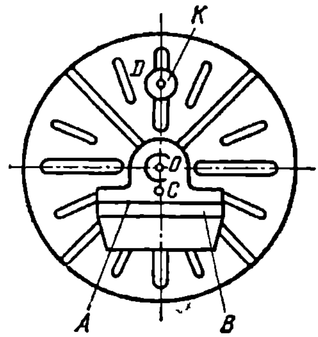


Fig. 139



140

nevertheless when the angle bar and the workpiece are mounted on the faceplate the centre of gravity will shift to position C and throw the whole system out of balance. To prevent the spindle supports from being subjected to centrifugal force, the system must be balanced. This is done by attaching a counterweight K along the diameter passing through O and C. By denoting the weight of the angle bar plus the workpiece as $\mathbf{G}$, and of the counterweight K as $\mathbf{G}_1$, the following condition must be observed: $GOC = G_1OD$; that is, the moments of these two forces with respect to axis O must be equal.

112. Questions for Review

1. A train leaves the station, travels on a straight and horizontal track and gradually gathers speed until a certain moment when it will have attained a constant speed. Answer the following questions:

a) What forces are acting on the locomotive's coupling and on the couplings between carriages, and how are these forces directed?

b) Are these forces equal in magnitude?

Answer separately for acceleration and constant speed.

2. Assume that a train, whose carriages have no brakes, was travelling at a certain speed along a straight and horizontal track when the locomotive brakes were applied. What forces would arise between the carriages and what direction would they have? Would these forces have the same

while the second member represents its kinetic energy at the initial moment t_0. Hence Eq. (88) may be formulated as follows:

The work done by a motive force in causing a given displacement, is equal to the increase in kinetic energy during that displacement.

Let us investigate some specific cases.

1. The above body is under the action of force **P** which coincides with the direction of its motion and the resisting force **F**; then the work done by the resultant of these two forces is equal to $(P-F)s$, and Eq. (88) becomes

$$(P-F)s = \frac{mv_t^2}{2} - \frac{mv_0^2}{2}$$

or

$$Ps = Fs + \frac{mv_t^2}{2} - \frac{mv_0^2}{2} \qquad (89)$$

that is, *the work of a motive force is equal to the sum of work accomplished by the force of resistance during displacement, plus the increase in kinetic energy of the body.*

2. If the body possesses uniform motion, according to which $v_t = v_0$, then in the right side of Eq. (89) the difference $\frac{mv_t^2}{2} - \frac{mv_0^2}{2} = 0$, whence Eq. (89) becomes

$$Ps = Fs, \qquad (90)$$

that is, *when there is uniform motion of translation, the work of the motive force is equal to that done by the force of resistance, in which case the kinetic energy remains constant.*

3. If the body starts from a state of rest, i.e., when $v_0 = 0$ and the force of resistance must be coped with, then Eq. (89) becomes

$$Ps = Fs + \frac{mv_t^2}{2}, \qquad (91)$$

that is, *when initial speed is zero, the work of the motive force is equal to the sum of the work accomplished by the force of resistance plus the kinetic energy developed during displacement.*

This case corresponds to the first phase in the motion of a train (when tractive force is in action) as already discussed in the preceding section 126.

4. The body has acquired a definite speed and then proceeds further under the force of resistance, according to which $Ps = 0$, and Eq. (89) becomes

$$0 = Fs + \frac{mv_t^2}{2} - \frac{mv_0^2}{2},$$

from which

$$\frac{mv_0^2}{2} = Fs + \frac{mv_t^2}{2}, \qquad (92)$$

that is, *the initial kinetic energy of the body* (at the moment the motive force ceases) *is equal to the sum of the work done by the* *force of resistance during the given interval of time plus the kinetic energy the body possesses at the end of that interval.*

From Eq. (92) it is apparent that $\frac{mv_t^2}{2} < \frac{mv_0^2}{2}$. Whence $v_t < v_0$, which means that the body possesses retarded motion. When all its kinetic motion has been expended, then $\frac{mv_t^2}{2} = 0$ and correspondingly $v_t = 0$, that is, the final speed of the body becomes zero and it stops. From Eq. (92) we obtain $\frac{mv_0^2}{2} = Fs$, which means that all the initial kinetic energy has been expended in overcoming the force of resistance.

As is apparent from the left side of Eq. (87), kinetic energy must be expressed in units of work (kilogramme-metres) and indeed

$$\frac{mv^2}{2} = \text{kg} \times \text{m}^{-1} \times \text{sec}^2 \times \text{m}^2 \times \text{sec}^{-2} = \text{kg-m}.$$

Kinetic energy is of tremendous importance in engineering. Some illustrations of its use will be investigated later.

Oral Exercises

1. Can two bodies of different mass have the same kinetic energy? On what condition?

2. If the speed of a body possessing uniform motion of translation is increased n times, what change will there be in its kinetic energy?

Illustrative Problem 64. If the speed of a train is v_0, what distance s will it travel after the brakes have been applied?

Solution: when the brakes are applied the train's kinetic energy $T = \frac{mv_0^2}{2}$, or, if we denote the weight of the whole train as **G**, then its mass $m = \frac{G}{g}$ and $T = \frac{Gv_0^2}{2g}$.

By denoting the force of friction as F and the coefficient of friction as f we obtain from Eq. (92)

$$Fs = \frac{Gv_0^2}{2g}.$$

Inasmuch as $F = fG$ which gives $fGs = \frac{Gv_0^2}{2g}$, then after cancelling G it becomes

$$s = \frac{v_0^2}{2gf}.$$

We thus see that the distance s required to stop the train, by application of its brakes, does not depend on the mass or the weight of the train but merely on its speed and coefficient of friction.

Illustrative Problem 65. A body sliding down an inclined plane AB (Fig 155) halts at point C, a distance of s from B along the horizontal surface. Find the coefficient of friction f if the motion began without an initial velocity at point A which lies a distance l from B.

Solution: the velocity of the body at both positions A and C is zero, therefore its kinetic energy at these positions is also zero. Along segment

AB the body is controlled by the force of gravity **G** and the force of friction **F**. Along segment *BC*, only the force of friction is acting upon the body.

We first find the kinetic energy that the body acquired at the time it reached *B*. The work W_1 done by force **G** during displacement l is expressed by $Pl = Gl \sin \alpha$, and the work done by the force of friction **F** is expressed by $W_2 = Fl = fQl = -fG \cos \alpha\, l$. Hence the sum of work of the motive force and the force of resistance $W_1 + W_2 = Gl \sin \alpha - Gfl \cos \alpha$. By equating the sum of work and kinetic energy at point *B*, we obtain

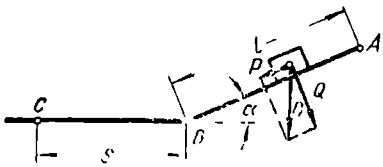


Fig. 155

$$Gl \sin \alpha - Gfl \cos \alpha = \frac{mv^2}{2}.$$

Since the body expends this kinetic energy completely in moving along distance s, we obtain $fGs = \frac{mv^2}{2}$ and finally evolve

$$Gl \sin \alpha - Gfl \cos \alpha = fGs,$$

which, after cancelling G, becomes

$$f = \frac{l \sin \alpha}{l \cos \alpha + s}.$$

128. The Energy of a Body Moving Under the Force of Gravity. Potential Energy

The law of the transformation of kinetic energy is obviously also applicable to the force of gravity.

When we throw a stone upwards, we impart a definite velocity to it or an amount of kinetic energy corresponding to the initial velocity. This energy, if there were no resistance from the air, would be expended entirely in raising the ston to a definite height, that is, would be disbursed in work overcome the force of gravity. When the stone has risen height it will have lost all its kinetic energy and its velocity will become zero. After this the stone will begin to fall, its kinetic energy increasing in proportion to its velocity and it will strike the ground with the same velocity that it had at the beginning of its upward motion, as already explained in kinematics (Sec. 67).

By employing Eq. (88) and denoting the upward motion as $v_t = 0$ (the velocity at the highest point) and the downward motion as $v_0 = 0$ (the initial velocity when the stone begins to fall), we obtain the following two equations:

$$Ph = \frac{mv_0^2}{2} \text{ during the upward motion}$$

and (93)

$$Ph = \frac{mv_t^2}{2} \text{ during downward motion,}$$

wherein h is the height the stone reaches.

This upward motion is analogous to the motion of a train without tractive force *expending* all its kinetic energy in overcoming resistance. The downward motion is similar to the motion of a train travelling with an excess of tractive force overcoming resistance, and as a result *acquiring* kinetic energy.

Assume that a body of weight G and mass m is falling to the ground from a given height. We denote two positions O_1 and O_2 of the centre of gravity of the body (Fig. 156). We also denote h_1 as the height of the centre of gravity when at position O_1, and h_2 as the height at position O_2. If the velocity of the falling body at O_1 is equal to v_1 and at O_2 is equal to v_2, then the kinetic energy equation (88) for position O_2 will be

$$G(h_1 - h_2) = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} \qquad (94)$$

or

$$Gh_1 + \frac{mv_1^2}{2} = Gh_2 + \frac{mv_2^2}{2}. \qquad (95)$$

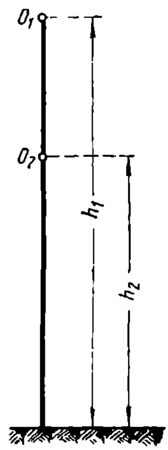


Fig. 156

Wherefore, *when a body is falling under the force of gravity, the sum of the product of the weight of the body multiplied by its height from the ground plus the kinetic energy the body possesses at that height, is a constant quantity.*

The first item of the above sum, which represents the amount of work expended to raise the body to the given height, is called the *potential energy* of the body. The magnitude of this potential energy depends upon the height, for which reason it may also be called the *energy of position*. The magnitude of kinetic energy is determined by the velocity, hence it represents the *energy of motion*.

In more detailed studies on mechanics it is demonstrated that when a body is moving under the force of gravity, Eq. (95) holds true not only for the body's vertical direction but also for any other trajectory.

Wherefore, *when a body is hurled upwards, the sum of its potential and kinetic energy is constant at any height, independent of the shape of the trajectory through which it is moving.*

From Eq. (95) we see that at the moment a body starts to rise, all its energy is in the form of kinetic energy ($h_1 = 0$), and when it has reached its greatest altitude ($v_2 = 0$) all its energy has been converted into potential energy which is again changed into kinetic energy when the body falls to the ground.

Thus, when a body moves upward and then falls back again, its energy remains constant in magnitude but changes from kinetic to potential and then back again to kinetic. This transition of mechanical energy from one form to another is a part of the general principle of the conservation of energy, first discovered by the great Russian scientist M. Lomonosov.

The kinetic energy of a moving body is made wide use of in driving piles, forging metals, and many other kinds of work in engineering. Sometimes the work is done entirely by kinetic energy of a freely falling body (the head of a pile driver, drop hammer, and the like). At other times, besides this energy, additional kinetic energy is imparted to the body during its fall (steam hammer, hammer, etc.).

The transformation of kinetic energy into potential energy and back again is not restricted to rising and falling bodies; by expending work in compressing a spring, we impart a certain amount of potential energy to it through its internal forces of resilience, which energy again becomes kinetic as the spring returns to its original form.

Oral Exercises

1. What is the difference between potential and kinetic energy?
2. If two bodies of the same mass are at different heights h_1 and h_2, which will have the greater potential energy and how much greater?
3. The velocity with which one body falls to the ground is n times greater than that of another. How much more kinetic energy has the first than the second?

Illustrative Problem 66. Water enters a hydraulic engine at a high level and at a speed $v_1 = 4$ m/sec and emerges at a lower level $h = 1.8$ m at a speed $v_2 = 1$ m/sec. The quantity of water passing through the engine per second $Q = 6$ m³. What is the horsepower of the engine?

Solution: the engine receives its power firstly from the potential energy of the water and secondly from its kinetic energy. The potential energy is equal to 1,000 Qh and the kinetic energy is equal to

$$\frac{mv_1^2}{2} - \frac{mv_2^2}{2} = \frac{1{,}000Q}{2g}(v_1^2 - v_2^2).$$

Hence the energy used by the engine in one second

$$N = 1{,}000Q\left(h + \frac{v_1^2 - v_2^2}{2g}\right) = 1{,}000 \times 6\left(1.8 + \frac{4^2 - 1^2}{2 \times 9.81}\right) \text{ kg-m/sec}$$ or,

in horsepower, $N = \frac{1{,}000 \times 6}{75}\left(1.8 + \frac{4^2 - 1^2}{2 \times 9.81}\right) = 205$ hp.

129. Kinetic Energy of a Body Rotating Around a Fixed Axis

Assume that a body, to which any number of forces are applied, does not have motion of translation but rotates about a fixed axis. Let us see how we can apply Eq. (88) as derived for a material point:

$$Ps = \frac{mv_t^2}{2} - \frac{mv_0^2}{2}.$$

Since in motion of translation all points of a body move in one way, this equation is applicable to the motion of a body of mass m as a whole. In rotation, motion is more complex because different points of the body, instead of moving in one way, describe various trajectories and possess different velocities and accelerations at one and the same time.

Expressed on the left side of Eq. (88) is the work performed by the force along a distance s. When applied to rotation, this work is given by Eq. (77), in which work is determined by the turning moment and angular displacement. As for the right side of the equation, the velocities v_t and v_0 at the final and initial moments respectively, as well as mass m, must embrace each separate point of the body. Since the velocity of a point, as already explained in kinematics, is proportional to the radius of rotation, then the right side of the equation must contain the sum of the product of the mass of the particles multiplied by the square of the distance from the axis of rotation. This sum embraces all the points of the body and is called the *moment of inertia* in respect to the axis of rotation. It is evident from the above that the unit of the moment of inertia is the product of the unit of mass multiplied by the square of the unit of length, i.e.,

$$\text{kg-m}^{-1}\ \text{sec}^2 \times \text{m}^2 = \text{kg-m} \times \text{sec}^2.$$

In order to understand the physical meaning of the moment of inertia, let us consider the following example. Assume that two cylinders of similar weight and material but of different diameters are fixed to similar shafts. We impart to both shafts an identical angular velocity and when the turning moment ceases to act, each shaft will continue to rotate at the expense of the kinetic energy imparted to it by each cylinder. If we observe the time consumed by each cylinder to come to a standstill (or count the total number of revolutions made by each) we shall see that, with equal resistance for each specimen, the shaft to which is fixed the cylinder of greater diameter will rotate longer. This means that the kinetic energy of this cylinder is greater although its mass is the same as that of the other. This is because the cylinder of greater diameter has a greater moment of inertia. A very narrow disc will revolve even longer.

The moment of inertia of a cylinder rotating about its geometrical axis $I = \frac{mR^2}{2}$, R being its radius*.

From this we see that when the radius of a rotating cylinder increases n times, its kinetic energy increases n^2 times. It likewise follows that in order to impart to a cylinder of greater diameter the same angular velocity as to one of smaller diameter, there must be a greater turning moment for the former, or if both have an equal turning moment, it must be applied for a longer period of time.

* The letter I is the usual symbol for the moment of inertia.

130. Governing an Engine. The Function of the Flywheel

Kinetic energy is of great importance in the work of machines.

As already explained, the less the work of the force of resistance, the greater will be the kinetic energy a body acquires under the action of a motive force. This also holds true for the work of a machine as a whole, inasmuch as a machine is made up of a number of interrelated moving parts. For example, let us consider a steam engine imparting motion to a dynamo generating electricity. If the amount of expended electric energy diminishes, the load on the engine will also diminish. Hence, if the turning moment on the engine's main shaft remains constant, there will be a surplus of energy over the work of the forces of resistance. This will cause an increase both in velocity and kinetic energy. It therefore follows that the engine must be equipped with a device making it possible to maintain the desired rpm. Such a device is called a *governor*.

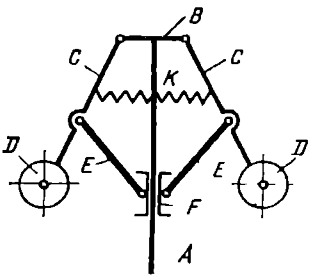


Fig. 157

There are different kinds of governors. Fig. 157 shows schematically one of the types of a centrifugal governor. To spindle A, which is rotated by the engine's shaft, is connected the crossbar B to which the arms C—C are connected through pivots. Arms C—C, which bear the balls D—D, are in their turn attached through pivots to arms E—E. These are connected at their other ends to the sleeve F which may slide freely on the spindle. The arms C—C are drawn together by the spring K. Thus each arm C is acted upon by the following forces: its own weight and the weight of its connections (the arms E—E, etc.), the weight of the ball, the pulling force of the spring K, and the centrifugal force developed by the ball. The spring can be regulated in such a way that at a prescribed number of revolutions the arms C—C will be in equilibrium and the sleeve F will maintain its position in respect to the spindle. If the rpm of the spindle increase, there will occur a corresponding increase in centrifugal force developed by the balls, arms C—C will stretch outward and arms E—E will raise the sleeve F. The sleeve is connected through a special mechanism with the steam throttle or fuel intake. Thus any change in the position of the sleeve F will cause a corresponding change in the supply of steam or fuel. In this way the rpm of the engine shaft are kept at the prescribed rate.

In piston engines (steam engines and internal combustion engines) a slider-crank mechanism is used (turn back to Fig. 122) in which the piston assumes the role of a slider. As will be explained later in Part II of this book*, the specific character attending the transmission of motion from the slider to the crank is that the latter rotates with variable angular velocity, the variations being periodic. In order to overcome this fault, a *flywheel* is fitted on the main shaft of the engine. This flywheel accumulates mechanical energy during one period and gives it up the next, thereby making rotation of the main shaft almost uniform.

Flywheels are also used when it is necessary to do work in a short time which otherwise would require a considerable increase in the power of the machine (for example, in heavy presses, giant shears for cutting metal, etc.).

It is therefore apparent from what has been said that the amount of kinetic energy that a flywheel can accumulate depends on its moment of inertia—on its mass, diameter, and on how its mass is distributed; the further a certain mass is situated from the axis of rotation, the greater will be the moment of inertia. For this reason the rim of a flywheel, unlike that of an ordinary sheave, is made massive.

131. Mechanical Efficiency

All machines are intended to overcome useful resistance (the resistance of metal to cutting, the resistance of a load to being displaced, etc.). We shall denote work done in overcoming useful resistance as W_u. There are also various kinds of harmful resistance in a machine (friction, resistance of the air). We shall denote the work done in overcoming this resistance as W_h. If a machine is to run uniformly, the work of the motive force W_{mf} must be equal to that required to overcome all resistances, that is,

$$W_{mf} = W_u + W_h. \tag{96}$$

The motion of a machine is said to be established if the velocity of all its moving elements remain unchanged after each revolution of the shaft.

If $W_{mf} > W_u + W_h$, then the surplus work is expended on increasing kinetic energy with a corresponding increase in velocity. This occurs when an engine is being started, in which case $W_u = 0$ because there is no useful resistance.

When motive force is cut off, the continued motion will be due to the kinetic energy the machine has accumulated. If useful resistance also ceases, the energy will be absorbed in overcoming harmful resistance and, after a given time, the machine will stop**.

* See Part II, Sec. 186 (p. 267)

** The science dealing with the forces acting upon the various links of machines is called dynamics of machines. Extensive research in this field has been done by the Russian scientists, N. Zhukovsky, K. Rerikh, N. Mertsalov, and others.

Therefore a machine possessing established motion must satisfy Eq. (96).

By dividing both sides of this equation by W_{mf}, we obtain

$$\frac{W_u}{W_{mf}} + \frac{W_h}{W_{mf}} = 1. \quad \text{(a)}$$

The first member of the left side of the equation denotes the share of the machine's work in doing useful work (work for which the machine is designed). This expression represents *mechanical efficiency*, which is a *measure of the useful expenditure of mechanical energy*. By denoting it in the accepted manner by the letter η, we obtain

$$\eta = \frac{W_u}{W_{mf}}. \quad (97)$$

The second member of the equation expresses the part of the work expended in overcoming harmful resistance. Accordingly, Eq. (a) may be given as

$$\eta = 1 - \frac{W_h}{W_{mf}}. \quad (98)$$

Thus we see that efficiency is always less than 1.

Oral Exercises

1. When will the work done by a machine satisfy the equation $W_{mf} = W_u$? What would its efficiency be equal to?
2. Can a machine do useful work if $W_h = W_{mf}$? What would its efficiency be equal to in this case?

Illustrative Problem 67. Under the action of force $\mathbf{P}_1$ a body of weight $\mathbf{G}$ is displaced at a constant speed from position A to B on an inclined plane (Fig. 158). Find its efficiency if the coefficient of friction $f = 0.1$, and the angle of inclination $\alpha = 27°$.

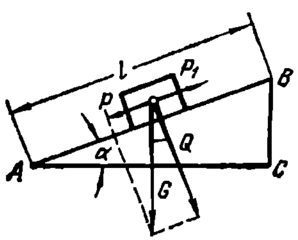


Fig. 158

Solution: if motion is uniform, the condition $W_{mf} = W_u + W_h$ must be satisfied. The useful work done in overcoming the force of gravity $W_u = Pl = G \sin \alpha l$. The magnitude of the work done against the force of friction $W_h = Qfl = G \cos \alpha fl$. Hence,

$$W_{mf} = Gl \sin \alpha + Gfl \cos \alpha,$$

and

$$\eta = \frac{Gl \sin \alpha}{Gl (\sin \alpha + f \cos \alpha)} = \frac{\sin \alpha}{\sin \alpha + f \cos \alpha}.$$

By dividing the numerator and denominator of the right side by $\cos \alpha$, we obtain

$$\eta = \frac{\tan \alpha}{\tan \alpha + f} = \frac{0.51}{0.51 + 0.1} \approx 0.84.$$

132. "Perpetual Motion" as an Impossibility

For many centuries fruitless attempts have been made to invent a machine which, if once started, would continue to run without a further supply of energy—the "perpetual motion" machine that would run without the application of any motive force. If such a thing were possible Eq. (96) would read

$$W_{mf} = W_u + W_h = 0.$$

From this it follows that such a machine would work for an indefinite length of time at a constant speed if $W_u = 0$ and $W_h = 0$, that is, if no work had to be done in overcoming resistance. Let us concede that a machine does no useful work ($W_u = 0$), in which case $W_h = 0$; this infers that neither should there be any work to overcome harmful resistance. This is impossible, for any movement of contiguous bodies relative to each other is always accompanied by harmful resistance. Accordingly, however small such resistance may be, the machine will expend the kinetic energy of starting in order to overcome this resistance and will inevitably come to a standstill.

Hence we see it is impossible to make a machine which would do useful work, or even only the work of overcoming harmful resistance, for an indefinite length of time without a further supply of energy.

133. Impact

If a body in motion comes into contact with another body (either moving or at rest) the interaction between them is called *impact*.

Experiment has shown that impact is accompanied by a change in form (deformation) of the colliding bodies. The magnitude of deformation depends upon the physical properties of the bodies. After impact, some bodies recover their original form, while others remain deformed. The ability of a body to resume its original form is called *elasticity*. It must be noted here that there are no perfectly elastic materials, just as there are no absolutely hard materials. However, some materials may be considered elastic (ivory, tempered steel) and others inelastic (clay, for example). Accordingly, there may be either an elastic or inelastic impact, depending upon the materials of the colliding bodies.

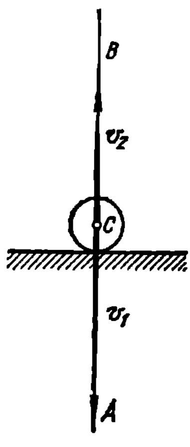


Fig. 159

Let us assume that a ball of mass m (Fig. 159) is falling freely. After it comes in contact with a horizontal surface it becomes deformed for an instant. If the ball and the horizontal surface were both absolutely inelastic, the ball would remain motionless. If the ball possesses a velocity of v_1 when it falls on the surface, its kinetic energy would be $\frac{mv_1^2}{2}$ and would be expended in the work of deformation.

Now let us assume that both the ball and surface are made of absolutely elastic materials. In this case the ball's kinetic energy would be expended very rapidly in overcoming its internal forces of elasticity, that is, it would be expended in deformation. Kinetic energy would be converted into potential energy of the deformed body, after which the reverse would happen: the two bodies would recover their original form under the action of the force of elasticity, potential energy would again be transformed into kinetic energy whose magnitude $\frac{mv_2^2}{2} = \frac{mv_1^2}{2}$ and the ball would move in the opposite direction with a velocity of $\mathbf{v}_2$ equal in magnitude to the velocity $\mathbf{v}_1$ which it had at the moment of impact. Hence, when such an impact is absolutely elastic, *the velocity of rebound is equal to the velocity of the fall.* If two elastic balls of the same mass are moving towards each other with equal velocities, after rebound they will move in reverse directions with the same velocities.

Now let us assume that impact is not absolutely elastic. This means that the kinetic energy of the ball before impact $\frac{mv_1^2}{2}$ will not be fully regained after rebound, i.e., $\frac{mv_2^2}{2} < \frac{mv_1^2}{2}$, from which it follows that $v_2 < v_1$ and the ball will rebound with a smaller velocity. The relationship $\frac{v_2}{v_1} = k$, called the *coefficient of restitution*, describes the elasticity of materials. For example, if the balls are of wood, k 0.5; if of steel, $k = 0.77$, etc.

134. Impact of a Freely Falling Hammer

Impact is a phenomenon that is taken advantage of extensively in industry since it makes it possible for one of two colliding bodies, if it has a small mass but a great velocity, to do a large amount of work with a small displacement. The work of a sledge hammer or a pile driver illustrates this.

Let us examine the work of the drop hammer shown in Fig. 160, the ram D of which and its die E drop freely under the action of the force of gravity. We denote the weight of these dropping units as G and the height of their fall as H. The velocity v_1 which they have upon dropping is, according to Eq. (37), $v_1 = \sqrt{2gH}$, hence they acquire kinetic energy of $\frac{mv_1^2}{2} = GH$.

As has been explained in the preceding section, when impact is inelastic, the velocity after rebound $\mathbf{v}_2$ is less than the velocity of the fall $\mathbf{v}_1$, according to which $\frac{mv_2^2}{2} < \frac{mv_1^2}{2}$, that is, part of the kinetic energy is expended in the deformation of the mutually colliding bodies. Since it is the aim in the process of forging to deform the workpiece as much as possible, therefore the greater the hammer's expended kinetic energy, the more efficient it will be.

During forging, the workpiece K (Fig. 160) lies on the anvil B which is mounted on a massive steel block C, which in turn rests upon a foundation. When the die hits the workpiece, it not only deforms it but shakes all the undersupports, which means that a portion of the kinetic energy is expended in displacing these undersupports. Obviously, the smaller this displacement, the more effective will be the hammer's energy. From this it follows that all the undersupports of a drop hammer should be made as heavy as possible. In more detailed studies of mechanics it is proved that the efficiency of a drop hammer is expressed by the equation

$$\eta = \frac{1}{\frac{G}{G_0} + 1}(1 - k^2), \qquad (99)$$

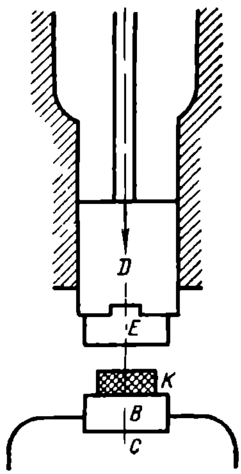


Fig. 160

in which G is the weight of the dropping units of the hammer, G_0 the weight of the workpiece and its supports, and k the coefficient of restitution. It is evident that with a greater G_0 there will be a smaller denominator and hence the hammer will be more efficient. Usually with a freely falling ram, the weight of the steel block is made ten to fifteen, and even twenty, times heavier than the weight of the ram.

Illustrative Problem 68. A forging hammer, whose dropping units weigh $G = 2{,}250$ kg and fall from a height $H = 1.5$ m, forges a workpiece in ten strokes. Find the amount of useful mechanical energy W, if the weight of the steel block $G_0 = 45$ tons, the coefficient of restitution $k = 0.4$, and the friction loss of energy in the guides is 5%.

Solution: the kinetic energy of one stroke $W_1 = 0.95\frac{mv^2}{2} = 0.95\,GH =$ $= 3{,}206$ kg-m. Hence in ten strokes the energy expended usefully $W =$ $= 3{,}206 \times 10 = 32{,}060$ kg-m. We then find the efficiency of the hammer through Eq. (99):

$$\eta = \frac{1}{\frac{2{,}250}{45{,}000} + 1}(1 - 0.4^2) = 0.8.$$

Whereupon, the energy spent on forging alone $W_u\eta = 32{,}060 \times 0.8 \approx$ $\approx 25{,}650$ kg-m.

135. Questions for Review

1. Explain why railway carriages and locomotives are equipped with bumpers.
2. It occurred that the last few carriages in a train had no brakes.

What will happen when the train's brakes are applied? Will the bumper springs between these carriages be compressed (deformed) all to the same extent?

3. A body of weight G falls from height h_1 to height h_2. What change is there in its potential energy?

4. If the shaft of a machine must change its direction of rotation at brief intervals of time, should it be equipped with a flywheel?

5. Why is not the steam engine of a locomotive in need of a flywheel?

6. What kind of motion will a machine have if at a certain moment $W_{mf} < W_u + W_h$?

7. Is an efficiency $\eta \geqslant 1$ possible?

8. A body which comes into collision with an immovable barrier remains motionless. What is its kinetic energy expended on?

9. Explain why it is more advantageous, when cutting a workpiece, to use a heavy vise and a heavy workbench.

10. One of two drop hammers has a heavier anvil and foundation than the other. Which of the two will work more productively?

136. Exercises

72. A locomotive with a tractive force of 15,000 kg pulls a train weighing 1,500 tons along a horizontal track. Considering that the resistance to motion is 0.005 of the weight of the train, find the kinetic energy it accumulates after an elapse of two minutes of starting, and the work performed during that time, assuming the tractive force to be constant.

73. After an elapse of six minutes the same train reached an upgrade, moving against a resistance of 0.075 of its weight. If steam is cut off at the beginning of the upgrade, how long will it take the train to stop and what distance will it have covered from that point.

74. After starting from the station, a train weighing 400 tons develops a speed of 72 km/hr when it had covered a distance $s = 1{,}600$ m. Find the tractive force **P**, assuming it to be constant, and also the braking force F, if upon cutting off steam and applying the brakes, the train travels another 2,000 m (assuming resistance without braking to be 0.005 of the weight of the train).

75. The weight of the dropping units of a drop hammer $G = 3$ tons, and of the workpiece, anvil, and other undersupports 40 tons. Find the efficiency of the hammer if the coefficient of restitution $k = 0.4$.

PART TWO

THE THEORY OF MACHINES AND FUNDAMENTAL CONCEPTS OF STRAIN

THE THEORY OF MACHINES

INTRODUCTION

137. Machines and Mechanisms

Assume that a threading lathe is cutting a thread on a workpiece. The rotation of the electric motor is transmitted to the spindle of the lathe and then to the lead screw. The rotation of the screw is converted into motion of translation of the carriage. By setting the lathe properly, we may obtain the required rotating speed of the spindle as well as the motion of translation of the carriage.

A system of interconnected bodies performing prescribed motions is called *a mechanism.*

Each moving part making up a mechanism is called *a link.*

That link of a mechanism which imparts motion to other links is called the *driver*, while those to which the motion is imparted are called the *followers*, or *driven* links.

A metal-cutting lathe is put in motion by an electric motor. The motor receives electricity from the local supply and converts it into mechanical energy which the lathe expends performing mechanical work to overcome useful resistance (resistance to cutting). The electric motor in its turn receives electricity generated by a dynamo which is also put in motion by a unit of some kind (a hydroturbine, an internal combustion engine, etc.) which is run either by the mechanical energy of a hydraulic engine, or thermal energy derived from fuel in an internal combustion engine, etc.

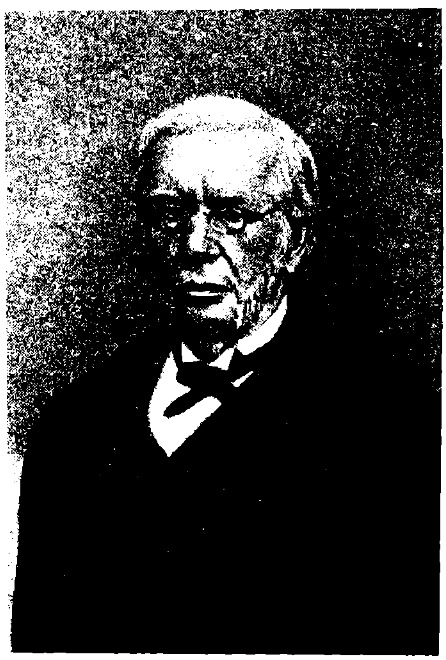
In all these instances we find that the unit either receives mechanical energy and transforms it into some other form of energy (a dynamo), or receives some form of energy and transforms it into mechanical energy (an electric motor, internal combustion engine, steam turbine), or performs useful mechanical work by means of mechanical energy supplied to it (hydroturbine and metal-cutting lathe).

A combination of mechanisms designed to transform energy into the form required and thus to do useful work is called *a machine.*

Mechanisms are not only incorporated into machines, they are also used independently. For example, a clock is not a machine since it is not intended to transform energy or to overcome useful resistance.

138. Historical Survey of Machine Engineering in Russia

Long ago, in an age when machine construction was still in its infancy, talented Russians skilfully achieved practical solutions to complex mechanical problems. This was especially true at the time of Peter the Great, who encouraged many outstanding inventors in their work, such as A Nartov, N Pilenko, M. Sidorov and others; Nartov invented the first lathe with a carriage and the first duplicating lathe Of the numerous Russian mechanics



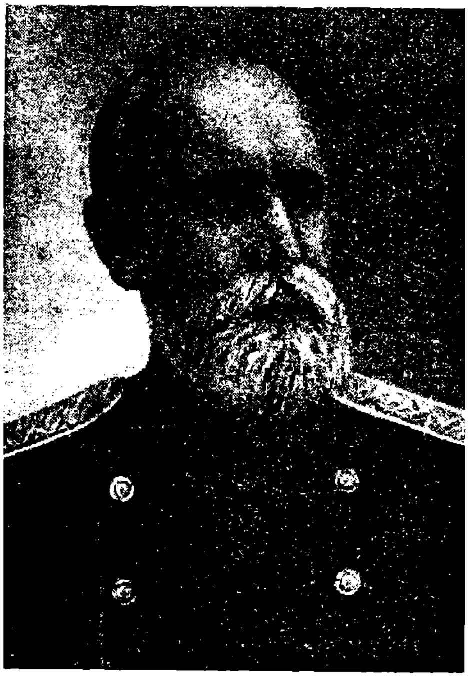
I. Vyshnegradsky

of the 18th century, particular note must be made of I. Polzunov (1728-1766) for his steam engine.

The brilliant Russian scholar M. Lomonosov combined his many world-famous purely scientific researches with inventions in machine engineering, such as the spherolathe, a grinding machine and a facing lathe.

I. Kulibin (1735-1818) became well known for his major inventions in various branches of technology, particularly in the construction of different kinds of instruments.

Neither was theoretical work neglected in the 18th century; the first treatise on mechanics to be published in Russia was compiled by G. Skornyakov-Pisarev and appeared in 1722,



N. Petrov

containing calculations for the construction of levers, windlasses and other simple mechanisms.

Beginning with the end of the 18th century, engineering mechanics began to progress rapidly in Russia— a development which continued into the 19th and 20th centuries. Among the eminent scientists responsible for this advance were P. Chebyshev, I. Vyshnegradsky, N. Petrov, M. Ostrogradsky, V. Kirpichev, N. Zhukovsky and a host of others.

These achievements of Russian scientists and inventors in the field of engineering did not receive proper support in pre-revolutionary Russia. But the Great October Socialist Revolution, which swept away capitalism and placed the former privately-owned means of production into the hands of all the people, completely changed this situation.

In his closing address before the Third All-Russian Congress of Soviets, delivered after his brilliant analysis of the historic significance of the Great October Socialist Revolution, V. I. Lenin said:

"In the past man's mind and genius provided a chosen few with all the benefits of technology and culture, while most others were deprived of the essentials of education and development. But now all the wonders of engineering, all the achievements of culture, will be within the reach of all the people, and never again will the mind and genius of man be turned into a means of coercion and exploitation"*.

M. Ostrogradsky

Lenin's profound words are turning into reality before our very eyes. Each year labour-consuming processes are being mechanised on an ever-widening scale in the U.S.S.R., where engineering is creating highly productive machines. This work is in close harmony with the policy of extensive automation — the highest stage of mechanisation.

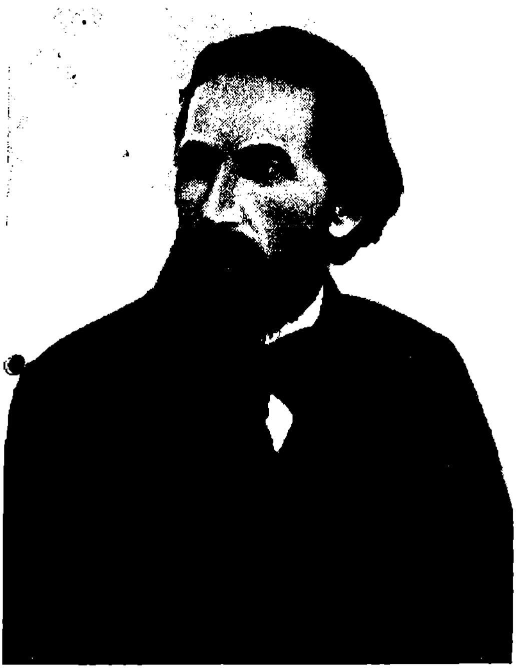
In machine building, efficient Soviet-made automatic lathes, as well as entire production lines of unique design, are already in extensive use in the manufacture of machine parts.

On construction sites, walking draglines with a 25 m^3 (and

* V. I. Lenin, *Collected Works*, Russ. ed., Vol. 26, p. 436.

more) capacity may now be seen; a small but efficient crew on one of these machines displaces the work of from seven to nine thousand hand labourers.

Many efficient mining machines, particularly for the coal fields, were first designed in the Soviet Union. At the present time the coal-combine takes the place of several machines heretofore used



V. Kupichev

separately in the operations of cutting, blast-hole drilling, and loading of the coal upon the conveyors. The U.S.S.R. now takes first world place in the mechanisation of coal mining.

Great strides are being made in the Soviet Union in the production of equipment for electric stations, metallurgical plants, highly-efficient machine tools, automatic production lines, forges, all types of unique instruments and other machines.

These mechanisation processes, which are doing away with former labour-consuming hand operations, not only make work easier but also raise productivity to a very high level.

And now still greater events have occurred in the development of Soviet science and technique—the launchings of Soviet manned rockets into the outer space. For these space ships—Vostok-1 and Vostok-2 — are the forerunners of man's flights, in the not-too-distant future, to the moon and the planets of the solar system—Venus, Mars and others.

CHAPTER XV

THE INCLINED PLANE, THE PULLEY, AND THE WINDLASS

The inclined plane, the pulley, and the windlass (also known as a winch) were among the very first engineering contrivances in technical history. They are still used as integral parts of various machines and mechanisms and for that reason we shall begin with them in making our acquaintance with the theory of machines and mechanisms. Until recently the inclined plane, the pulley, and the windlass were called "simple machines"

139. The Inclined Plane

Assume that a body of weight **G** is lying on an inclined plane KM (Fig. 161*a*). We resolve the force of gravity **G** as represented by vector $\overline{CA}$, into component $\overline{CD}$ perpendicular to KM, and component $\overline{CB}$ parallel to KM. The force $\overline{CD}$ is balanced by the reaction **N** directed in the opposite direction. Hence, the body's motion on the inclined plane will take place under the action of force $\overline{CB}$. If there were no friction between the body and the inclined

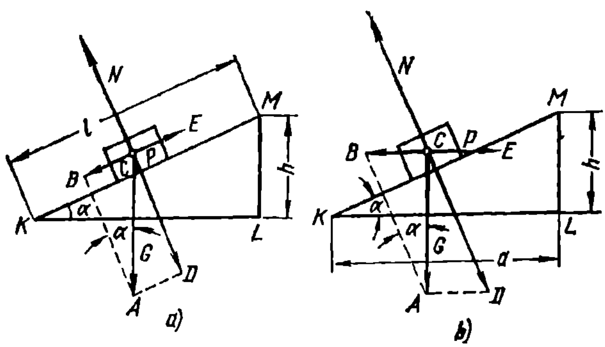


Fig. 161

plane, the body would slide down with a definite acceleration. In order for the body to be in equilibrium (to either remain at rest or to be displaced along the plane at a constant speed), a force **P** represented by vector $\overline{CE}$ and equal and opposite to vector $\overline{CB}$ would have to be applied to it. Thus the body can be in a state of equilibrium under the action of three forces — **G**, **N**, and **P**.

Let us determine the magnitude of force **P**.

By denoting the length KM of the plane as l and the height LM as h, we obtain, from the similarity of the right triangles

KLM and *ABC*,

$$\frac{CB}{ML} = \frac{CA}{KM}, \text{ from which } CB = CA\frac{ML}{KM} = CA\frac{h}{l}.$$

And since force **P** is represented by vector $\overline{CE}$ which is equal in magnitude to vector $\overline{CB}$, we obtain

$$P = G\frac{h}{l}. \quad (100)$$

This equation can be given another form. By denoting the angle of inclination *LKM* of the plane as α, we obtain from ΔKLM

$$h = l \sin \alpha,$$

$$\text{from which } \frac{h}{l} = \sin \alpha, \quad \text{and}$$

$$P = G \sin \alpha. \quad (101)$$

Let us look into a case when force **P** is not parallel to the length of the inclined plane but to its base *KL* (Fig. 161*b*). In this case we resolve force **G** into two components — one component $\overline{CD}$ perpendicular to the length of the inclined plane, and another $\overline{CB}$ parallel to its base *KL*. Just as before, from the similarity of the right trangles *KLM* and *ABC*, we arrive at the relationship

$$\frac{CB}{ML} = \frac{CA}{KL}, \text{ from which } CB = CA\frac{ML}{KL} = CA\frac{h}{a},$$

hence

$$P = G\frac{h}{a}, \quad (102)$$

in which *a* is the base of the inclined plane.

From ΔKLM we obtain

$$h = a \tan \alpha, \text{ from which } \frac{h}{a} = \tan \alpha,$$

$$\text{and } P = G \tan \alpha. \quad (103)$$

A comparison of Eqs (100) and (102) will show that the first way of applying force **P** is the more advantageous since its magnitude is less, the same being evident from Eqs (101) and (103), because $\sin \alpha < \tan \alpha$.

Let us assume that the body is moving uniformly up the same inclined plane. In this case the weight **G** of the body constitutes a useful resistance which is overcome by the motive force **P**. Assume this force to be parallel to the length of the plane (Fig. 161*a*). Since the sine of the angle cannot exceed 1, it follows from Eq. (101) that when $\alpha = 90°$, inevitably $P < G$, that is, when force **P** is parallel to the length of the inclined plane, the inclined plane gives an advantage in *force*. This advantage is determined by the ratio of the magnitude of the force of resistance **G** to the magnitude of the motive force **P**, which according to Eq. (100), is represented by

$$\frac{G}{P} = \frac{l}{h}.$$

Thus, in order to raise a body to a height $h = ML$, force **P** must be exerted through the entire displacement $l = KM$. We could raise the body to the same height h without the inclined plane if we applied a vertical force to it, equal and opposite to the weight **G** of the body.

From this relationship it follows that *the greater the gain in force, the greater the loss in displacement, and vice versa.*

This is the "ABC" of mechanics.

The conclusions thus reached are also applicable to the second case examined above, when force **P** is parallel to the base of the inclined plane. It should only be noted that since the $\tan 45^\circ = 1$, force **P**, as is apparent form Eq. (103), will be smaller than force **G** when $\alpha < 45^\circ$, whereas when $\alpha > 45^\circ$ the two forces will be equal, and when $\alpha - 45^\circ$ force $P > G$.

Now let us compare the work performed by the forces applied to the body when its motion along the inclined plane is uniform. As already noted, the body is under the action of forces **G**, **P**, and **N**. From Fig. 161*a* it is evident that force **G** forms an angle $ACB = 90^\circ - \alpha$ to the incline. By employing Eq. (76) we obtain the work W_G performed by this force through displacement l:

$$W_G = Gl \cos (90^\circ - \alpha) = Gl \sin \alpha.$$

The work performed by force **P**

$$W_P = Pl = G \sin \alpha l = Gl \sin \alpha.$$

The work performed by force **N** directed perpendicular to the motion, is zero. Thus we see, $W_G = W_P$, that is, *the work of the motive force is equal to the work of the force of resistance.*

Heretofore we have limited ourselves to uniform motion of a body up an inclined plane without taking friction into account. In actuality friction diminishes any advantage gained in force. Therefore besides force $\overline{CB}$, the force of friction $F = fN$ (in which f represents the coefficient of friction) is also directed opposite to the motion.

When force **P** is directed parallel to the inclined plane, force $\mathbf{N} = G \cos \alpha$, which means that the force of friction $F = Gf \cos \alpha$.

If the body is to have uniform motion upwards, force **P** must be equal to the sum of the forces of resistance, i.e.,

$$P = G \sin \alpha + Gf \cos \alpha = G(\sin \alpha + f \cos \alpha). \qquad (104)$$

Illustrative Problem 69. It is necessary to raise load $G = 400$ kg a distance of 0.5 m along two parallel inclined beams each 5 m in length. Find the force required to do this work if the coefficient of friction $f = 0.15$.

Solution: with $h = 0.5$ m, and $l = 5$ m (Fig. 161*a*), we have $0.5 = 5 \sin \alpha$, from which $\sin \alpha = 0.1$, $\alpha = 5°45'$, $\cos \alpha = 0.995$, and the force required

$$P = 400\,(0.1 + 0.15 \times 0.995) \approx 100 \text{ kg}.$$

140. The Wedge

The wedge is one form of the inclined plane and possesses the shape of a triangular prism (Fig. 162*a*). In a longitudinal cross-section of this prism the angle $\alpha = \angle KML$ is considerably smaller than either of the two other angles. Edge KL is called the *head* of the wedge, while the side edges KM and LM are its *cheeks*.

Assume that the wedge, under the action of force **P**, is penetrating into another body at a constant speed. The body resists the motion of the wedge. This is expressed by the reactions $\mathbf{N}_1$ and $\mathbf{N}_2$ perpendicular to the cheeks of the wedge. When examining the equilibrium of the wedge without taking friction into account, we find that forces **P**, $\mathbf{N}_1$ and $\mathbf{N}_2$ balance each other. We delineate these three forces from any arbitrary point O, and on vectors $\overline{OC_1}$ and $\overline{OD_1}$, representing forces $\mathbf{N}_1$ and $\mathbf{N}_2$, we construct the parallelogram $OC_1E_1D_1$ (Fig. 162*b*). If the system is in equilibrium, the diagonal $\overline{OE_1}$ must be equal in magnitude and opposite in direction to vector $\overline{OE}$ representing force **P**. A comparison of triangles OC_1E_1 and KLM will show that they are similar because their angles are formed by mutually perpendicular sides; from this it follows that

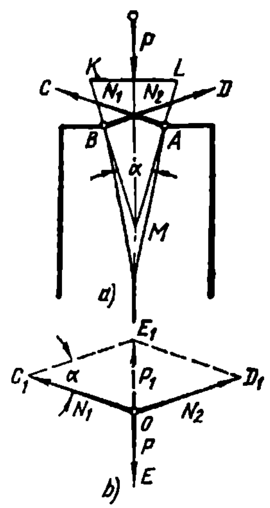


Fig. 162

$$P : N_1 : N_2 = KL : ML : KM. \tag{105}$$

If the wedge has equal edges as shown in the figure ($KM = ML$), then

$$\frac{N}{P} = \frac{KM}{KL}, \tag{106}$$

that is, *the mechanical advantage in force is equal to the ratio of the length of the cheeks to the thickness of the head.* The smaller the angle of the wedge and the thinner the head, the greater will be the gain in force.

The properties of the wedge are used to advantage in various splitting and cutting tools. Later (Sec. 201) we shall study the use of the wedge in the fastening of machine parts.

It must be noted that the force of friction increases as the angle of the wedge decreases. For example, the splitting of wood with a thick-headed axe, instead of with an ordinary carpenter's hatchet, is easier because the additional weight lends more kinetic striking energy and also because of the greater ease with which the axe can be pulled out if the wood is not entirely split.

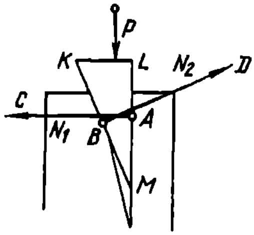


Fig. 163

Illustrative Problem 70. What would be the magnitudes of forces $\mathbf{N}_1$ and $\mathbf{N}_2$ during the uniform displacement of a wedge KLM (Fig 163) possessing a thickness $KL = 25$ mm and length $LM = 200$ mm when under the action of force $P = 50$ kg, if there were no friction?

Solution: from Eq. (105) we obtain $\frac{P}{N_1} = \frac{KL}{LM} = \frac{25}{200} = \frac{1}{8}$ from which $N_1 = 8P = 400$ kg.

From the same equation we obtain

$$N_2 = P\frac{KM}{KL} = 50\cdot\frac{\sqrt{25^2 + 200^2}}{25} \approx 403 \text{ kg.}$$

141. The Lever

Let us examine the simple case of a straight lever (Fig. 164) to which are applied two parallel forces **P** and **Q** acting perpendicular to the longitudinal axis AB. Point O, called the *fulcrum*, is at distances a and b from the points of application of forces **P** and **Q**.

Two conditions stated in Sec. 34 must be observed to keep the lever in equilibrium: they are a) Eq. (12) - the algebraic sum of all forces must be zero, and b) Eq. (13) — the algebraic sum of the moments of the forces must also be zero.

The first condition is expressed as

$$P + Q - R = 0, \text{ from which } P + Q = R,$$

in which R is the reaction at the fulcrum*.

Since the algebraic sum of the moments of all the forces with respect to fulcrum O is zero, then

$$Pb - Qa = 0,$$

or

$$\frac{Q}{P} = \frac{b}{a} \qquad (107)$$

* The weight of the lever is ignored in this case.

that is, *the forces are inversely proportional to the arms of the lever.*

Now let us take a more complex example when the forces **P** and **Q** are not directed perpendicular to the axis of the lever (Fig. 165). We resolve force **P** into two components — $\overline{BL}$ acting along the axis of the lever and $\overline{BK}$ acting perpendicular to the axis. Repeating the process with force **Q**, we obtain forces $\overline{AE}$ and $\overline{AF}$. If the fulcrum is constructed so that the lever cannot be displaced in a horizontal direction, the resultant of forces $\overline{BL}$ and $\overline{AE}$ will be balanced by the horizontal component of reaction **R** at the fulcrum. Therefore if the lever is to

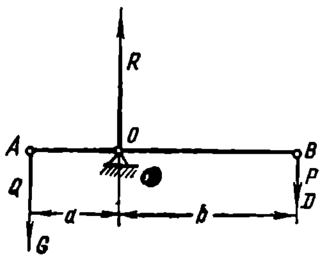


Fig. 164

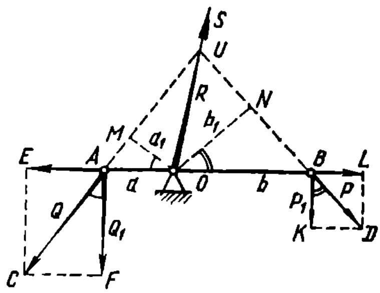


Fig. 165

remain in equilibrium, it is required that the algebraic sum of the moments of the other forces with respect to any point should be zero. By taking point O as the centre of the moments, we obtain

$$P_1 b = Q_1 a. \qquad (a)$$

From point O we delineate lines $OM = a_1$ and $ON = b_1$ perpendicular to the lines of action of forces **P** and **Q**. Then comparing the right triangles OMA and AFC and also ONB and BKD, we see that they are similar pairs because their acute angles have mutually perpendicular sides: $\triangle OMA \backsim \triangle AFC$, and $\triangle ONB \backsim \triangle BKD$, from which it follows that

$$\frac{OA}{AC} = \frac{OM}{AF} \quad \text{or} \quad \frac{a}{Q} = \frac{a_1}{Q_1}, \text{ whence } Q_1 a = Q a_1.$$

In the same way we obtain $P_1 b = P b_1$.

Substituting these expressions for $P_1 b$ and $Q_1 a$ in the above Eq. (a) we obtain

$$Pb_1 = Qa_1, \quad \text{or} \quad \frac{Q}{P} = \frac{b_1}{a_1}. \qquad (108)$$

We thus see that we have obtained an expression analogous to Eq. (107), the only difference being that included in it are the arms of the moments of forces **P** and **Q** with respect to the fulcrum, instead of the arms of the lever a and b.

Now let us investigate a general case when the lever is not straight (Fig. 166).

We resolve the forces **P** and **Q** respectively into the components $\overline{BL}$, $\overline{BK}$ and $\overline{AF}$, $\overline{AE}$, of which BK is perpendicular to OB and AF is perpendicular to OA. Then, reasoning as before, we arrive at the same equation (108)*.

In the above cases the fulcrum O was situated between the points of application of the forces. This type of lever is called a *lever of the first kind* as distinguished from one of *the second kind* when the points of application of the forces are on the same

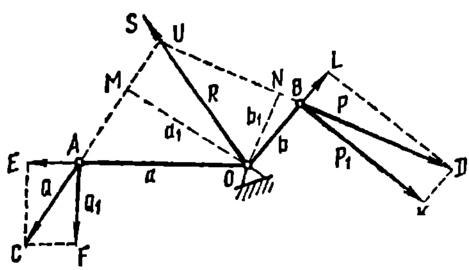


Fig. 166

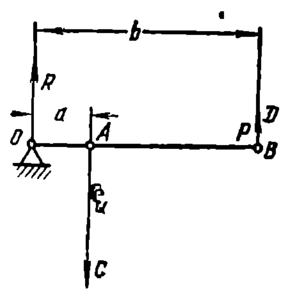


Fig. 167

side as the fulcrum (Fig. 167). By applying Eq. (12) to a lever of the second kind, we determine the reaction of the fulcrum **R** from the equation

$$Q+P-R=0,$$

whence

$$R=Q+P. \qquad (109)$$

Then taking the algebraic sum of the moments of the forces with respect to point O, we obtain $Qa-Pb=0$, whence

$$Pb=Qa. \qquad (110)$$

If the lines of action of the forces were not perpendicular to the axis of the lever, or if the axis of the lever were not straight, we should have obtained the same result as for a lever of the first kind.

Wherefore, *in all cases when a lever is in equilibrium, the forces* **P** *and* **Q** *applied to it are inversely proportional to the distances between their lines of action and the fulcrum.*

* The reaction of the fulcrum may be determined as follows. As already shown in Sec. 24, the lines of action of forces **P**, **Q**, and **R** intersect at one point U. Hence the line of action of the reaction is known. By constructing a parallelogram on the force **P** and **Q** we obtain their resultant. The reaction $\overline{OS}$ will be equal and opposite to it.

From this it is apparent that the lever, in allowing a lesser force to balance a greater one, achieves a mechanical advantage. It is also easily understood that the displacement of the point of application of the lesser force **P** will be as much greater than that of the point of application of force **Q**, as the magnitude of **Q** is greater than that of **P**; here again the "ABC" of mechanics is valid.

Bearing in mind that there is friction between the fulcrum and the lever, we conclude that the useful work the latter performs is somewhat less than the work performed by the motive force.

Levers are not only used to convert a lesser force into a greater one, but also for advantage in displacement. For example, by displacing point A a certain distance (Fig. 167), we displace point B a distance as many times greater as arm b is greater than arm a. This property of a lever is frequently utilised in the construction of measuring instruments.

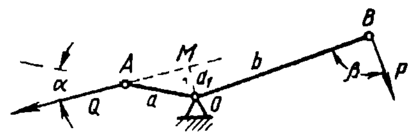


Fig 168

The lever is extensively used in machines and other mechanisms, and also in devices of all kinds.

Illustrative Problem 71. Arm a of the bent lever AOB in Fig. **168** is 80 mm in length, and arm b is 300 mm. What should be the magnitude of force **P** acting at an angle of $\beta = 90°$ to arm OB in order to balance force $Q = 120$ kg acting at an angle of $\alpha = 30°$ to arm OA?

Solution: In employing Eq. (108) we must take $b_1 = b = 300$ mm, $a_1 = a \sin \alpha = 80 \sin 30° = 80 \times 0.5 = 40$ mm, and $Q = 120$ kg. After substituting these values in the equation we obtain

$$P = \frac{Qa_1}{b_1} = \frac{120 \times 40}{300} = 16 \text{ kg}$$

142. A System of Levers. The Differential Lever

The mechanical advantage obtained from a lever can be increased considerably by using a system of several interconnected levers.

Let us consider the two levers forming the system shown in Fig. 169. To the end B of lever AB with fulcrum O_1 a second lever with fulcrum O_2 is fastened by means of strap BC attached to its end C. By applying force **P** to end A we obtain force Q_1 on end B, equal to the relationship

$$\frac{Pb_1}{a_1}.$$

This force is transmitted to end C of lever CO_2 on which the acting forces will be determined according to the relationship

$Q_1b_2 = Q_2a_2$, from which force Q_2 obtained at point D is

$$Q_2 = Q_1 \frac{b_2}{a_2}.$$

Replacing Q_1 in this expression with the value just evolved for it, we obtain

$$Q_2 = P \frac{b_1 b_2}{a_1 a_2}. \tag{111}$$

If there had been three levers then

$$Q_3 = P \frac{b_1}{a_1} \times \frac{b_2}{a_2} \times \frac{b_3}{a_3}, \text{ etc.}$$

Thus the mechanical advantage obtained by a whole system of levers is equal to the product of the numbers expressing the mechanical advantage produced by each lever in the system.

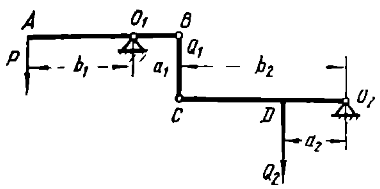


Fig. 169

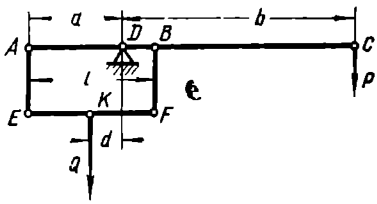


Fig. 170

If we took two levers with a ratio between the arms of $\frac{b_1}{a_1} = \frac{b_2}{a_2} = 10$, the mechanical advantage obtained by the system would be $\frac{Q}{P} = 10^2 = 100$. Accordingly, a displacement of 0.1 mm of point D would displace point A a distance of $0.1 \times 100 = 10$ mm.

However, such a system of levers is extremely cumbersome. For this reason a variation is used, called *a differential*, or *floating coupling*.

Assume that lever AC (Fig. 170) with a fulcrum D has a crosspiece EF suspended from it by two straps AE and BF. A force **Q** is applied at point K in the middle of the crosspiece, and force **P**, its equilibrant, is applied to the long arm of the lever at point C. Let us determine the relationship between these two forces.

Since force **Q** is applied at the middle of the crosspiece EF, a force $\frac{Q}{2}$ is acting on each strap — one at point A and another at point B. Let us write the conditions required for the lever to be in equilibrium, using Eq. 12, since all the forces are parallel:

$$-\frac{Q}{2} a + \frac{Q}{2}(l - a) + Pb = 0,$$

or

$$Q(l-a-a)+Pb=Q\left(\frac{l}{2}-a\right)+Pb=0,$$

whence

$$Q=P\frac{b}{a-\frac{l}{2}}.$$

By denoting the distance between the line of action of force **Q** from the fulcrum as d, i.e., $d=a-\frac{l}{2}$, we finally obtain

$$Q=P\frac{b}{d}. \tag{112}$$

It is seen that the mechanical advantage will be equal to the ratio of the bigger arm CD of the lever to the distance between the two vertical straight lines delineated through the middle K of the crosspiece and the fulcrum. Since this distance can be made infinitely small, theoretically an infinitely great mechanical advantage can be obtained.

Systems of floating couplings are used, for example, in decimal and centesimal scales.

Illustrative Problem 72. In the floating lever just studied, the arm $b=1{,}000$ mm, arm $a=251$ mm, and $l=500$ mm; then $d=251-\frac{500}{2}=1$ mm. Substituting these values in Eq. (112), we obtain $Q=1{,}000\ P$.

143. Fixed and Movable Pulleys

A pulley is a sheave on the rim of which there is a groove for a rope (or sprocket teeth for a chain). The simplest type is the *immovable pulley*, the geometrical axis of which remains fixed when it is in operation (Fig. 171).

Assume that the rope (or chain) has a load to be raised that exerts a force **Q** at one end of it. To determine the force **P** which must be applied to the other end of the rope in order to balance force **Q**, we may regard the pulley as a bent lever AOB having arms of equal length because $AO=OB=R$, R being the radius of the sheave. The conditions for equilibrium of this lever with respect to its fulcrum O, is $PR=QR$, from which

$$P=Q.$$

Thus in an immovable pulley neither the force nor the velocity changes in magnitude; only the direction of the force changes. This is advantageous in many cases. For instance, instead of raising a load by pulling it upwards, it is much more convenient to use such an immovable pulley which makes it possible to do the same work by applying to the rope the same force directed downwards. Due to harmful resistance, the efficiency of this pulley is ordinarily from 0.8 to 0.9.

A *movable pulley*, so called because its axis is displaced when it is in use, is shown schematically in Fig. 172. A rope, one end of which is fastened to a stationary hook K, passes round the sheave L from below; a motive force **P** acts on its other end*. The force of resistance **Q** (such as the weight of the load) is applied to the casing of the movable pulley in which its axis is rotating.

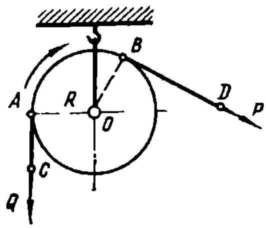


Fig. 171

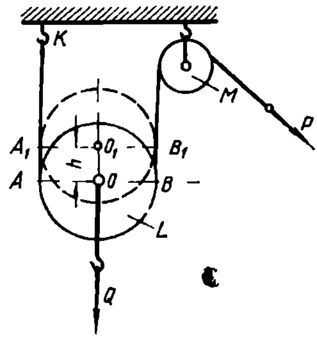


Fig. 172

Let us work out the relationship between the motive force **P** and the force of resistance **Q**. Let us consider the diameter AB of the pulley to be a lever of the second kind, turning about point A under the action of force **P**. By applying Eq. (110), in which we substitute diameter AB instead of b, and radius AO instead of a, we obtain $\frac{P}{Q} = \frac{R}{2R}$, whence

$$P = \frac{Q}{2}, \tag{113}$$

that is, *the motive force is equal to half the force of resistance.*

Obviously in this case also, the gain in force is lost in displacement. Indeed, in order to raise the centre of the pulley to a height of $OO_1 = h$, the free end of the rope must be pulled a distance of $AA_1 + BB_1 = 2h$, which means that the point of application of force **Q** receives a displacement only half of that received by the point of applications of force **P**. Furthermore, the work performed by force **Q** is $Qh = 2Ph$, and the work performed by force **P** is $P2h$; in other words, the work performed by the motive force is equal to the work performed by the force of resistance, which is as it should be.

* Since the movable pulley is ordinarily used to raise loads by means of a force acting downwards, a second fixed pulley M is shown in the illustration.

144. Systems of Pulleys and the Differential Pulley Block

Just as in levers, pulleys are combined into systems to increase their mechanical advantages. Fig. 173 represents one of these systems: it consists of several (in this case three) fixed pulleys rotating in the casing K, and the same number of movable pulleys rotating in the second casing L. The rope, one end of which is fastened to the hook of the first casing, is passed round all the pulleys in succession, while to its free end M the motive force **P** is applied. In the present case force **Q** is distributed among six segments of one and the same rope, in which the tension must obviously be the same throughout the entire length. It follows that a load $\frac{Q}{6}$ is acting on each segment of the rope, and the force which must be applied to the free end of the rope to keep the system in equilibrium will be

$$P = \frac{Q}{6} = \frac{Q}{2 \times 3}.$$

If there had been four pairs of pulleys in the system, force **P** would be $\frac{Q}{8} = \frac{Q}{2 \times 4}$. Thus we see that the mechanical advantage is equal to twice the number of movable pulleys. And if the movable block had n pulleys, the motive force would be

$$P = \frac{Q}{2n}. \tag{114}$$

But, according to the rule already learnt, the displacement of the point of application of **P** will be $2n$ times the displacement of the point of application of the force of resistance **Q**.

Instead of having the pulleys on separate axes and arranged vertically one above the other, they are usually arranged several in each casing and on one horizontal axis (Fig. 174).

Systems of pulleys (fixed and movable) grouped in blocks and with a rope or chain wound about them are called *tackle.*

Just as there is a differential lever, there is also a differential pulley block as shown in (Fig. 175). The upper fixed block is made double with two stages of sheaves of radius R and r. As is evident from the illustration, this block and the lower movable block are connected by an endless chain; from the lower block the chain is passed to the larger sheave in the upper pulley and then goes down in the form of a freely swinging loop M, one segment of which is meant to be pulled by hand. Then the chain is passed upward and around the smaller of the sheaves in the upper block and down again to the movable block.

Let us see what forces are acting on the upper block so as to find the relationship between the motive force **P** and the force of resistance **Q**.

Under the action of force **Q**, forces $P_1 = P_2 = \frac{Q}{2}$ are created in each segment *I* and *II* of the chain. Assume that the upper block makes one revolution at which time the work of the motive force **P** is $W = P\,2\pi R$. During the same interval force $\mathbf{P}_1$ performs work

$$W_1 = P_1 2\pi r = \frac{Q}{2}\cdot 2\pi r = Q\pi r.$$

Finally, the work of force $\mathbf{P}_2$ is

$$W_2 = P_2 2\pi R = \frac{Q}{2}\, 2\pi R = Q\pi R.$$

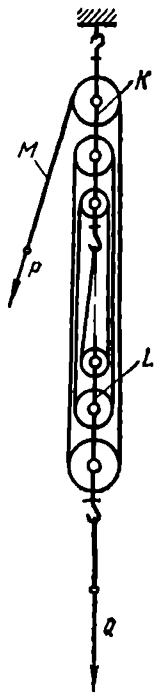


Fig. 173

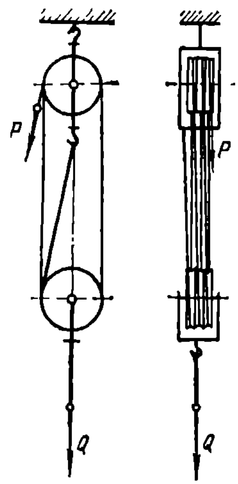


Fig. 174

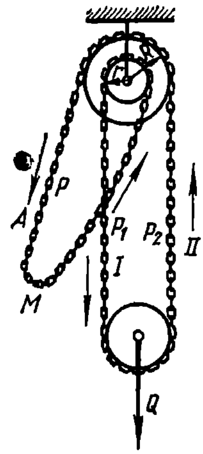


Fig. 175

The first two forces are motive forces, while the third is the force of resistance. Since the work of the motive forces must be equal to the force of resistance, then

$$2\pi PR + \pi Qr = \pi QR, \quad \text{or} \quad 2PR + Qr = QR,$$

from which we obtain the force acting on segment *A* of the loop:

$$P = Q\,\frac{R-r}{2R} = Q\,\frac{R-r}{D}, \tag{115}$$

in which *R* and *r* are the respective radii of the larger and smaller sheaves of the fixed block and *D* is the diameter of the larger sheave.

Since the difference between *R* and *r* can be made infinitely small, a great mechanical advantage may be obtained with this block.

Illustrative Problem 73. What must be the diameter of the smaller sheave of the fixed double block in a differential block to obtain a mechanical advantage of $\frac{Q}{P} = 8$, if the diameter of the larger sheave $D = 200$ mm and efficiency $\eta = 0.8$?

Solution: from Eq. (115) we obtain $\frac{Q}{P} = \frac{D}{R - r}$, and after taking harmful resistance into account, the mechanical advantage will be

$$\frac{Q}{P} = \frac{D}{R - r} \eta,$$

which, after substituting numerical values, becomes $8 = \frac{200 \times 0.8}{100 - r}$, in which $r = 80$ mm, and the diameter of the smaller sheave will be 160 mm.

145. Simple and Differential Windlasses

A simple device for obtaining mechanical advantage is the windlass (Fig. 176); a drum K (Fig. 176*a*) is fixed to a shaft rotating in two bearings. The shaft is rotated by the crank L fastened to one end of it. As the shaft rotates, the rope M, one end of which in fastened to the surface of the drum, is wound around the latter and overcomes the force of resistance **Q**. Let D denote the diameter of the drum, and a the length of the crank at whose end the force **P** is applied (Fig. 176*b*). The relationship

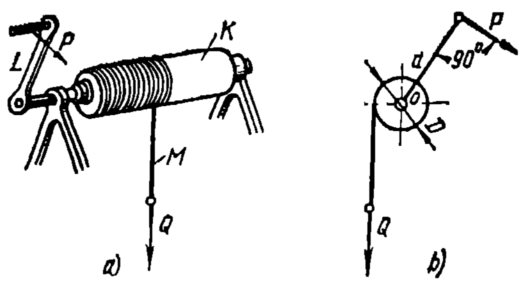


Fig. 176

between forces **P** and **Q** can be found by equating the amount of work each executes during one revolution of the shaft. The work of force **P** is expressed as $W_P = P2\pi a$, and the work of **Q** as $W_Q = Q\pi D$, where D is the diameter of the drum. Accordingly, $P2\pi a = Q\pi D$, whence

$$P = Q \frac{D}{2a} \tag{116}$$

The *differential windlass* with its stepped drum (Fig. 177) gives a much greater mechanical advantage than the simple type. Let D denote the diameter of the larger step, and d that of the smaller.

When the crank is turned clockwise, the rope will be wound on the larger step and unwound from the smaller. By disregarding as negligible the converging lines of the segments of rope dropping to the movable block, we shall assume that each of them is subjected to a force $P_1 = P_2 = \frac{Q}{2}$. Let us formulate an equation for the work of the motive power and that of the force of resistance.

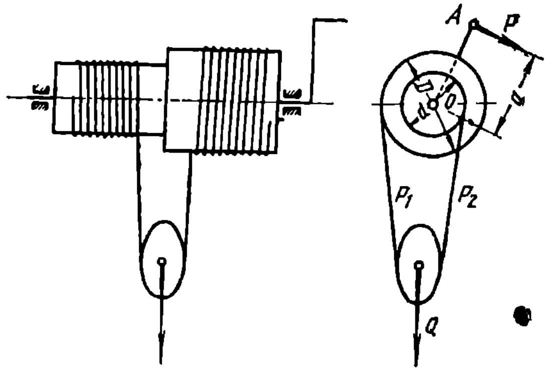


Fig. 177

The work performed by force $\mathbf{P}_1$ during one revolution of the drum $W_1 = P_1 \pi D = \frac{Q}{2} \pi D$, the work of force $\mathbf{P}_2$ as applied to the smaller step $W_2 = P_2 \pi d = \frac{Q}{2} \pi d$, and the work of force $\mathbf{P}$ as applied to the crank $W = P 2\pi a$. Hence

$$2\pi P a + \frac{\pi Q}{2} d = \frac{\pi Q D}{2},$$

from which

$$P = Q \frac{D - d}{4a} = Q \frac{R - r}{2a}, \qquad (117)$$

whence R and r are the radii of the larger and smaller steps of the drum, respectively. Thus we see that we have formulated the same expression as for the differential block.

Illustrative Problem 74. A differential windlass has a two-step drum of diameters $D = 350$ mm and $d = 300$ mm. What length must the crank be in order to raise at a constant speed a load $Q = 200$ kg with a force $P = 16$ kg, if the efficiency of the windlass $\eta = 0.6$?

Solution: by including the force of friction in Eq. (117), the latter becomes

$$P\eta = Q \frac{R - r}{2a},$$

in which $P = 16$ kg, $\eta = 0.6$, $Q = 200$ kg, $R = 175$ mm, and $r = 150$ mm.

By restating the equation and then substituting corresponding values, we solve for the length of the crank a:

$$\frac{Q(R-r)}{2P\eta} = \frac{200 \times 25}{2 \times 16 \times 0.6} \approx 260 \text{ mm.}$$

146. Questions for Review

1. If a body on the inclined plane shown in Fig. 161a is moving up the plane at a constant speed and the plane is lengthened but with the height h remaining the same, what change will there be in force **P**?
2. What change will there be in the magnitude of force **P** exerted on the wedge in Fig. 162 if the thickness of the wedge head is decreased but with the length of the wedge and the speed of its application remaining the same?
3. Which will be the greater mechanical advantage: when force **P** is applied perpendicularly, or at an angle, to the arm of a lever?
4. If the length of the arms of the bent lever AOB (Fig. 166) are equally increased, will there be any change in the magnitude of the force **P** required to keep it in equilibrium?
5. What will be the total mechanical advantage obtained by a system of three levers, one of which gives a three-fold, the second a five-fold, and the third a seven-fold mechanical advantage?
6. State the advantages of the differential lever.
7. What difference is there between the mechanical advantage obtained by a fixed and a movable pulley block?
8. What is a tackle?
9. What are the advantages contained in the differential block; in the differential windlass?

147. Exercises

76. A load $G = 200$ kg is moving uniformly up an inclined plane with an angle of inclination $\alpha = 30°$. What must be the magnitude of the motive force **P** directed parallel to the incline if the coefficient of friction $f = 0.10$?

77. Using the data in Ex. 76, determine the efficiency of the inclined plane.

78. Two loads of weight $G_1 = 10$ kg and $G_2 = 15$ kg are on inclined planes with angles of inclination of α_1 and α_2 and are connected with each other with a cord passed through a fixed pulley (Fig. 178). If angle $\alpha_1 = 30°$ and the two loads are in equilibrium (neglecting the force of friction), what is angle α_2?

Hint to solution: remember that forces $\mathbf{P}_1$ and $\mathbf{P}_2$ are equal in magnitude.

79. If the angles of inclination α_1 and α_2 in Fig. 178 are 30° and 45°, respectively, and the force of friction is disregarded, what must be the ratio between the weights $\mathbf{G}_1$ and $\mathbf{G}_2$ when they balance each other?

80. In order to find the distance x from the end A to the centre of gravity C of the rod AB in Fig. 179, the end A was suspended to a fixed point E and then the rod placed so that it rested on scales at point D. Find the distance x if $a = 300$ mm, the weight

of the rod $G = 1.5$ kg, and if weight G_1 balancing the rod on the other pan of the scales is 1.0 kg.

81. Derive Eq. (108) for a straight lever of the second kind.

82. Derive the same Eq. (108) for a bent lever of the second kind.

83. What force **P** must be exerted on the differential lever in Fig. 170 to balance a force $Q = 1$ ton, if $AD = DB = 250$ mm, $EK = 249$ mm, $KF = 251$ mm, and $DC = 1{,}000$ mm?

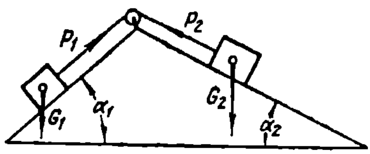


Fig. 178

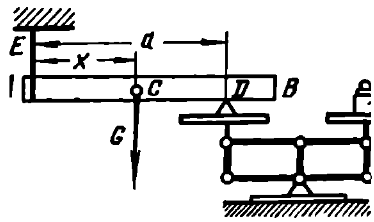


Fig. 179

84. Assume that the tackle in Fig. 174 has five movable blocks. What force **P** would be required to raise a load of 200 kg?

85. What mechanical advantage would the differential block in Fig. 175 give if the diameters of the sheaves are $D = 360$ mm and $d = 320$ mm?

86. In Fig. 177, showing a differential windlass, $D = 300$ mm, $d = 250$ mm, $a = 400$ mm, and its efficiency $\eta = 0.7$. What force **P** is needed to raise a load of 500 kg?

CHAPTER XVI

TRANSMISSION OF POWER BETWEEN PARALLEL SHAFTS

148. General Principles of Transmission

In order to transmit motion to the moving links of a machine, mechanical energy is needed. This energy may be imparted to the machine in different ways. But usually it is done by an adjacently installed electric motor, in which case it is said that the machine has an *individual drive*. But sometimes mechanical energy is imparted to several machines at once through a single shaft known as the *transmission shaft* acting as a *group drive*. And frequently one machine is driven by several electric motors, as in very large machine tools and other kinds of giant machinery. Both in individual and group drives, devices whose function it is to impart diverse angular velocities (rpm) to the driving shafts of the machine are sometimes mounted as intermediary links between the electric motor and the machine.

In short, various mechanisms which are referred to under the general term of *transmission* are used to impart mechanical energy both to machines as a whole and to their individual links.

The most common kind of transmission is that which transmits rotational motion from one shaft to another.

The position of the shafts in relation to each other may differ: their axes may lie in the same plane, or in different planes. If the shafts are in the same plane, they may either intersect or be parallel to each other.

Let us begin our study of the various kinds of transmissions with the simplest form — when the axes of the shafts lie parallel to one another.

149. Transmission Through Pliant Connectors

Flat belts, sometimes ropes, are used to transmit rotational motion between parallel shafts; these belts are wound about wheels, called sheaves, which are fixed to the shafts.

Assume that the rotation of shaft O_1 in Fig. 180 is to be transmitted to shaft O_2. We fasten two sheaves, opposite to each other, to the shafts and wrap an endless belt $ABDFECA$ round the two in such a manner that it is stretched tightly about their rims. With ample friction between the belt and the sheaves, the rotation of one shaft will be transmitted to the other. The shaft (and sheave) O_1 which

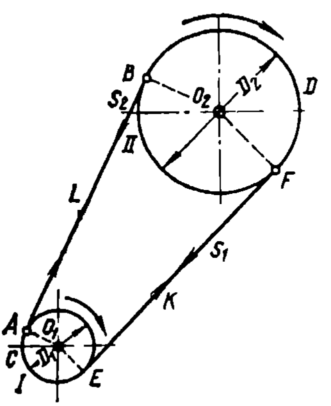


Fig. 180

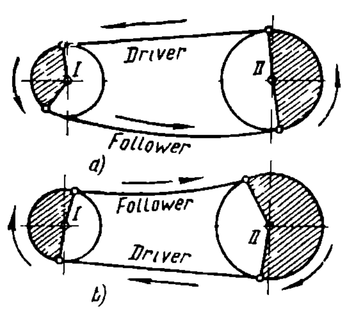


Fig. 181

causes the motion is called the *driver*, while the shaft (and sheave) O_2 which receives the motion is called the *follower*, or *driven* unit.

Angles AO_1E and BO_2F subtending arcs ACE and BDF where the belt is in contact with the rims of the sheaves, are called the *angles of contact*.

The greater the angle of contact, the better will be the transmission of rotation, inasmuch as the arc of contact between the sheaves and the belt will be greater. For this reason belt transmissions are always designed so that the angle of contact is as large as possible.

Assume that shaft *I* (Fig. 181*a*) is the driver and shaft *II* is the follower. With the direction of motion as shown in the drawing, the upper segment of the belt will be pulled taut and lie almost in a straight line since it is transmitting the force that is rotating the follower, whereas the lower segment will be slack and sag under its own weight. If the direction of motion is changed, as shown in Fig. 181*b*, it will be just the opposite—the upper side will sag. A comparison of the angles of contact on the driving and driven shafts in the two drawings will show that it is greater in the second case. Hence, here transmission of rotation will be more efficient. It follows that the lower segment of the belt should always be the driver.

The belt connecting the sheaves should be as pliant as possible; this type of transmission is called *transmission through pliant connectors.*

150. The Speed Ratio and Transmission Number in Transmission Through Pliant Connectors

In making calculations corcerning transmission of rotational motion, a coefficient showing the ratio between the angular speeds of the two given shafts or, in other words, between their rpm, is used. This ratio of rpm (or ratio of angular speeds) of two shafts between which motion is transmitted is called the *speed ratio* and is denoted by the letter *i*.

Of the two connected shafts, one is the driver and the other the follower. Therefore the speed ratio must be so stated as to indicate the order in which the shafts are referred. For this purpose indices, consisting of the numbers of the two shafts, are affixed to the letter representing the speed ratio. If it is a ratio of rpm of the driven shaft to rpm of the driving shaft, it is stated as

$$i_{2,1} = \frac{\omega_2}{\omega_1} = \frac{n_2}{n_1}. \qquad (118)$$

If on the contrary the ratio is that of the rpm of the driving shaft to the rpm of the driven shaft, it will be stated as

$$i_{1,2} = \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2}. \qquad (119)$$

The latter ratio, that is, the ratio of rpm of the driver to rpm of the follower, is called the *transmission number.*

It is thus apparent from Eq. (118) that when $n_1 = 1$, $i_{2,1} = n_2$; we may therefore say that $i_{2,1}$ *shows the number of revolutions of the follower to one revolution of the driver.*

Finally, by multiplying Eqs (118) and (119), we obtain

$$i_{2,1} \times i_{1,2} = \frac{n_2}{n_1} \times \frac{n_1}{n_2} = 1, \qquad (120)$$

whence

$$i_{2,1} = \frac{1}{i_{1,2}}$$

that is, *the speed ratio of the driving shaft to the driven shaft and of the driven shaft to the driving shaft are reciprocal to each other.*

Oral Exercises

1. State which shaft of the following three cases has the greater angular speed: when $i_{2,1} > 1$; when $i_{2,1} < 1$; when $i_{2,1} = 1$.
2. What is the transmission number when $i_{2,1} = 1$?

151. Kinematics of Transmission with One Pair of Sheaves

Let us return to Fig. 181 and assume that the belt, wound about the two sheaves, neither stretches nor slips. Under such conditions the motion of the belt will be the same at all its points and be equal to the speed of any point on the rims of either of the sheaves. In other words, the peripheral speed of sheave *II* will equal the peripheral speed of the driving sheave *I*, from which we evolve the following equation.

$$\frac{\pi D_1 n_1}{60} = \frac{\pi D_2 n_2}{60}, \quad \text{or} \quad D_1 n_1 = D_2 n_2, \qquad (121)$$

that is, *the product of the diameter of the driver and its rpm is equal to the product of the follower and its rpm.* From this we may

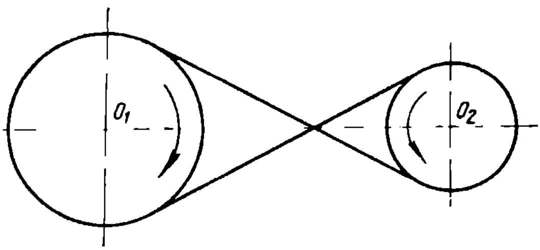


Fig. 182

determine the speed ratio of $i_{2,1}$ (the relationship between rpm n_2 of the follower sheave and rpm n_1 of the driving sheave):

$$i_{2,1} = \frac{n_2}{n_1} = \frac{D_1}{D_2}, \qquad (122)$$

that is, *the speed ratio of the two sheaves is in inverse ratio to their diameters.*

As is apparent from Fig 180, the driving and driven shafts both revolve in the same direction. This type of transmission is called an *open-belt* drive, as distinguished from the *crossed-belt* drive when the belt is crossed in the form of a figure 8 (Fig. 182). In the latter case the two sheaves will revolve in opposite directions.

Eq. (122) shows the relations between four quantities: the diameters D_1 and D_2 of the two connected sheaves and their respective rpm. If three of these quantities are known the fourth can be evolved.

Illustrative Problem 75. The driving sheave on the shaft of an electric motor has a diameter of 180 mm and rotates at 1,000 rpm. If it were required to drive another sheave that must rotate $n_2 = 320$ rpm, what must be the diameter of this follower sheave?

Solution: from Eq. (122)

$$D_2 = D_1 \frac{n_1}{n_2} = 180 \times \frac{1,000}{320} \approx 560 \text{ mm}.$$

Illustrative Problem 76. If an electric motor attached to a sheave of 300 mm in diameter transmits $n = 400$ rpm to a driven (follower) sheave of 560 mm in diameter, how many rpm will the sheave on the motor attain?

Solution: from Eq. (122)

$$n_1 = n_2 \frac{D_2}{D_1} = 400 \times \frac{560}{300} \approx 750 \text{ rpm}.$$

152. Kinematics of Transmission with More than One Pair of Sheaves

We can determine the speed ratio for any number of sheaves by calculating it consecutively for each sheave in the train of sheaves.

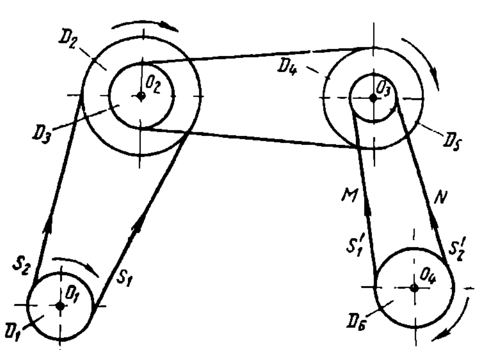


Fig. 183

Assume that rotation is transmitted from shaft O_1 to shaft O_4 (Fig. 183) by means of sheaves D_1 and D_2, D_3 and D_4, D_5 and D_6. It is seen from the drawing that sheaves D_1, D_3, and D_5 are drivers while sheaves D_2, D_4, and D_6 are followers. The speed ratio between shafts O_2 and O_1 is $i_{2,1} = \frac{D_1}{D_2}$ and rpm n_2 of shaft O_2 is $n_2 = n_1 i_{2,1} = n_1 \frac{D_1}{D_2}$.

In the same way we may find the speed ratio between shafts O and O_2 which is $i_{3,2} = \frac{D_3}{D_4}$ and the rpm n_3 of shaft O_3 is

$$n_3 = n_2 i_{3,2} = n_1 i_{2,1} i_{3,2} = n_1 \frac{D_1}{D_2} \times \frac{D_3}{D_4}.$$

Finally, the speed ratio between the shafts of the last pair of sheaves is $i_{4,3} = \frac{D_5}{D_6}$ and the rpm n_4 of the last driven shaft O_4 is

$$n_4 = n_3 i_{4,3} = n_1 i_{2,1} i_{3,2} i_{4,3} = n_1 \times \frac{D_1}{D_2} \times \frac{D_3}{D_4} \times \frac{D_5}{D_6}.$$

By denoting the speed ratio between this shaft and the driving shaft O_1 as $i_{4,1}$ we obtain

$$n_4 = n_1 i_{4,1} = n_1 \frac{D_1}{D_2} \times \frac{D_3}{D_4} \times \frac{D_5}{D_6} \quad (123)$$

in which

$$i_{4,1} = i_{2,1} i_{3,2} i_{4,3} = \frac{D_1}{D_2} \times \frac{D_3}{D_4} \times \frac{D_5}{D_6}. \quad (124)$$

Wherefore, *the total speed ratio is equal to the product of all the individual speed ratios* (i.e., the speed ratios between adjacent shafts). *The rpm of the driven shaft is equal to the rpm of the driving shaft multiplied by the ratio of the product of the diameters of all the driving shafts to the product of the diameters of all the driven shafts.*

Of course, changing the order of the multipliers and multiplicants will make no difference in the final product. From this it follows that we can change the places of any two sheaves whose diameters are in the numerator or denominator of the right part of Eqs (123) and (124). This means that the rpm of the driven shaft will not change if either the driven or driving sheaves are rearranged among themselves. But it is also obvious that *driving sheaves cannot be put in the place of driven sheaves or vice versa.* For instance, the sheave of diameter D_1 cannot be put in the place of that with diameter D_2, or sheave D_6 in the place of D_5, etc., for this would change the total speed ratio and consequently the rpm of the driven shaft.

Illustrative Problem 77. Shaft O_1 receives rotational motion from an electric motor with a sheave D_1 having a diameter of 180 mm and which attains $n_1 = 1{,}500$ rpm through sheaves $D_2 = 540$ mm, $D_3 = 160$ mm, and $D_4 = 400$ mm. Find the total speed ratio $i_{3,1}$ and the rpm of shaft O_3.

Solution: according to Eqs (123) and (124) we evolve $i_{3,1} = \frac{180}{540} \times \frac{160}{400} = \frac{2}{15}$, and $n_3 = n_1 i_{3,1} = 1{,}500 \times \frac{2}{15} = 200$ rpm.

153. Statics of Sheave Transmission

Now that we have grasped the kinematics of the transmission of rotational motion by means of sheaves, let us turn to the statics of such transmission so as to determine the relationship between motive forces and forces of resistance.

Let us return to Fig. 180. In order that there should be sufficient friction between the belt and the rims of the sheaves, a definite tautness must be maintained in the belt. After the sheaves have begun rotating, the driving segment K of the belt becomes still more taut, while the follower segment L of the belt loses some of its tautness. Let S_1 represent the pull on the tight side and S_2 the pull on the slack side. Both these forces act on the driven sheave and consequently two similar forces of the same magnitude but of opposite direction are acting on the driving sheave.

The turning moment or, as we shall henceforth call it, the *torque*, which imparts rotational motion to shaft O_2 will be

$$M_2 = S_1 \frac{D_2}{2} - S_2 \frac{D_2}{2} = (S_1 - S_2) \frac{D_2}{2}.$$

The difference in tautness $S_1 - S_2$ is called the *effective pull* of the belt and is denoted by the letter P.

Thus we find that the torque on the driven shaft

$$M_2 = P \frac{D_2}{2}. \tag{125}$$

From Eq. (122) we obtain

$$D_2 = D_1 \frac{n_1}{n_2} = D_1 \frac{1}{i_{2,1}} \text{ and } M_2 = P \frac{D_1}{2i_{2,1}}.$$

As already stated, two similar forces S_1 and S_2 are acting on O_1; hence, the torque on the driving shaft will be

$$M_1 = P \frac{D_1}{2},$$

and after equating the expressions for M_1 and M_2, we finally obtain

$$M_2 = \frac{M_1}{i_{2,1}}. \tag{126}$$

Wherefore, *the torque on the driven shaft is equal to the torque on the driving shaft divided by the speed ratio $i_{2,1}$ between them.*

It is simple to prove that Eq. (126) similarly applies to any number of pairs of sheaves.

Assume that the first driving sheave of diameter D_1 (Fig. 183) makes one revolution. Eq. (124) shows that this would cause the last driven sheave of diameter D_6 on shaft O_4 to execute $i_{4,1} = i_{2,1} i_{3,2} i_{4,3}$ revolutions. The work done by forces S_1 and S_2 on the driving sheave will be $W_1 = (S_1 - S_2)\pi D_1 = P_1 \pi D_1$.

The work done on the driven sheave D_6 at the same time by forces S_1' and S_2' will be $W_4 = (S_1' - S_2')\pi D_6 i_{4,1} = P_4 \pi D_6 i_{2,1} i_{3,2} i_{4,3}$.

And since $W_4 = W_1$, we obtain $P_1 D_1 = P_4 D_6 i_{4,1}$, from which

$$M_4 = \frac{M_1}{i_{4,1}} = \frac{M_1}{i_{2,1} i_{3,2} i_{4,3}}. \tag{127}$$

Wherefore, *the torque on the last driven shaft is equal to the moment on the first driving shaft divided by the total speed ratio between* *them, or, in other words, by the product of all the individual speed ratios.*

Eqs (126) and (127) do not take into account the loss due to harmful resistance in the drive. Such resistance reduces the mechanical energy imparted to the driven shaft, and consequently decreases the torque and effective pull. If these losses are taken into account, Eq. (126) becomes

$$M_2 = \frac{M_1}{i_{2,1}}\eta, \tag{128}$$

in which η is the efficiency of transmission.

For belt transmission the value of η ranges from 0.94 to 0.985.

•

Oral Exercises

1. If the speed ratio $i_{2,1} < 1$, what can be said of the torque on the driven shaft — will it be greater or less than the torque on the driving shaft?

2. Answer Question 1 if $i_{2,1} = 1$.

Illustrative Problem 78. If the electric motor in Illustrative Problem 77 transmits power $N = 7.4$ kw, find the torque on shaft O_1 and the effective pull on sheave D_1.

Solution: If the motor power $N = 7.4$ kw $= 7.4 \times 1.36 \approx 10$ hp and if $n = 1{,}500$ rpm, the torque on the driving shaft will be, according to Eq. (83),

$$M_t = 716.2\,\frac{10}{1{,}500} = 4.775 \text{ kg-m}.$$

By applying Eq. (127) we obtain the torque on shaft O_4:

$$M_4 = \frac{M_1}{i_{3,1}} = 4.775 : \frac{2}{15} = 35.812 \text{ kg-m}$$

and the effective pull P_4 on sheave D_4 will be

$$P_4 = \frac{2M_4}{D_4} = \frac{2 \times 35{,}812}{400} = 179 \text{ kg}.$$

154. Belt Transmission with Variable Speed Ratios

It is frequently necessary that a driving shaft, rotating at a constant speed, transmit varying speeds to the driven (follower) shaft. One of the widely applied methods to achieve this is the use of stepped pulleys.

Let us fix two stepped pulleys, with steps of different diameters, opposite each other on the driving shaft *I* and the driven shaft *II* as shown in Fig. 184. With this arrangement the belt can be shifted so as to run on any pair of steps d_1 and D_1, d_2 and D_2, etc. In this way different speed ratios are obtained: $i_{2,1} = \frac{d_1}{D_1}$, $\frac{d_2}{D_2}$ etc.

There will be as many speed ratios as there are steps on the pulley. It is readily understood that for a drive of this kind the belt must be the same length no matter which of the paired pulleys it runs on. To achieve this, the following equation must hold true:

$$d_1 + D_1 = d_2 + D_2 = \dots d_5 + D_5. \qquad (129)$$

Wherefore, *the sum of the diameters of the steps opposite each other must be the same in all cases.*

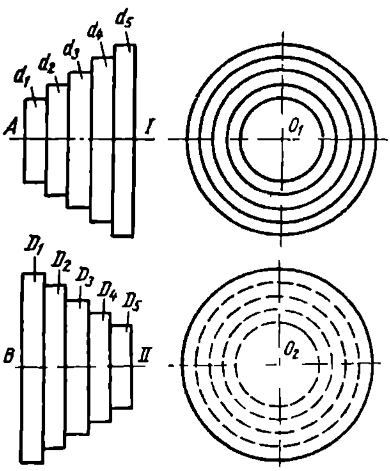


Fig. 184

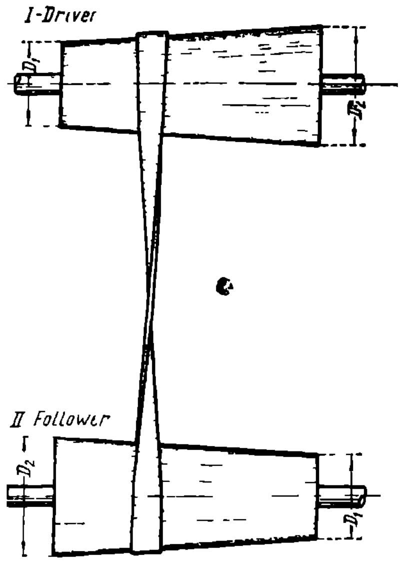


Fig. 185

Let n represent the rpm of the driving shaft. With the use of five-step pulleys we can transmit five different speeds to the driven shaft, as follows

$$n_1 = n\frac{d_1}{D_1}, \quad n_2 = n\frac{d_2}{D_2}, \quad n_3 = \frac{d_3}{D_3}, \quad n_4 = \frac{d_4}{D_4},$$

$$\text{and } n_5 = \frac{d_5}{D_5}.$$

But it must be understood that an *unlimited* variation of speeds cannot be obtained between n_5 and n_1. In other words, the speed variations imparted to the driven shaft will differ sharply from each other instead of being gradual. Other methods are used to shift speeds gradually. Fig. 185 illustrates one such method.

We connect the belt to two frusta-cone drums arranged in opposite directions and with base diameters of D_1 and D_2. When the belt is at the extreme left, the speed ratio will be $i_{2,1} = \frac{D_1}{D_2}$, while when at the extreme right it will be $i_{2,1} = \frac{D_2}{D_1}$. Thereby, the speed ratio may be made to range anywhere from $i_{2,1} = \frac{D_1}{D_2}$ to $i_{2,1} = \frac{D_2}{D_1}$.

A variant of this method is to make the drums with curved sides instead of the straight-lined frusta-cone.

There are also other methods of achieving infinitely-variable speeds in transmitting rotational motion between parallel shafts.

155. Transmission with a Belt Tightener

Very often the distance between the driving and driven sheaves of a machine is made as small as possible so as to decrease the general size of the machine. But this has a bad effect on the belt drive inasmuch as it leads to a decrease in the arc of contact on the smaller sheave (usually the driver), and which, in its turn, results in slip.

The arc of contact of the smaller sheave is decreased also because of the increase in the transmission number.

For satisfactory operation, the ordinary belt drive must have a transmission number of not more than 3 (in exceptional cases it may be 5), but often the rpm must be slowed down to less than one third. This has resulted in the introduction of drives with belt tighteners.

Assume shaft O_1 in Fig. 186 to be the driver and shaft O_2 the follower. With rotation in the direction shown, segment K of the belt will be the taut side, and L will be the slack segment. An idler-pulley M is carried on arm A of a bent lever, and to arm B of the same lever a weight N is fixed. The lever balances freely on its axis O. Since the centre of gravity of the lever is situated to the right of axis O, the arm B of the lever is pulled clockwise and the idler-pulley presses against L and tightens it.

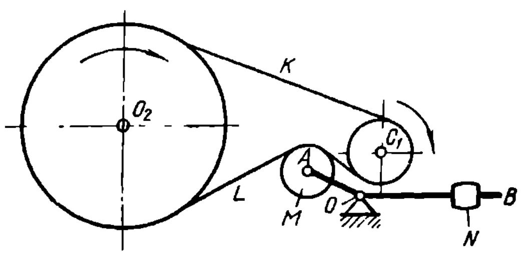


Fig. 186

It can be seen that the idler-pulley increases the arc of contact on both sheaves and reduces slip. Load N can be shifted to any position on the arm of the lever to regulate the tautness of L as desired. The use of the belt tightener has another advantage: ordinarily, any belt will stretch with use and must be often shortened. But the employment of a belt tightener makes this unnecessary because tautness is kept uniform in the belt.

But the greatest advantage of a transmission with a belt tightener, as compared to an ordinary belt transmission, is that it allows an increase in the transmission number (up to ten and sometimes even more) and at the same time keeps the whole drive compact. Belt tighteners are designed in various ways. Axis O of the lever is often made to coincide with the geometrical axis of shaft O_1; this is better to a certain extent. In small power transmission a spring is often used in place of the weight N.

156. Flat and V-Shaped Belts

Belting is made of different materials and varied cross-sections—either *flat* or *V-shaped*. Inasmuch as belting is subject to tension it is made in different thicknesses *single-ply* and *double-ply* — depending on the effective pull it must undergo.

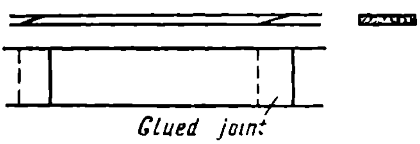


Fig. 187

Single-ply leather belts are made of strips of leather glued together into a continuous length (Fig. 188) and ranging from 3.0 mm to 5.5 mm in thickness and as much as 300 mm in width. If calculations show that single-ply belting will not be strong enough, double-ply is used. This consists of two layers of single-ply belting either glued, or sewn and glued, along its entire length.

At the present time, flat textile belts, impregnated with rubber, are in wide use. They are made of different kinds of fibres (cotton or wool).

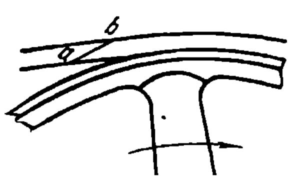


Fig. 188

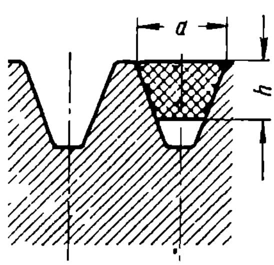


Fig. 189

Three methods are used to fasten the ends of flat belts: *gluing, lacing,* or *metal connections.* The ends of a leather belt are scarfed for a length of 100-200 mm and when put on the sheaves must be placed as shown in Fig. 188, in which the letters *ab* mark the glued joint. For textile belts impregnated with rubber, the joint is cut with a step.

V-belts, which occupy a special place in transmission, consist of one or several bands of trapesoidal section (Fig. 189) and are used instead of flat belts. Cross-sectional area varies, depending on the dimensions *a* and *h*; the smallest dimensions are 10 and 6 mm, and the largest 50 and 30 mm, respectively. V-belt drives are used when centre distance between shafts is short and transmission numbers are large.

157. Chain Transmission

Chain transmission is a special variation of the pliant connector; the belt is replaced by a chain whose links mesh with the teeth of a sprocket wheel, prevent slipping, and ensure a constant speed ratio. Chains are used for high transmission numbers (up to 15) and can impart as much as 5,000 hp. They are mostly

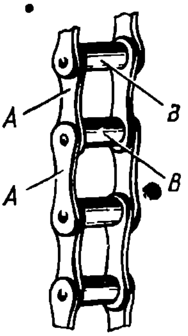


Fig. 190

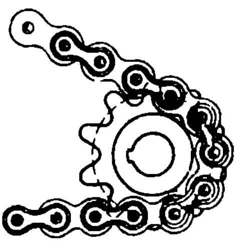


Fig. 191

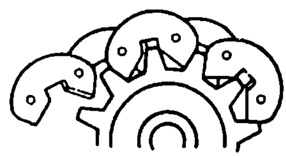


Fig. 192

used when the distance between centres is short. But they are also employed when the centre distance is as much as 8 *m*.

Various types of construction are used for the chains, depending on their intended function. Fig. 190 shows a type of *roller chain*. The drawing shows that the chain consists of flat pin-connected links *A* and rollers *B*. The rollers are freely mounted on bushings and when the drive is in operation they mesh with the teeth of the sprocket wheel (Fig. 191). Double- and multiple-width chains of this kind are used for heavy-duty transmission.

The *toothed chain* shown in Fig. 192 is an improved type which works very smoothly and makes great speeds possible. It is also called the noiseless chain.

The possibility of regulating tautness is also incorporated into the construction of chain drives by means of tightening-pulleys and other devices.

158. Friction Transmission Between Parallel Shafts

The belt drives we have studied thus far utilise friction between the belt and the rim of the sheave. But the force of friction can act directly without recourse to a pliant connector if the contacting parts are pressed to each other with sufficient force, resulting in a *friction transmission.*

Fig. 193 represents two smooth cylindrical rollers fixed to parallel shafts O_1 and O_2. If two equal and opposite forces **Q** and **Q'** are applied to the shaft centres, they will cause friction between the surfaces of the rollers, the magnitude of which will depend on the amount of applied pressure, the material of which the rollers are made, and the condition of their surfaces. This friction contact will cause the driven shaft to revolve. If friction is insufficient to overcome the resistance of the driven shaft, the cylinders will slip against each other. Accordingly, if the drive is to work satisfactorily, it must be so built as to create the

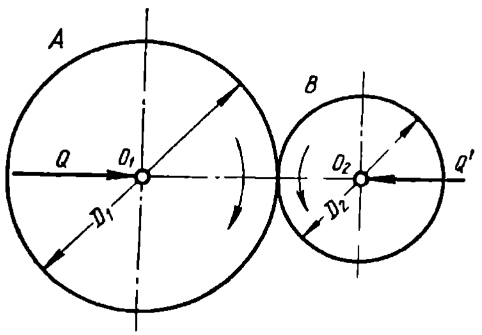


Fig. 193

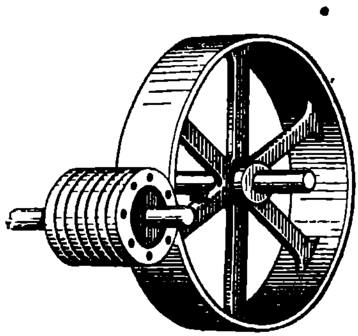


Fig. 194

greatest amount of friction. Various materials are used in the construction of the rims: both may be of cast iron or one may be of cast iron or steel while the other of "textolite", etc. Fig. 194 shows a pair of friction wheels of which the smaller is made of leather rings compressed longitudinally by means of two washers.

When there is no slip, the peripheral speed of both drums will be alike. Hence in this case Eqs (121) and (122), which were evolved for drives with pliant connectors, are fully applicable without reservation.

Eq. (129), in which M_1 is the torque on the driving shaft and i_{21} is the speed ratio between the two shafts, is also applicable.

Friction rims can likewise operate without being in immediate contact with each other. For instance, rotation can be transmitted through a steel or leather ring pressed between the two rims (Fig. 195).

Illustrative Problem 79. Power $N = 1.5$ hp is transmitted by shaft O_1 to shaft O_2 (Fig. 195). The diameter of the driving wheel, which attains $n_1 = 200$ rpm, is $D_1 = 400$ mm. Both rollers are of cast iron (coefficient of friction $f = 0.15$).

Find the diameter D_2 of the driven wheel if it must attain $n_2 = 1{,}000$ rpm, the required pressure **Q**, and the torque on the driven shaft.

Solution: through Eq. (122) we find the diameter of the driven wheel:

$$D_2 = \frac{D_1}{i_{2,1}} = D_1 \frac{n_1}{n_2} = 400 \frac{200}{1{,}000} = 80 \text{ mm}.$$

To determine force **Q**, first the effective pull **P** transmitted by the wheels must be evolved; to find this, we must calculate the torque. From Eq. (84) we obtain the torque on the driving shaft:

$$M_1 = 71{,}620 \frac{N}{n_1} = 71{,}620 \frac{1.5}{200} = 537.15 \text{ kg-cm}.$$

Hence the effective pull

$$P = \frac{M_1}{R_1} = \frac{2M_1}{D_1} = \frac{1{,}074.3}{40} = 26.86 \text{ kg}.$$

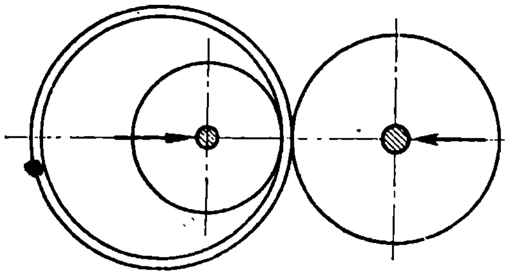


Fig. 195

Pressure **Q** is determined through the equation $P = fQ$, from which

$$Q = \frac{P}{f} = \frac{26.86}{0.15} = 179 \text{ kg or} \approx 180 \text{ kg}.$$

It should be noted that 180 kg is the minimum possible pressure. Depending on working conditions, a reserve must be added to **Q** so as to make up for irregularities in the work of the drive. This required reserve force may be as much as 100%, in which case force **Q** must be twice 180 kg, that is, 360 kg.

Torque M_2 on the driven shaft can be determined in various ways:

a) since effective pull is alike for both wheels when there is no slip, we find M_2 by multiplying the effective pull P by the radius of the driven wheel:

$$M_2 = P \frac{D_2}{2} = 26.86 \times \frac{8}{2} = 107.44 \text{ kg-cm};$$

b) we can obtain the same result by using Eq. (126):

$$M_2 = \frac{M_1}{i_{2,1}} = \frac{M_1}{\frac{n_2}{n_1}} = \frac{537.15}{5} = 107.43 \text{ kg-cm*};$$

c) finally, we may find the torque through Eq. (84):

$$M_2 = 71{,}620 \frac{N}{n_2} = 71{,}620 \times \frac{1.5}{1{,}000} = 107.43 \text{ kg-cm}.$$

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* The negligible discrepancy of 0.01 kg-cm is caused by the round numbers used in determining **P**.

159. Friction Transmission with a Variable Speed Ratio

Friction transmission is especially practical when it is employed to give the driven shaft variable speeds from a driving shaft revolving at a constant speed.

Assume that the two cones in Fig. 185 are mounted one above the other on parallel shafts with a small intervening space. Instead of a belt we will use a ring on the lower cone. When the ring is pinched between the two cones (as shown in Fig. 195) the rotation of the driver will be imparted to the follower. By sliding the ring along the length of the cone we can obtain any rpm of the driven shaft, ranging from $n_1 \frac{D_1}{D_2}$ to $n_1 \frac{D_2}{D_1}$.

Fig. 196 represents another type of infinitely-variable friction transmission between parallel shafts. Assume shaft I to be the driver and shaft II the follower. Discs A_1 and A_2 are fixed to the ends of the shafts. Between the discs there is an idler-pulley B which can be moved along the shaft on which it is mounted and fastened in the position required. Assume that shaft I executes n_1 rpm. If there is no slip between the discs and the pulley, the peripheral speed of the pulley (when it is in the position shown in the drawing) will be equal to the speed of any point on disc A_1 lying on a circle with a radius of R'; that is, its peripheral speed

$$v_1 = \frac{2\pi R' n_1}{60} \ mm/\text{sec}.$$

The same speed will be attained on disc A_2 at any point lying on a circle with radius R''; this speed, at n_2 rpm of the disc, will be

$$v_2 = \frac{2\pi R'' n}{60} \ \text{mm/sec}.$$

Since $v_1 = v_2$, then

$$\frac{2\pi R' n_1}{60} = \frac{2\pi R'' n_2}{60}$$

or $R' n_1 = R'' n_2$ from which

$$i_{2,1} = \frac{n_2}{n_1} = \frac{R'}{R''}.$$

Thus we see that the speed ratio is equal to the inverse ratio of the distance of the middle section of the pulley from the geometrical axes of the shafts. The greatest possible speed ratio $i_{2,1}$ is $\frac{R_1}{r_2}$, while the smallest possible is $\frac{r_1}{R_2}$. With the aid of this mechanism it is possible to obtain any speed of the driven shaft, ranging from $n_1 \frac{r_1}{R_2}$ to $n_1 \frac{R_1}{r_2}$.

It is easy to understand that the driven shaft will rotate in the same direction as the driver.

Assume that Fig. 197 represents two pairs of frusta-cones A_1 and A_2, and B_1 and B_2, fixed to driving shaft I and driven shaft II, respectively. The cones are mounted in sliding keyways and the distance between each pair can be adjusted by a special device. Both pairs of cones are in contact with a steel ring C (shown in cross-section). The driving cones, when pressed to the ring, will make it rotate through friction and the ring will transmit the rotation to the driven cones and through them to the driven shaft II. When the ring is in the position shown in the drawing, the speed ratio

$$i_{2,1} = \frac{R_1}{R_2}.$$

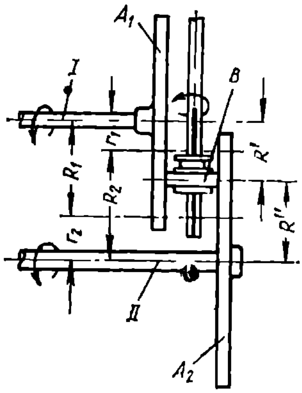


Fig. 196

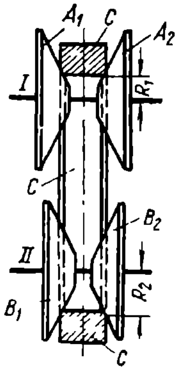


Fig. 197

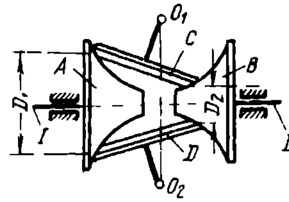


Fig. 198

If the cones on the driving shaft are moved further apart and the second pair of cones moved closer together, radius R_1 will decrease and radius R_2 will increase and the speed ratio will diminish correspondingly. In this way, within certain limits we can obtain any rpm on the driven shaft although the driving shaft is rotating at a constant speed*.

Sometimes it is required that the rpm. transmitted by the driving shaft to another shaft on the same axis, be changed. Such a transmission is shown schematically in Fig. 198: driving shaft I transmits rotation at variable speeds to shaft II, lying on the same axis. Two friction cones A and B with concave sides are fastened to the shafts. Two rollers C and D are clamped between the sides of the cones. The driving cone A transmits rotation to the driven cone B by means of these rollers which rotate about their axes. The shafts on which the rollers are mounted can be adjusted to any required angle with respect to O_1 and O_2, contact between the rollers and the cones taking place along circles of different radii on the side surfaces A and B

* Transmissions of this construction are also made with special kinds of V-belts, chains, etc. in place of the ring.

and with a corresponding change in speed ratio. The speed ratio for the position of the rollers, as shown in the drawing, is

$$i_{2,1} = \frac{D_1}{D_2}.$$

The above are various examples of friction transmission in mechanisms used for infinitely-variable speeds of rotation and are called *friction speed variators*. They are widely used, particularly in machine tools.

Oral Exercises

1. Does the speed ratio of the drives shown in Figs. 195 and 197 depend on the diameter of the ring?
2. Does the speed ratio of the drive in Fig. 196 depend on the diameter of the roller B?
3. In what direction does the driven cone rotate in relation to the driving cone represented in Fig. 198?

160. Spur Gears

If we take a cylinder and cut regularly-shaped grooves at equal distances from each other around its surface, we shall have a *spur gear*.

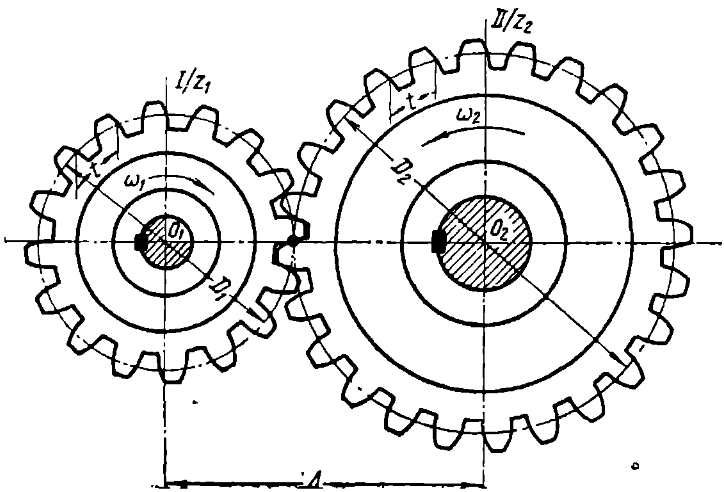


Fig. 199

If we put two such gears together so that the teeth of one mesh into the spaces of the teeth of the other and mount both on shafts O_1 and O_2 (Fig. 199) rotating in stationary bearings, one of them, the driver, will put into motion the second, the follower. In this instance the teeth are cut on the external surfaces of the cylinders; such gears are called *external gears*, as distinguished from *internal gears*, such as shown in Fig. 200, where the teeth on gear I mesh with gear II whose teeth are on the internal surface of the cylinder.

When these gears rotate, it is as if two circles with centres O_1 and O_2 are rolling against each other without slipping and always coming into contact at a certain point P lying on the line of centres O_1 and O_2. These circles bear the name of *pitch circles* and correspond with the circumferences of the friction wheels already shown in Fig. 193. They differ from the latter, however, in that there may occur a slip between the friction wheels, whereas there can be no slipping along the pitch circles of spur gears since the teeth prevent it. From this it is clear that toothed gearing is more dependable when torque is great and the speed ratio must be maintained with precision.

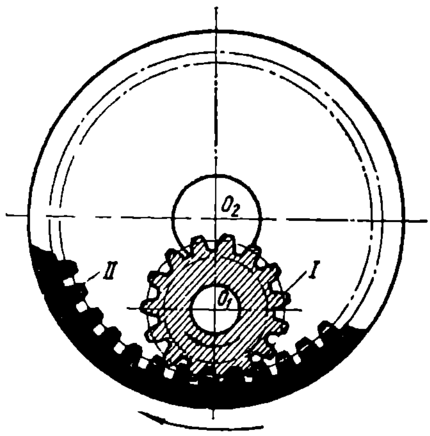


Fig. 200

161. Speed Ratio and the Transmission Number of Toothed Gears

Since there is no slipping between pitch circles when toothed gears rotate, we may therefore apply the same principles in determining their speed ratio as for finding the speed ratio of a belt or friction drive and thereby obtain the same Eq. (122):

$$i_{2,1} = \frac{n}{n_1} = \frac{D_1}{D_2},$$

in which, in the given case, D_1 and D_2 correspondingly represent the diameters of the pitch circles of the driver and follower gears.

It is clear that diameters D_1 and D_2 must be known to determine the speed ratio. But pitch circles are not visible on gears and it would be very intricate to measure their diameters. Hence the formula must take a different form.

Since the teeth of the gear are arranged round its circumference at equal distances, these distances correspond to the arc of the pitch circle stretching from a point on one tooth to a corresponding point on the next tooth, or (which is the same), from the centre or edge of one tooth to the centre or edge of the next. This distance is called the *tooth pitch* and is designated by the letter t (Fig. 199). Obviously gears that mesh must have the same pitch. The tooth pitch is equal to the length of the pitch circle divided by the number of teeth. Thus by denoting the number of teeth as z, we obtain

$$t = \frac{\pi D}{z}.$$

By equating the tooth pitch of the driving gear and the tooth pitch of the driven gear, we evolve

$$\frac{\pi D_1}{z_1} = \frac{\pi D}{z}, \text{ or } \frac{D_1}{D_2} = \frac{z_1}{z_2} \tag{130}$$

whereupon the said Eq. (122) becomes

$$i_{2,1} = \frac{n}{n_1} = \frac{z_1}{z_2} = \frac{D_1}{D_2}. \tag{131}$$

Wherefore *the speed ratio of a pair of gears is inversely equal to the ratio of the number of their teeth, or, which is the same thing, inversely equal to the ratio of the diameters of their pitch circles.*

This applies both to external and internal gears, the only difference being that in external gears the driver and follower rotate in opposite directions, whereas in internal gear they rotate in one direction.

162. Kinematics of Drives Possessing More than One Pair of Gears

We will henceforth schematically represent a gear by a circle corresponding to its pitch circle (Fig. 201), and the letter denoting the gear will also denote the number of its teeth. If the gear is fixed immovably to the shaft, we shall mark its rim with a cross (Fig. 201*a*).

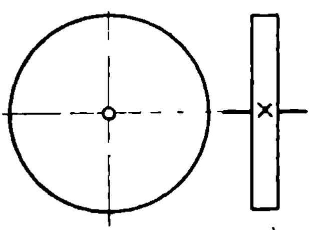


Fig. 201

Gears need not necessarily be immovably fixed to the shaft; they are often mounted on a key which moves in a keyway in the shaft, or the gear may be moved along a spline fastened to the shaft. In both such cases the gear rotates with the shaft but can be fixed at any point along its length*. The conventional indication for this method of mounting is shown in Fig. 201*b*.

Fig. 202 represents a train of gears, from z_1 to z_6 in which z_1 is the driver. For conventional brevity we shall put a sign × be-

* This method of fastening gears to shafts is frequently met with in machine tools.

tween the letters representing gears that are meshed together, and a long dash – between those representing gears on one shaft or on a common bushing. Accordingly, the chain of gears shown in Fig. 202 may be written schematically in the following way:

$$z_1 \times z_2 - z_3 \times z_4 - z_5 \times z_6.$$

Assume that the driver z_1, attached to shaft O_1, makes n_1 revolutions per minute and it is necessary to find the rpm n_6 of the last driven gear z_6.

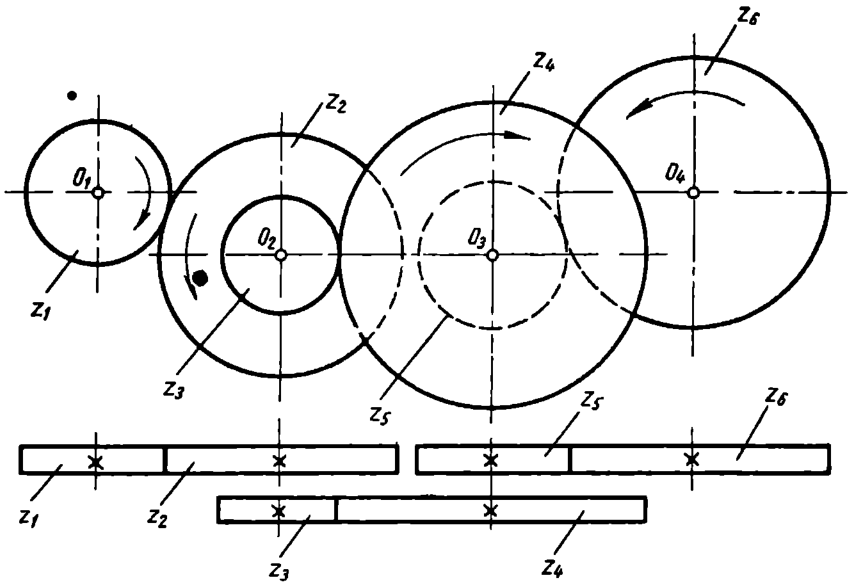


Fig. 202

We obtain the rpm of shaft O_2 through Eq. (131)

$$n_2 = n_1 i_{2,1} = \frac{n_1 z_1}{z_2}.$$

On examining shaft O_3 we see that it receives rotation by means of gears z_3 and z_4, of which the first is a driver. Their speed ratio, therefore, is $i_{3,2} = \frac{z_3}{z_4}$, and the rpm of shaft O_3 is

$$n_3 = n_2 i_{3,2} = n_1 i_{2,1} i_{3,2} = n_1 \frac{z_1}{z_2} \times \frac{z_3}{z_4}.$$

Gear z_6 receives rotation from gear z_5, their speed ratio is $i_{4,3} = \frac{z_5}{z_6}$, and the rpm of shaft O_4 is

$$n_4 = n_3 i_{4,3} = n_1 i_{2,1} i_{3,2} i_{4,3} = n_1 \frac{z_1}{z_2} \times \frac{z_3}{z_4} \times \frac{z_5}{z_6}. \qquad (132)$$

The quotient obtained by dividing the rpm n_4 of the last driven shaft O_4 by the rpm n_1 of the first driving shaft will be the *total speed ratio* $i_{4,1}$; hence it will be

$$i_{4,1} = i_{2,1} i_{3,2} i_{4,3}. \tag{133}$$

According to our line of reasoning it is therefore apparent that Eqs (132) and (133) can be applied to any number of pairs of gears.

Wherefore, *the total speed ratio is equal to the product of the individual speed ratios of all the pairs of gears in the train.*

But it must be noted that the direction of rotation of the last driven gear is to be taken into account: for it is clear that if there is an even number of axes between the first driver and the last driven gear, the former and the latter will rotate in opposite directions; and if there is an odd number of axes between the said extremes, they will rotate in the same direction. In the train of gears we have just considered there are two intermediary pairs of gears ($z_2 z_3$ and $z_4 z_5$), therefore gear z_6 rotates in the opposite direction to driver gear z_1.

A comparison of the above equations with Eqs (123) and (124) will show that the kinematics of toothed gears and of drives with pliant connectors are alike. That which was said in Sec. 152 concerning the arrangement of the driver and the driven wheels also applies to the trains of gears we have just considered.

Oral Exercises

1. If we reverse the places of gears z_1 and z, will it change the rpm of shaft O_1 shown in Fig. 202? Will it change the rpm of shaft O_2?

2. Will the rpm of shaft O_4 be changed if z_1 and z_3 are each increased m times; or if z_1 is increased m times and z_5 is decreased by the same amount; or if z_3 and z_6 are each increased m times?

Illustrative Problem 80. The train of gears shown in Fig. 202 consists of a gear possessing $z_1 = 20$ teeth mounted on driving shaft O_1, and of five other gears whose number of teeth are $z_2 = 50$, $z_3 = 30$, $z_4 = 60$, $z_5 = 25$, and $z_6 = 100$. What are the rpm n_3 and n_4 of shafts O_3 and O_4, if n_1 is equal to 1,500 rpm?

Solution:

$$n_3 = n_1 \frac{z_1 z_3}{z_2 z_4} = 1{,}500 \times \frac{20 \times 30}{50 \times 60} = 300 \text{ rpm}$$

and

$$n_4 = n_1 \frac{z_1 z_3 z_5}{z_2 z_4 z_6} = 1{,}500 \times \frac{20 \times 30 \times 25}{50 \times 60 \times 100} = 75 \text{ rpm}.$$

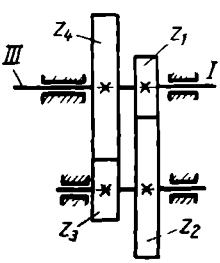


Fig. 203

Illustrative Problem 81. The driving gear on shaft *I* in Fig. 203 possesses $z_1 = 14$ teeth. The number of teeth on the other gears is $z_2 = 70$, $z_3 = 15$, and $z_4 = 45$. If the driver shaft attains $n_1 = 750$ rpm, what are the rpm of shaft *III*?

Solution: $n_3 = n_1 \dfrac{z_1 z_3}{z_2 z_4} = 750 \times \dfrac{14 \times 15}{70 \times 45} = 50$ rpm.

163. Statics of Toothed-Gear Transmission

Now let us determine the relationship between torque and effective pull in parallel-shaft gear drives, just as we did for drives with pliant connectors.

Assume that shaft O_1 transmits rotation to shaft O_3 (Fig. 204) according to the scheme $z_1 \times z_2 - z_3 \times z_4$. Let us find the torque on shaft O_3 if the torque on shaft O_1 is M_1. By denoting the pitch-circle diameter of the gear on this shaft as D_1, we obtain the effective pull $\mathbf{P}_1$ of this pitch circle as

$$P_1 = \frac{M_1}{R_1} = \frac{2M_1}{D_1}.$$

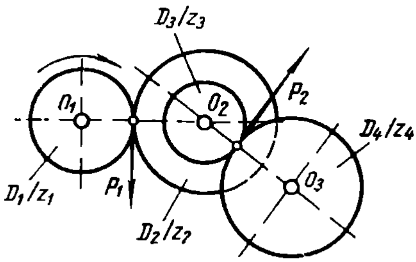


Fig. 204

This effective pull will be transmitted to the teeth of the driven gear z_2. Hence the torque on shaft O_2

$$M_2 = P_1 \frac{D_2}{2} = \frac{2M_1}{D_1} \times \frac{D_2}{2} = M_1 \frac{D_2}{D_1},$$

while the effective pull $\mathbf{P}_2$ on the pitch circle of the second driver gear z_3 will be equal to the torque M_2 divided by the radius of the gear, i. e.,

$$P_2 = \frac{2M_2}{D_3} = 2M_1 \frac{D_2}{D_1} \times \frac{1}{D_3}.$$

The same effective pull is transmitted to gear z_4 of pitch-circle diameter D_4. Therefore the torque on shaft O_3

$$M_3 = 2M_1 \frac{D_2}{D_1} \times \frac{D_4}{2D_3} = M_1 \frac{D_2}{D_1} \times \frac{D_4}{D_3}.$$

From the above Eq. (131) it follows that the diameters of two meshing gears are proportional to the number of their teeth, i.e., $\frac{D_2}{D_1} = \frac{z_2}{z_1}$ and $\frac{D_4}{D_3} = \frac{z_4}{z_3}$, from which we finally obtain

$$M_3 = M_1 \frac{z_2}{z_1} \times \frac{z_4}{z_3}. \qquad (134)$$

But $\frac{z_2 z_4}{z_1 z_3}$ is inversely equal to the speed ratio $i_{3,1}$. Therefore

$$M_3 = \frac{M_1}{i_{3,1}}, \qquad (135)$$

in which M_1 is the torque of the first driver, M_3 is the torque of the last driven shaft, and $i_{3,1}$ is the speed ratio.

Wherefore, *the torque on the driven shaft of a gear drive is equal to the torque on the driving shaft divided by the speed ratio.*

If we take harmful resistance into account, we must include the efficiency of the drive in the equation. Accordingly, it becomes

$$M_3 = \frac{M_1}{i_{3,1}} \eta. \qquad (136)$$

The efficiency will depend on the workmanship of the teeth, shafts, and bearings in which the shafts rotate. Loss due to friction between well-meshed teeth is not more than 1 per cent.

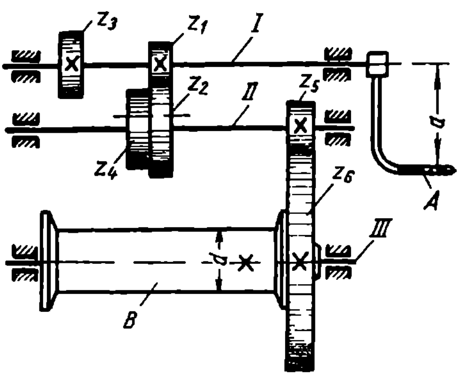


Fig. 205

Illustrative Problem 82. Fig. 205 represents the kinematic scheme of a winch with a hand crank. Shaft *I* is rotated by crank *A*. There are two gears on this shaft, $z_1 = 12$ and $z_3 = 22$. A block of two gears $z_2 = 36$ and $z_4 = 88$ is key-mounted on shaft *II*; z_2 can mesh with z_1 and z_4 with z_3. Gear $z_5 = 12$ meshes with the big gear $z_6 = 72$ on shaft *III* which carries the drum *B* upon which the rope is wound. The drum can be rotated by either of two schemes: shaft $I - z_1 \times z_2 - z_5 \times z_6 - B$; or $I - z_1 \times z_4 - z_5 \times z_6 - B$.

Determine the following when the winch is working according to the first scheme: a) the effective pull P that must be applied to crank *A* to raise, with the aid of a fixed pulley, a load $G = 0.6$ tons; b) the speed v at which the load will rise if the crank is turned at the rate of $n_1 = 25$ rpm; c) the power expended on the crank (the arm of the crank $a = 300$ mm, the diameter of the drum $d = 200$ mm, and the efficiency of the winch $\eta = 0.9$).

Solution: 1. According to Eq. (136) the torque on shaft *III*

$$M_3 = \frac{M_1}{i_{3,1}} \eta,$$

whence the torque on shaft *I* $M_1 = Pa = P \times 0.3 = 0.3P$ kg-m;

$$i_{3,1} = \frac{z_1}{z_2} \times \frac{z_5}{z_6} = \frac{12}{36} \times \frac{12}{72} \times \frac{1}{18}$$

and $\eta = 0.9$. By restating the equation, we obtain $M_3 = \dfrac{0.3P}{\frac{1}{18}} \times 0.9$.

But on the other hand, $M_3 = G\dfrac{d}{2} = 600 \times \dfrac{0.2}{2} = 60$ kg-m. Hence, $60 = 0.3 \times 18 \times 0.9P$, from which the effective pull $P = 12.3$ kg.

2. If the crank attains $n_1 = 25$ rpm, shaft *III* will receive $n_3 = n_1 i_{3,1} = \dfrac{25}{18}$ rpm, and the speed at which the load is raised will be equal to

the peripheral speed of the drum, i.e.,

$$v = \frac{\pi d n_3}{60}\ \text{m/sec} = \frac{\pi 0.2 \times 25}{60 \times 18} = 0.015\ \text{m/sec} \approx 15\ \text{mm/sec}.$$

3. The power expended, as found by Eq. (82), is $N = \frac{Pv}{75}$, in which P is the force applied to the crank and v is the linear velocity of a point on the crank describing a circle of radius a and which is equal to

$$\frac{2\pi a n_1}{60 \times 1{,}000} = \frac{\pi 0.3 \times 25}{30}.$$

$$\text{Accordingly, } N = \frac{12.5\pi \times 0.3 \times 25}{30 \times 75} \approx 0.13\ \text{hp}.$$

164. Idler Gears

Fig. 206 represents three intermeshing gears z_1, z_2, and z_3, the former being the driver. Let us determine the speed ratio between shafts O_3 and O_1.

The speed ratio between shafts O_2 and O_1

$$i_{2,1} = \frac{z_1}{z_2}.$$

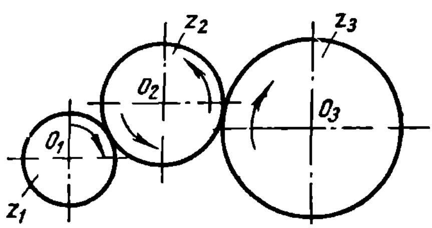


Fig. 206

In comparing shafts O_2 and O_3, we see that of the mating pair of gears z_2 and z_3, the former is the driver and the speed ratio

$$i_{3,2} = \frac{z_2}{z_3}.$$

Consequently, the total speed ratio

$$i_{3,1} = i_{2,1} i_{3,2} = \frac{z_1}{z_2} \times \frac{z_2}{z_3} = \frac{z_1}{z_3}.$$

Thus we see that the speed ratio between shafts O_3 and O_1 does not depend on the number of teeth in gear z_2 on the middle shaft O_2. Hence z_2 is known as an *idler gear*. By comparing it with the other two, we find that it is simultaneously a follower with respect to gear z_1 and a driver in relation to gear z_3. This is the distinguishing feature of an idler gear. Whether a gear is an idler or a working gear depends, of course, on the role it plays in a given chain of gears.

Idler gears are used in two instances. In the first place, if motion is to be transmitted between two shafts spaced so far apart that the gears would have to be made very large, one or more idlers are used. With their aid rotation can be transmitted through any intervening distance irrespective of the diameters of the working gears. In the second place, when gears z_1 and z_3 mesh together directly, their shafts will turn in opposite directions. But if an idler gear is used between them, the driven gear will rotate in the same direction as the driver. Accordingly, in the second instance idler gears are used when it is necessary to change the direction of rotation of the driven gear.

It therefore follows that *an idler gear is a gear which simultaneously meshes with two other gears, and is a follower in relation to one of the gears and a driver with respect to the other. An idler gear does not change the speed ratio between the other two gears, but it does change the direction of rotation of the driven gear.*

Oral Exercises

1. It is necessary for shaft O_1 in Fig. 206 to transmit rotation to shaft O_3 in a direction opposite to its own. Gears z_1 and z_3 do not mesh with each other. How many idler gears will be needed?

2. Will the speed ratio $i_{3,1}$ (Fig. 206) change if gears z_1 and z_2 or z_2 and z_3 are interchanged?

Illustrative Problem 83. Fig. 207 represents a train of gears in which shaft O_1 transmits rotation to shaft O_4 in the following way: a plate and its handle A turn freely on shaft O_1. The plate carries, on pins O_2 and O_3, two gears z_2 and z_3 which are in constant mesh with each other. z_2 is also constantly in mesh with gear z_4 on shaft O_4. When the mechanism is in the position shown in the drawing, rotation from shaft O_1 is not transmitted because z_1 is not in mesh with any of the other gears. If we pull the handle A in the direction of arrow 1, gear z_2 will mesh with the driving gear z_1 and the mechanism will work according to scheme $z_1 \times z_2 \times z_4$. Rotation of the driven shaft will be in the direction of arrow $1'$.

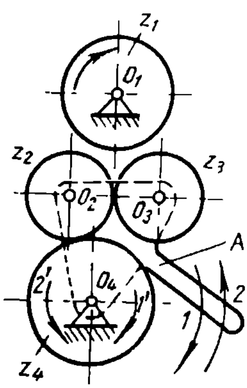


Fig. 207

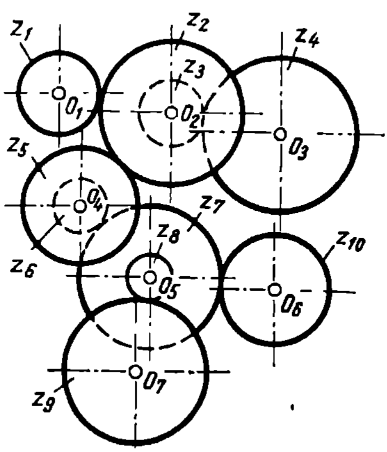


Fig. 208

If we pull the handle A in the direction of arrow 2, the driving gear z_1 will be in mesh with gear z_3 and the mechanism will work according to scheme $z_1 \times z_3 \times z_2 \times z_4$; gear z_4 will rotate in the direction of arrow $2'$ (opposite to that in the first example). We thus see that in the first case there is one idler gear and the speed ratio $i_{4,1} = \frac{z_1}{z_4}$; while in the second case, with two idler gears the speed ratio $i_{4,1} = \frac{z_1}{z_4}$ but rotation is in the opposite direction This arrangement is called a *reversing mechanism* and is used in thread cutting lathes to reverse the direction of the carriage and also to disengage it from the transmission.

Illustrative Problem 84. In the train of gears illustrated in Fig 208, the driving gear z_1 transmits rotation to three gears z_4, z_9, and z_{10} in accordance with the following schemes 1) z_1 – $z \times z_4$, 2) $z_1 \times z \times z_5$ $z_6 \times z_7 \times z_{10}$, 3) z_1 – z_5 – z $z_8 \times z_9$ Find the rpm of shafts O, O_6, and O if the rpm of shaft O_1 equals n_1.

Solution 1 The rpm of shaft O

$$n = n_1 \frac{z_1 z}{z z_4}$$

2 The rpm of shaft O_6

$$n_6 = n_1 \frac{z_1 z_6}{z z_{10}} \quad \text{(gears } z \text{ and } z \text{ are idlers)}$$

3 The rpm of shaft O_7

$$n = n_1 \frac{z z_1 z_8}{z z z_9} \quad \text{(gear } z \text{ is an idler)}$$

165. Spur-Gear Differential Mechanisms

In the gear transmissions we have thus far investigated all the component gears rotate about fixed axes and motion is transmitted by one driver A more complex drive shall now be examined

In Fig 209, representing such a mechanism, the pair of gears A and K are mounted as follows gear A revolves around the fixed axis O_1, while around the same axis but independent of gear A, an arm B (called a *spider*) may turn in either direction. To arm B gear K is mounted on a pin (axis O) around which it freely turns and simultaneously meshes with gear A

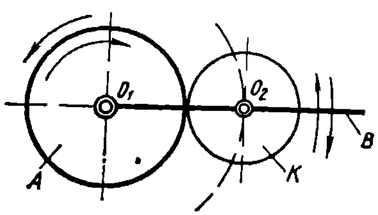


Fig 209

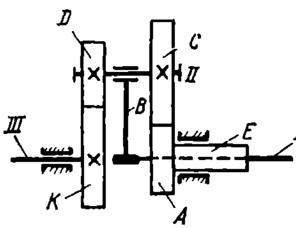


Fig 210

Thus the rotation of gear K is a combination of two rotations. it rotates together with arm B and it also rotates in relation to arm B. This arrangement allows us to select the number of revolutions of gear A and arm B, the direction of rotation of each of them and the number of teeth on A and K, thus obtaining any desired rpm and any direction of rotation of the driven gear K. Such a mechanism, which can combine several independent motions, is called a *differential*.

The above described differential is the simplest type. A more complex mechanism of this kind is shown schematically in Fig. 210. Gear A, which is part of bushing E, receives rotation from one source, while shaft I which receives rotation from a second source turns freely within bushing E. Fastened to the left end of shaft I is crank B on the end of which is a bushing and in which shaft II rotates. Gears C and D are fixed to either end of shaft II. C meshes with gear A, and D meshes with gear K which is on a separate shaft III whose axis coincides with shaft I. When shaft I rotates, gear C rolls around gear A and rotation of the desired speed and direction is transmitted through gears D and K to shaft III. The intermediate gears C and D are called *planetary gears*. Gears A and K, around which the planetary gears roll, are known as *solar* or *central gears*.

There is a variation of this mechanism: gear A does not revolve, whereupon rotation is transmitted to shaft III from shaft I alone. This type of transmission is called a *planetary gear train*.

The ability of these mechanisms to transmit rotation from a number of sources, the possibility of their adjustment to obtain very low speed ratios as well as rotation in any direction, and also their compactness, has brought them into wide use in machine tools.

In the above examples the central and planetary gears are external, but similar drives can also be arranged with internal gears.

166. The Geometry of Toothed Gearing

To express the pitch-circle diameter D in relation to the tooth pitch t we use Eq. (130):

$$t = \frac{\pi D}{z},$$

from which

$$D = \frac{t}{\pi} z. \tag{137}$$

Accordingly, the distance A between axes O_1 and O_2 of the two meshing gears, as shown in Fig. 199, is:

$$A = O_1O_2 = \frac{D_1 + D_2}{2} = \frac{t}{\pi} \times \frac{z_1 + z_2}{2} \tag{138}$$

But when this centre distance is expressed through the incommensurable quantity π, it cannot be calculated exactly and the fraction obtained is clumsy and inconvenient for practical use. Nevertheless, this measurement must be obtained with great precision when assembling a gear mechanism. For this reason a quantity called the *module*, expressing the relationship of the tooth pitch to π has been introduced. Since the tooth pitch is expressed in millimetres, whereas π is an abstract quantity, the module is therefore also expressed in millimetres and denoted by the letter m. Accordingly,

$$m = \frac{t}{\pi} \text{ mm} \qquad (139)$$

and the tooth pitch

$$t = \pi m \text{ mm.} \qquad (140)$$

By adopting this quantity, Eq. (137) offers the following expression for the diameter of the pitch circle

$$D = mz, \qquad (141)$$

that is, *the diameter of the pitch circle in gears, expressed in millimetres, is equal to the module multiplied by the number of teeth.*

From this a simple expression is evolved for the centre distance

$$A = m\frac{z_1 + z_2}{2} \qquad (142)$$

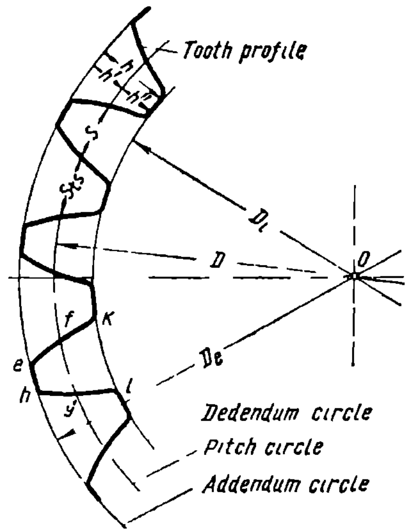


Fig. 211

that is, *the centre distance in millimetres is equal to the module multiplied by half the number of teeth of the meshing gears.*

The portion of the tooth, extending beyond the pitch circle *efgh* (Fig. 211), is called its *point* while the part lying within the pitch circle *fklg* is known as the *root*. And correspondingly, the radial distance h' from the pitch circle to the top of the point is called the *addendum*, and the radial distance h'' from the pitch circle to the bottom of the root is called the *dedendum*. These distances, relative to the module, are

$$h' = m, \text{ and} \qquad (143)$$

$$h'' = 1.2\ m, \qquad (144)$$

hence the whole height of the tooth $h = h' + h'' = 2.2\ m$ (145)

Knowing the addendum of the tooth, then the diameter D_e of the circle against which the tips of all the teeth lie and which is called the *addendum circle*, can be expressed as

$$D_e = D + 2h',$$

which, after substituting the values of D and h' from Eqs (141) and (143), becomes

$$D_e = mz + 2m = m(z + 2), \tag{146}$$

that is, *the diameter of the addendum circle is equal to the module multiplied by the number of teeth plus 2.*

We find diameter D_i of the *dedendum circle* in the same way:

$$D_i = D - 2h'' = mz - 2.4\,m = m(z - 2.4). \tag{147}$$

It is easy to see from the above that the following relationships are obtained for internal gearing:

$$D_0 = D - 2h' = mz - 2m = m(z - 2) \tag{148}$$

and

$$D_i = D + 2h'' = mz + 2.4\,m = m(z + 2.4). \tag{149}$$

The tooth pitch t is measured along the pitch circle and is equal to the *thickness of the tooth* s plus *the width of the tooth space* s_{ts}, in which the thickness of the tooth is equal to the width of the tooth space, i.e.,

$$s = s_{ts} = 0.5\,t = 0.5\,\pi\,m. \tag{150}$$

Besides the gear dimensions indicated above, there is also the *face width* b (i.e., the width of the rim of the gear). There is no exact standard for this dimension; it is selected in each individual case according to the load to be borne by the tooth.

In the U.S.S.R. there is an approved standard of modules (see Supplement III).

In the United States and Great Britain, *diametral pitch* is used instead of the module. Diametral pitch is expressed in inches and is the quotient obtained by dividing the number of teeth in a gear by the diameter of the pitch circle. In other words, it may be said that *diametral pitch is the ratio of the number of teeth in a gear per inch of its diameter of pitch circle.*

By denoting diametral pitch as p we therefore obtain

$$p = \frac{z}{D} \text{ (in inches).} \tag{151}$$

If D and t be expressed in inches in the equation $z = \frac{\pi D}{t}$ and this equation be placed in the above Eq. (151), then

$$p = z\frac{\pi D}{tD} = \frac{\pi}{t} \text{ (in inches),} \tag{152}$$

that is, *diametral pitch is equal to* π *divided by the tooth pitch expressed in inches.*

To find the relationship between diametral pitch and module, we place $D = mz$ (mm) into Eq. (141) and, bearing in mind that one inch = = 25.4 mm, we obtain

$$p = z : \frac{mz}{25.4} = \frac{25.4}{m}. \tag{153}$$

We thus see that the module is the reciprocal of diametral pitch: the larger the one, the smaller the other. It may likewise be said that as the module increases, the tooth pitch also increases, but with an increase in diametral pitch the tooth pitch decreases.

Oral Exercises

1. Calculate the tooth pitch for modules of 2 mm, 5 mm, and 10 mm, respectively.

2. If $t = 15$ mm is the result evolved from calculation of a tooth pitch what is the nearest value of the module that corresponds to this pitch (see Supplement III)?

Illustrative Problem 85. Calculations show that the tooth pitch of a gear of $z = 60$ teeth should be approximately, but not less than, 15 mm. Calculate the chief elements of the gear.

Solution: the module $m \approx \frac{15}{\pi} \approx 4.7764$. By choosing the nearest larger module as $m = 5$ mm, we find that the addendum $h' = 5$ mm, the dedendum $h'' = 1.2 \cdot 5 = 6$ mm, the height of the tooth $h = 11$ mm. The thickness of the tooth and width of the tooth space are each equal to $s = s_0 = 0.5\pi m \approx 7.85$ mm. The diameter of the addendum circle $D_e = 5(60 + 2) = 310$ mm.

Illustrative Problem 86. Find the module of a gear by making the required measurements.

Solution: we measure the diameter of the addendum circle and find that it is, for example, 126 mm. If the number of teeth are, let us say, 36, then the module will be $m = \frac{126}{36} = 3.5$ mm.

Illustrative Problem 87. A gear of $z = 45$ teeth and a module of 4 mm is to be made. What must be the diameter of the finished blank, and the cutting depth of the milling machine?

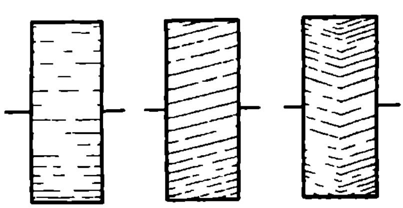
Solution: the lathe operator must machine the blank according to the diameter of the addendum circle; this must be $D_e = 4(45 + 2) = 188$ mm.

The milling machine operator must cut the tooth spaces to a depth equal to the full height of the teeth $h = 2.2 \times 4 = 8.8$ mm.

167. Chief Forms of Spur-Gear Teeth

In order that a mating pair of gears operate satisfactorily, the sides of the teeth on both gears are given precisely the same form. The curve of the side surface of a tooth is called its *profile* (Fig. 211). The profile for the teeth of a pair of mating gears must be designed so as to ensure uniformity of speed ratio for all moments of time. The most common curve for this profile is the involute curve. Teeth of this shape are called *involute teeth*.

Gears are also distinguished according to their form along the face width, the most com-



a) b) c)

Fig. 212

mon form being the *straight spur gear* shown in Fig. 212*a*. If the lines along the face width are slanting, the gear is called a *helical gear* (Fig. 212*b*). Often helical gears are cut as shown in Fig. 212*c*, where each tooth line along the face width is formed of two slanting segments meeting at an angle. This type is called *a herringbone gear*. Both helical and herringbone gears result in smoother transmission, and the herringbone type of teeth lend particular strength to the gear.

168. Intermittent Transmission of Rotation

In transmitting rotational motion it is sometimes required that the continuous rotation of the driver shaft be changed to intermittent rotation of the driven shaft, the latter pausing fully a number of times during the course of each revolution. One of the mechanisms used for this purpose is the *Geneva wheel*, a simple type of which is shown in Fig. 213.

The continuously rotating crank A, which is fixed fast to shaft O_1, has a driving pin D made to fit into the radial slots C in disc B which is part of shaft O_2. As the pin enters one of these slots, the rotating crank forces disc B to turn until the pin abandons the slot, at which moment disc B stops turning and dwells

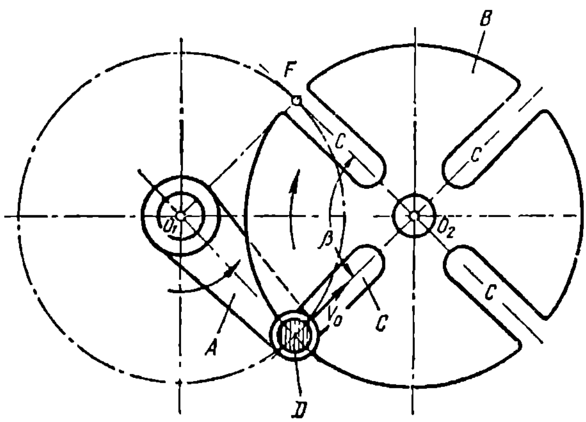


Fig. 213

in this position. But as the crank continues to rotate, the pin enters the next slot and again imparts rotation to the disc as before. In this way as the centre of the driving pin D describes a circle around axis O_1 as it rotates, it will successively enter all the slots in the disc in a radial direction, first approaching axis O_2 and then receding from it. The number of pauses (periods of dwell) made by disc B will depend on the number of slots in the disc. It there are three slots, the disc will rotate between each period of dwell through an angle $\beta = \frac{360°}{3} = 120°$; if there are four slots, it will rotate through an angle $\beta = \frac{360°}{4} = 90°$, etc.

Thus, whereas the driver crank A will rotate uniformly, the follower disc B will turn intermittently. When the pin first enters a slot, the velocity v_0 of the centre of the pin will be directed towards the centre of the disc and the speed of the disc will be zero. The disc will subsequently rotate with increasing speed till it reaches its maximum when the crank coincides with the centre line O_1O_2. Then a slowing down will occur, reaching a full stop when the crank is in position O_1F and the pin abandons the slot.

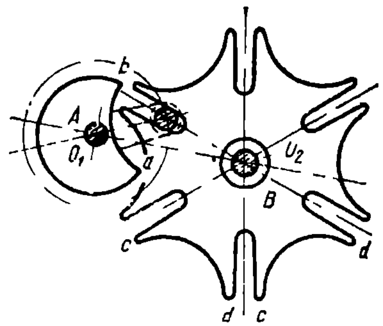


Fig. 214

However, this simple type of Geneva wheel is not entirely satisfactory. If for some reason the disc should turn ever so slightly after the pin leaves a given slot, all the slots will be thrown out of line with the crank, and when the pin is again ready to enter a slot, the latter will not be in its desired position and the mechanism will break. To prevent this, the mechanism must be constructed so that the disc is locked in position during each period of dwell.

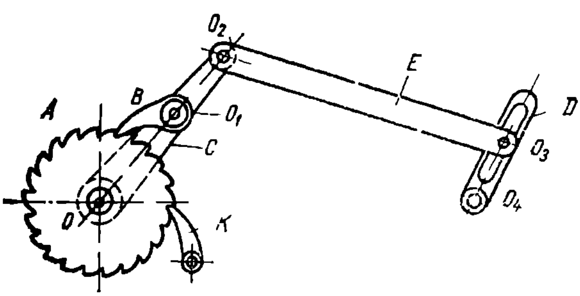


Fig. 215

A mechanism of this kind, in which the follower shaft O_2 makes one full revolution with six periods of dwell equal to six revolutions of the driver shaft O_1, is shown in Fig. 214. Disc A and the crank are fixed fast to shaft O_1. Disc B has radial slots, between which it is cut away by arcs *cd*, the radii of which are equal to the radius of disc A. Disc A is also cut away (arc *ab*), making it possible to clear disc B and rotate unhindered together with the crank, as shown in the drawing. As the pin abandons a slot in disc B, the convex side of disc A slides into one of the hollows *cd*, thereby locking disc B in position. Disc A itself, however, continues to rotate, its convex side sliding through the hollow in disc B. This arrangement is used in cinema apparatus, in the reversing mechanisms of machine tools, etc.

Another type of mechanism for transmitting intermittent rotational motion is the *ratchet-and-pawl* (Fig. 215). A toothed wheel A, called a *ratchet*, is fixed fast to shaft O which is to rotate intermittently. The *pawl* B turns freely on pin O_1 of the lever C and is pressed to disc A by a spring (not shown in the drawing). Lever C is pin-jointed by means of O_2 to slider E which, in its turn, is pin-jointed by means of O_3 to the crank D rotating around the fixed axle O_4. If the teeth are shaped as shown in the drawing, the pawl will be driven into a tooth space when the lever C swings counter-clockwise and will turn the wheel through an arc depending upon the amplitude of swing. When the lever swings in the other direction, the pawl will slide over the teeth of the ratchet without causing the latter to move. In order to ensure that shaft O will dwell absolutely motionless during the given moment, a second pawl K, on a fixed axle, is introduced. The pin O_3 can be set to any position in the slot of crank D for the purpose of regulating the amplitude of swing of lever C.

Ratchet-and-pawl mechanisms are used a great deal in machinery, particularly in planing and other machine tools.

169. Questions for Review

1. What is the difference between the speed ratios $i_{2,1}$ and $i_{1,2}$?

2. If the speed ratio $i_{2,1} = \frac{1}{3}$, what is the transmission number?

3. The rpm of the driven shaft in a drive with pliant connectors must be increased m times. What change must be made in the diameter of the driving sheave? In the diameter of the driven sheave?

4. If it were necessary to change the direction of rotation of shaft O_4 in the belt drive shown in Fig. 183 while maintaining the same direction of motion of shaft O_1, how should it be done?

5. If slip is ignored, is there any difference in the speed of the belts between shafts O_1 and O_2, O_2 and O_3, and O_1 and O_4 in Fig. 183, when the sheaves are of different diameters?

6. Are the torque and the power on shafts O_1, O_2, O_3, and O_4 (Fig. 183) the same? (Neglect harmful resistance.)

7. If the rpm of the driver are constant, will the speed of the belt on the different steps of stepped cones be uniform?

8. Given two pairs of gears—one external and the other internal. The number of teeth on the driver and follower of the first pair are each equal to the number of teeth on the corresponding gears of the second pair. What will be the difference in the rotation of the driven shafts?

9. What rearrangement can be made in a train of several pairs of gears without changing the full speed ratio of the train?

10. How can one tell the difference between an idler gear and a working gear in a train of gears? When are idler gears used?

11. Will the rpm of shafts O_4 and O_5 in Fig. 208 change if gears z_3 and z_5 are interchanged?

170. Exercises

87. Fig. 216 shows a belt drive: between shafts O_1 and O_3 is situated shaft O_2, to which is fixed a sheave of diameter D_2 connected by a belt with the driver sheave D_1 on one side and by another belt with the follower sheave D_3 on the other. What are the rpm of the driven shaft if the rpm of the driving shaft are n_1.

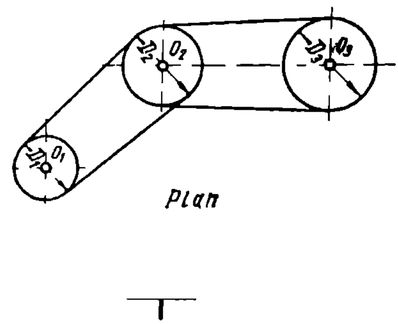


Fig. 216

88. Shaft O_1 (Fig. 183) executes 1,500 rpm. Calculate the rpm of shaft O_4 and also the torque on that shaft, if the following data is given: power $N = 22.5$ kw, diameters of the sheaves $D_1 = 300$, $D_2 = 150$, $D_3 = 200$, $D_4 = 800$, $D_5 = 200$, and $D_6 = 250$ mm.

89. Using the same data given in Ex. 88, determine the rpm of shaft O_2 and the torque of that shaft.

90. Given the rpm of shaft I $n_1 = 750$ and 800 mm as the distance between two shafts (Fig. 196) At what distance R_1 from the axis of shaft I must the roller B be mounted if shaft II is to attain $n_2 = 250$ rpm?

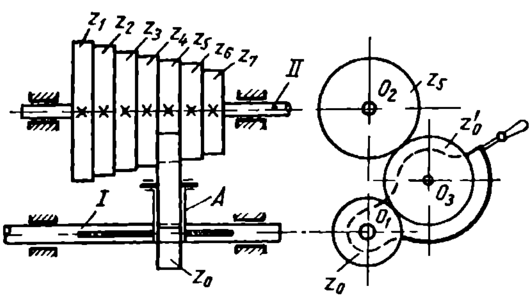


Fig. 217

91. Shaft I (Fig. 217) transmits rotation to shaft II on which are fixed gears z_1 to z_7. On shaft I, gear z_0 slides in a keyway and is permanently meshed with gear z_0' which rotates on an axis fixed to the housing A. By moving this housing along axis O_1 so that it is opposite any one of the gears z_1 to z_7 and then bringing gear z_0' into mesh with it, it is possible to transmit rotation from shaft I to shaft II at the required speed ratio. Write all the speed ratios that can be obtained with this gear train.

92. In the train of gears shown in Fig. 218 shaft O_1 transmits $n_1 = 150$ rpm to shafts O_2, O_3, and O_4. Calculate the rpm n_2, n_3, and n_4 of these shafts if the number of teeth on the gears is as follows: $z_1 = 30$, $z_2 = 50$, $z_3 = 20$, $z_4 = 50$, $z_5 = 25$, $z_6 = 50$, $z_7 = 20$ and $z_8 = 45$.

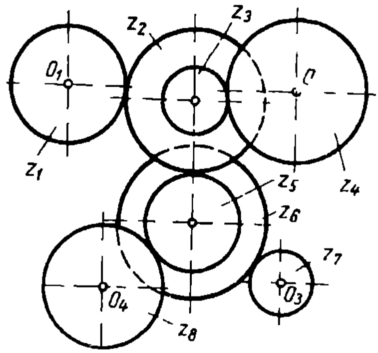


Fig. 218

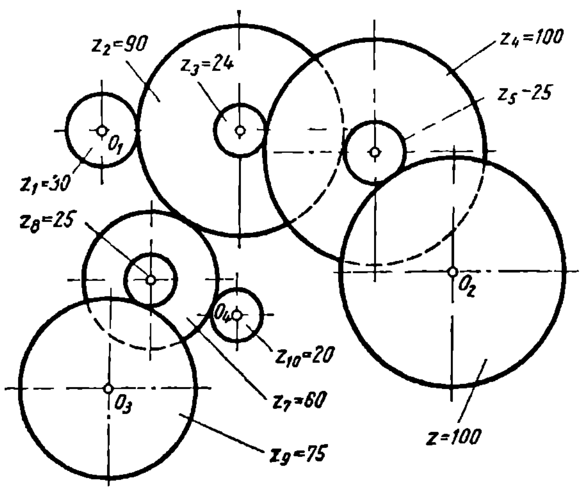


Fig. 219

93. Calculate the torque on shaft O_4 in Ex. 92 if the power transmitted $N = 1.5$ hp.

94. Calculate the rpm of shafts O_2, O_3, and O_4 of the mechanism represented schematically in Fig. 219, assuming that shaft O_1 executes $n_1 = 300$ rpm.

CHAPTER XVII

TRANSMISSION BETWEEN NON-PARALLEL SHAFTS

171. Transmission of Rotation Between Non-Parallel Shafts Through Pliant Connectors

Now that we have studied the main types of transmission for transmitting rotational motion between parallel shafts, we shall investigate transmission of rotation between non parallel shafts that intersect at one level and also those that intersect at a distance.

Fig. 220 shows a belt transmission between two shafts that intersect at a distance and form an angle of 90°. In this transmission the centre lines of the belt segments advancing upon the pulleys A and B must lie approximately in the mid-planes of these respective pulleys. Such an arrangement is classified as a *quarter-turn transmission*. Experience shows that this kind of transmission operates properly if the segment of the belt receding from the driver forms an angle α not greater than approximately 25° to the mid-plane of the pulley. This kind of transmission is also used between non-parallel shafts that intersect each other at a distance at an angle other than 90°, in which case guide pulleys are sometimes used.

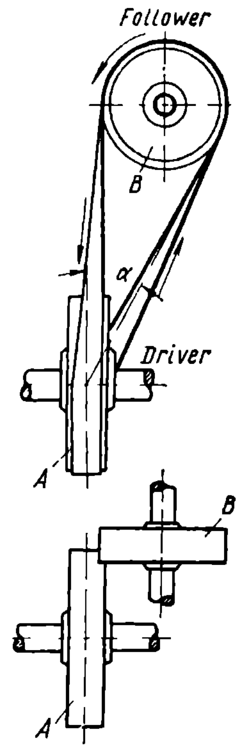


Fig. 220

172. Friction Transmission Between Non-Parallel Shafts

Transmission between non-parallel shafts can also be accomplished through friction gearing. Fig. 221 shows a transmission of this kind called *rolling cones:* on the ends of shafts I and II, whose axes lie in the same plane and intersect at an angle at point O, are situated two rollers in the form of frusta-cones. If sufficient friction is created under the action of axial forces Q_1 and Q_2 the frusta will rotate without slipping. Let us see how to determine their speed ratio.

Assume that at a given moment the two frusta-rollers are in contact along line Bb. We shall take any arbitrary point M along the line of contact (Fig. 222), where two points on the surfaces of the two rollers coincide. The point on the driving roller K

lies at a distance of MN_1 from the axis of rotation. If this roller executes n_1 rpm, the velocity at this point will be, according to Eq. (54),

$$v_1 = 2\pi MN_1 n_1.$$

In the same way the velocity of point L on the surface of the driven roller

$$v_2 = 2\pi MN_2 n_2.$$

If there is no slip, the velocities of the two points will be equal, i. e., $2\pi MN_1 n_1 = 2\pi MN_2 n_2$ from which the speed ratio is

$$i_{21} = \frac{n_2}{n_1} = \frac{2\pi MN_1}{2\pi MN_2} = \frac{MN_1}{MN_2}. \qquad \text{(a)}$$

Let us denote D_1 as the diameter AB of the base of the driving cone, and d_1 as the diameter ab of its apex. The right trian-

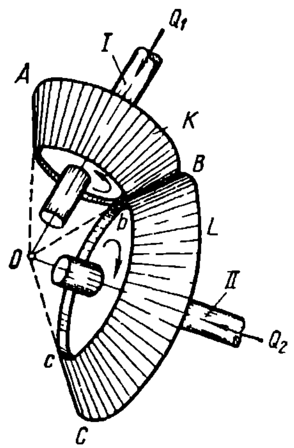


Fig. 221

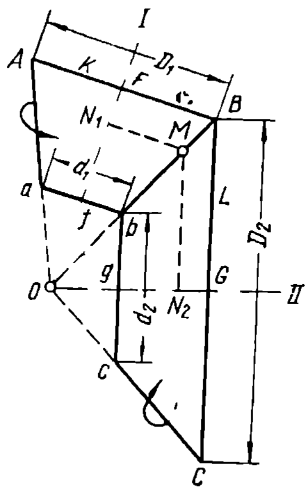


Fig. 222

gles OMN_1, OBI, and Obf are similar, from which it follows that

$$\frac{MN_1}{OM} = \frac{BI}{OB} = \frac{bf}{Ob}.$$

Likewise from the similarity of triangles OMN_2, OBG, and Obg we obtain

$$\frac{MN_2}{OM} = \frac{BG}{OB} = \frac{bg}{Ob}.$$

If we devide the first group by the second, we obtain

$$\frac{MN_1}{MN_2} = \frac{BI}{BG} = \frac{bf}{bg}.$$

By equating this expression with the above (a), we evolve

$$i_{2,1} = \frac{BF}{BG} = \frac{bf}{bg}.$$

But $BF = \frac{D_1}{2}$, and $BG = \frac{D_2}{2}$, while $bf = \frac{d_1}{2}$, and $bg = \frac{d_2}{2}$, whereupon we finally obtain

$$i_{2,1} = \frac{D_1}{D_2} = \frac{d_1}{d_2}, \tag{154}$$

that is, *the speed ratio between two rolling frusta is inversely equal to the diameters of their bases or their apices.*

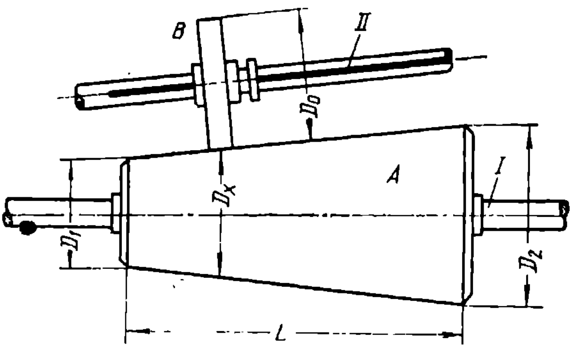


Fig. 223

Friction transmission between non-parallel shafts can also be accomplished with variable speed ratios. Assume it necessary that shaft *I* (Fig. 223) with a constant rpm transmit rotation to shaft *II*, and that shaft *II* rotate at varying angular speeds as needed. We mount the cone *A* on the driver shaft with its slanting side parallel with shaft *II* to which wheel *B* is mounted on a sliding keyway, thus making it possible to set it into any position. If we denote D_x as the diameter of the cone in the section corresponding to the centre line of wheel *B*, then the speed ratio between the shafts

$$i_{2,1} = \frac{D_x}{D_0} \tag{155}$$

Therefore at n_1 rpm of shaft *I*, the latter can transmit varying rpm to shaft *II*, ranging from a minimum of $n_1 \frac{D_1}{D_0}$ to a maximum of $n_1 \frac{D_L}{D_0}$.

Fig. 224 schematically represents a friction transmission with a variable speed ratio for geared shafts whose axes intersect at right angles. Disc *A*, which is fixed to the driving shaft *I*, is pressed to the friction wheel *B* which moves in a keyway and can thus be set into any position along shaft *II*. Accordingly, it is possible to obtain a circle of contact between disc *A* and wheel *B* of any radius R_x With the wheel in the position shown in the drawing, the speed ratio

$$i_{2,1} = \frac{R_x}{R_0}. \quad (156)$$

It is obvious that if the wheel is moved to the right, the speed ratio will diminish; if it is set opposite to the axis of shaft *I*,

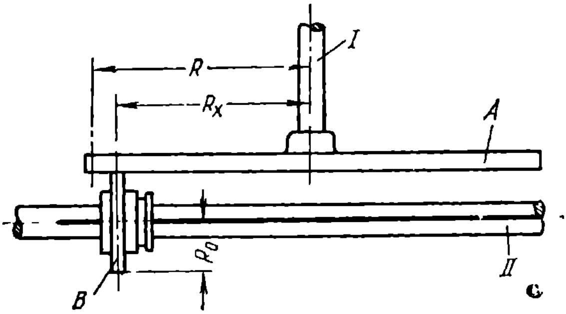


Fig. 224

the speed ratio will be zero and the driven shaft will not rotate; if it is moved still further to the right beyond the centre of the disc, the direction of rotation of the driven shaft will be the opposite to that when the wheel was to the left of the centre and the speed ratio will increase as the wheel is moved further from the centre. Hence if the driving shaft is rotating at n_1 rpm, the rpm of the driven shaft n_2 will range from 0 to $n_1 \frac{R}{R_0}$ in either direction.

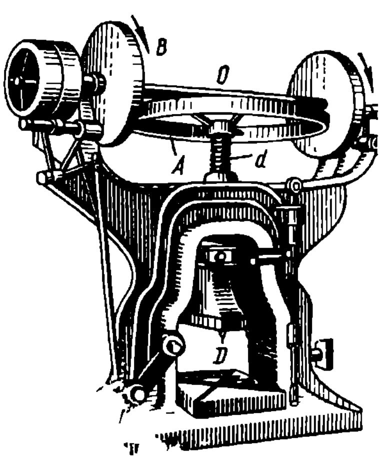


Fig. 225

Illustrative Problem 88. Fig. 225 is a general view of a friction press. On the driving shaft *O* which can move somewhat in an axial direction, are fixed two friction wheels *B* and *C*. The rim of friction wheel *A*, which is fixed fast to screw *d*, is covered with leather. Screw *d* turns in a threaded bushing fixed in the frame of the press, and to its lower end is attached the ram *D* in such a way that the screw can turn about its axis. The ram slides in guides.

Let wheel *A* be pressed against wheel *B*, and if the driving shaft is moved in the direction shown by the arrows and the screw has a right-hand thread, it will screw into the threaded bushing and impart downward motion to the ram with an increasing speed as the distance of wheel A from the centre of wheel B increases. When this stamping operation is finished, driver shaft O is shifted in an axial direction to the right with the aid of a special mechanism and wheel C becomes pressed against A. Then the screw will begin to turn in the opposite direction and the ram will rise with decreasing speed.

173. Bevel-Gear Transmission

Let us assume that we have cut teeth on a pair of rolling frusta-cones in such a way that if their edges were prolonged beyond the apices of the cones they would intersect at point O (Fig. 226). We would then have a pair of *bevel gears*. Axes O_1 and O_2 of the gears in Fig. 227 intersect at O, forming the angle δ. Bevel gears are mostly used between shafts that are perpendicular to each other.

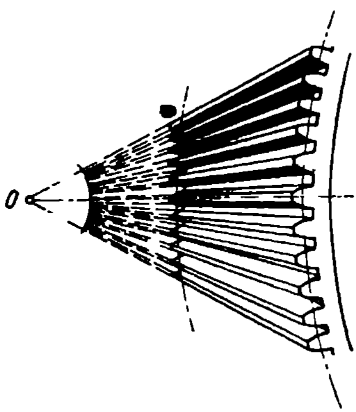


Fig. 226

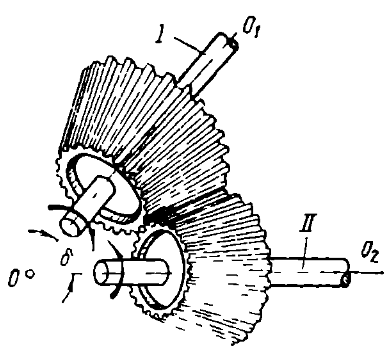


Fig. 227

Assume that the gear on shaft I has z_1 teeth, that the one on shaft II has z_2 teeth, and that the driver gear makes z_2 rpm, i. e., $n_1 = z_2$; thereby $z_2 \times z_1$ teeth would pass an immovable mating point of the gears and the driven wheel would therefore execute $\frac{z_2 \times z_1}{z_2} = z_1 = n_2$ rpm. From this it follows that

$$i_{2,1} = \frac{n_2}{n_1} = \frac{z_1}{z_2}. \tag{157}$$

Wherefore, *the speed ratio $i_{2,1}$ of bevel gears, just as of spur gears, is equal to the ratio of the number of teeth on the driving gear to the number of teeth on the driven gear.*

As concerns the direction of rotation of bevel gears, it is determined either with respect to their bases or their apices. If the bevel gears are external, the driven gear will rotate in the opposite direction to the driving gear.

Bevel gears may be internal (Fig. 228) as well as external. If internal, the rotation of the driving and driven gears will be in the same direction. Internal gears are little used due to the difficulty of cutting bevel gears with the teeth on the inside.

Fig. 229 shows another type of bevel gear in which a conical gear A is in mesh with a toothed disc B.

Differential mechanisms are made with bevel gears just as they are with spur gears. Fig. 230 represents a simple type *bevel-gear differential*. Shaft I, which is in one piece with spider B, passes freely

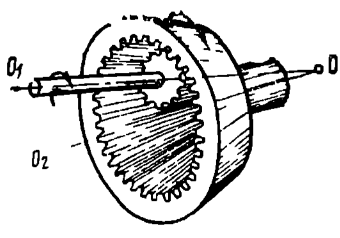


Fig. 228

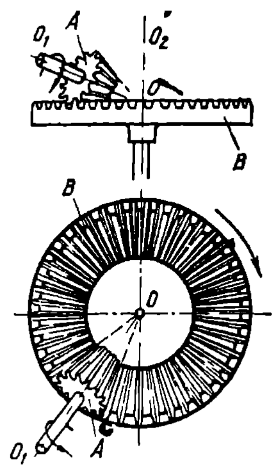


Fig. 229

through the hub of gear A. Gear K is mounted on the spider and meshes simultaneously with gears A and L, the latter being a part of shaft II. When gear A and shaft I, together with its spider, rotate, the two motions combine to rotate gear L together with shaft II. If gear A is prevented from rotating, shaft II will receive rotation from one source of motion only—from shaft I. Gear K is a planetary gear.

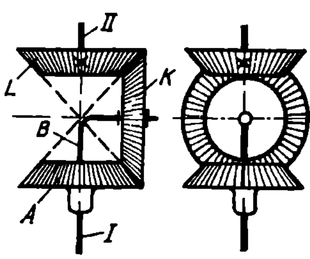


Fig. 230

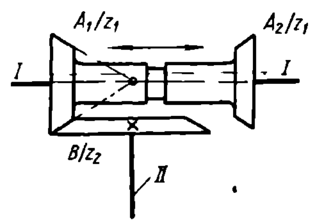


Fig. 231

Illustrative Problem 89. Fig. 231 represents bevel gears which allow the direction of rotation of the driven shaft to be changed. A_1 and A_2, whose apexes face each other, form a double bevel gear capable of sliding along a key on driving shaft I. Gear B is part of the driven shaft II which is perpendicular to the driving shaft. In the position of the double gear shown in the drawing, gears A_1 and B are in mesh. But if the double gear is moved to the extreme left, gears A_1 and B will be disengaged and

A. will be brought into mesh with gear B. It is evident that shaft II will then rotate in the opposite direction, although the driving gear will continue to rotate in the same direction. The speed ratio in both cases will be the same, i.e., $\frac{z_1}{z_2}$. When the double gear is in the central position, shaft II will not rotate.

Illustrative Problem 90. Fig. 232 illustrates a bevel-gear drive intended to impart two angular velocities of different magnitude and direction to the driven shaft II from the uniformly rotating driving shaft I. Gears A_1 and A_2 possess different numbers of teeth z_1 and z_3 (thus differing from Ex. 89), and there are two gears on the driven shaft B_1 and B_2 with z_2 and z_4 teeth. In the position shown in the drawing the speed ratio $i_{2,1} = \frac{z_1}{z_2}$, but when A_1 and A_2 are at the extreme left it will become $i_{2,1} = \frac{z_3}{z_4}$ and rotation will be in the opposite direction. When they are in the central position, shaft II will not rotate.

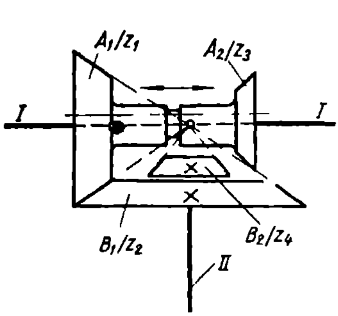


Fig. 232

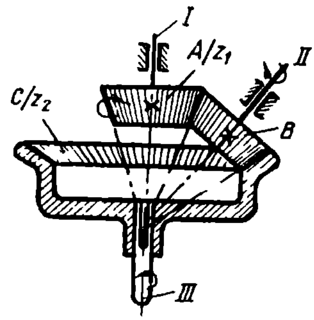


Fig. 233

Illustrative Problem 91. Fig. 233 shows a mechanism with an idler gear. Driver shaft I transmits rotation to shaft III by means of gears A and B. B meshes internally with gear C, which is part of shaft III. Shafts I and III are coaxial (they rotate about one axis).

It is seen that gear B is an idler. Hence the speed ratio $i_{3,1} =$ z_2 and rotation of shaft III is opposite to that of the driving shaft I.

174. The Screw

Let us cut out of paper a right triangle ABC (Fig. 234); the leg AB will be equal to the circumference of the cylinder shown in plan and elevation in Fig. 234a. Let us wrap the triangle about the cylinder, whose diameter is denoted by d, in such a way that its apex A will coincide with some arbitrary point K on the cylinder's base, and leg AB will lie along the base. Since AB is equal to the circumference of the cylinder, point B will coincide with point K and the initial point A, and the hypotenuse AC will rise around the side of the cylinder in a three-dimensional curve called a *helix*. Angle BAC is formed by the tangent to the helix and the plane of the cross-section of the cylinder and is known as the *lead angle* α *of the helix*. Leg BC is perpendicular to the base of the cylinder and occupies

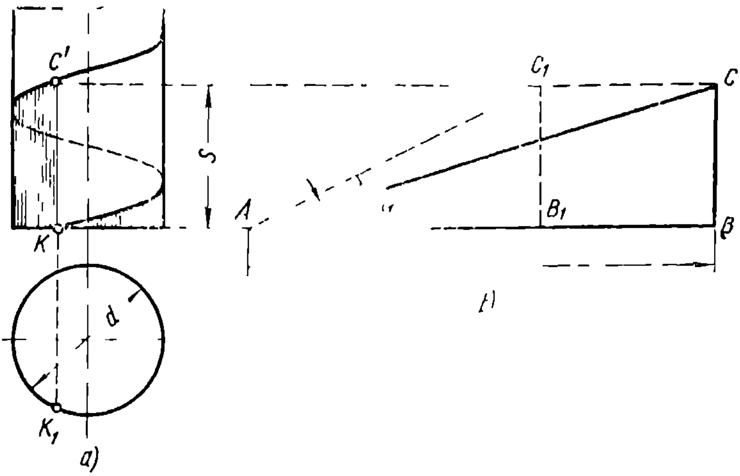


Fig. 234

position KC'. We thus see that the distance between two turns of the helix, measured along a line perpendicular to the base of the cylinder, is a constant quantity called the *lead of the helix* and is designated by s.

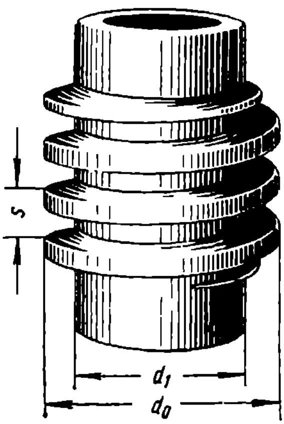


Fig. 235

From triangle ABC we obtain the relationship

$$s = \pi d \tan \alpha, \tag{158}$$

that is, *the lead of the helix is equal to the circumference of the cylinder multiplied by the tangent of the lead angle.*

It is evident from triangles ABC and AB_1C_1 in Fig. 234*b* that if the lead remains the same, the smaller the diameter of the helix the greater will be the lead angle α_1.

If we cut a groove of definite profile along the line of the helix, we shall obtain a *threaded screw*. The groove, or thread, may be triangular, rectangular, or square in profile, known correspondingly as V-thread, flat thread, and square thread. A screw has external and internal diameters d_0 and d_1 respectively (Fig. 235). It is apparent from what has been said above, that the lead angle of a screw with a given lead s will differ in its internal and external cylinders, for which reason it is classified on the basis of its average diameter, denoted by d.

Assume that after having delineated one helix $KA_1A_2A_3A_4$, we delineate another $LB_1B_2B_3B_4$, with the same lead angle (Fig. 236). If the second helix is started at point L_1 exactly opposite the starting point K_1 of the first helix, it will occupy

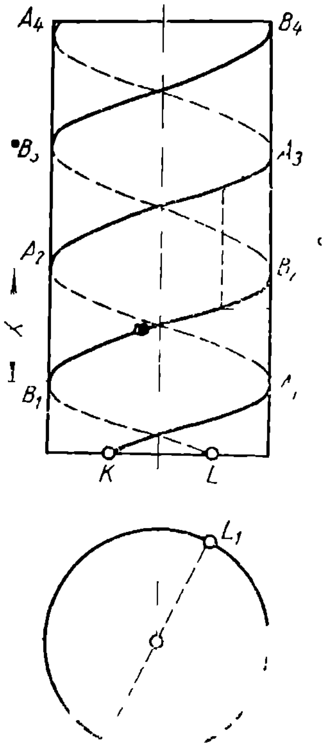


Fig. 236

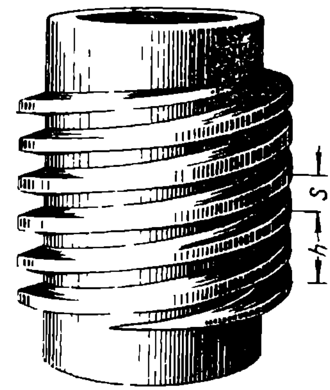


Fig. 237

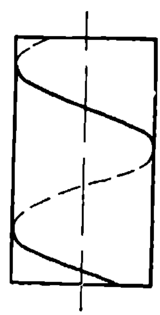


Fig. 238

a position between the turns of the first helix and cut its lead in half. A screw threaded in this manner is said to have a *double thread* (Fig. 237). *Triple-threaded* screws are made in the same way: between the turns of the first thread, two more threads are cut at equal distances from each other and from the first thread, their angular distances from each other being $\frac{360}{3} = 120°$. In a *quadruple-threaded* screw the angular distance between threads would be $\frac{360°}{4} = 90°$, and so forth with additional threads.

In a multiple screw, the pitch is the distance s between corresponding points on two adjacent threads and the distance between corresponding points on one and the same thread will be the *lead*. Hence by denoting the lead as h and the number of threads as z, then

$$h = sz. \tag{159}$$

Accordingly, for a multiple screw, Eq. (158) becomes

$$h = \pi d \tan\alpha \tag{160}$$

while the pitch becomes

$$\pi d \tan \alpha \tag{161}$$

In all the above cases the thread of the screw rises from left to right. Such a screw is said to have a *right-hand* thread. If the thread rises from right to left (Fig. 238), the screw is said to have a *left-hand* thread.

Oral Exercises

1. The lead angle of the thread on two cylinders of different diameters is the same. What can be said of the lead?
2. The threads on two cylinders have the same lead angle but a different lead. What can be said of the diameters of the cylinders?

175. Helical-Gear and Worm-Gear Transmission

We shall now pass on to the study of gear transmission between shafts whose axes intersect at a distance and for which purpose *helical gears* are used (Fig. 239). A helical gear may be regarded as a multiple-threaded screw with involute teeth, the number of threads of which is equal to the number of teeth (Fig. 240). Helical gears are mostly used between shafts which cross at a distance and form an angle of 90°.

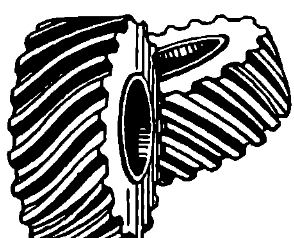


Fig. 239

Reasoning as in the case of bevel gears, we come to the conclusion that while the driving gear makes one revolution, the driven gear executes a turn of in which z_1 is the number of teeth on the driving gear and z_2 the number of teeth on the driven gear. Therefore the speed ratio for helical gear is

$$i_{21} = \frac{z_1}{z_2}.$$

In helical gears one must understand the difference between normal and circumferential pitch. Let AB and CF (Fig. 241) represent the pitch elements of two adjacent teeth on a gear, The distance between them $t_n = BD$ and is measured perpendicular to their length; this is called the *normal pitch*. The distance $t_s = BF$ and is measured along the pitch circle; this is called the *circumferential pitch*. By denoting, as we did with the ordinary screw, the lead angle of the thread as α, we obtain from triangle BDF the relationship between these two pitches:

$$t_n = t_s \sin\alpha. \qquad (162)$$

A variant of the helical gear is the *worm gear*. The worm A (Fig. 242) is part of the driving shaft and transmits rotation

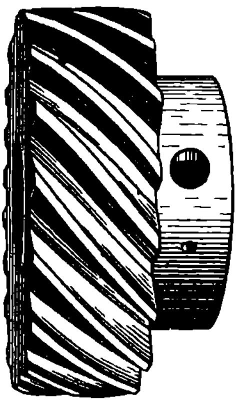


Fig. 240

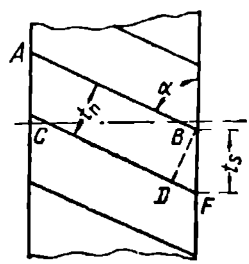


Fig. 241

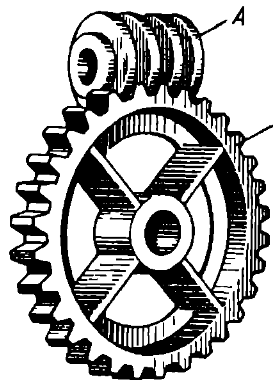


Fig. 242

to the worm gear B, which is part of the driven shaft. It is clear from the illustration that the worm is a cylinder with a screw thread cut into it, which fits into the tooth spaces of the mating worm gear. The worm may be single- or multiple-threaded and either left-hand or right-hand. It is obvious that the pitch of the worm and the worm gear are the same.

Let us denote the number of threads on the worm as z_w and the number of teeth on the gear as z_g. If $z_w = 1$, which means that the worm is single-threaded, in one revolution it will turn one tooth of the mating gear, that is, the gear will turn $\frac{1}{z_g}$ of one revolution and the speed ratio

$$i_{gw} = \frac{1}{z_g}.$$

If the worm is multiple-threaded, it will turn z_w teeth of the mating gear when it executes one revolution, i. e., the gear will turn $\frac{z_w}{z_g}$ of one revolution; hence, the speed ratio

$$i_{gw} = \frac{n_g}{n_w} = \frac{z_w}{z_g}, \qquad (163)$$

that is, *the ratio of rpm of the worm gear to rpm of the worm is equal to the ratio of the number of threads on the worm to the number of teeth on the worm gear.*

It is thus clear that the speed ratio of worm-gear mechanisms is expressed similarly to the speed ratio of spur gears, the only difference being that the number of teeth on the driving gear is replaced by the number of threads on the worm. The special feature of the worm-gear drive is its possibility of obtaining very small speed ratios.

The direction of rotation of the gear depends on the direction of rotation of the worm and direction of the thread, i.e., whether it is right- or left-hand.

It is not always possible to transmit motion from a worm gear to a worm; it depends upon the lead angle of the thread on the worm and the coefficient of friction between this thread and the teeth of the gear. The greater the coefficient of friction, the greater the lead angle must be.

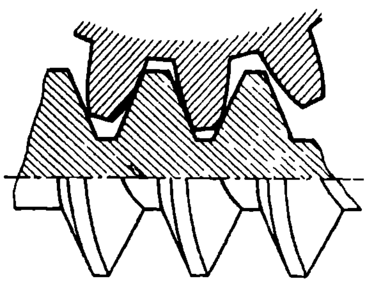


Fig. 243

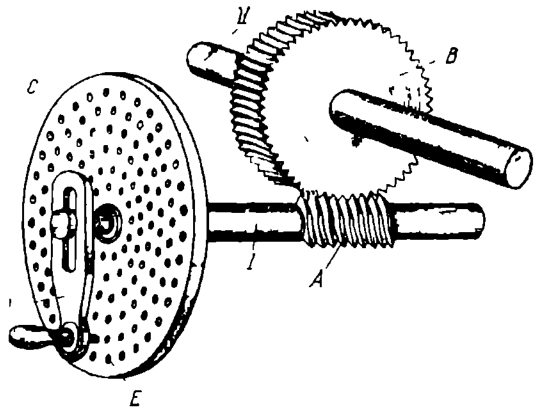


Fig. 244

Fig. 243 illustrates the meshing of the worm and worm gear, where it can be seen that the thread of the worm in cross-section possesses the form of an equilateral trapezoid.

Illustrative Problem 92. Fig. 244 shows schematically an ordinary index head of a horizontal milling machine. The worm *A*, which is part of shaft *I*, meshes with the worm gear *B* mounted on spindle *II* with which the workpiece is connected. Shaft *I*, to the front end of which is fixed the handle *D*, passes freely through the rigidly fixed index disc *C*. On disc *C* there are perforations arranged at equal distances in concentric circles. The handle *D* can be set on the shaft *I* so that its dowel *E* aligns with any one of the perforated concentric circles. Assume it necessary that a workpiece executes $\frac{1}{z}$ of a turn. By setting the dowel to align with the circle with q holes and by turning the handle along that circle for a distance of p holes, we transmit $\frac{p}{q}$ turns to shaft *I* carrying the worm. If the number of threads on the worm is z_w and the number of teeth on the gear is z_g, we obtain

$$\frac{p}{q} \times \frac{z_w}{z_g} = \frac{1}{z},$$

$$\text{from which } \frac{p}{q} = \frac{z_g}{z_w z}.$$

The worm in index heads is made single-threaded as a rule, and the number of teeth on the worm gear is usually 40, i.e., $z_w = 1$ and $z_g = 40$. Accordingly, $\frac{p}{q} = \frac{40}{z}$.

Assume it is necessary to mill a gear with 28 teeth. By giving z its numerical value of 28 in this equation, we obtain

$$\frac{p}{q} = \frac{40}{28} = \frac{10}{7} = 1\frac{3}{7}.$$

Accordingly, since we must give the handle $1\frac{3}{7}$ turns, we choose a perforated circle on the disc corresponding to the number of holes divisible by 7, for example, 49. We set the handle with the dowel E to align with that circle and subsequently give the workpiece $1\frac{3}{7}$ $1\frac{21}{49}$ turns each time, i.e., we give it one full turn plus 21 divisions in addition.

176. The Universal Joint

The *universal joint* is another mechanism that serves to transmit rotation between non parallel shafts. Fig 245 represents one such mechanism schematically: the ends of shafts I and II rotate in bearings M and N (Fig. 245*a*). Shackles C and A are fixed to the ends of the shafts in such a way that the axes III and IV passing through the shackle holes are perpendicular to the corresponding shafts. The fitting of the ends of a right-angle spider into these holes completes the universal joint.

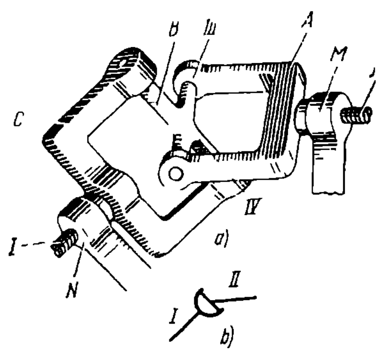


Fig. 245

When shaft I carrying shackle C rotates, the shackle also rotates while its ends turn about axes III and IV and transmit motion to shackle A which is part of shaft II. The driven shaft makes one turn to each turn of the driving shaft. Fig. 245*b* illustrates the symbol used to represent this mechanism is kinematic diagrams.

However, the angular velocity of the driven shaft is not constant, because *while the driver rotates at uniform speed, the follower rotates at a variable speed.*

Sometimes a *double universal joint* (Fig. 246) is used to transmit motion between non-parallel, non-intersecting shafts (in automobiles, machine tools, etc.). In the double universal joint the two shafts *I* and *II* are connected by an intermediate shaft *III* by means of two joints $A_1B_1C_1$ and $A_2B_2C_2$. The axes of the shackles A_1 and A_2 attached to the ends of the intermediate shaft must both be in one plane, while the axes of shafts *I* and *II* must be parallel to each other or be in a symmetrical position with respect to axis O_1O_2 connecting the centres of the joints.

It is frequently necessary to transmit rotation to a driven shaft whose position is not permanent. Fig. 247 is a diagram of a mechanism used in such cases. Assume that the driven shaft *II* changes its position in relation to the driving shaft *I* when the machine is in operation, thus causing the distance between the two joints to vary. To provide for this situation, link *III* must be able to vary in length: spindle *D*, which carries on one of its ends the shackle of universal joint *B*, is made to slide in an axial direction into the cylinder *C* which is part of the shakle of the second universal joint *A*. There is a keyway in spindle *D* in which a key, fastened to the wall of the cylinder, slides freely. With this construction, shaft *II* can change its position while receiving rotation through the variable-length link *III*, which is known as a *telescopic joint*. This type of mechanism with its two universal joints and telescopic joint, is used in certain kinds of machine tools.

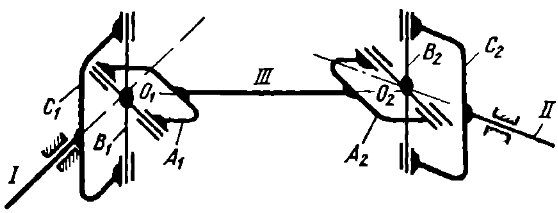


Fig. 246

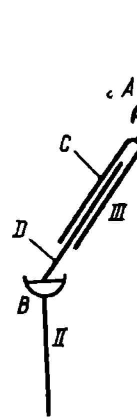


Fig. 247

177. Questions for Review

1. Are the diameters of the friction frusta, represented in Fig. 222, the same at points *B*, *M*, and *b*?
2. Which of the mechanisms shown in Figs 223 and 224 makes it possible to change the direction of rotation of the driven shaft while maintaining a constant direction of rotation in the driving shaft?
3. Is the pitch of a bevel gear the same, no matter at what point along the pitch elements it is measured?
4. Which is larger in a helical gear, the circumferential or the normal pitch?

5. What change occurs in the normal pitch of a helical gear if the lead angle is increased while the circumferential pitch remains the same? What will the normal pitch be when the lead angle $\alpha = 90°$?

6. In one worm-gear transmission the worm is single-threaded, in another it is double-threaded. If the number of turns on the worms and the number of teeth of the worm gears are the same, which of the driven shafts will rotate faster, and how much faster?

178. Exercises

95. The diameter of the apex of the friction cone, shown in Fig. 223, $D_1 = 280$ mm, the diameter of its base $D_2 = 400$ mm,

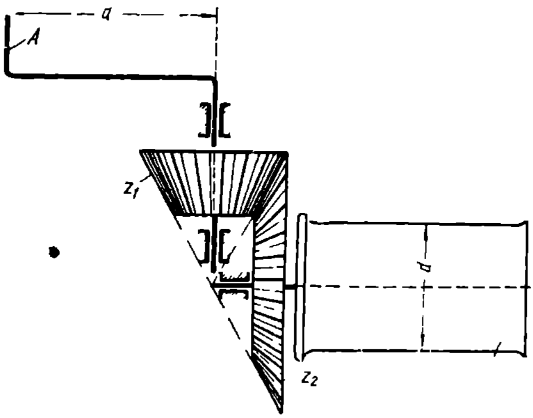


Fig. 248

and of the roller $D_0 = 300$ mm. The rpm of the driver shaft I is $n_1 = 350$. What are the maximum and minimum rpm that can be attained on the driven shaft?

96. In the friction transmission shown in Fig. 221, the greatest possible distance R obtainable between the roller B and the centre of disc A is 250 mm; the diameter of the roller is 125 mm. If the rpm of shaft I is $n_1 = 800$, what is the maximum rpm that can be obtained on shaft II of the drive?

97. Crank A of the windlass in Fig. 248 turns with a peripheral velocity $v_0 = 0.785$ m/sec. Calculate the speed with which it can move a load on the

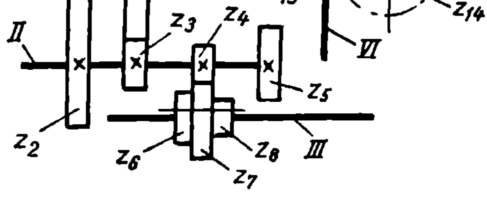


Fig. 249

cable that winds about its drum if $a = 250$ mm, $z_1 = 15$, $z_2 = 45$, and $d = 180$ mm.

98. A single-threaded worm executes 900 rpm and its mating worm gear possesses 45 teeth. Find the rpm of the gear.

99. Solve Ex. 98 for a triple-threaded worm.

100. From the driving shaft *I* in Fig. 249 rotation is transmitted as follows: *a)* to shaft *III* through shaft *II* according to the schemes $z_1 \times z_2 - z_3 \times z_6$, or $z_4 \times z_7$, or $z_5 \times z_8$; *b)* to shaft *VII* through shafts *II*, *IV*, *V*, and *VI* according to the scheme $z_1 \times z_2 - z_3 \times z_9 \times z_{10} - z_{11} \times z_{12}$ - worm with threads $z_{13} \times z_{14}$. Shaft *I* executes n_1 rpm. Find the rpm of shafts *III* and *VII*.

CHAPTER XVIII

CONVERSION OF ROTATION INTO LINEAR TRANSLATION AND VICE VERSA

179. Conversion of Rotation into Linear Translation

Motion in engineering is not limited to rotation. In machine tools the basic motion is rotation, but it is also converted into other kinds of required motion. For instance, the rotation of the driving shaft of a thread-cutting lathe is converted into motion of translation for its carriage by means of a train of gears and racks (for longitudinal machining) or with the aid of a screw and nut (for cutting threads). The rotation of a sheave ultimately becomes linear translation for the table of a planing machine, for the cutter of a shaper, etc. The conversion of linear translation into rotation is exemplified in piston engines, but on the whole is less frequently applied.

There are even more complex forms of motion often met with in machines, but in this chapter we shall study the chief ways of converting rotation into linear translation, and vice versa.

180. Friction Mechanisms for Obtaining Linear Translation

A friction mechanism employed to obtain linear translation is, for example, one that transmits motion to the head of a friction stamping hammer (Fig. 250): the head *B* of the hammer is suspended from a board *A* of hard wood (usually beech or hornbeam) which is held pressed between rotating rollers and guided by slides. If the force of friction between the rollers and board is greater than the weight of the hammer and board, the board will rise when the rollers revolve in the direction shown in the drawing. The speed **v** of the board (if there is no slipping) will be equal to the peripheral speed of the rollers and is therefore

$$v = \frac{\pi D n}{60 \times 1{,}000} \qquad (164)$$

in which D is the diameter of the roller in mm, n is its rpm, and v is the speed of the hammer in m/sec.

The downward movement of the hammer occurs under the action of its own weight. As it falls the rollers are moved apart by a mechanism not shown in the drawing.

The motion of translation of the ingot held between the rollers of a rolling mill, or of logs in a sawmill, etc., is based on the same principle.

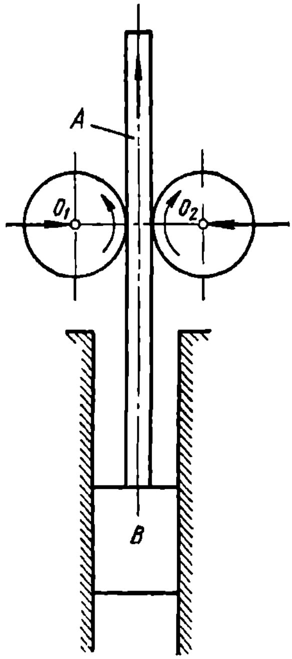


Fig. 250

Illustrative Problem 93. The weight of the dropping parts of a friction hammer which is raised by two rollers is $G = 450$ kg, the coefficient of friction between the rollers and the lifting board $f = 0.45$, the diameter of the roller $D = 350$ mm, the rpm of each roller $n = 135$, and the force raising the lifting board and the hammer must be double their combined weight. What pressure Q must be exerted by the rollers on the lifting board and at what speed will the board rise?

Solution: the friction F between the rollers and the lifting board is $2fQ$, wherein $Q = \frac{F}{2f}$. And since the force of friction must be double the weight of G,

$$Q = \frac{2G}{2f} = \frac{G}{f} = \frac{450}{0.45} = 900 \text{ kg}$$

The speed at which the load is raised

$$v = \frac{\pi 350 \times 135}{60 \times 1{,}000} \text{ m/sec} \approx 2.5 \text{ m/sec.}$$

181. The Rack-and-Pinion

In the transmission just previously presented, motion was imparted under the action of friction. Now let us assume that we have cut teeth into the surface of the aforementioned lifting board and its rollers. We would then have a toothed mechanism consisting of a spur gear A (Fig. 251) and rack B. This kind of transmission is used to impart motion to the table of planing machines, the spindle feed of a drilling machine, etc.

It is obvious that the speed of the rack is equal to the peripheral velocity of the pitch circle of the gear, for which reason the former Eq. (164) is applied, but in a slightly changed form than in the case of friction transmission. By bearing in mind that the pitch diameter $D = mz$, then the speed of the rack

$$v = \frac{\pi D n}{60} = \frac{\pi m z n}{60} \text{ mm/sec} = \frac{\pi m z n}{60 \times 1{,}000} \text{ m/sec}, \qquad (165)$$

in which

m — module of engagement;
z — the number of teeth on the pinion;
n — rpm of the pinion.

The force **P** which transmits motion of translation to the rack is easily expressed. If we denote the torque on shaft O of the pinion as M_t, then the effective pull on the pitch circle will be $\frac{M_t}{R}$, in which R is the radius of the pitch circle, hence

$$P = \frac{2M_t}{D}. \qquad (166)$$

We have been assuming that the pinion is transmitting motion to the rack. The opposite is also possible when the rack, possessing motion of linear translation, transmits rotation to the pinion. And obviously the relationship just obtained likewise holds true here: knowing the speed of the rack we can calculate the rpm of the pinion by Eq. (165) and the effective pull on the pinion, according to the force applied to the rack, by Eq. (166). In both presented cases the pinion rotates about a fixed axis O.

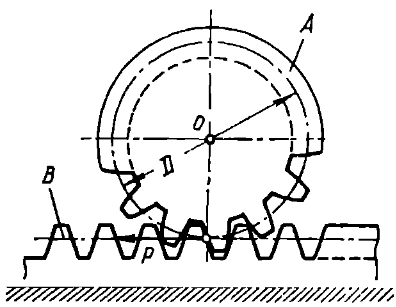


Fig. 251

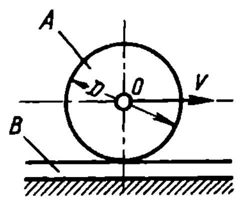


Fig. 252

Now assume that the pinion A in Fig. 252 is rolling on an immovable rack B. Such a transmission is similar to the rolling of a wheel on an immovable surface. It is easy to see that in one revolution the pinion's axis will move a distance $l = \pi D$, which is equal to the length of its pitch circle, while in n revolutions (n may be a whole number or a fraction) it will attain a distance $l = \pi D n$. An example of a transmission of this kind is found in the automatic longitudinal feed of a lathe where, geared to an immovable rack rigidly fastened to the frame of the machine, is a pinion which is part of the shaft in the apron of the carriage. As the pinion receives rotation from the feed mechanism, it rolls on the rack and thereby transmits motion of translation to the carriage.

Finally there is the *rack-and-worm* transmission in which the worm is the driving link instead of the pinion.

Illustrative Problem 94. Fig. 253 is a kinematic diagram of a rack-type jack. When crank A is rotated, shaft O_1 carrying rotate. Gear z_1 transmits motion to rack B according to the scheme $z_1 \times z_2 - z_3 \times z_4 - z_5 \times$ rack B. Find the time t required to raise a load vertically to a height $h = 220$ mm, if the peripheral speed of the crank $v_0 = 0.8$ m/sec, the length O_1C of the handle $a = 250$ mm, the number of teeth on the gears $z_1 = 5$, $z_2 = 20$, $z_3 = 5$, $z_4 = 20$, $z_5 = 5$, and the module of the mating rack and gear $m = 14$ mm. Also determine the lifting capacity $\mathbf{Q}$ of the jack if the force exerted on the crank $P = 35$ kg, and the efficiency of the jack $\eta = 0.75$.

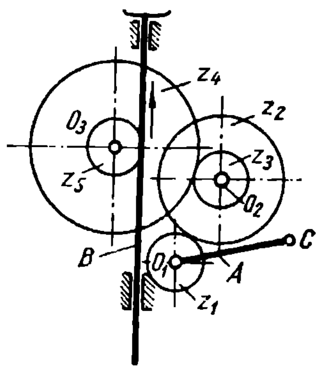


Fig. 253

Solution: with a peripheral speed $v_0 = 0.8$ m/sec, the crank executes $n_1 = \frac{30v_0}{\pi a} = 30.5$ rpm. The speed ratio between shaft O_1 and shaft O_3 via shaft O_2 is $i_{3,1} = \frac{5 \times 5}{20 \times 20} = \frac{1}{16}$. Hence, as the crank C attains n_1 rpm, gear z_5 attains $n_3 = n_1 i_{3,1} = 30.5 \times \frac{1}{16}$ rpm. Correspondingly, the vertical displacement of the rack per minute $h_1 = \pi m z_5 n_3 = \frac{\pi \times 14 \times 5 \times 30.5}{16} = 420$ mm, and the time needed to raise a load to a height $h = 220$ mm is

$$t = \frac{220}{420} = 0.525 \text{ min.} = 31.5 \text{ sec.}$$

To determine the lifting capacity $\mathbf{Q}$, we use Eq. (136) from which we find the torque on shaft O_3:

$$M_3 = \frac{M_1}{i_{3,1}}\eta = \frac{35 \times 250}{\frac{1}{16}} \times 0.75 = 35 \times 250 \times 16 \times 0.75 \text{ kg-mm.}$$

The diameter of gear z_5 is $D_5 = mz_5 = 14 \times 5$ mm and, according to Eq. (166),

$$Q = \frac{2M_l}{D} = \frac{2 \times 35 \times 250 \times 16 \times 0.75}{14 \times 5} = 3{,}000 \text{ kg} = 3 \text{ tons.}$$

182. Kinematics of the Screw-and-Nut Drive

The transformation of rotation into linear translation is widely achieved through a mechanism consisting of a screw and nut. Fig. 254 is a diagram of such an arrangement: a single-threaded screw *2* rotates in fixed bearings; the screw carries a nut *1* which slides in guides but cannot turn. When the screw turns once, the nut is displaced for a distance equal to the pitch s of the screw; when the screw revolves half a turn, the nut is displaced for a distance of $0.5s$; and at a quarter of a turn the displacement is for a distance of $0.25s$, etc. From this we may say that when the screw executes $\frac{p}{q}$ turns, the nut moves for a distance

$$S = \frac{p}{q} s. \tag{167}$$

There is a mechanism of this kind, for example, in a thread-cutting lathe, where the rotational motion of the lead screw is transformed into linear translation of a nut connected with the apron.

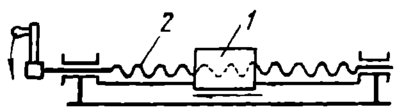


Fig. 254

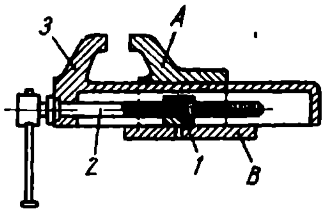


Fig. 255

The principle of the screw-and-nut drive is used in other devices for the transformation of rotation into linear translation, an instance being the parallel vise illustrated in Fig. 255: screw *2* turns within nut *1*, which is immovably fixed to the base of the vise *B*. Furthermore, screw *2* turns freely in the movable part of the jaw *3* and transmits linear translation to it, thus pulling it so as to pinch the workpiece between the immovable and movable jaws *A* and *3*, respectively.

In the above illustrations the screw is the driving link. However, the opposite is also possible, where the nut acts as the driver. Fig. 256 is a diagram of a screw jack which works on this principle: the nut *3* can turn freely in base *1* but cannot move axially. Screw *2*, passing through the nut, can move axially, but cannot turn. Accordingly, by turning the nut we impart linear translation to the screw.

Fig. 256.

It is quite obvious that in all these cases we may apply Eq. (167), in which $\frac{p}{q}$ may denote either the turning of the screw in the nut, or the turning of the nut on the screw. The direction in which the screw (or the nut, as the case may be) moves, depends evidently on whether the thread is right-hand or left-hand, and in which direction the screw or the nut is being turned.

The screw-and-nut drive may also be used to convert linear translation into rotation. For instance, by moving a nut in an axial direction, we can impart rotation to a screw if the lead angle is sufficiently great. The hand-drill shown in Fig. 257 operates on this principle: the screw *2* rotates together with the chuck *1* when the nut *3* is moved along its axis.

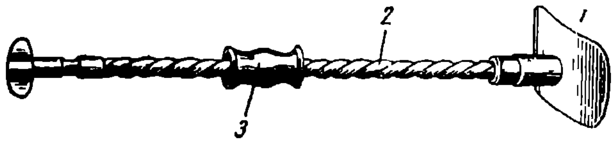


Fig. 257

Incidentally, a screw-and-nut drive of this kind will not work if the lead angle of the thread is small. A screw mechanism in which the screw cannot rotate under pressure of the nut, is called a *self-locking* mechanism.

Fig. 258 is a diagram of what is called a *differential screw.* Screw *1* has a pitch of s_a along part *a*, and a pitch of s_b along part *b*. Part *a* of the screw rotates within the immovable nut *2*, and part *b* rotates within nut *3* which cannot turn but can move in an axial direction. Assume that the direction of the thread on parts *a* and *b* is the same. By giving the screw one turn, we displace it axially within nut *2* for a distance equal to the pitch s_a. If nut *3* had turned with the screw it would also have moved in an axial direction for a distance of s_b. However, since the nut cannot turn, it moves along the screw in the opposite direction for a distance equal to the pitch s_b. Consequently the absolute displacement of the nut with respect to the immovable guides is $s_a - s_b$. If the threads *a* and *b* were dissimilar, the displacement of nut *3* would be $s_a + s_b$.

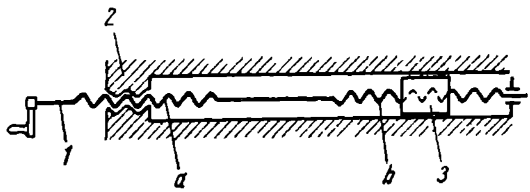


Fig. 258

From this it follows that if the screw rotates for $\frac{p}{q}$ turns the nut will be displaced for a distance

$$s = \frac{p}{q}(s_a \pm s_b). \tag{168}$$

The minus sign is used when threads *a* and *b* have the same direction, and the plus sign when they have opposite directions.

It is readily understood that when both threads have the same direction, the displacement of the nut will be small because forteach revolution it will only amount to the difference between the pitches.

All these simple screw mechanisms can be used in a great variety of combinations. Take Fig. 259 for example: screw 2 is prevented from moving in an axial direction by bearing 1; on parts a and b of the screw the threads have the same pitch but are opposite in direction. Nuts 3′ and 3″ cannot rotate and when the screw is turned they will either move closer or further apart, in either case with equal speed. This kind of mechanism is used in a double-jawed drill chuck.

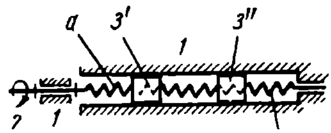


Fig. 259

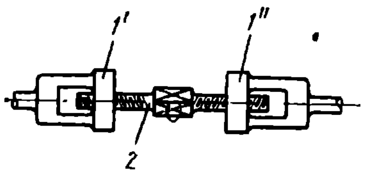


Fig. 260

The *turnbuckle* shown in Fig. 260 works on the same principle. When the screw 2, which has a right-hand thread at one end and a left-hand thread on the other, is turned, the stirrups 1′ and 1″ and the rods (or ropes) connected with them will be pulled together. Fig. 261 also shows a turnbuckle, but with another arrangement of parts.

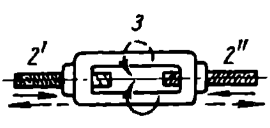


Fig. 261

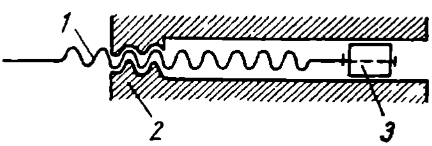


Fig. 262

Illustrative Problem 95. The screw 1 in Fig. 262 has a right-hand thread with a pitch $s = 2.5$ mm. One of its ends is within the immovable nut 2 and the other rotates freely in the block 3 which slides in fixed guides. How many times, and in what direction, must the screw be turned to displace the slideblock for a distance $S = 81$ mm from left to right?

Solution: $\frac{p}{q} = \frac{S}{s} = \frac{81}{2.5}$ turns $= 32.4$ turns $= 32$ full turns plus 144°, all clockwise.

Illustrative Problem 96. The screw mechanism shown in Fig. 258 has a right-hand thread with a pitch $s_a = 4$ mm on its length a, while on length b it has a thread of the same direction but with a pitch $s_b = 3.5$ mm. If the screw is turned 45° clockwise, how far will the slide block 3 be displaced, and in what direction?

Solution: by applying Eq. (168) (with the minus sign because the threads are in the same direction) we obtain

$$S = \frac{45}{360}(4 - 3.5) = 0.0625 \text{ mm}.$$

Since the screw is turned clockwise, the slider moves from left to right.

183. Statics of the Screw-and-Nut Drive

Assume that a force **P** is applied at point A of the lever 3 fixed to screw 1 and having an arm a (Fig. 263). Under the action of this force the screw will turn in the rigidly fixed nut 2, moving upwards and overcoming useful resistance **Q**. Express the relationship between forces **P** and **Q**.

As we have already pointed out several times, the work of the motive force must be equal to the total work done by the forces of resistance. For the time being we shall assume that harmful resistance is negligible and can therefore be ignored.

Now let us equate the work of the motive force **P** and the force of useful resistance, during one turn of the screw.

When the screw is turned once, the point of application A of force **P** describes a trajectory equal to $2\pi a$. Hence the work performed by force **P**

$$W_P = 2\pi Pa.$$

During one turn, the screw moves axially for a distance equal to its pitch. Accordingly, the work performed by the force of resistance **Q** is $W_Q = Qs$. By equating the amount of work we get

$$2\pi Pa = Qs,$$

whence

$$Q = 2\pi \frac{a}{s} P. \tag{169}$$

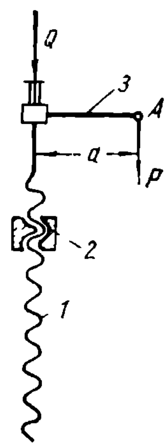


Fig. 263

From this we conclude that *the longer the arm of application of the motive force and the smaller the pitch of the screw, the greater the mechanical advantage.*

In order to actually express the obtained force **Q**, we must multiply the right-hand part of this equation by efficiency η:

$$Q = 2\pi \frac{a}{s} P\eta. \tag{170}$$

Eq. (169) may be presented differently. By expressing the pitch of the screw in terms of its average diameter d_{av} according to Eq. (158) we obtain

$$s = \pi d_{av} \tan \alpha.$$

By substituting this value for s in Eq. (169) we obtain

$$Q = \frac{2a}{d_{av} \tan \alpha} P, \quad (171)$$

that is, *the longer the arm of application of the motive force and the smaller the average diameter of the screw and the tangent of the lead angle, the greater the mechanical advantage.*

Accordingly, when taking the force of friction into account, Eq. (171) becomes

$$Q = \frac{2a}{d_{av} \tan \alpha} P\eta. \quad (172)$$

Finally, the relationship we seek can be obtained in another form: since Pa is the moment of force of **P** relative to the axis of the screw (the torque M_t), we may, thereby, write

$$Q = \frac{2M_t}{d_{av} \tan \alpha} \eta, \quad (173)$$

that is, *the magnitude of the force acting on the screw in an axial direction is equal to twice the torque multiplied by the efficiency coefficient and divided by the average diameter of the screw thread and the tangent of the lead angle corresponding to this average diameter.*

Since the lead angle of the screw may be made sufficiently small, a great mechanical advantage can be obtained with a screw transmission. With the aid of the screw and nut, we can make very strong fastenings with comparatively small physical effort, can hold workpieces in a vise, and apply the same principle to jacks, screw presses, etc.

The efficiency coefficient is calculated for each individual case, depending on the lead angle of the screw and the coefficient of friction.

Illustrative Problem 97. In the screw jack represented in Fig. 263 the arm $a = 800$ mm, efficiency $\eta = 0.4$, and the pitch of its screw $s = 8$ mm. What force **P** must be expended in order to raise a load $Q = 3$ tons at a constant speed*?

Solution: from Eq. (170) we obtain

$$P = \frac{Qs}{2\pi a \eta} = \frac{3{,}000 \times 8}{2\pi 800 \times 0.4} \approx 12 \text{ kg}$$

184. Thread Profiles of Principal Types of Transmission Screws

If we cut a screw across a longitudinal plane coinciding with its axis, the section thus obtained will be through the turns of its thread. A thread receives its name in accordance with the

* Screw jacks must be self-locking, which means that the screw must not turn under the action of an axial load. For this reason the efficiency of a screw jack is always less than 0.5.

profile thus revealed in section. There are various types of profiles, corresponding to intended use.

If the screw is to transmit motion, it is obvious that it must possess the greatest efficiency possible and the least mechanical loss. If the screw and nut are to be used for the fastening of all kinds of parts, they must be constructed so as to create the greatest possible amount of friction between their contact surfaces to keep the nut from unscrewing.

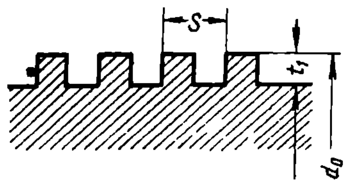


Fig. 264

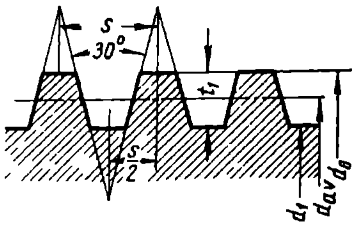


Fig. 265

In more detailed courses of engineering mechanics it is proved that, other geometrical elements being equal, the loss due to friction is the least when the thread is *rectangular* (Fig. 264) and the depth of the thread t_1 is equal to half the pitch, i. e., when $t_1 = \frac{s}{2}$. Such a rectangular thread is called a *square thread.*

The square thread has certain disadvantages, the greatest being the difficulty of achieving precision in its manufacture, for which reason it is being displaced by the Acme thread shown in Fig. 265. In cross-section this thread is an *equilateral trapezoid* with the inclination of its sides forming an angle of 30° with each other. The technical terms of other elements of threads shown in Figs. 264 and 265 are described in Sec 200 and illustrated in Fig. 293.

185. Slider-Crank Mechanism

The *slider-crank mechanism* shown schematically in Fig. 266 is another means of transforming rotation into linear translation. The crank *2* which is part of shaft *A* turning in fixed bearings in the frame *1*, is jointed to the connecting rod *3* by the crankpin *B*. The other end of the connecting rod is jointed by means of a wrist pin *C* to the slider *4* which moves in straight fixed guides. Thus we see that when the crank is continuously rotating, the slider will achieve *reciprocal motion of translation* and reverse its direction at the end of each stroke. Accordingly, during one revolution of the crank the slider will execute two strokes, first in one direction and then in the other—a feature of this mechanism which chiefly distinguishes it from other mechanisms presented in this chapter.

The slider-crank mechanism is also employed for converting reciprocal linear translation into rotation, as for instance in steam engines and internal combustion engines, where the driving link is the piston which, with the aid of a connecting rod, causes the crankshaft to rotate. In this arrangement another specific factor must be coped with: when the slider *4* moves from left to right, the crank will rotate clockwise and when the slider has travelled as far as it can go, the crank will occupy position AB_0' before the slider begins travelling in the opposite direction. This position of the crank is called the *dead centre*. In order that the crank continue revolving past the dead centre when it is the driving link of a mechanism, a *flywheel* is used, which is a wheel with a heavy rim and mounted on the crankshaft. The kinetic energy of the flywheel keeps the mechanism in constant motion.

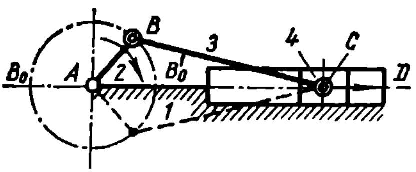


Fig. 266

186. Kinematics of the Slider-Crank Mechanism

Now let us study the motion of the slider when the crank is rotating uniformly.

Assume that the crank in Fig. 267*a* is rotating uniformly in a clockwise direction. We shall take AB_0 as the initial position of the crank. From B_0 we mark off with a compass a distance equal to the length of the connecting rod along the line on which the wrist-pin centre C moves, and obtain point C_0 which at the given moment coincides with the centre of the pin. This point is the extreme left position of the slider. To find the position of wrist-pin centre C at other moments of time, we divide the circle described by the centre of crankpin B into several equal parts, let us say 12. Then each part will represent an arc equal to $\frac{1}{12}$ of the circle through which the crank moves at equal intervals of time while executing one revolution (provided it rotates uniformly). In the course of its movement it will occupy, in turn, position AB_1, AB_2, AB_3, ..., AB_{11}, and finally return to AB_0 (its initial position). Now, with a radius equal to the length of the connecting rod BC, we will mark off points from B_1, B_2, B_3, etc., on the straight line along which the wrist-pin centre C moves. As a result, we find that after $\frac{1}{12}$ of a turn of the crank, point C is at C_1, having moved from its initial position for a distance $S_1 = C_0C_1$; after two such intervals, the displace-

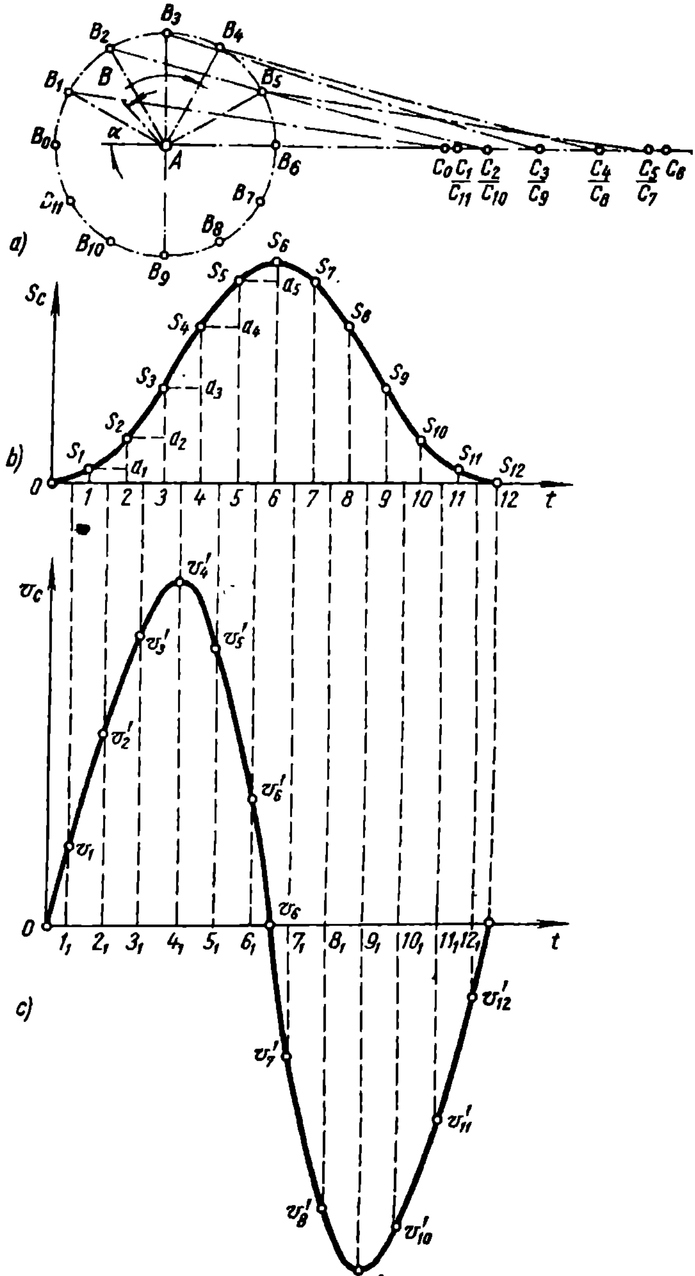


Fig. 267

ment of point C will be $S_2 = C_0C_2$; after three intervals — $S_3 = C_0C_3$, etc. When the crank reaches position AB_6, point C will be at position C_6 at a distance equal to the length of the connecting rod from point B_6 and corresponding to the extreme right position of the slider. Then the slider begins to move in the opposite direction (from right to left) and its distance from its initial position steadily diminishes. Thus, position AB_7 of the crank corresponds to the position of wrist-pin centre C at point C_7 and which coincides with point C_5; position AB_8 of the crank corresponds to the position of wrist-pin centre C at point C_8, etc. When the crank returns to its initial position, point C will be at C_0.

We thus see from this diagram that the linear segment C_0C_6 equals segment B_0B_6, i.e., the diameter of the circle described by the crankpin B, while the diameter of this circle is equal to twice the length of the crank. Therefore, by denoting the length of the crank as r and a stroke of the slider as H, we find that

$$H = 2r, \tag{174}$$

that is, *in a slider-crank mechanism the stroke of the slider is equal to twice the length of the crank.*

From what has been said it follows that in order to find the position of the slider at a given moment, we must mark off from the crankpin centre at that moment (using a radius equal to the length of the connecting rod) a point on the line described by the wrist-pin centre. And, vice versa, if the position of the slider is given, the position of crank can be found by marking off from the wrist-pin centre (using the same radius) a point on the corresponding semicircle described by the crankpin centre and connecting this point with the centre of this circle.

Having located the centre of wrist pin C, we can now plot a curve representing its distance from the initial position C_0, by the method explained in kinematics (Sec. 57). By adopting a right-angle system of coordinates as shown in Fig. 267*b*, we lay out equal segments according to a chosen scale along axis O_t, each segment representing the time during which the crank achieves $\frac{1}{12}$ of a turn. Then constructing perpendiculars at points *1*, *2*, *3*, etc., and laying out segments *1* — S_1, *2* — S_2, *3* — S_3, etc., representing the distances C_0C_1, C_0C_2, C_0C_3, etc., from the initial position, we obtain a line of points S_1, S_2, S_3, etc., which we connect with a curved line. In this way we obtain a displacement-time graph for the centre of wrist-pin C and can find its position for any given moment of time.

From this curve we see that displacement of the slider differs for equal intervals of time although the crank rotates uniformly. For instance, when the crank turns through the angle B_0AB_1, the slider moves a distance of *1* — S_1; when it turns through angle

B_1AB_2 the slider is displaced for a distance of $a_1 - S_2$; while through angle B_2AB_3 it is displaced a distance of $a_2 - S_3$, etc. Thus the displacements first increase and then decrease.

From this we come to the conclusion that *when the crank is the driver and rotates uniformly, the slider moves non-uniformly; and vice versa, when the slider is the driver, its motion is uniform and the crank's rotation is not uniform.* This is an important feature of the slider-crank mechanism.

Having solved the displacement-time graph, we can now plot the velocity curve which makes it possible to determine the velocity of the slider for any moment of time. As already explained, while the crank is moving through the angle B_0AB_1 (Fig. 267*a*), the slider is displaced for a distance of $1 - S_1$ (Fig. 267*b*). By dividing this displacement by its executed time, we obtain the average velocity of the slider during that interval; similarly, by dividing the displacement $a_1 - S_2$ by the same interval of time (for we have already divided one revolution of the crank into even parts), we obtain its average speed for that interval, etc. Thus we may calculate the average velocity of the slider during a 180° turn of the crank.

Now let us draw a right-angle system of coordinates at a suitable scale, and lay out the time along axis *Ot* and the average velocity of point *C* of the wrist-pin on axis *Ov* (Fig. 267*c*). We mark off these speeds on perpendiculars constructed on the time axis *Ot* at points 1_1, 2_1, 3_1, etc., and lying between the segments *0—1*, *1—2*, *2—3*, etc. (Fig. 267*b*). As a result we obtain points v_1, v_2', v_3', etc., (Fig. 267*c*) through which we draw a line $Ov_1v_2'v_3'v_4'v_5'v_6'v_6$ which constitutes the velocity-time curve of the slider during the first half-turn of the crank (the time consumed in turning from position AB_0 to AB_6).

From this moment the slider starts moving in the opposite direction, from right to left, and its velocity is directed in the opposite direction; therefore we construct a second leg of the curve, symmetrical with the first but below the time axis.

When analysing the velocity-time graph thus obtained, we see that when the crank is at the left dead centre AB_0, the velocity of the slider is zero (point *O* on Fig. 267*c*). As it rotates further, the velocity of the slider grows and reaches its maximum when the crank is between AB_3 and AB_4 (Fig. 267*a*). Then its velocity begins to decrease till it again becomes zero when the crank is at its right dead centre AB_6. Then, as the crank executes the second half of its turn, the curve is repeated in reverse order.

Oral Exercises

1. What is the sum of the segments (Fig. 267*b*), $1 - S_1$, $a_1 - S_2$, $a_2 - S_3$, $a_3 - S_4$, $a_4 - S_5$, $a_5 - S_6$?
2. Indicate these sums on Fig. 267*a*, on both the left and right sides of the diagram.

3. Indicate on Fig. 267*a* the position of the crank corresponding to point v_4 on the time axis in Fig. 267c.

Illustrative Problem 98. The length of the crank *AB* (Fig. 266) is 120 mm and the length of the connecting rod *BC* is 420 mm. The crank attains $n = 180$ rpm. Plot the displacement- and velocity-time curves for point *C* and find its velocity at the moment the crank forms an angle $\alpha = 50°$ with the left dead centre.

Solution: we draw a diagram of the mechanism similar to Fig. 267*a* at a scale of 1 : 8, then divide the circle described by point *B* into 12 equal parts. Setting our compass at a radius of $\frac{420}{8} = 52.5$ mm, we mark off points C_0, C_1, C_2, etc., and then delineate a displacement-time curve (Fig. 267*b*). At 180 rpm the segments *0 - 1*, *1 - 2*, *2 - 3*, etc., on the time axis (5 mm each) represent intervals of time equal to $\frac{60}{180 \times 12} =$

$= \frac{1}{36}$ sec, or a 30° turn of the crank.

By measuring the displacements $1 - S_1$, $a_1 - S_2$, $a_2 - S_3$, etc., and multiplying them by the scale of 8 and dividing by $\frac{1}{36}$ sec, we obtain the average velocity for each interval of time. Then establishing a scale of velocities of 50 mm/sec = 1 mm, we lay out points 1_1, 2_1, 3_1, etc., as the ordinates representing the velocity at this scale and then connect the points with a curve.

The position of crank *AB*, forming an angle $\alpha = 50°$ with AB_0 will correspond with a point on Fig. 267*b*, lying at a distance of $\frac{5 \times 20}{30} =$

$= 3\frac{1}{3}$ mm to the right of point *1*. By plotting a line from this point to its intersection with axis *Ot* on the velocity-time curve, we obtain an ordinate of 28.5 mm in length representing the sought velocity which, at the chosen scale, is $v_c = 28.5 \times 50 = 1{,}425$ mm/sec $= 1.425$ m/sec.

187. The Eccentric Mechanism

Assume that we increase the dimensions of crankpin *B* shown in Fig. 266 to the size illustrated in Fig. 268, and that now crankpin B_1 is part of the crank while its bushing B_2 is part of the connecting rod. It is evident that the mechanism is still a slider-crank in which the length of the crank *2* is equal to *AO*, the length of the connecting rod *3* is equal to *OC*, and in which *O* remains the centre of the main bearing. By still further increasing the diameter of the crankpin, we obtain a mechanism whose skeleton outline is shown in Fig. 269: a round disc B_1 turns freely within the bushing B_2, which latter is part of the connecting rod *3* rotating around axis *A*. This mechanism works similar to

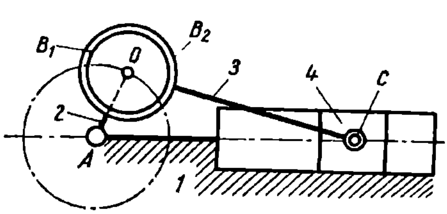


Fig. 268

a slider-crank which has a crank *2* whose length AO equals the distance between the axis of rotation A of disc B_1 and the geometrical axis of the disc, and a connecting rod *3* whose length OC equals the distance between this axis and the wrist-pin centre on slider *4*. This is called an *eccentric mechanism*, its disc B_1 is known as the *eccentric*, the connecting rod *3* is the *eccentric rod*, and the segment OA is the *eccentricity*.

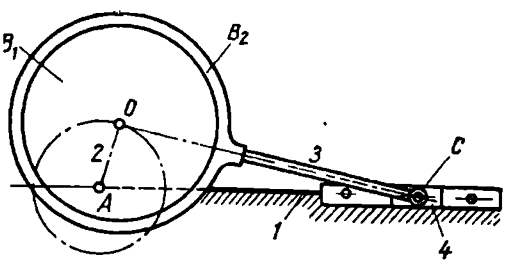


Fig. 269

It is apparent from the above that *an eccentric mechanism operates like a slider-crank whose crank length is equal to the eccentricity, and the length of the connecting rod is equal to the distance between axes O and C of the eccentric rod.*

The special feature of this mechanism is that its slider possesses a short stroke and the diameter of the crankpin is large enough to withstand great pressure. The eccentric mechanism is widely used in stamping and forging presses, etc.

188. The Rocker-Arm Mechanism

Fig. 270 is a scheme of a mechanism with a crank *1* which rotates about the fixed axis O. On the crank's end is a pin, centred on A, upon which is freely mounted the slide-block *2* which slides in a straight longitudinal guide cut into the arm *3*. This arm, known as a *rocker-arm*, can swing from the fixed axis O_1 when the crank rotates: swinging is caused when the slide-block *2* slides in the guide of the rocker-arm.

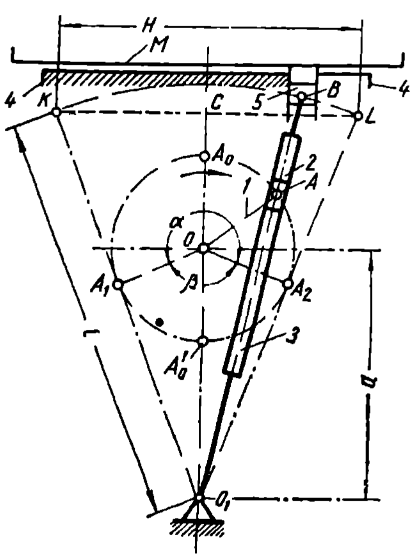


Fig. 270

Assume that the crank turns in the direction shown by the arrow. After an interval of time, the axis O_1B of the rocker-arm will be in position O_1L, tangent to the circle $A_1A_0A_2A_0'$ described by the centre of the crankpin A. At this moment the crank OA_2 occupying a radial line of this circle will be perpendicular to the axis of the rocker-arm. Obviously this will be the extreme right position of the arm, since as the crank continues turning as before, the arm will begin moving in the opposite direction — from right to left — and when the crank has turned through the angle $A_2OA_1 = \beta$, the arm will be in its extreme left position O_1K and perpendicular to the crank OA_1. Then the arm will again move from left to right and when the crank has turned through the angle $A_1OA_2 = \alpha$, it will return to position O_1L.

Thus, while the crank in its continuous rotation executes one turn, the rocker-arm oscillates, with axis O_1 as its centre, passing from its extreme left position to the extreme right and back again.

Now let there be a pin B at the upper end of the arm, around which slider *5* turns freely as it moves in straight guides which are part of the slider M which, in its turn, moves in immovable guides *4*. When the arm oscillates with this arrangement, slider *5* will transmit motion to slider M as it moves in the guides which are part of it. M will move from one end position to the other and back.

Thus *with the aid of the rocker-arm, the rotational motion of a crank is converted into reciprocal motion of translation of the slider.* The crank is usually made so that its length can be changed, thereby changing the length of the slider stroke.

This rocker-arm mechanism is used in a number of machines, including planers.

189. Kinematics of the Rocker-Arm Mechanism

We have shown that the rocker-arm mechanism converts rotation into reciprocal translation. In this it is similar to the slider-crank, but there is a good deal of difference between them in other respects.

First let us take up the method for determining the length of the stroke of the slider *2* in relation to the geometrical elements of the mechanism. We will denote the length of the crank OA as r, the length of the rocker-arm $O_1B = O_1K = O_1L$ as l, and the distance OO_1 between the axis of rotation of the crank and the axis of oscillation of the rocking arm as a.

Since the right triangles O_1CK and O_1A_1O have a common acute angle, they are similar, from which it follows that $\frac{O_1K}{O_1O} = \frac{KC}{OA_1}$, or $\frac{l}{a} = \frac{H}{2r}$, in which $\frac{H}{2} = KC$ and represents half a stroke of the slider. From this the length of the stroke is

$$H = \frac{2rl}{a}. \qquad (175)$$

It becomes evident that the stroke of the slider is directly proportional to the length of the crank and the length of the rocker-arm, and inversely proportional to the distance between their axes of rotation*.

Hence, in order to determine the distance $r = OA$ for a given stroke, we evolve the relationship

$$r = \frac{Ha}{2l}. \qquad (176)$$

The slider makes its stroke from left to right in the time interval that the crank turns through angle α, and executes its return stroke during the time the crank turns through angle β. We will denote the time it takes the crank to turn through angles α and β as t_α and t_β, respectively. Then the average velocity of the slider from left to right $v'_{av} = \frac{H}{t_\alpha}$, while from right to left $v''_{av} = \frac{H}{t_\beta}$, and the relationship between these velocities will be

$$\frac{v'_{av}}{v''_{av}} = \frac{H}{t_\alpha} : \frac{H}{t_\beta} = \frac{t_\beta}{t_\alpha}. \qquad (177)$$

Inasmuch as the crank rotates uniformly, the time spent by it to turn through angles α and β is directly proportional to the angles:

$$\frac{t_\beta}{t_\alpha} = \frac{\beta}{\alpha},$$

which when placed into Eq. (177) gives

$$\frac{v'_{av}}{v''_{av}} = \frac{\beta}{\alpha}. \qquad (178)$$

If we name the left-to-right stroke, during which the crank turns through angle α, the *advance stroke*, and the right-to-left stroke when the crank is turning through angle β, the *return stroke*, then on the basis of Eq. (178) we may state that *the average speeds of the advance and return strokes of the slider are inversely proportional to their corresponding angles.*

Hence the time taken by the return stroke is less than that of the advance stroke in the same ratio as the angle α is greater than angle β. This is the chief difference between the rocker-arm mechanism and the slide-crank mechanism, and it is this very feature of quick return that explains its use in shapers where the advance stroke is slower because it is limited by the cutting speed, and where it is desirable to make the return stroke as fast as possible.

* In the mechanism used in shapers, the length of the crank is adjustable; this is done by shifting the crankpin A in a radial slot provided in the disc of the gear fixed to shaft O and which operates as the arm of the crank.

Heretofore the inferred speeds of the slider have been *average*. But the actual speeds are not constant: at the end position the slider speed is zero, from where it gradually increases its speed till it reaches the centre position and then its speed again falls off till it reaches zero at the opposite end. Thus *in the rocker-arm mechanism, just as in the slider-crank mechanism, the slider possesses non-uniform motion and the crank has uniform motion.*

There is a variant of the rocker-arm mechanism in which the arm rotates instead of oscillating.

Illustrative Problem 99. If it is necessary to set the stroke of a shaper at $H = 400$ mm (Fig. 270), at what distance $OA = r$ must the slide-block *2* on the arm be set from the axis of rotation and what will be the average speed of the working (advance) and of the return strokes v_w and v_{ret} if $l = 900$ mm, $a = 540$ mm, and the rpm of the crank is 40?

Solution: we find the length r of the arm through Eq. (176):

$$r = \frac{Ha}{2l} = \frac{400 \times 540}{2 \times 900} = 120 \text{ mm}.$$

We find angles α and β.

From triangle A_1O_1O we evolve $OA_1 = OO_1 \cos \frac{\beta}{2}$, whence

$$\cos \frac{\beta}{2} = \frac{OA_1}{OO_1} = \frac{r}{a} = \frac{120}{540} = 0.222, \text{ and } \frac{\beta}{2} = 77°10'.$$

$\beta = 154°20'$, and $\alpha = 360° - 154°20' = 205°40'$.

At 40 rpm the crank executes one turn in $\frac{1}{40}$ min. $= \frac{60}{40} = 1.5$ sec. The time spent to turn through the angle $\alpha = 205°40'$ will be $t_\alpha = \frac{1.5 \times 205°40'}{360°} = 0.857$ sec. Hence $t_\beta = 1.5 - 0.857 = 0.643$ sec.

The average speed of the working stroke $v_{av.w} = \frac{400}{0.857} = 467$ mm/sec $= \frac{467 \times 60}{1,000}$ m/min $= 28.02$ m/min.

The average speed of the return stroke $v_{av.ret} = \frac{400}{0.643} = 622$ mm/sec $= \frac{622 \times 60}{1,000}$ m/min $= 37.32$ m/min.

190. The Cam Mechanism

Let slider A (Fig. 271) execute reciprocal motion of translation and move in fixed guides. To the slider is fastened a fascia piece K called a *cam*. A rod B, called *the stem of the follower* and which moves in fixed guides, has its end pressed against the cam by means of a spring C. Assume that the cam is in the position indicated by the dotted line and the end of the follower-stem is in contact with the surface of slider A. As the slider moves from left to right, the tip of the follower-stem will rise along the incline nm of the cam, then from surface m to l the follower will remain motionless if this surface is parallel to the axis of the slider, and from l to k the tip of the follower will move down to the end of the incline. When the cam moves in the reverse direction, the follower tip will slide along the cam's surfaces in the opposite order. If the cam had no horizontal segment, there would be no prolonged pause of the follower-stem at the apex of its position. If the slider moves uniformly, the character of motion (speed and acceleration) of the follower will depend on the profile of the cam and the speed of the slider.

Direct contact between the follower and the cam would create friction and subsequent wear of the two contiguous surfaces. To avoid this, a roller, which rolls on the surface of the cam, is usually attached to the end of the follower.

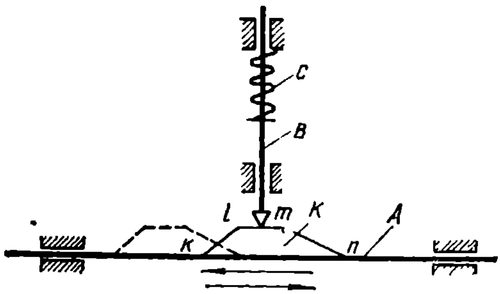


Fig. 271

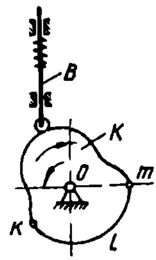


Fig. 272

If the follower were given the form of a lever with one end pressed to the cam by means of a spring, it would achieve oscillating motion around its rotational axis.

In the above cases we have dealt with linear translation of one direction being transformed into linear translation of another direction or into oscillating motion.

Fig. 272 is a diagram of a cam mechanism transforming rotational motion into linear translation. The cam K, which rotates round axis O, imparts to the follower B reciprocal motion of translation, the nature of which is determined by the profile of the cam. If the curve klm were a circle with its centre at O, the follower would remain motionless as it skirts along this part. Fig. 273 shows the skeleton outline of a mechanism in which the rotation of the cam K causes oscillating motion of the follower B.

In the cam examples presented thus far, their profiles and the trajectory of the various points of their followers lie in a single plane or in parallel planes. This type is called a *disc cam*, as distinguished from a *cylindrical cam*, which does not answer to the enumerated conditions. Fig. 274 is a schematic view of a cylindrical cam K, ringed with a *groove* that is not parallel with any cross-section of the cylinder. A roller *C* that moves in the groove, turns freely on a spindle which is part of follower *B*. Follower *B* moves in fixed guides parallel to the axis of rotation of the cam. When the cam rotates, the follower receives reciprocal motion of translation

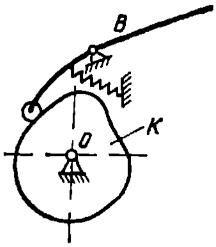


Fig 273

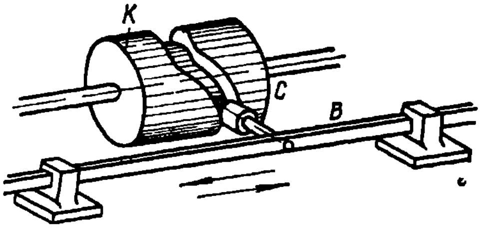


Fig 274

It is clear from what has been presented that in a cam mechanism the motion of the follower is determined by the profile of the cam Cams are extensively used to impart many kinds of motion, particularly in automatic machines and machine tools.

191. Determining the Working Surface of a Disc Cam

To determine the required profile of a cam, it is necessary to first know the required motion of the follower, or as they say, the "specification" of the follower's motion

Assume that the diagram shown in Fig 275*b* is just such a specification The angles α of the turn of the cam are laid out on the axis of the abscissae $O\alpha$, and the corresponding distances between *A* of the follower *B* and the rotational axis O_1 of the cam are plotted on the axis of the ordinates *OS* It is furthermore specified that the cam rotate uniformly and that its rotational axis intersect the axis of the follower (Fig 275*a*) It is seen that the curve representing one revolution of the cam has been equally divided into 16 parts

From the displacement diagram (Fig 275*b*) we find that when the cam makes its first two-sixteenths of a turn, the distance between the contact surface of the follower and the axis of rotation remains the same, as shown by the equal segments $0 - a_0 = 1 - a_1 = 2 - a_2$, and then this distance increases. At the end of the third-sixteenth of a turn, it is equal to the linear segment $3 - a_3$, at the end of the fourth- and fifth-sixteenth it will be equal to segments $4 - a_4$ and $5 - a_5$. Then the follower remains motionless during two-sixteenths of a turn, after which the distance grows until a moment corresponding to linear segment *10*. During turns *10—11* the follower again has a period of dwell; then beginning with moment *11* the distance decreases, i.e., the follower moves in the reverse direction; at the moment corresponding to the completion of the full turn of the cam the distance between the end of the follower and the axis of rotation is represented by the linear segment $16 - a_{16}$ which is equal to the initial distance $0 - a_0$ and means that the follower has returned to its initial position. As the cam continues to rotate, the follower again executes its motions in the same order, or, as they say, repeats the cycle.

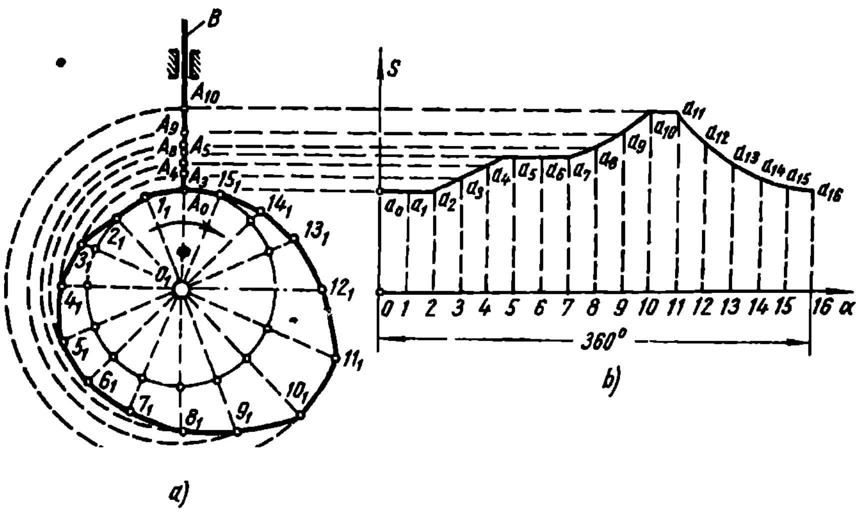


Fig. 275

Now let us proceed with the construction of the working surface of the cam (Fig. 275*a*). We extend axis $O\alpha$ to the left and mark upon it an arbitrary point O_1. This we shall consider as the rotational axis of the cam. Assume that the follower *B* is moving vertically upwards. Since it has been stipulated that its axis intersect the axis of the cam, we delineate a vertical line through point O_1, and by laying out on it the linear segment $O_1A_0 = Oa_0$, equal to the initial distance between the contact surface of the follower and the axis of rotation O_1, we obtain the initial position of the contact surface of the follower A_0. At moment *3* the contact surface of the follower *A* will be at a distance of $3 - a_3$ from the rotational axis; by laying out this segment on the axis of the follower we obtain position A_3 of its contact surface. Transferring the other points a_4, a_5, a_6, etc., on the displacement diagram, we obtain a number of positions for the contact surface of the follower, to wit, A_4 and A_5, which coincide with positions

A_6 and A_7, and A_8, A_9, A_{10}, which coincide with position A_{11}, etc.*.

Now we draw a circle with point O_1 as its centre and a radius equal to the shortest distance O_1A_0 and, dividing it also into sixteen equal parts corresponding to sixteen cam revolutions, delineate radial segments through points 1_1, 2_1, 3_1, 4_1, 5_1, etc.

When the cam turns through an angle of $22.5° \times 2 = 45°$, the contact surface of the follower A will remain in its initial position A_0; by the end of the next turn through the angle $2_1O_13_1$ of 22.5°, the contact surface of the follower will be at position A_3. Hence, in order to find the point on the surface of the cam corresponding to point A_3 of the follower, we plot an arc from centre O_1, with a radius equal to O_1A_3, to the point where it intersects the radius O_13_1. In the same way position A_4 of the contact surface of the follower will coincide with point 4_1 of the profile, etc. Repeating this process for all the other positions of the follower we obtain the other points on the working surface, which we unite with a curved line and thus obtain the profile of the cam. Between points 5_1 and 7_1, and 10_1 and 11_1 (just as between A_0 and 2_1) it will consist of arcs of a circle.

Fig. 276

If the follower had a roller on its end, we should first have found the curve corresponding to the motion of the axis of the roller and then, having delineated from various points on this curve a number of circles with radii equal to the radius of the roller, we would plot the profile of the cam, tangent to all these circles (or arcs) (Fig. 276).

192. Questions for Review

1. In the rack-and-pinion transmission represented in Fig. 277, the pinion z_1 transmits motion to the rack B through the idler gear z_2. What change would there be in the speed and direction of the rack if pinion z_1 was in mesh directly with the rack?

2. Two screws of different diameters are threaded to the same pitch. What can be said about the lead angle of their threads?

3. In two screws of the same diameter the distance between turns of the thread is the same, but one is single-threaded and the other multiple-threaded. What can be said about their lead angles?

4. In the mechanism diagrammed in Fig. 258 the threads on lengths a and b have the same direction and pitch. What is the absolute displacement of nut 3 when screw 1 is rotated?

5. What would be the answer to question 4 if the threads on lengths a and b possessed different directions?

* Positions A_{12}, A_{13}, A_{14} and A_{15}, are not shown so as not to complicate the drawing.

6 What change would there be in the mechanical advantage obtained with a screw-and-nut if the lead angle were decreased?

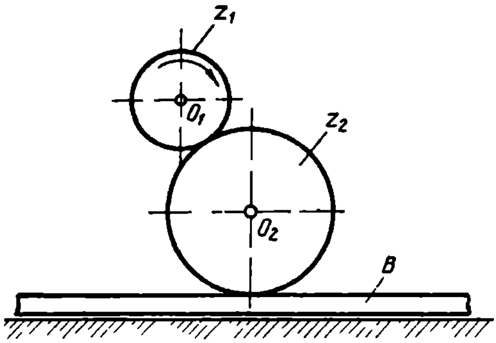


Fig. 277

7. What kinematic differences are there between the rocker-arm mechanism and the slider-crank mechanism? Why is the slider-crank mechanism impractical for use in a shaper?

193. Exercises

101. A load of $Q = 1.5$ tons is raised to a height $h = 180$ mm in 25 sec by the screw jack whose principal properties were enumerated in Ex. 97 (Fig. 263). What force **P** is exerted on its handle and what power expended if the efficiency of the jack $\eta = 0.4$?

102. The rack B in Fig. 278 is put in motion by the train of gears z_1, z_2, z_3, and z_4 in which z_1 is the driver The power on shaft O_1 is $N = 12$ hp. Find the force transmitted to the rack and also its speed if the rpm of the driving shaft $n_1 = 900$, the number of teeth on gears $z_1 = 24$, $z_2 = 60$, $z_3 = 25$, $z_4 = 75$, the module of the last gear $m = 5$ mm, and the efficiency of the drive $\eta = 0.85$.

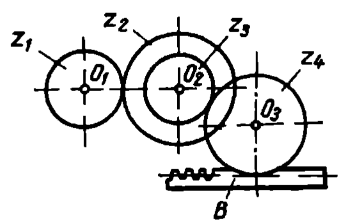


Fig. 278

103. The driving gear z_1 in Fig. 279 transmits motion 1) to rack A according to the scheme $z_1 \times z_2 - z_3 \times z_4 - - z_5 \times$ rack, 2) to the screw mechanism with nut C which cannot turn and according to the scheme $z_1 \times z_2 - z_3 \times z_4 \times z_6$ — screw $B \times$ nut C, and to 3) shaft VI according to scheme $z_1 \times z_2 \times z_7$ — worm $D \times$ worm gear z_8.

Given the following. $z_1 = 30$, $z_2 = 60$, $z_3 = 25$, $z_4 = 80$, $z_5 = = 40$, $z_6 = 50$, $z_7 = 40$ teeth; module of the rack $m_5 = 4$ mm, pitch of screw $B = 5$ mm, the worm D is triple-threaded, the number of teeth on the worm gear $z_8 = 60$, the rpm of the driving shaft $n = 400$, Find the speed v of the rack A, the speed v_c of the nut, and the rpm n_6 of shaft VI.

104. Fig. 280 shows the skeleton profile of a slider-crank mechanism with the following working properties. The slider

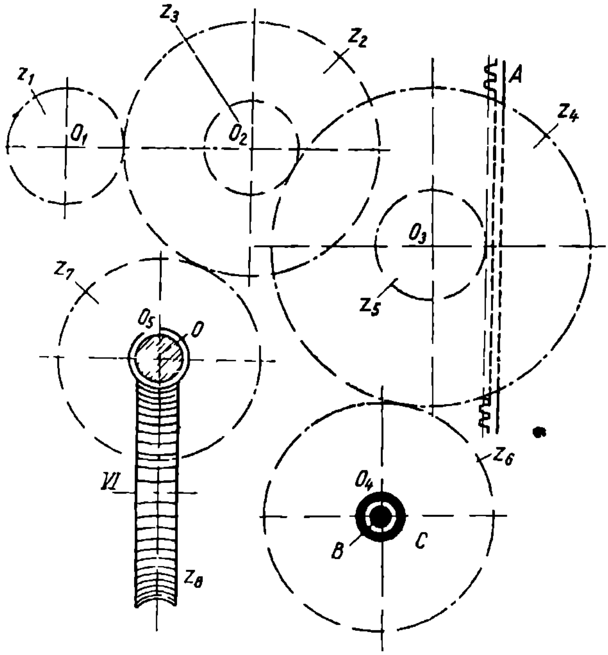


Fig. 279

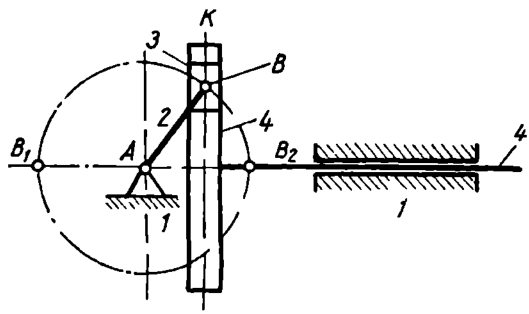


Fig. 280

3 turns freely on crankpin B of crank *2* which rotates about the fixed axis A. When the crank rotates, the slider receives relative reciprocal motion of translation in the slot of arm *4* which slides in fixed guides *1*. The length of the crank $AB = 120$ mm and it achieves 180 rpm. Plot the displacement and velocity curves for the mechanism.

CHAPTER XIX

AUXILIARY PARTS EMPLOYED IN TRANSMITTING ROTATION

194. Axles and Shafts and Their Components

In order that sheaves, gears, cams, etc., achieve rotation, they are mounted on parts called *axles* and *shafts*. Assume that sheave K (Fig. 281*a* and *b*) receives rotational motion from a belt and transmits this rotation further through sheave L. Rotation is imparted by the effective pull $\mathbf{P}_1$ which creates the torque $M_1 = P_1 \frac{D_1}{2}$, whence D_1 is the diameter of the sheave K (Fig. 281*b*). Acting against this torsion is the moment $M_2 = P_2 \frac{D_2}{2}$, equal in magnitude and opposite in direction, in which $\mathbf{P}_2$ is the pull transmitted to the sheave connected by belt with sheave L, and D_2 is the diameter of sheave L. In this way the part of the shaft *2* situated between sheave K, which receives the pull, and sheave L which imparts it, tends to twist under the action of two equal and opposite moments. Furthermore, the shaft tends to bend because it is subjected to its own weight, the weight of the sheaves, and the stretching action of the belts. Due to all these factors we may say that *under operation a shaft is subject to combined torsion and bending*.

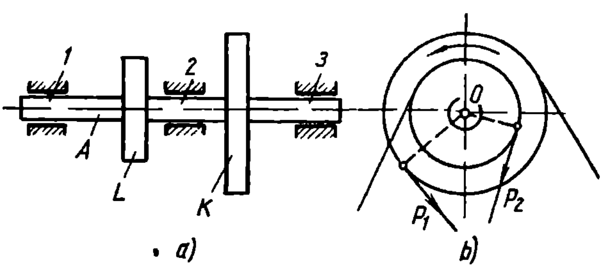


Fig. 281

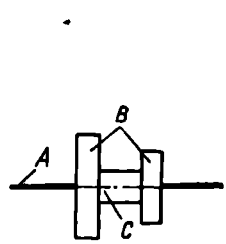


Fig. 282

Now let us assume that a unit consisting of two sheaves B is turning freely on a cylindrical shaft A (Fig. 282). It is clear that in this case the cylindrical shaft will be subject only to bending, inasmuch as torque will be imparted from one sheave to the other via the bushing C. This type of detail, whose geometric axis coincides with the axis of the revolving part that it carries, is known as an *axle*. Accordingly, *the principal difference between an axle and a shaft is that an axle is subject only to bending*, whereas a shaft bears both bending and torsion forces.

In the above example there is no need for the axle to revolve in order that the sheaves turn. But there are other instances where the axle must turn, an example of which is the axle of a railway carriage: as the wheels revolve, the axle upon which they are mounted also revolves but is not subject to torque.

Shafts and axles are held in supports the construction of which corresponds to the given function of the detail. Thus, the transmission shaft *A* schematically shown in Fig. 281*a* is installed in three supports, or sliding bearings. The part of the shaft within the bearing is known as the *journal*. Shaft *A* thereby possesses three journals *1*, *2*, and *3*. Journals *1* and *3* at the end of the shaft (or axle) are called *pivots*, while the intermediate journal (numbered *2* in Fig. 281*a*) is a *neck journal*. If the longitudinal (axial) forces acting upon the shaft are very great (Fig. 283), abutting journals are used to bear the thrust and they are therefore known as *thrust bearings*.

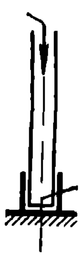


Fig. 283

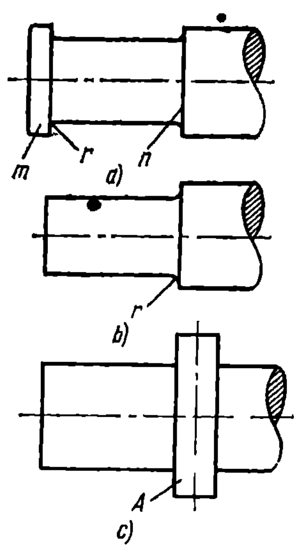


Fig. 284

The shaft and supports must mate in such a way as to prevent any motion of the shaft in an axial direction. There are various ways of doing this. One method is to cut out a part at the end of the shaft or the axle so as to have a cylindrical portion of less thickness than the rest, thus making a pivot with two shoulders (*m* and *n* in Fig. 284*a*) which will, it is clear, prevent axial displacement. A simpler construction is a pivot with a single shoulder (Fig. 284*b*).

The transition from a surface of greater diameter to one of lesser diameter (*r* in Fig. 284*a* and *b*) and called a *hollow chamfer*, is made in the form of an arc, of definite radius for every shaft diameter. The chamfer is indispensable for long service of shaft or axle.

Often *collars* are used to prevent axial displacement of the shaft (*A* in Fig. 284*c*). These are fastened to the shaft with set-screws. When necessary, collars are also employed together with neck journals.

195. Main Types of Sliding Bearings

The support in which a shaft or axle rotates is called a *bearing.* In some bearings the force from the shaft is perpendicular to the axes of the bearing (Fig. 281) and then they are called *radial bearings*; in other cases the force is directed parallel to the axle or shaft, in which case they are known as *thrust bearings* or *step bearings*. Some bearings may be of the combined *radial-thrust* type.

Bearings for the axles of rolling stock of railways are called *journal boxes*.

The choice of bearings depends on the conditions under which they are to work; the most important factors are the applied load and the rpm of the axle, or shaft.

In sliding bearings, the journal of the axle is in contact with the inner surface of the bearing and slides over it, thus creating the first kind of friction.

196. Antifriction Bearings

As already explained in Sec. 50, the loss from rolling friction is much less than that from sliding friction and explains the wide use of antifriction bearings.

Antifriction bearings are of various types, depending upon the direction of the acting forces. Figs 285 and 286 show one type — ball bearing — and its components (the numerals for the respective parts are repeated in both illustrations).

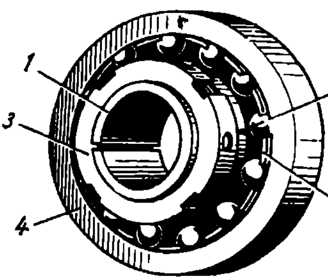


Fig. 285

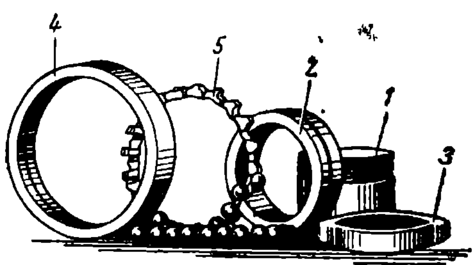


Fig. 286

The split sleeve *1* is mounted on the journal of the axle, or shaft. The outer surface of the sleeve is slightly conical and carries the inner race *2*. The outer race *4* is concentric with the inner one and between them are the steel balls. When the round nut *3* is screwed on to the sleeve *1*, the latter is drawn into the inner race *2*, thus fastening it on the journal. When one race turns relative to the other, the balls roll in the grooves in the outer surface of the inner race and the inner surface of the outer race. To keep the balls at a constant distance from one another they are placed in the cage *5*.

When the outer race *4* is fastened to the shaft housing (Fig. 287)*, we get a bearing to support a rotating axle or shaft. Fig. 288 shows a similar arrangement for a sheave; the ball bearings are mounted in the hub.

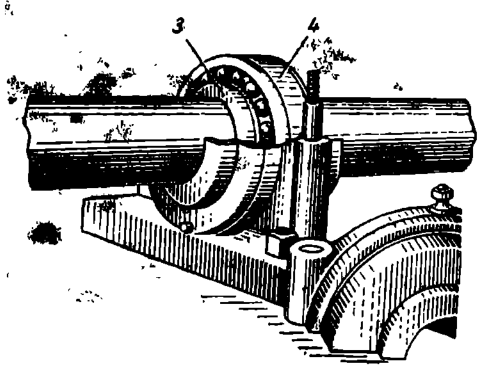


Fig. 287

In the bearing just presented, the balls are arranged in one row and from which it derives its name — *single-row* ball bearing; whereas Fig. 289 shows a cross-section of a *double-row* ball bearing.

Another type of antifriction bearings is where rollers are used instead of balls, these are called roller bearings.

Besides radial ball- and roller-bearings, *thrust* and combined *radial-thrust bearings* are also used. Fig. 290 shows a cross-section of one type of thrust ball bearings: the shoulders of shaft *A* rest on race *1* which is fastened tightly to the pivot; race *2* is immovable and the balls, rolling in the grooves between the two races, take up the thrust transmitted by the shaft.

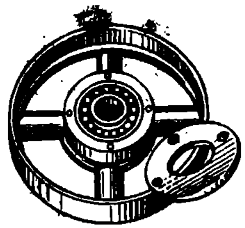


Fig. 288

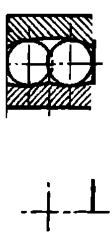


Fig. 289

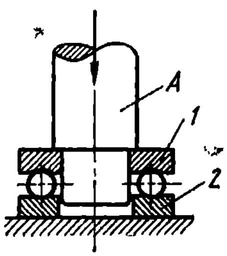


Fig. 290

Antifriction bearings are better than sliding bearings in several ways: there is less friction loss, a smaller amount of lubricant is required, a smaller longitudinal clearance is achieved, etc. But they also have a number of objectionable features, amongst which are their larger diametrical clearance and at times the complication of their assembly.

* The housing is shown with the cover removed and lying alongside.

197. Couplings

Coaxial shafts are joined by a machine part called coupling*. The simplest couplings, used for rigid connections of the ends of two shafts, are called *rigid couplings*. Fig. 291 is a cross-section of a *flanged rigid coupling:* two flanges *1* and *3* are keyed to the ends of the shafts and bolted together. To ensure alignment, one of the flanges has a centring projection *2* which fits into a corresponding depression in the other flange.

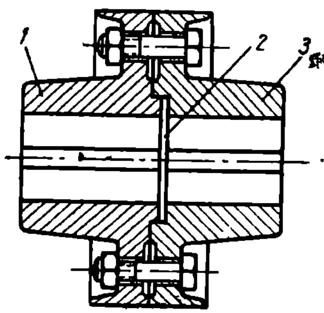


Fig. 291

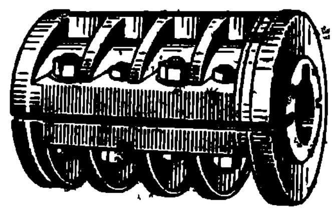


Fig. 292

Another type of coupling is the *ribbed coupling* shown in Fig. 292. This consists of two longitudinal halves mounted simultaneously on the ends of both shafts and then tightened with bolts. To preclude the possibility of the shaft twisting in the coupling, the ends of both are keyed together.

It is often necessary to engage or disengage two coaxial shafts during their rotation. Couplings used for this purpose are called *clutches*.

198. Questions for Review

1. What is the main difference between an axle and a shaft?
2. What is the difference between neck journals of an axle or shaft, and pivots?
3. When is a bearing called a step bearing?
4. What are collars used for?
5. How are bearings classified as to load and type of friction?

CHAPTER XX

DEMOUNTABLE CONNECTIONS

199. Threaded Connections

Every machine or assemblage of engineering equipment consists of parts joined into units. In some instances the parts forming such units move relative to one another, in other cases they

* The connections used in joining pipes, spars, tie rods and other similar equipment are, incidentally, also called couplings.

comprise a fixed whole without any movement relative to each another.

Often the parts are so joined that when necessary (as during repairs or overhauling) they may be taken apart without damaging the joint. Such connections are said to be *demountable*, as distinguished from *permanent* connections which cannot be separated without destroying some of the members.

The most prevalent demountable connection is the threaded type. Its construction depends on the parts to be joined and on the expected load. The threads are either cut into the parts to be joined, or are prepared on special fastening details — screws, bolts, nuts, etc.

Examples of permanent connections are those that are riveted or welded; the only way such assemblies can be taken apart is to destroy the basic elements forming the riveted or welded seam.

200. Threads for Connections

The chief element in a threaded connection is a helical thread classified by its diameter, pitch and profile. In the U.S.S.R., government standards (GOST and OST) have been established which must be strictly observed.

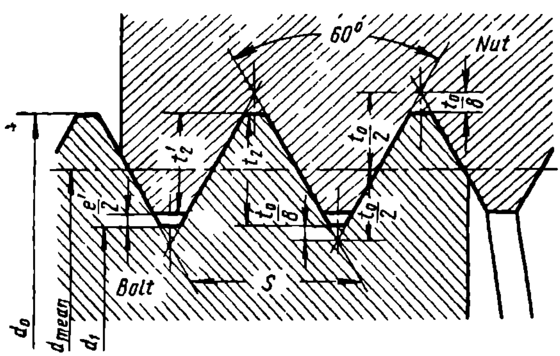


Fig. 293

The reliability of a threaded connection depends on the magnitude of friction acting between its elements; the greater the friction, the more reliable the connection. Since the greatest friction is obtained with a triangular thread, it is the one chosen for holding purposes. Fig. 293 shows a thread dimensioned by the metric system and designed for thread diameters of from 1 to 600 mm. The drawing shows that the apex angle of the thread is 60°, and since the sides of the thread form equal angles with the axis of the bolt, the thread has the form of an equilateral triangle. The height of the triangle

$$t_0 = 0.866\ s, \qquad (179)$$

in which s is the pitch of the thread.

Since the amount cut off at the thread's apex and base is $\frac{t_0}{8}$ the actual height of the thread

$$t_2 = t_0 - 2\frac{t_0}{8} = 0.75\,t_0 = 0.6495\ s. \qquad (180)$$

Further examination will show that between the roots of the thread on the bolt (diameter d_1) and the points of the thread on

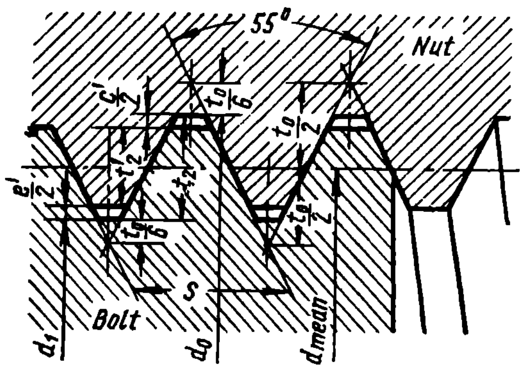


Fig. 294

the nut, a radial clearance has been left the magnitude of which

$$\frac{e'}{2} = t_2 - t_2', \qquad (181)$$

whence t_2' represents the depth of the working profile of the thread. OST Specifications for each size of screw include the dimensions of the external diameter d_0, average (mean) diameter d_{av}, internal diameter d_1, pitch s, height of thread t_2, and the clearance e'.

When dimensions are given in the metric system, they are marked with an M followed by the external diameter and pitch. For instance, M 30 × 3.5 infers that the screw has an external diameter of 30 mm and a pitch of 3.5 mm. Screws with dimensions based on the inch are also used in the U.S.S.R. Fig. 294 shows such a thread with a profile angle of 55°, used for diameters of from 3/16″ to 4″ (OST 1260). The cross-section of the thread is in the form of an isosceles triangle with an angle of 55° at its apex. The altitude of this triangle

$$t_0 = 0.96049\ s, \qquad (182)$$

in which s is the pitch of the thread.

The distance of the cut-off of this triangle from the apex is $\frac{t_0}{6}$ from which we obtain

$$t_2 = t_0 - 2\frac{t_0}{6} = \frac{2}{3}t_0 = 0.6403\ s. \qquad (183)$$

Another difference of this thread as from one of the metric type is that there are two clearances $\frac{e'}{2}$ and $\frac{c'}{2}$ between the bolt and the nut at the root and apex of the profile. The rest of the notations in Fig. 294 are the same as for the metric-type thread.

The number of threads per inch ranges from 3 (when the diameter is 4″) to 24 (when the diameter is 3/16″). The inch-style thread is forbidden* in the manufacture of new articles. Among other threads for connections is the GOST 6357-52 for piping with diameters of 1/8″ to 18″.

201. Tapered-Pin Connections

Some demountable connections are implemented with tapered pins and are therefore called *pin connections*. Assume it is necessary to join two details *1* and *2* as in Fig. 295*a*. After drilling holes to fit the exact shape of the pin in both details (Fig. 295*b*),

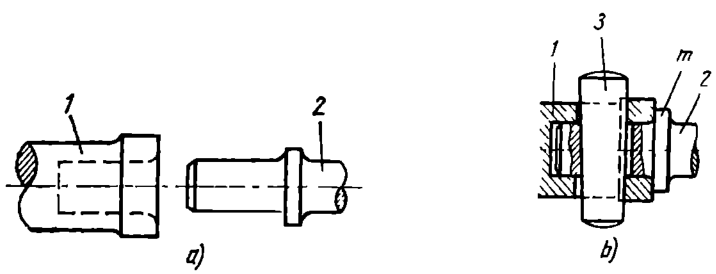


Fig. 295

we drive in the pin with a hammer or a press. Under the action of the tapered pin *3*, the end of detail *2* will be drawn into the socket in detail *1*. If there is a flange *m* and the pin is driven in firmly, a reliable connection is formed which, if necessary, can easily be taken apart by forcing the pin out in the opposite direction.

* In the U.S.S.R. — *Editor's note.*

Such connections may also be made as shown in Fig. 296*a*: the two parts to be joined are enclosed in a common bushing and the connection is made with two pins.

Tapered pins are held in place by friction, which increases as forces N_1 and N_2 increase (see Fig. 162) As already explained in

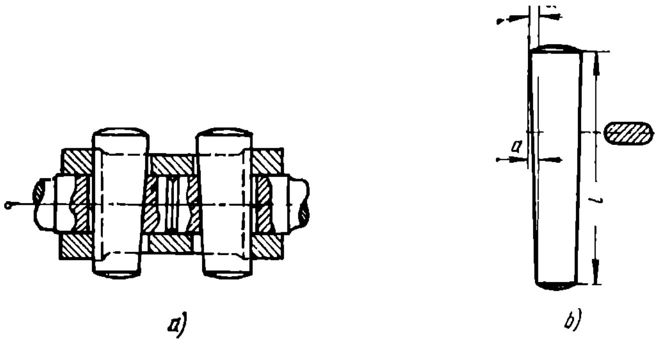


Fig 296

Sec. 140, these forces increase as the angle α formed by the sloping sides of the pin decreases. Hence angle α is made as small as possible, its ratio $\frac{a}{l}$ (Fig. 296*b*) to the length being $\frac{1}{100}$, $\frac{1}{40}$, or $\frac{1}{30}$ and very seldom greater.

STRENGTH OF MATERIALS

CHAPTER XXI

BASIC PRINCIPLES

202. Stress and Strain in a Body Under the Action of External Forces

Assume that a rectilinear bar is resting with its ends on two supports and we exert a force on it, at some point between the supports, by hanging a definite weight to it. Under the action of this force the bar will bend and become curvilinear in form. By repeating this experiment with weights of various magnitude applied at the same point, we will find that the bar bends more as we increase the force. Such a change in form, or dimensions, of a body under the action of applied forces (called loads) is known as *strain*

By observing the form of the bar after removing the load causing the strain we will see one of two things: either the bar will return entirely to its original shape, or its form will be only partially restored. In the first instance the strain is called *elastic* while in the second case, when the bar remains partially deformed, it is said to have attained plastic strain or a *permanent (residual) set*. We thus see that when a body is subjected to the action of a load under certain conditions elastic strain may be accompanied by a permanent set. If the load is still further increased, strain will become so great that the bar will fail.

Strain is not only caused by direct action of one body upon another, but also by a body's own weight. If we place a metal bar on two supports and it is very long in comparison to its thickness, we shall see that its own weight and corresponding reactions at the supports will cause it to become curvilinear in form.

In the above examples, strain is so great that it becomes visible; such strain is not permissible in engineering structures or machines except where special parts are used (springs of various kinds), meant to absorb strain of considerable magnitude. However, all parts of any structure or machine may become somewhat strained under the action of applied external forces; and though such strain may not be visible, it can be measured with precise instruments.

203. External and Internal Forces, and the Cross-Section Method

We know from experiment that the greater the force applied to a body, the greater the strain. For instance, in the case of the bar just mentioned as being subject to bending, the extent of curvature depends on the magnitude of the applied forces at a given cross-section.

In all cases when external forces cause strain in some member, internal forces arise, in proportion to the magnitude and direction of the forces, to resist the external forces. How can we determine the magnitude of internal forces caused by the action of external forces?

Assume that a beam *ABCD* (Fig. 297) is under the action of a system of balanced forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, and $\mathbf{P}_5$. Let us find the internal forces acting in the cross-section *mn*. To do this, we reason as follows: if the beam as a whole is in equilibrium, then all parts of it are also in equilibrium.

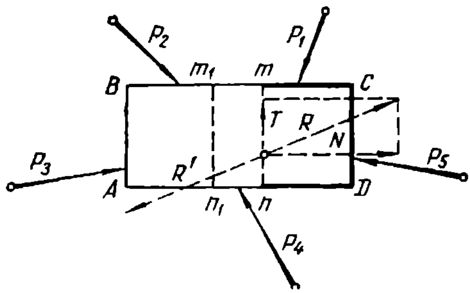


Fig. 297

Let us try to determine, for example, what forces would be acting on the part *mCDn*. In the first place there are external forces $\mathbf{P}_1$ and $\mathbf{P}_5$, and then there are, apparently, some forces acting from portion *BmnA*. Let us imagine that we cut the beam along line *mn*, and remove the portion *BmnA*. If we denote **R** as the resultant of the elementary forces acting from the discarded portion upon the remaining free body, then we may say that the free body *mCDn* is in equilibrium under the action of forces $\mathbf{P}_1$, $\mathbf{P}_5$, and **R**. This force **R** is the internal force that balances forces $\mathbf{P}_1$ and $\mathbf{P}_5$. As already explained in Statics, when there is a system of forces in equilibrium, any one of the forces balances all the others. Hence the force **R** is equal and opposite in direction to the resultant of forces $\mathbf{P}_1$ and $\mathbf{P}_5$.

Reasoning in the same way with respect to portion *BmnA* (that is, discarding portion *mCDn*), we come to the conclusion that it is under the action of the internal force **R'** which is equal to the resultant of forces $\mathbf{P}_1$ and $\mathbf{P}_5$, or what is just the same, force **R'** balances the external forces $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$, as applied to portion *BmnA*.

We therefore find that the internal forces in the right and left portions of the beam are equal in magnitude (which is as it should be according to the law stating that action and reaction are equal and opposite), but the direction of these forces depends on which of these portions we consider as the given free body.

As we shall eventually see, the direction of internal forces determines the nature of strain that the body undergoes.

In this case the system of external forces is such that the resultant **R** of the internal forces is not perpendicular to the given cross-section *mn*. We therefore resolve **R** into two components – **N**, perpendicular to the vertical plane and **T**, lying within it. We may thus replace **R** with these two components. The first is called the *normal* force, and the second the *tangential* force.

Now let us find the answer to the question as to whether the internal force will be the same in all sections of the beam. Assume that we cut the beam along line m_1n_1, parallel to *mn*, and discard the left portion Bm_1n_1A. With the given distribution of forces we see that there are now three forces instead of two – $\mathbf{P}_1$, $\mathbf{P}_5$, and $\mathbf{P}_4$ — acting on the remaining free body. Hence the force balancing them and equal to the resultant of the discarded forces $\mathbf{P}_2$ and $\mathbf{P}_3$ will differ from them and thereby the internal force in this cross-section will also differ.

Furthermore, if instead of a vertical plane we had taken our section of the beam in some direction other than perpendicular to its axis, the force **R** would be directed towards this plane at a different angle and consequently the components **N** and **T** would have been different. Therefore in general the internal forces differ at different sections of the beam.

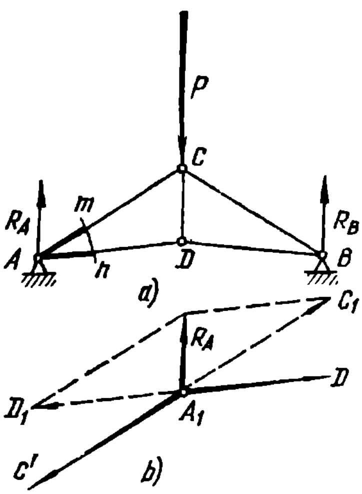


Fig. 298

This method for determining internal forces in a strained body is called the *cross-section method* and is made wide use of in solving strain problems of bodies.

Illustrative Problem 100. A roof truss is resting on two supports *A* and *B* (Fig. 298*a*). A vertical force **P** is applied at *C*. Since the truss is symmetrical with respect to the king-post *CD*, the reactions of the supports will each be $\frac{P}{2}$, i.e., $R_A = R_B = 0.5P$. Find the internal forces acting in the rafter *AC* and the tie beam *AD*.

Solution: Let us assume that, by cutting through plane *mn*, we have procured a free body at joint *A*. *A* is in equilibrium under the action of the reaction $\mathbf{R}_A$ and the internal forces in both *AC* and *AD*. We begin with a point A_1 below the diagram (Fig. 298*b*), and after delineating the vector of force $\mathbf{R}_A$, we resolve it into two components A_1D_1 and A_1C_1 directed towards *AD* and *AC*. In order that these two forces be balanced by force $\mathbf{R}_A$, they must be directed in opposite directions. In this way we obtain a balanced system of forces $\mathbf{R}_A$, $\overline{A_1D'}$, and $\overline{A_1C'}$. From this we find that the tie beam *AD* is under tension, while the rafter *AC* is under compression

204. Internal Forces of Elasticity

When a body is strained under the action of external forces the points of application of these forces are more or less displaced, with the result that these forces perform a definite amount of work in the strained body. This work is equal to the negative work of the internal forces resisting strain. If strain is elastic, the work of the internal forces is equal to the potential energy accumulated by the strained body. This energy is returned when the body assumes its original form.

This factor connected with the potential energy of strain is frequently utilised in engineering - among other things in machines and instruments employing springs, membranes, and similar resilient parts.

205. Stress in Strained Bodies

The internal forces **R** and **R**′ as presented in Sec. 203 are resultants of the elementary forces of interaction between the particles of two parts of a strained body. Thus if the strained body represented in Fig. 299 were imagined to be cut as shown, then we may assume such an elementary force acting on each particle.

Let the elementary force Δ**P** be acting on some small area ΔF of portion *I* of the body. It is obvious that the greater the force, the greater the internal forces set up in the material. These forces are measured by a quantity called *stress*, which is found by dividing the elementary force Δ**P** by the area ΔF. By denoting the force as σ, we obtain

$$\sigma = \frac{\Delta P}{\Delta F} \qquad (184)$$

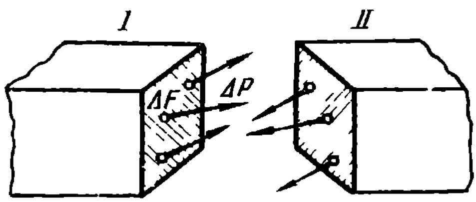


Fig. 299

In general, stresses differ in different parts of a strained body.

Inasmuch as force is expressed in kilogrammes and area in centimetres or millimetres, then stress is expressed in kg/cm² (read, kilogrammes per square centimetre), or kg/mm².

If internal forces are distributed evenly over the given cross-section, the stress will be the same at all its points and can be determined by dividing this force **P** by the whole area of the cross-section F:

$$\sigma = \frac{P}{F} \cdot \qquad (185)$$

206. Ultimate Strength and Safe Stresses

If we take two pieces of steel wire of the same quality and length but of different diameters and hang them so as to support equal loads, we shall find that the one of the smaller diameter will have the greater elongation. Thus, under the same load, the strain of the wires will differ. If we further load the wires, we shall find that at a certain load the thinner wire will acquire a permanent set and will not recover its initial shape when the load is removed, whereas strain in the thicker wire will remain elastic. Finally, at a certain further load the thin wire will snap, whereas the thick one will remain unbroken under the same load.

Thus we see that strain in the two wires differs under one and the same load. The reason is that the stress is less in the cross-section of the thicker wire because its area is greater. From which it follows that it is not the magnitude of the exerted force that determines the nature of strain, but the magnitude of stress. However, stress alone does not determine the character of strain. Indeed, if we repeat the experiment with two pieces of copper wire of the same diameters, we would find that they acquire greater elongation under the same loads and break under smaller loads. This means that the nature and magnitude of strain depends also on the physico-mechanical properties of the material.

Strains of various kinds find practical application in engineering. For instance, in the forging or rolling of metal we force it to undergo non-elastic strain which enables us to give it the required shape. In the cutting of metals, forces required for separating the chips are put into action. In all these cases we create stresses which correspond to the produced strain.

Where machines and other engineering equipment are concerned, however, the problem is entirely different. Obviously in their case there must be no permanent set, since such strain would cripple normal work of the parts. Accordingly, all parts of machines and other equipment must be subject to only elastic strain, to disappear when the action of the load is removed. Whereupon machine parts are so constructed as to keep strain within permissible limits, from which it is clear that the created stress under a load must also be within permissible limits. To accomplish this, calculations in design are based on certain *safe stresses* which are established as a definite fraction of the stress that would break the member. Obviously the magnitude of a safe stress depends firstly on the material of which the part is to be made.

All these factors concerning determination of strain and stress under the action of external forces, as well as calculations of the strength of elements of machines and all other types of structures, are treated in the science called Strength of Materials,

207. Static and Dynamic Loads

There are various kinds of action of a load on a structure. Let us consider a bridge for example; here the load fluctuates within narrow limits. Its own weight is constant, but the load created by the traffic passing over it changes gradually in the course of comparatively prolonged intervals of time. The floor load in an apartment house changes in a similar way, as does also the hydraulic pressure exerted upon a dam, etc. This kind of load, which grows gradually and then either remains constant or undergoes comparatively little change, is called a *static load.*

There are other cases where external forces applied to a body do not increase slowly but act with a force that grows quickly to its maximum; and finally there are cases where the whole load is applied simultaneously and produces impact. These are called *dynamic loads.* Wagon couplings undergo such loads when a train starts suddenly, and such loads occur when a forge is working, and when badly centred workpieces are being machined on a lathe.

A dynamic load produces greater strain and stress than a static load. Hence, in the designing and operation of machines and other engineering structures, everything is done to avoid dynamic loads except when impact is needed to obtain greater effect (the blow of a forge hammer, pile-driver, etc.).

208. Chief Types of Strain

All parts of a structure act upon each other in various ways and, accordingly, the forces exerted by one part upon another cause different kinds of strain: the cable of a hoisting machine is stretched, the foundation of a structure is compressed, a horizontal beam is bent, etc.

Strain is divided into the following categories:

1. *Tension* and *compression* (Fig. 300*a* and *b*): such strains occur in the elements of bridge trusses, forge hammers (compression), the shank of a bolt when tightened by a nut, etc.

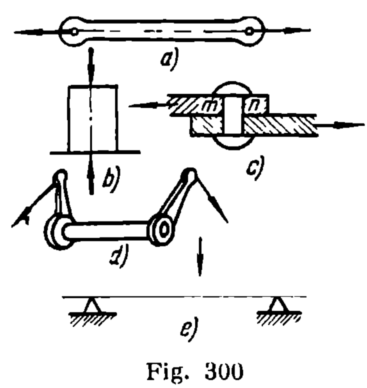


Fig. 300

2. *Transverse displacement (shear)* (Fig. 300*c*): under the action of equal and opposite forces a rivet may shear along the line *mn*; a too great tightening of a nut and bolt will cause the thread to strip off the shank of the bolt along the internal diameter, etc.

3. *Twist* (Fig. 300*d*): twist is a strain that occurs in all parts that undergo torsion. Shafts of machines are most subject to twist.

4. *Bending* (Fig. 300*e*): beams and girders of all kinds are subject to bending, any axle may bend under its own weight, the weight of other parts mounted on it, and the action of applied forces.

All the above strains are classified as *simple*; but very often machine parts are subjected to several stresses at once. Shafts, for instance, are acted upon by torsion and tension at the same time, resulting in *combined strain.*

209. Questions for Review

1. What is meant by strain and upon what factors does its magnitude depend?
2. Is it possible to judge the magnitude of strain by only the magnitude of the load acting on the member under consideration?
3. In what cases is it necessary to induce a permanent set in a material?
4. In what kind of calculations is the cross-section method used? Explain it in brief.
5. State the kinds of strain produced in the following components:
a) an ordinary cutter when a surface is being machined;
b) a drill in operation;
c) the lead screw of a thread-cutting lathe when in operation;
d) the jaws of the chuck on a lathe when the outer surface of a workpiece is being fastened;
e) the screw of a parallel vise holding a workpiece;
f) the shank of a rivet in its cross-sections directly beneath the heads where they hold the rivetted parts.

CHAPTER XXII

TENSION AND COMPRESSION

210. Tension. Absolute and Unit Elongation

Assume that a force **P**, constant in magnitude and axial in direction, is applied to the lower end of the immovable prismatic bar shown in Fig. 301. The bar is in equilibrium under the action of two equal and opposite forces — **P** and the reaction **P'***. Let us take a cross-section *mn* perpendicular to the axis of the bar at an arbitrary plane; by imagining that we have discarded the lower part of the bar, we come to the conclusion that the upper part as a free body is in equilibrium under the action of force **P** and its opposite force **P'**. Accordingly, the bar will stretch along its whole length under the action of force **P**. If the cross-sectional area of the bar is equal to F, the stress

* In the given example the weight of the bar is ignored.

in the cross-section

$$\sigma = \frac{P}{F}$$

from which

$$P = \sigma F. \tag{186}$$

If we increase the magnitude of force **P**, then the length of the bar also increases. Assume that with a certain magnitude of force **P**, the length of the bar $l_1 = l + \Delta l$, in which l is the original length, and Δl is its elongation when in a state of strain. This final length is called *absolute elongation*. But absolute elongation does not tell us all about the character of strain. To illustrate this let us take a rubber band, cut it into two unequal lengths and hang equal weights on them; it will be found that their absolute elongations differ in magnitude. Hence, absolute elongation is no indication of proportionate strain under the action of a given tensile force. However, if we compare the elongation per unit length of the two bands of rubber, we shall find that it is the same for both. This is called *unit elongation* and is a ratio of absolute elongation to original length. By denoting unit elongation as ε, we obtain

$$\varepsilon = \frac{\Delta l}{l} \tag{187}$$

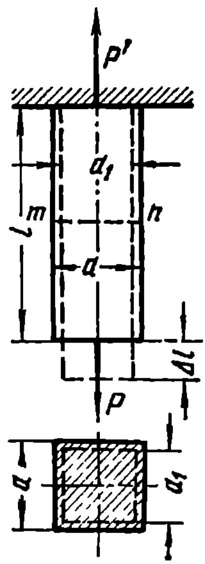


Fig. 301

Since the numerator and denominator of this fraction are both expressed in units of length, it is an abstract number. Ordinarily, unit elongation is expressed in percentage, then

$$\varepsilon\% = \frac{\Delta l}{l} 100. \tag{188}$$

Wherefore, *longitudinal strain of a body under the action of a tensile force is measured by its unit elongation.*

211. Transverse Strain of a Body Under the Action of a Tensile Force

Experience has shown that the elongation of a bar under a tensile force is accompanied by transverse contraction, that is, by a decrease in cross-sectional area perpendicular to the line of action of the force. Assume that the bar in Fig. 301 is being stretched by two equal and opposite forces **P** and **P'**, that it is square in cross-section, and that one of its sides is equal to a.

When the bar is elongated by a length equal to Δl, the dimension a is decreased to a_1, and the difference between a and a_1 is the *absolute transverse contraction* Δa, i. e.,

$$\Delta a = a - a_1. \qquad (189)$$

Transverse strain of a bar is measured by *unit transverse compression:*

$$\varepsilon' = \frac{\Delta a}{a} \qquad (190)$$

The relationship between unit transverse compression ε' and unit elongation ε is

$$= \mu\varepsilon, \qquad (191)$$

that is, transverse strain is proportional to unit elongation. The coefficient μ is known as the *coefficient of transverse compression* and is a constant determined empirically for each material. The coefficient for carbon steels, for instance, is from 0.24 to 0.28; for copper it is from 0.31 to 0.34; for aluminium it is from 0.32 to 0.36, etc. It is less than 0.5 for most materials, while for rubber it is almost 0.5. Unit elongation for most materials is three to four times more than transverse compression.

212. The Tensile-Stress Diagram

In investigating the properties of metals in the study of materials, changes are examined in the length of a bar when it is stretched in a tensile testing machine. In the tensile diagram the loads are plotted along the vertical axis and corresponding elongations marked off along the horizontal axis.

Up to a certain load P_p, elongation is proportional to the load. Under greater loads, elongation increases faster in proportion, and at a still greater load P_s elongation continues without any further increase of the load: the material begins to "yield". Then resistance to strain increases until a moment is reached corresponding to a certain maximum load. At this stage a cross-sectional reduction (a "bottleneck") becomes apparent at some place along the specimen's length. This is the beginning of complete failure: for henceforth the bottleneck narrows rapidly even with a decreased load and finally the specimen snaps.

With such a tensile diagram we can make a *tensile-stress diagram* to show the relationship between stress and unit elongation.

On a rectangular system of coördinates (Fig. 302) we plot unit elongation in per cent (ε%) along the axis of abscissae, and corresponding stresses σ along the axis of ordinates. The form of the curve obtained will be similar to that in the tensile diagram. Point A on the curve will correspond to a stress beyond which unit elongation ceases to be proportional to the stress. The stress at this point is the *limit of proportional elongation* and is denoted as σ_p. Point B on the curve will correspond to the stress σ_s and is called the *yield point*. Point C will correspond to the stress σ_b when a neck begins to form on the test bar and will show the moment when failure begins; this stress is called *ultimate strength**. And finally, D will mark the point at which the metal snaps.

It must be additionally noted that slightly higher than point A is a point corresponding to the stress at which strain passes from elastic to plastic: this is called the *elastic limit*. However, since this stress is very close to the limit of proportional elongation, the two may be considered the same for practical purposes.

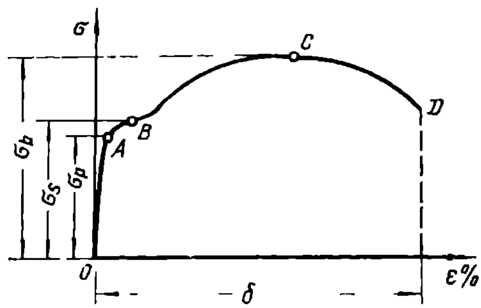


Fig. 302

The magnitude of stresses σ_b, σ_s, and σ_p characterise the mechanical properties of a material, i. e., the capacity for resisting the action of external forces causing strain and failure. For steel containing 0.15% carbon, $\sigma_b = 35$—45 kg/mm² and $\sigma_s = 20$ kg/mm², for steel containing 0.6% carbon, $\sigma_b = 61$—87 kg/mm² and $\sigma_s = 50$ kg/mm²; for chromium-nickel steel $\sigma_b = 90$ kg/mm² and $\sigma_s = 75$ kg/mm². From this we see that the strength of steel increases as its carbon content increases, and also as special alloying elements are added.

The unit elongation of a material undergoing a tensile force is denoted as δ, is expressed as a percentage, and characterises the elasticity of the material. The smaller it is, the more brittle the material.

An illustration is grey cast iron which fractures before hardly receiving any elongation or transverse contraction. The ultimate strength of cast iron is considerably lower than that of steel: for grey cast iron, $\sigma_b = 18$—27 kg/mm². Thus the tensile-stress diagram for brittle materials is quite different from the one characterising the previously-examined materials.

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* Sometimes this stress is called *temporary resistance*.

213. Relationship Between Stress and Unit Elongation. The Modulus of Elasticity

The reader is reminded that up to the limit of proportional elongation σ_p, elongation of a test bar is proportional to its stress and the linear segment OA is a straight line, as is seen from the diagram in Fig. 302. Within these limits, if under the stress σ_1 the bar receives a unit elongation of ε_1 and under the stress σ_2 receives an elongation of ε_2, etc., we obtain the following identities:

$$\frac{\sigma_1}{\varepsilon_1} = \frac{\sigma_2}{\varepsilon_2} = \frac{\sigma_3}{\varepsilon_3} = \text{, etc.}$$

This means that the relationship between stress and corresponding unit elongation is a constant, which, if denoted as E, assumes the general form of

$$\frac{\sigma}{\varepsilon} = E,$$

or

$$\sigma = \varepsilon E. \qquad (192)$$

This coefficient E is called the *modulus of elasticity*; given an equal strain, then the greater the stress the greater the coefficient E. Since stress is expressed in kg/cm² or kg/mm², and unit elongation is an abstract number, the modulus of elasticity is expressed in the same units as stress - ordinarily as kg/cm².

As already noted, the limit of proportional elongation may be considered the same as the elastic limit and, hence, Eq. (192) may be used either to find the magnitude of elastic strain under a given stress or to obtain the stress corresponding to a given strain. The numerical value of the modulus of elasticity has been determined empirically for different materials with the aid of the equation $E = \frac{\sigma}{\varepsilon}$, by measuring their elongation under a given stress.

For example, the moduli of elasticity, in kg/cm², of carbon steel is from 2,000,000 to 2,200,000, for steel castings 1,750,000, for cold-drawn brass from 910,000 to 990,000, for wood along the grain from 90,000 to 120,000, for wood across the grain from 4,000 to 10,000, and for leather belting from 2,000 to 6,000.

Illustrative Problem 101. A steel specimen, pulled with a force $P = 500$ kg, received an elastic elongation $\Delta l = 0.0272$ mm. Its cross-sectional area $F = 181.2$ mm² and length $l = 200$ mm. Find its modulus of elasticity E.

Solution: we find that the stress $\sigma = \frac{P}{F} = \frac{500}{181.2}$ kg/mm² and unit elongation $\varepsilon = \frac{\Delta l}{l} = \frac{0.0272}{200}$, and by applying Eq. (192) we obtain

$$E = \frac{\sigma}{\varepsilon} = \frac{Pl}{F\Delta l} = \frac{500 \times 200}{181.2 \times 0.0272} = 20{,}250 \text{ kg/mm}^2 = $$
$$= 2{,}025{,}000 \text{ kg/cm}^2.$$

Illustrative Problem 102. Determine the absolute elongation of a steel bar of length $l = 2$ m and cross-sectional area $F = 2$ cm² under a load $P = 3$ tons.

Solution: Eq. (192) indicates that unit elongation $\varepsilon = \frac{\sigma}{E}$ and stress $\sigma = \frac{P}{F}$. Therefore $\varepsilon = \frac{P}{EF} = \frac{\Delta l}{l}$, from which $\Delta l = \frac{Pl}{EF}$. Taking E as 2,000,000 kg/cm² and substituting numerical values, we obtain

$$\Delta l = \frac{3{,}000 \times 200}{2{,}000{,}000 \times 2} = 0.15 \text{ cm} = 1.5 \text{ mm}.$$

Illustrative Problem 103. Fig. 303 shows a shaft of length $l = 20$ m and diameter $d = 50$ mm. It revolves in three bearings with collars A and B at the ends of the shaft to prevent longitudinal displacement. The collars were installed on the shaft in close contact with the bearings during the summer months when the temperature was 30°C. What axial force will the collars exert on the bearings in winter when the temperature in the building is 10°C?

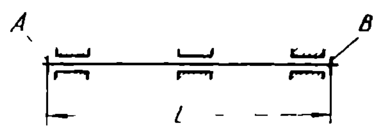

Fig. 303

Solution: the coefficient of linear expansion of steel is 0.0000125. Since the temperature falls 30° − 10° = 20°, the shaft contracts 0.0000125 × × 20 = 0.00025 of its length. This will give rise to a tensile stress in the shaft which we find through Eq. (192) and by taking the modulus of elasticity $E = 2{,}000{,}000$ kg/cm² and $\varepsilon = 0.00025$. Then by substituting numerical values we obtain

$$\sigma = 2{,}000{,}000 \times 0{,}00025 = 500 \text{ kg/cm}^2.$$

In order to find the force **P** acting on the shaft axially, we multiply this stress by the cross-sectional area of the shaft:

$F = \frac{\pi d^2}{4} = \frac{\pi 5^2}{4} = 19.6$ cm², whence the required force $P = 500 \times$ $\times 19.6 = 9{,}800$ kg.

If nothing prevents the shaft from shortening, it will attain its absolute longitudinal contraction which, according to Eq. (187), will be

$$\Delta l = \varepsilon l = 0.00025 \times 20 = 0.005 \text{ m} = 5 \text{ mm}.$$

We thus see that the collars are exerting too great a pressure on the bearings to achieve normal operation, and the work of the drive will be disrupted.

Illustrative Problem 104. Assume that the bar vertically suspended in Fig. 301 is not under the action of any external force **P**, but only of its own weight. At a freely-chosen cross-section mn the internal forces will be equal to the weight of the part of the bar below it. Obviously the higher the cross-section, the greater the internal forces, and the greatest force will be in the uppermost cross-section where the bar is suspended. The greatest stress will also be in this cross-section, i.e., $\sigma = \frac{G}{F}$, in which **G** is the weight of the bar and F is the cross-sectiona area. By denoting specific gravity as γ, then $G = Fl\gamma$ and $\sigma = \frac{Fl\gamma}{F} = l\gamma$. We thus see that in this example the stress does not depend on cross-sectional area.

214. Compression

Compressive strain is the opposite of tensile strain. All the relationships that have been given for tension are also applicable to compression: if the bar in Fig. 304 were under the action of compressive forces **P** and **P'**, at any cross-section perpendicular to the line of action of the load there would be compressive stresses $\sigma = \frac{P}{F}$, in which **P** is the load and F is the cross-sectional area. The bar will strain longitudinally along the direction of the load and its length will diminish as expressed by $\varepsilon = \frac{\Delta l}{l}$, or in percentage as expressed by $\varepsilon\% = \frac{\Delta l}{l} \times 100$.

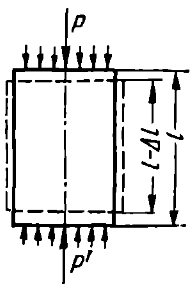

Fig. 304

Compressive stresses are proportional to unit longitudinal strain, i. e., $\sigma = E\varepsilon$. The modulus of elasticity E for compression is the same as for tension for most materials.

Under compression, contrary to tension, the lateral dimensions of a specimen bar will increase. Eq. (191) expresses the relationship between the magnitude of these dimensions and longitudinal strain.

215. Design Formulae for Allowable Tensile and Compressive Stresses

Tensile or compressive stresses, occurring in the same direction as the load, are determined by Eq. (186):

$$P = \sigma F.$$

As already explained in Sec 206, machine parts are designed so that their stresses do not exceed safety limits. By substituting the allowable tensile or compressive stress for the stress σ in Eq. (186), we can determine cross-sectional area for a given load to ensure the strength of a machine part.

The allowable tensile stress is denoted as R_z and Eq. (186) becomes

$$P = R_z F. \tag{193}$$

We replace stress σ by the allowable compressive stress R_d in the same equation, which becomes

$$P = R_d F. \tag{194}$$

Eqs (193) and (194) are for *solving allowable tension and compression.*

In determining the cross-sectional area of a part, the magnitude of the tensile or compressive load and the allowable stresses must be known. As already noted, the allowable stress is a fraction of ultimate strength and may be expressed as

$$R = \frac{\sigma_b}{n}, \tag{195}$$

in which R is the allowable tensile or compressive stress (R_z or R_d), σ_b is the ultimate strength, and n is a number indicating how much larger the second is than the first and called the *factor of safety.*

This safety factor is not a constant, it must ensure the strength of the given part against permanent set and depends on a number of circumstances. For instance, the safety factor for brittle materials is larger than for elastic materials, and larger for a dynamic load than for a static load, etc.

Illustrative Problem 105. Arm AB of the bracket in Illustrative Problem 6 (Sec. 21) is to be made of mild steel and round cross section. Calculate its diameter d if the allowable stress R_z is 1,400 kg/cm².

Solution. In the quoted problem it was found that force $\mathbf{P}_1$ acting along arm AB was equal to 900 kg. Whereupon

$$F = \frac{\pi d^2}{4} = \frac{P}{R_z},$$

whence $d = \sqrt{\frac{4P}{\pi R_z}} = 9$ mm.

Illustrative Problem 106. Eyebolt *2* in Fig. 305 is passed freely through the wooden beam *1* and has a rod *6* suspended from pin *5* passed thoough its eyes. A tensile force $\mathbf{P}$ is applied to the rod. Denote the internal diameter of the bolt as d_1 and calculate the dimensions required for the various members of this assembly.

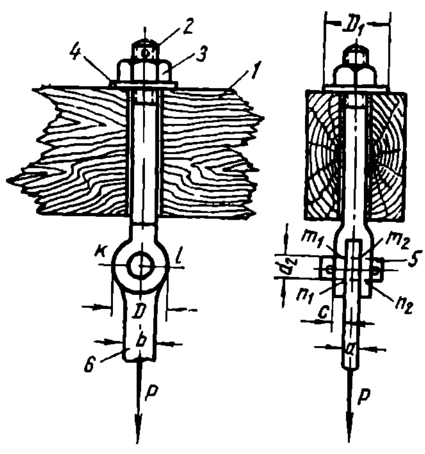

Fig 305

Solution: from Eq. (193),

$$F = \frac{P}{R_z} = \frac{\pi d_1^2}{4},$$

whence

$$d_1 = 2\sqrt{\frac{P}{\pi R_z}},$$

which, when the numerical value of the allowable stress enters the equation, gives us the internal diameter of the bolt, and from Standards Tables (OST) we then find the corresponding external diameter of the bolt.

Then we calculate the dimensions of the eyes of the bolt. We see that failure may take place along cross-section kl, the area of which for the two eyes is $2(D - d_2)c$, in which d_2 is the diameter of the hole for the pin, and c is the thickness of the eyes. Hence the design equation will be

$$P = 2(D - d_2)cR_z.$$

Lastly, we calculate the size of rod 6. With a thickness of a and a width of b, the area resisting fracture $F_1 = ba$. The design equation is $baR_z = P$, from which, having found the area ba, we can take some suitable value either for a or b, and calculate the other dimension. Since at cross-section kl the rod is weakened by the hole required for pin 5, it must satisfy the equation

$$(D - d_2)aR_z \geqslant P.$$

216. Compression and Buckling

Try the following experiment: place a thin steel bar in a vertical position on spring scales as shown in Fig. 306. Press your hand vertically down upon the upper end of the bar, thus gradually increasing the axial compressive force **P** but maintaining the bar's vertical position. By observing the pointer on the scales we will see that force **P** increases but the bar remains straight. As we increase the pressure, it reaches a point at which the bar begins to bend, but when we release the pressure, it recovers its original shape. If we increase the force beyond the point where the bar just begins to bend, bending will increase and, with further pressure, the bar will acquire a permanent set and then break.

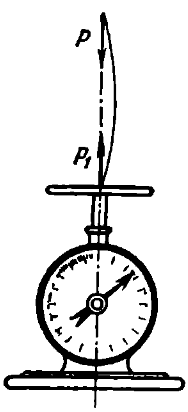

Fig. 306

If we repeat this experiment with bars of different lengths but with their cross-sectional dimensions and their material remaining the same, we shall see that the magnitude of the compressive force at which the deflection of the bar ceases to be elastic depends upon its length, i. e., as the length increases, the magnitude of this force decreases. This is called the *critical load* and is denoted as $\mathbf{P}_{cr}$.

Thus, the failure of an axially-compressed bar may occur not because compressive stresses exceed their allowable magnitude, but because of longitudinal distortion, technically known as *loss of stability* and resulting in *buckling*.

The eminent Russian scientist Academician L. Euler was the first to investigate buckling; the formula determining the critical, or buckling, load for slender columns is known as Euler's Equation.

Columns, compression struts of various types of trusses, the connecting rods of piston engines, and other machine elements and members of engineering structures are all subject to buckling.

217. Questions for Review

1. Two bars of the same material and similar cross-sectional area become elongated to different extents under equal loads. What is the explanation for this? What can be said about their unit elongations?

2. Can it be said that absolute elongation is proportional to unit elongation?

3. Why must machine parts be made so they can acquire only elastic strain?

4. Does the load alone indicate the magnitude of the stress in a material? Does the absolute strain indicate it?

5. What can be said of the modulus of elasticity of materials that have different unit elongations under the same load?

6. Under what condition will a vertically suspended bar fail because of the action of its own weight? At what cross-section will the fracture take place?

218. Exercises

105. A chain made of 16-mm round steel links is under a load of two tons. Calculate the tensile stress in its links.

106. What will be the absolute elongation of a steel rod 8 m long and 60 mm in diameter under the action of a load $P = 30$ tons? (The modulus of elasticity $E = 2{,}000{,}000$ kg/cm².)

107. What stress is created in a steel bar 6 m long and 25 mm in diameter if its absolute elongation under a load is 3 mm? (The modulus of elasticity $E = 2{,}000{,}000$ kg/cm².)

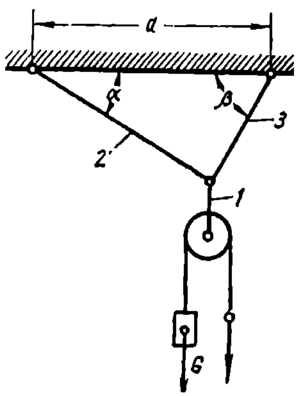

Fig. 307

108. A copper bar 100 mm in diameter was tightly fixed between two immovable walls when the temperature was 15°C. What stress will be created in the bar and what pressure will it exert on the walls at a temperature of 50°C? (The coefficient of linear expansion of copper is 0.0000167; and $E = 1{,}000{,}000$ kg/cm².)

109. At what length would a bar of mild steel break under the action of its own weight if its ultimate strength is 4,000 kg/cm²? (Specific gravity $\gamma = 7.85$ g/cm³.)

110. A copper bar that is being tested in a tensile testing machine has an initial length, between two marks, $l = 200$ mm. Under a load $P = 500$ kg the marked length elongates 0.032 mm. Find its modulus of elasticity if the diameter of the bar was 20 mm before the test began.

111. The chain of a hoisting machine is under a load $G = 500$ kg. Find the tensile stress in the links of the chain if it is made of 8-mm round steel.

Hint to solution. It must not be forgotten that the load is distributed between two cross-sections.

112. What must be the diameter of rods *1*, *2*, and *3* (Fig. 307) whose allowable stress is 1,400 kg/cm², $\alpha = 30°$, $\beta = 60°$, and from which a fixed pulley is hung and with the aid of which a load $G = 2$ tons is raised?

CHAPTER XXIII

SHEAR AND TORSION

219. Shear (Strain in Lateral Displacement)

A sheave, prismatically keyed to a shaft revolving as shown by the arrow in Fig. 308*a*, transmits rotation to another (driven) sheave mounted on a parallel shaft. A force **P** is acting from the first-mentioned shaft (right to left) upon the lower part of the key, while an equal and opposite force **P′** is exerted on the part of the key entering the key-seat in the sheave's hub. If the key is not strong enough it will shear along line *mn* as shown in Fig. 308*b*. From this it follows that internal stresses, due to the interaction of these forces, are set up along line *mn* that resist the shear.

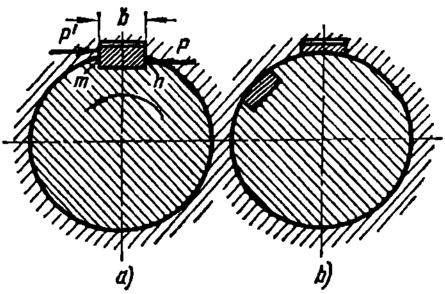

Fig. 308

By dividing the internal force, equal to force **P**, by the area F of the cross-section *mn*, we obtain the stress τ:

$$\tau = \frac{P}{F} \tag{196}$$

or, by denoting the width of the key as b and its length as l,

$$\tau = \frac{P}{bl}$$

The stress τ is the shearing stress and is expressed in kg/cm² or kg/mm². This stress acts along the plane of strain *mn* and is a *tangential stress.* Depending on the magnitude of the tangential stress, strain may be either elastic or plastic, or it may even result in complete shearing of the member.

220. Determining the Amount of Shear Strain. The Modulus of Elasticity for Shear

Let us investigate, as we did in the case of tension, what quantity may be used to measure the magnitude of shear strain.

Assume that two equal and opposite forces **P** and **P′** are acting on a portion of a beam (Fig. 309*a*) at cross-sections AC and BD, situated at a small distance x from each other. Under the action of these forces the parallelepiped $ABDC$ (shown at an enlarged scale in Fig. 309*b*) will become twisted and take the form of the parallelogram AB_1D_1C in Fig. 309*c*. Hence, points B and D, and any other point lying along segment BD, will be shifted with respect to segment AC to the extent of $BB_1 = DD_1 = s$.

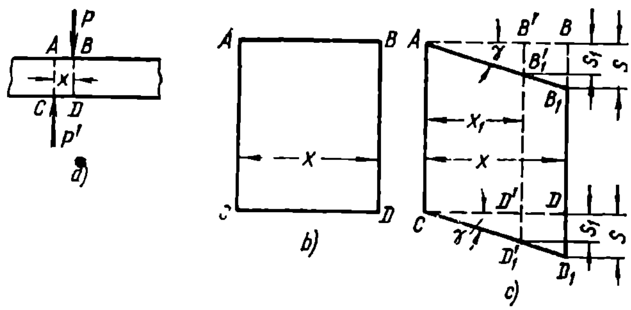

Fig. 309

This quantity s may be considered to be the absolute shear strain in the section of beam under consideration. However, just as with absolute elongation, absolute shear does not give a full picture of the degree of strain, since this latter depends on the dimensions of the body.

If we take cross-section $B'D'$ at a distance of x_1 from AC, absolute shear strain will be $B'B_1' = D'D_1' = s_1$. From the similarity of triangles ABB_1 and $AB'B_1'$ we obtain

$\frac{s}{s_1} = \frac{x}{x_1}$ or $\frac{s}{x} = \frac{s_1}{x_1}$ which denotes a certain value of γ.

Thus we see that the ratio between absolute shear strain and the distance between cross-sections is a constant and for that reason is used as a measure of shear. The quantity γ is called *unit shear strain* (the angle of shear).

Wherefore, *unit shear strain is equal to absolute shear strain divided by the distance between the planes of shear.*

When investigating elongation we found that the stress σ is proportional to unit elongation within the limits of elastic strain. Theory confirmed by empirical research show that the same relationship exists with respect to shear, that is,

$$\tau = G\gamma, \qquad (197)$$

in which τ is the shearing stress, γ is the unit shearing stress, and G is a coefficient called the *modulus of elasticity for shear.* This equation is analogous to Eq. (192). And since τ is expressed in kg/cm^2 (or kg/mm^2) while γ is an abstract number, the modulus of elasticity G for shear is expressed in the same units as stress (usually in kg/cm^2).

Several values for the modulus G, given in kg/cm^2, are: carbon steel – 810,000; aluminium – 260,000 to 270,000; copper – 400,000, etc.

221. Allowable Shear

Let us denote R_s as the *allowable shearing stress* considered necessary to ensure the strength of a part. By assuming that the stresses are equally distributed over the whole cross-section, we obtain the following design equation for shear:

$$P = R_s F \tag{198}$$

with which, knowing the given force **P** and the allowable shear stress R_s, we can determine the area of the cross-section required to ensure necessary strength of a part. The value for allowable shear R_s, just as for tensile and compressive stresses, varies in each case and depends on the material and specific conditions that must be provided for. Allowable shear is smaller than allowable tensile stress R_z. It may be approximately considered that R_s for plastic materials ranges from 0.55 R_z to 0.7 R_z and for brittle materials from 0.8 R_z to R_z.

Illustrative Problem 107. In Fig. 305 the pin *5*, referred to in Illustrative Problem 106, has a diameter $d_2 = 15$ mm. What is the maximum shear **P** that it can withstand if the allowable shear $R_s = 900$ kg/cm^2?

Solution: under the action of load **P** the pin may shear along two planes m_1n_1 and m_2n_2, corresponding to the planes of contact of the eyes and the rod *6*. This shear strain is resisted by two cross sections of area $\frac{\pi d_2^2}{4}$ each. Hence, through Eq. (198) we obtain

$$P = \frac{2\pi \times 1.5^2}{4} \times 900 \approx 3{,}200 \text{ kg}.$$

Illustrative Problem 108. A shaft transmits torque $M_t = 29$ kg-m. Find the shear exerted on the prismatic key (Fig. 308) along section *mn* if the diameter of the shaft $d = 45$ mm, the width of the key $b = 14$ mm, and its length $l = 70$ mm.

Solution: the plane of shear $F = bl = 14 \times 70 = 980\ mm^2 = 9.8\ cm^2$. The stress along plane *mn* is equal to $P = M_t : \frac{d}{2} = \frac{2{,}900*}{2.25}$ kg, and the shearing stress exerted on the key

$$\tau = \frac{P}{F} = \frac{2{,}900}{2.25 \times 9.8} = 132 \ kg/cm^2.$$

* We express torque in kg-cm, and diameter in cm.

Illustrative Problem 109. Tensile forces $P = P' = 2{,}900$ kg are acting in opposite directions on the lap joint shown in Fig. 310. It is held together by two rivets of diameter $d = 13$ mm. Find the shearing stress in the shanks of the rivets.

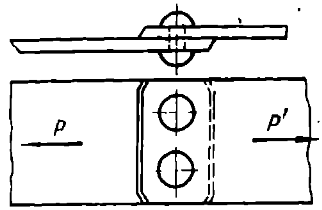

Fig. 310

Solution: the rivets may shear along the cross-sections of their shanks where both laps of the joint are in contact with them. The area of shear of the two rivets $F = 2\frac{\pi d^2}{4}$, and the shear in their shanks

$$\tau = \frac{P}{F} = 2{,}900 : \frac{\pi \times 1.3^2}{2} = \frac{2{,}900 \cdot 2}{\pi \cdot 1.3^2} = 1{,}094 \text{ kg/cm}^2.$$

222. Punching of Metals and Cutting Them with Steel Blades

Shear strain is taken advantage of in cutting metals by means of dies and steel blades. Unlike machines and other engineering structures where strain must not be allowed to exceed the elastic limit, strain in cutting is carried to the failure stage of the material along the plane of shear.

Fig. 311 is a schematic illustration showing strain in metal when it is being perforated by a punch. Under the pressure **P** of the punch A, the metal first begins to bend within the die B (Fig. 311a) at the same time acquiring compressive strain. As the punch subsequently presses harder on the metal, the stress becomes so great that the metal begins to shear, which is manifested by crack formation in the workpiece along the edges of the punch and die (Fig. 311b). The same thing occurs when metal is cut with steel.

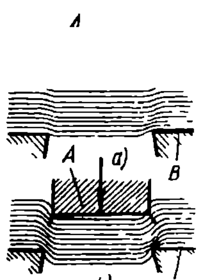

Fig. 311

Pressure **P** is determined according to the equation

$$P = \sigma F, \tag{199}$$

in which F is the area of shear (the product of the length of the shear l and the thickness of the metal δ), σ is the shear in the metal and is found empirically. For steel, for example, it has been thus determined as

$$\sigma = 11.0 + 0.56\,\sigma_b \text{ kg/mm}^2, \tag{200}$$

in which σ_b is ultimate strength.

Illustrative Problem 110. Find the pressure P required to punch a hole of diameter $d = 10$ mm in sheet steel of thickness $\sigma = 8$ mm and ultimate strength $\sigma_b = 60$ kg/mm².

Solution: the pressure (stress) $\sigma = 11.0 + 0.56 \times 60 \approx 45$ kg/mm.² The perimeter of the hole $l = \pi d = \pi \times 10 = 31.4$ mm; the area of shear $F = l\delta = 31.4 \times 8 \approx 251$ mm², and the pressure required is $P = F\sigma = 251 \times 45 \approx 11{,}300$ kg.

223. Torque

Assume that we have a cylinder on which we delineate a line A_0B_0 parallel to the axis on one side, and mark two plane circular sections mn and m_1n_1 (Fig. 312*a*) lying at a short distance x from one another. Assume that the cylinder is fixed at its left end and that torque M_t is applied to the right end. The torque is of course balanced by an equal and opposite moment acting on the fixed end of the cylinder. Under the action of these moments the cylinder experiences strain.

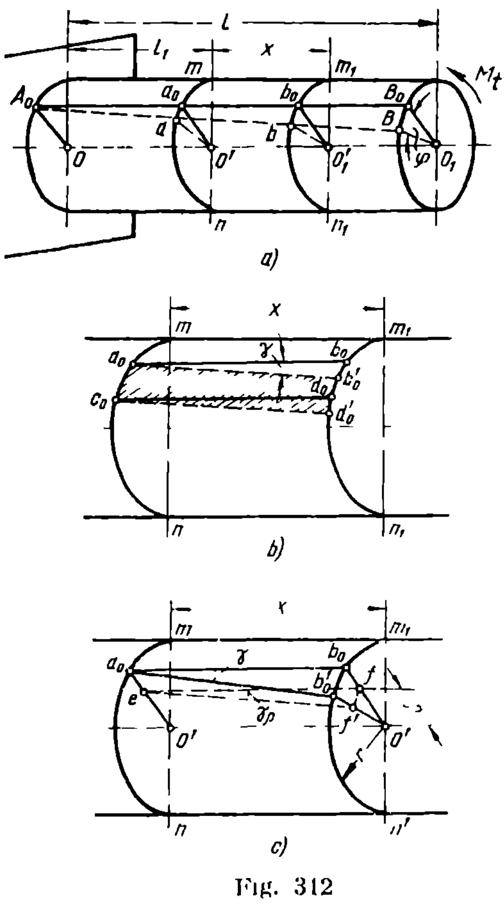

Fig. 312

This strain consists in the turning of sections mn and m_1n_1 with respect to each other around the axis of the cylinder OO_1. The line A_0B_0, which was perpendicular to the cross-section before torque was applied, becomes a helix whose tangent is inclined towards the cross-section. The points A_0 and B_0, which were mutually exactly opposite at different ends of the cylinder, are now shifted with respect to each other along a length of the arc B_0B which corresponds to the central angle $B_0O_1B = \varphi$. This is called the *angle of twist* and shows the absolute strain. By dividing this angle by the length l of the twisted cylinder, we obtain the *unit angle of twist*. By denoting this angle by the letter θ we obtain

$$\theta = \frac{\varphi}{l}. \tag{201}$$

If we express the angle φ in degrees, unit twist will be expressed in degrees divided by the length in metres or centimetres. By knowing the unit angle of twist θ, it is possible to calculate the angular displacement of one plane circular section with respect to another when the distance between them is given. Thus, the displacement of plane section m_1n_1 in relation to plane section mn will be $\varphi_x = \theta x$, and with respect to the left-end plane section it will be $\theta(l_1 + x)$.

224. Torque as a Form of Shear

Assume that two lines a_0b_0 and c_0d_0 (Fig. 312*b*) are delineated between sections mn and m_1n_1 on the already-mentioned cylinder before it was subjected to torsion. If the figure $a_0b_0d_0c_0$ were unfolded it would be a rectangle. And it is clear that after twisting, this rectangle will become a parallelogram $a_0b_0'd_0'c_0$. The line a_0b_0, which was at first perpendicular to the section mn, will become inclined through an angle $b_0a_0b_0'$, point b_0 will shift to b_0', line c_0d_0 will be inclined through $\angle d_0c_0d_0' = \angle b_0a_0b_0'$, and point d_0 will shift a distance $d_0d_0' = b_0b_0'$. By comparing this strain with the shear strain shown schematically in Fig. 309*c*, we see that it is similar except that in the given case the displacement of a point on plane section m_1n_1 is along a peripheral arc, whereas in the former case point B, like any other point on section BD, was displaced along a straight line. Therefore, we come to the conclusion that *torque is a form of shear*.

225. Distribution of Torsional Stress in a Plane Circular Section

If torque strain is distinguished by the turning of one plane section with respect to another, then the same relationship is valid for it as between stress and unit elastic strain, as already expressed in Eq. (197), i.e.,

$$\tau = G\gamma,$$

in which G is the modulus of elasticity for shear.

It need only be determined whether the tangential stress τ is the same at all points on the plane or whether it changes according to some specific principle. To answer this question we must first find if the unit strain γ is constant at all points on the plane section.

Let us assume as before that for the short distance x between circular plane sections mn and m_1n_1 (Fig. 312*c*), absolute strain is expressed by the arc b_0b_0'. The length s of this arc is related to the angle $b_0a_0b_0'$ as given in degrees and to the distance x through the identities

$$s = b_0b_0' = \frac{2\pi x}{360}\alpha = \frac{\pi x \alpha}{180}.$$

By entering the unit shear strain γ instead of $\frac{\pi\alpha}{180}$, we obtain

$$s = \gamma x. \tag{202}$$

On the other hand, we can find the length of this arc from plane section m_1n_1 in which it corresponds to the central angle $b_0O'_1b'_0$; this angle, as already explained in Sec. 223, is the angle of twist φ_x along the distance x and is equal to θx:

$$s = \frac{2\pi r\theta x}{360} = \frac{\pi r}{180}\theta x,$$

in which r is the radius of the plane circular section.

By equating its right member with Eq. (202), we evolve

$$\gamma x = \frac{\pi r}{180}\theta x,$$

whence

$$\gamma = \frac{\pi r\theta}{180}. \tag{a}$$

We now mark point f at a distance of ϱ from the axis on radius O'_0b_0 of the plane section m_1n_1 and determine its absolute displacement as expressed by the length of the arc ff'. According to Eq. (202) the length of this arc

$$s_\varrho = \gamma_\varrho x, \tag{b}$$

in which γ_ϱ is the unit torque at this point, as distinguished from γ (for s_ϱ is not equal to s although the length x is the same). On the other hand

$$s_\varrho = \frac{2\pi\varrho}{360}\theta x = \frac{\pi\varrho}{180}\theta x.$$

By equating its right member with Eq. (b) we obtain

$$\gamma_\varrho = \frac{\pi\varrho\theta}{180}.$$

Then by dividing each member of this equation by equation (a), we evolve the sought-for equation:

$$\frac{\gamma_\varrho}{\gamma} = \frac{\varrho}{r}. \tag{203}$$

Wherefore, *the unit strain at various points of the cross-section is proportional to the distance of these points from the axis of the cylinder subjected to torsion.*

By denoting τ as the shearing stress at a point on the side of the cylinder and τ_ϱ as a similar stress at a point lying on the same plane section at a distance of ϱ from the axis, then according to Eq. (197)

$$\tau = G\gamma, \tag{204}$$

$$\tau_\varrho = G\gamma_\varrho, \tag{205}$$

$$\frac{\tau_\varrho}{\tau} = \frac{\varrho}{r}. \tag{206}$$

Wherefore, ***the stress at different points on the cross-section of a cylinder is not constant under a given torque; it is proportional to the distance of the point from the axis of the cylinder.***

The greatest stress is at points farthest from the axis, i.e., on the surface of the cylinder. It follows that shearing strain is zero at points on the axis of the cylinder, and therefore the stress is also zero at these points. This can be expressed graphically as shown in Fig. 313: if we imagine that the cross-section is composed of an infinite number of concentric rings, we can then evolve the principle governing the change in stress by delineating the triangle ABC. The stresses are greatest at the surface of the cylinder and decrease as the diameters of the rings decrease.

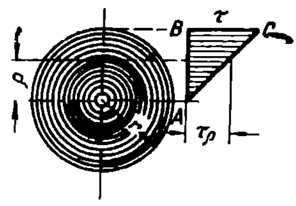

Fig. 313

Oral Exercises

1. Which angle in Fig. 312*c* is larger, γ or γ_ρ?
2. Is absolute shear strain the same at all points on a circular section? Is unit strain the same?

226. The Fundamental Equation for Torque

Heretofore we have established the relationship between stresses in a strained body and the external forces causing these stresses. Torque strain is caused by the action of the torsional moment. Now let us see how torsional stress is determined if the moments causing torque are known.

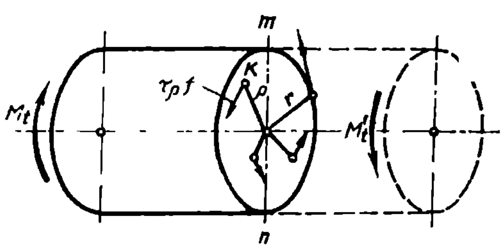

Fig. 314

We shall use the already-explained cross-section method. Let a cylinder be twisted by equal and opposite torques M_t and M'_t applied at its two ends (Fig. 314). Assume that we cut the cylinder at section mn; after discarding the right half, we examine the remaining half as a free body to see what forces or moments are acting upon it to keep it in equilibrium.

Moment M_t is applied to this half of the cylinder, and in cross-section mn there are elementary shearing forces acting opposite in direction to moment M_t. In order that this half of the cylinder be in equilibrium, the sum of the moments of the elementary shearing forces must be equal to the moment of the external forces M_t.

At an arbitrary point K we select an infinitesimally small area f at a distance of ϱ from the axis of the cylinder. By denoting τ_ϱ as the shear at this point, we obtain the elementary shearing force in this area, which is equal to $\tau_\varrho f$, and the moment of this force with respect to the axis of the cylinder, which is equal to $\tau_\varrho f \varrho$. There will be as many moments as there are such areas in the cross-section mn and the sum of these areas will be equal to the entire area of the cross-section. Whereupon, by indicing all these small areas, their distances from the axis, and their corresponding stresses, we obtain the following equation for equilibrium of the given free body of the cylinder:

$$M_t = \tau_{\varrho 1} f_1 \varrho_1 + \tau_{\varrho 2} f_2 \varrho_2 + \tau_{\varrho 3} f_3 \varrho_3 + \ldots \text{etc.} \tag{a}$$

By denoting, as before, τ as the stress in the plane circular section of radius r, we obtain, on the basis of Eq. (205),

$$\tau_{\varrho 1} = \tau \frac{\varrho_1}{r}; \quad \tau_{\varrho 2} = \tau \frac{\varrho_2}{r}; \quad \tau_{\varrho 3} = \tau \frac{\varrho_3}{r}, \text{ etc.,}$$

which, after it enters (a), becomes

$$M_t = \tau \frac{\varrho_1}{r} f_1 \varrho_1 + \tau \frac{\varrho_2}{r} f_2 \varrho_2 + \tau \frac{\varrho_3}{r} f_3 \varrho_3 + \text{etc.,}$$

or

$$M_t = \frac{\tau}{r} (f_1 \varrho_1^2 + f_2 \varrho_2^2 + f_3 \varrho_3^2 + \cdots \text{etc.}). \tag{b}$$

The expression enclosed in parenthesis comprises the sum of the products of all the elementary areas multiplied by the squares of their distances from the axis of the cylinder, and is called the *polar moment of inertia of the cross-section with respect to the axis.* Its denotation is J_p.

Accordingly, equation (b) acquires the form

$$M_t = \frac{J_p}{r} \tau. \tag{207}$$

This relationship, which is the fundamental equation for torque, links torque with the maximum stress τ on the surface of the torsion-subjected cylinder in the given section by means of the polar moment of inertia and the radius of the section.

The quotient obtained by dividing the polar moment of inertia by the radius of the plane section is the *cylinder's resisting moment under torque* and is denoted as W_p, i. e.,

$$W_p = \frac{J_p}{r} \tag{208}$$

and

$$M_t = W_p \tau. \tag{209}$$

Since the full deductions of this formula are extremely complicated in calculating the magnitude J_p of the cylinder, we have evolved it in its final form as follows:

$$J_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}, \tag{210}$$

where r is the radius and d is the diameter of the plane section.

After placing this value into Eq. (208), we obtain

$$W_p = \frac{\pi r^4}{2r} = \frac{\pi r^3}{2} = \frac{\pi d^3}{16} \tag{211}$$

and

$$M_t = \frac{\pi d^3}{16} \tau. \tag{212}$$

Of course both sides of this equation are to be expressed in the same units; if d is given in cm, and τ is in kg/cm², the right half of equation will be in cm³ $\times$ kg/cm² = kg-cm, and accordingly the left part of the equation will be expressed in kg-cm.

Having found the relationship between the stress τ and the torque we can now determine the angle of twist φ of the cylinder. From Eq. (207)

$$\tau = \frac{M_t r}{J_p}.$$

On the other hand, according to Eq. (204),

$$\tau = G\gamma.$$

Consequently $G\gamma = \frac{M_t r}{J_p}$, from which $\gamma = \frac{M_t r}{GJ_p}$.

By installing here the value of γ from Eq. (a), Sec. 225, we obtain

$$\frac{\pi r \theta}{180} = \frac{M_t r}{GJ_p}, \text{ from which } \theta = \frac{180}{\pi} \times \frac{M_t}{GJ_p}. \tag{213}$$

Angle θ is the angle of twist in degrees per unit of length. Hence the full angle of twist along the length l of the cylinder

$$\varphi = \frac{180}{\pi} \times \frac{M_t l}{GJ_p}. \tag{214}$$

It is fully evident that the greater the torque and the greater the length, the greater will be this angle; and the greater the modulus of elasticity for shear and the greater the diameter, the smaller will be the angle.

Oral Exercises

1. What points on the cross-section correspond to stress τ referred to in Eqs (207), (209), and (212)?

2. What change will there be in stress τ if the diameter of the cross-section is increased while the torque remains the same?

Illustrative Problem 111. A steel shaft of diameter $d = 60$ mm and length $l = 1{,}500$ mm is subject to torsion by two equal and opposite moments $M_t = M_t' = 8{,}600$ kg-cm. Determine the maximum stress on a cross-section of the shaft and the angle of twist φ.

Solution: we find the stress τ, acting on the surface of the shaft, through Eq. (212):

$$\tau = \frac{16M_t}{\pi d^3} = \frac{16 \times 8{,}600}{\pi \times 6^3} = 203 \text{ kg/cm}^2.$$

We find the angle of twist φ in degrees through Eq. (214). By entering $l = 150$ cm and the modulus of elasticity for shear $G = 800{,}000$ kg/cm² we calculate

$$\varphi = \frac{180}{\pi} \times \frac{8{,}600 \times 150 \times 2}{800{,}000 \times \pi \times 3^4} \approx 0.73^\circ.$$

227. Computing the Dimensions of Shafts for a Given Torsion

The stress τ, as we have learned, is the maximum shear in a cylinder under the action of torque M_t. If we replace τ in Eq. (212) by the allowable shear R_s, the equation can be used to calculate the dimensions of a shaft which is to transmit a definite torque M_t. Whereupon the equation becomes

$$M_t = \frac{\pi d^3}{16} \times R_s \approx 0.2d^3R_s, \tag{215}$$

from which the diameter

$$d = \sqrt[3]{\frac{M_t}{0.2R_s}}\,. \tag{216}$$

Another way to find the diameter would be to express it in relation to the power it transmits and the rpm, substituting $71{,}620\,\frac{N}{n}$ kg-cm for M_t in the equation:

$$d = \sqrt[3]{\frac{71{,}620}{0.2R_s} \times \frac{N}{n}}\,. \tag{217}$$

As for the allowable stress R_s, its magnitude depends on the material and conditions of service, for steel it ranges from 200 to 1,200 kg/cm². For instance, for steel transmission shafts subject to ordinary service conditions the accepted magnitude $R_s = 420$ kg/cm²; for short but not heavily loaded shafts $R_s = 600$ kg/cm²; and when the shaft is subject to impact, $R_s = 280$ kg/cm², etc.

When calculations are made on the basis of these figures, the strength of the shaft is ensured. Nevertheless, the diameter obtained by this method is often checked by means of a special calculation of the unit angle of twist of the shaft, which is ordinarily within the limits of 1/4°-1/2° per metre of length.

Calculating the dimensions of heavily loaded shafts (in steam engines, turbines, internal combustion engines, etc.) is considerably complicated because, aside from twist, such members are subject to extensive bending. Furthermore, in computing vital construction, it must also be remembered that additional margins must be included to make up for such shaft weakening factors as keyways, transition from one diameter to another, dynamic loads, and so forth.

Illustrative Problem 112. Sheave *1* mounted on a shaft as shown in Fig. 315 attains $n = 200$ rpm and receives power $N_1 = 18$ hp which is distributed through sheaves *2* and *3* to two other shafts (not shown) in the following way: sheave *2* transmits $N_2 = 12$ hp to a second shaft and sheave *3* transmits $N_3 = 6$ hp to a third one.

What must be the diameter of the first shaft if its allowable stress $R_s = 400$ kg/cm²?

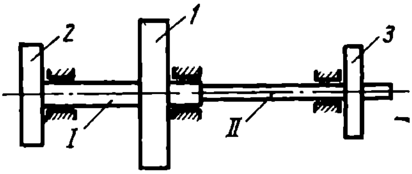

Fig. 315

Solution: as we see from the illustration, length *I* of the shaft imparts $N_2 = 12$ hp. The diameter of this length, according to Eq. (217),

$$d_1 = \sqrt[3]{\frac{71,620}{0.2 \times 400} \times \frac{12}{200}} \approx 3.8 \text{ cm} = 38 \text{ mm}.$$

Since the Standards (OST) do not include a shaft of exactly this diameter, we take the next largest, which is $d_1 = 40$ mm.

In the same way we calculate the diameter for length *II*:

$$d_2 = \sqrt[3]{\frac{71,620 \times 6}{0.2 \times 400 \times 200}} \approx 3 \text{ cm} = 30 \text{ mm}.$$

228. Questions for Review

1. How is the magnitude of shear strain measured?
2. In what plane does shear act in relation to the plane of action of external forces?
3. What is denoted by angle γ and segment s in Fig. 309*c*?
4. Analyse Eq. (203). Can it be used in cases of permanent set?
5. Why is torque regarded as a form of shear?

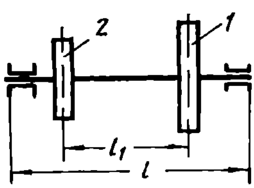

Fig. 316

6. What is the difference between torque and regular shear?
7. Name the following symbols in Fig. 312*c* and explain what they represent: γ, angle $b_0O_1'B_0'$ and the arc b_0b_0'.
8. The mechanical energy obtained by means of sheave *1* (Fig. 316) is transmitted to another shaft by means of sheave *2*. Along which length (l or l_1) is the shaft subject to torque?
9. In what direction will elementary forces $\tau_\varrho f$ (Fig. 314) act if we consider the right half of the torsion-subjected cylinder as a free body?
10. What change would there be in the solution to Illustrative Problem 112 if sheaves *1* and *3* transmitted power received from sheave *2*?

229. Exercises

113. In a rivetted joint (Fig. 317) the thickness δ of plates *1* and *2* is 8 mm, their width $b = 100$ mm, and the diameter of the rivets $d = 13$ mm. Find the shear τ in the shanks of the rivets and the tensile stress σ in the plates, assuming that all the rivets

are carrying the same load and that the external forces $P = P' = 4{,}000$ kg.

Hint to solution. The weakening of the lateral cross-section of the plates due to the holes for the rivets must be taken into account.

114. A sheave of diameter $D = 800$ mm rotates uniformly under the action of peripheral force $P = 50$ kg. Find the shearing stress exerted on the key in Fig. 308 if the diameter of the shaft $d = 50$ mm, and the dimensions of the key are: width $b = 16$ mm and length $l = 80$ mm.

115. Determine the maximum stress τ in the cross-section of the shaft in Ex. 114.

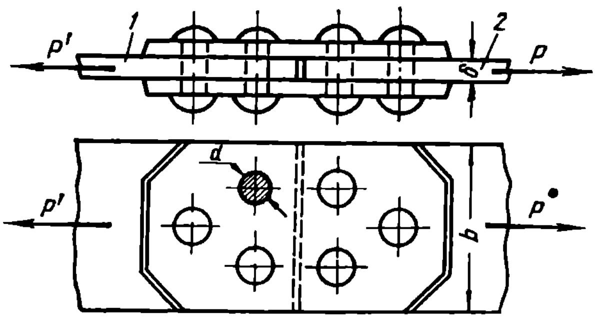

Fig. 317

116. Find the maximum stress in a shaft of diameter $d = 45$ mm which transmits $N = 18$ hp and attains $n = 180$ rpm.

117. Assume that sheave *1*, in Illustrative Problem 112, Fig. 315, receives the same power $N_1 = 18$ hp but changes its place with sheave *2* which transmits $N = 12$ hp. Calculate the diameter required for lengths *I* and *II* of the shaft with the same allowable stress of 400 kg/cm², and also state which of the two arrangements of the sheaves is more advantageous.

118. Solve Ex. 117 by assuming that sheave *1* ($N_1 = 18$ hp) is in the place of sheave *2*, sheave *3* ($N_3 = 6$ hp) is in the place of sheave *1*, and sheave *2* takes the place of sheave *3*.

CHAPTER XXIV

BENDING

230. The Nature of Bending Strain

Let us take a prismatic wooden beam in which several cuts *ab*, *cd*, *ef*, etc., have been made perpendicular to its axis and extending half-way up its height. We place it on supports *A* and

B with the cuts facing downward (Fig. 318*a*) and apply force **P** to it. Under the action of this force the beam bends, and the cuts *a*, *c*, *e*, etc., widen and take the form of trapezoids as shown in the same drawing at the right. As compared with a beam under the same load but without cuts, of course this beam will bend to a much greater extent and fail sooner.

The change in form of the cuts shows that the fibres in the convex side, curved from the bending of the beam, are stretched; the cuts on that side weaken the beam.

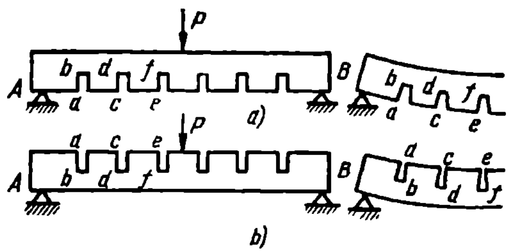

Fig. 318

Now let us repeat the experiment but place the beam with the cuts upward (Fig. 318*b*). Again applying force **P** we see that the cuts *a*, *c*, *e* in the concave side are now drawn together as is shown on the right, and when the force reaches a certain magnitude the cut edges will touch each other, after which greater resistance to bending will be set up in the beam. Then, removing the load from the beam, we fill the cuts snugly with little slabs of wood and again apply the same force with the result that the beam will resist the action of force **P** just as if it had no cuts and the little wooden slabs will be held tightly in place. From all this we must come to the conclusion that the fibres* in the concave part of the beam are under compression.

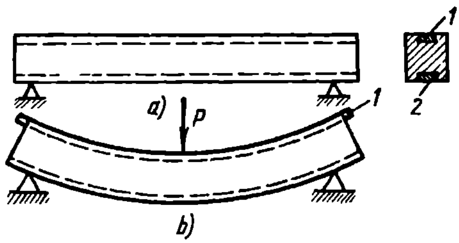

Fig. 319

In order to better understand the phenomenon just described, let us perform another experiment with another wooden beam. This time we make longitudinal dovetail grooves in opposite sides of the beam for its entire length (Fig. 319*a*), and insert into them planks of wood of the same shape as the cuts and the same length as the beam. When the beam has not yet been subjected to deformation, they fit exactly in place. But when the beam is bent under the action of force **P** (Fig. 319*b*) we shall see that the ends of plank *1* on the concave side of the beam protrude beyond the ends of the beam, whereas the ends of plank *2* on the convex side

* The term *fibres* is figuratively given to longitudinal elements, of infinitely small cross-section, in beams, bars, etc.

are drawn within the groove. From this we again conclude that the fibres on the concave side of the beam are compressed and those on the convex side are stretched.

Wherefore, *in a bent beam the fibres in its concave side undergo strain of tension while the fibres in the convex side are subjected to compressive strain.*

231. Distribution of Normal Stresses During Bending. The Neutral Plane

Our experiment has thereby shown that bending of a beam is accompanied by the elongation of some fibres and the shortening of others; from this it is evident that in a beam subjected to bending, tensile and compressive stresses are set up which cause this strain. In order to determine the magnitude of such stresses at various points along a cross-section of a beam, it is first necessary to determine how strain of the fibres of the beam varies at different heights along the cross-section.

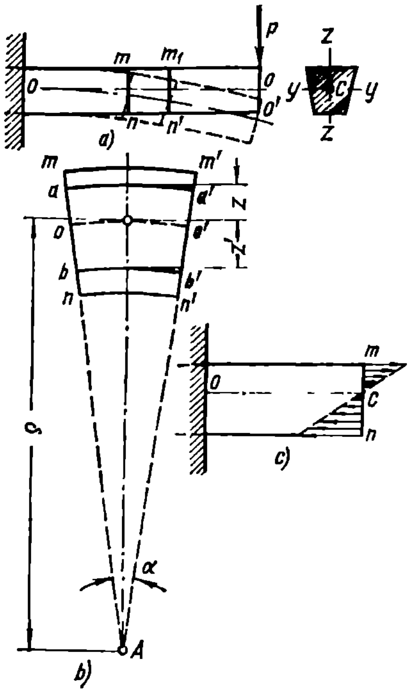

Fig. 320

Assume that a straight beam immovably fixed at one end is subjected at the other end to a force **P** applied in its plane of symmetry *zz* (Fig. 320*a*). As a result the beam bends and its axis becomes a curved line lying in the same plane of symmetry. Now let us assume that two straight lines *mn* and *m'n'* are delineated beforehand on the flank of the beam, perpendicular to its axis. Experiment has shown that when strain has occurred, these lines will remain straight but will no longer be parallel to each other. This means that, following strain, the cross-sections corresponding to these lines remain in the form of planes but turn relative to one another through a certain angle.

Now let us take two cross-sections *mn* and *m'n'* situated very closely together (Fig. 320*b*). When strain has occurred, these two sections will form a small angle $mAm' = \alpha$ with each other. It consequently follows that the fibres along the convex side of the beam — for example, in the same horizontal plane as mm', aa', etc., - are stretched while those in the same plane as nn', bb', etc., are shortened; and the further the fibre is from the convex surface the less it is stretched, and the further it is from the concave surface the less it is shortened. It is therefore evident that there should be fibres in some part of the beam that are not strained at all. There actually are such fibres in a beam; they lie in the plane yy (cross-section given in Fig. 320a) and coincide with axis oo which passes through the centre of gravity C of the cross-section*. The plane in which these unstrained fibres lie is called the *neutral plane*.

From all this there is no longer any doubt that all the fibres between the neutral plane and the convex side of the beam are elongated, and those between the neutral plane and the concave side are compressed.

Now let us see just how strain varies between fibres lying in different planes parallel to the neutral plane. Let us take fibre aa' in the strained section of the beam $mm'n'n$ (Fig. 320b), lying at a distance z from the neutral plane oo'. Since the neutral plane is neither elongated nor shortened, the length of a very small segment aa' between sections mn and $m'n'$ is the same before strain as that of oo'. Therefore the absolute elongation of this short length is equal to $aa' - oo'$

By denoting α as the angle mAm' formed by these sections, we obtain $aa' - oo' = \frac{2\pi(\varrho + z)\alpha}{360} - \frac{2\pi\varrho\alpha}{360} = \frac{\pi\alpha}{180} z$, in which ϱ is the radius of the small arc oo'.

Accordingly, the unit elongation of this segment

$$\varepsilon = \frac{aa' - oo'}{oo'} = \frac{\pi\alpha}{180} z : \frac{\pi\alpha}{180} \varrho = \frac{z}{\varrho} . \qquad \text{(a)}$$

By repeating this procedure for a very small segment bb' of the compressed fibre lying at a distance of z' from the neutral plane, we obtain the unit contraction of the segment as follows:

$$\frac{oo' - bb'}{oo'} = \frac{z'}{\varrho} . \qquad \text{(b)}$$

The radius ϱ may be considered to be constant when the angle α is very small. Therefore we deduce that the unit elongation and unit contraction are both proportional to the distance the fibre is from the neutral plane.

Eqs (a) and (b) may be combined into one as follows:

$$\varepsilon = \frac{z}{\varrho} . \qquad \text{(c)}$$

* It is seen that this plane is perpendicular to the plane of symmetry zz.

And since within the bounds of the elastic limit stress is proportional to strain, we obtain

$$\sigma = E\varepsilon = E\frac{z}{\varrho}, \qquad (218)$$

in which E is the modulus of elasticity and is alike for tensile or compressive strain.

This shows that in the given section stress is proportional to the distance of the fibre from the neutral plane. Therefore the fibres lying farthest from the neutral plane, along the convex or the concave surface of the strained beam, are subjected to the maximum stress. Fig. 320c shows graphically the distribution of normal stresses at section *mn* if we figuratively disregard the right half of the beam.

Another important point must finally be noted. Since the fibres lying in the neutal plane are not strained during bending, it follows that all cross-sections may be considered as turning about their corresponding axes oo' (Fig. 320b), i.e., around the straight lines along which these sections intersect the neutral plane. Each such straight line is the *neutral axis* of each given section and is perpendicular to the plane of symmetry (axis yy on the cross-section shown in Fig. 320a).

232. The Fundamental Equation for Bending

In using Eq. (218) to determine the normal stress σ at any point on a cross-section of a bent beam, we must not only know the modulus of elasticity E for the given material and the distance z of the point from the neutral plane, but also the radius ϱ of the arc oo' which is a segment of the bent longitudinal axis passing through the centre of gravity of the section. This radius we do not know, therefore this equation must be given a form in which the stresses causing the bending of the beam are used instead of radius ϱ. As an illustration, we shall continue the investigation of strain of the same preceding beam, using the cross-section method already familiar to us.

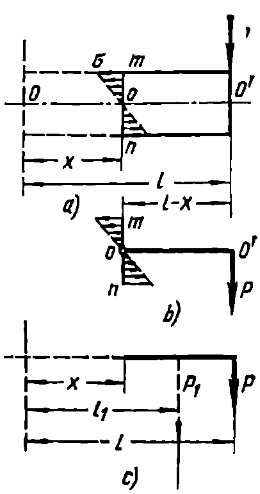

Fig. 321

Let us make a cross-section *mn* (Fig. 321a) through the beam at a distance of x from the plane of support. By figuratively discarding the left half of the beam, the right half as a free body is kept in equilibrium under the action of the external force **P** on the one hand, and by the internal normal forces directed perpendicular to the cross-section *mn* on the other. Thus we obtain the arrangement of forces *mnoO'* shown in Fig. 321*b*, constituting a three-member lever.

As already explained, section *mn* turns during deformation about the neutral axis which passes through point *o*. In order to express the conditions of equilibrium of such a lever, we must equate the algebraic sum of the moments of the internal forces with respect to the neutral axis and the moment of force **P** relative to the same axis. Taking a very small area f_1 on section *mn* at a distance of z_1 from the neutral plane, we find the moment of force acting on that area, which is equal to $\sigma_1 f_1 z_1$, and the algebraic sum of all the moments will be $\sigma_1 f_1 z_1 + \sigma_2 f_2 z_2 + \sigma_3 f_3 z_3$, etc.

The moment of the external force in relation to the neutral axis of the given section is the *bending moment* of that section. By denoting it as the letter M, we obtain

$$M = \sigma_1 f_1 z_1 + \sigma_2 f_2 z_2 + \sigma_3 f_3 z_3 + \ldots \text{etc.}$$

By applying Eq. (218), we may rewrite the above equation as follows:

$$M = \frac{E}{\varrho} f_1 z_1^2 + \frac{E}{\varrho} f_2 z_2^2 + \frac{E}{\varrho} f_3 z_3^2 + \cdots \text{etc.}$$

$$= \frac{E}{\varrho} (f_1 z_1^2 + f_2 z_2^2 + f_3 z_3^2 + \cdots \text{etc.}).$$

The sum in parenthesis is called the moment of inertia of the section about the neutral axis (i.e., axis *yy* in Fig. 320*a*) and is denoted by the letter J. Whereupon

$$M = \frac{EJ}{\varrho}. \qquad \text{(a)}$$

Eq. (218) gives us $\frac{E}{\varrho} = \frac{\sigma}{z}$.

Hence by substituting it in (a) we finally obtain

$$M = \frac{J}{z} \sigma, \qquad (219)$$

whence

$$\sigma = \frac{Mz}{J}. \qquad (220)$$

Thus, by knowing the bending moment and moment of inertia of the beam section, we can determine the normal stress at any point on that section which is at a distance of z from the neutral axis. It will be seen that the greater the moment M and the distance z, the greater the stress. At each section this stress attains its maximum when z is a maximum, that is, the maximum stress is at the points farthest from the neutral axis. When z is zero, the stress σ is zero, which is only to be expected, since there is no strain in the neutral plane.

Eqs (219) and (220) can be presented in different forms. The quotient, obtained in dividing the moment of inertia by the distance between the neutral axis and the fibre farthest from it, is the *moment of resistance to bending* W. Then Eq. (219) becomes

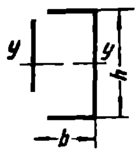

Fig. 322

$$M = W\sigma, \tag{221}$$

and Eq. (220) evolves into

$$\sigma = \frac{M}{W}. \tag{222}$$

The magnitude of the moment of inertia, just as the resisting moment, depends on the form and dimensions of the cross-section*.

For example, for a rectangular section of width b and height h (Fig. 322) the moment of inertia in relation to axis yy is $J = \frac{bh^3}{12}$, in accordance with which

$$z = \frac{h}{2} \text{ and } W = \frac{bh^3}{12} : \frac{h}{2} = \frac{bh^2}{6} \tag{223}$$

For a round section

$$J = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}; \text{ and since } z = r = \frac{d}{2},$$

hence

$$W = \frac{\pi r^3}{4} = \frac{\pi d^3}{32}. \tag{224}$$

By substituting this value for W in Eq. (222) we obtain the maximum stress σ in the given section, with the bending moment expressed in kg-cm and the dimensions of the cross-section in cm; thereby the stress σ is evolved in kg-cm/cm^3 — kg/cm^2, which is as it should be.

233. The Bending Moment

To determine the stress at any cross-section of a bent beam it is first necessary to know the bending moment, that is, the moments of external forces with regard to the neutral axis of the given section.

Let us examine a few simple cases.

By denoting x as the distance from the section to the plane of support of the beam (Fig. 321a) we obtain the moment of force **P** with respect to axis o lying in section mn and which is equal to $M = P(l—x)$. Then by taking succeeding sections to the left of mn we shall see that the arm $l—x$ increases with a

* All engineering handbooks contain formulae for calculating the moments of resistance to bending and torque for various cross-sections.

corresponding increase in the bending moment, which obviously reaches its maximum at the plane of support O where $x = 0$. It thus follows that the stress will be the greatest at the extreme left section of the beam where it can be determined by Eq. (222):

$$\sigma = \frac{M}{W} = \frac{Pl}{W}.$$

If the cross-section of the beam is rectangular, the bending stress

$$\sigma = Pl : \frac{bh^2}{6} = \frac{6Pl}{bh^2}. \qquad (225)$$

The section in which normal stresses reach their maximum is called *the critical section.*

Let us assume that another force $\mathbf{P}_1$ is applied to the same beam in addition to force $\mathbf{P}$, at a distance l_1 from the plane of support (Fig. 321c). As will be recalled from Statics, in this case the bending moment in relation to the same section mn will be equal to the sum of the moments of both forces, i.e.,

$$M = P(l - x) + P_1(l_1 - x).$$

And as before, the closer the section to the plane of support, the greater will be the moment.

Now let us find the bending moments for different sections of a beam lying on two supports (Fig. 323). Take a section lying at a distance of x_1 from the support at the left end, and by suppositionally disregarding the part of the beam to the right of the section, we obtain the bending moment for the remaining free body $M_{x1} = R_1 x_1$, in which $\mathbf{R}_1$ is the reaction of the left support. When $x_1 = 0$, the moment will be zero and there will be no bending moment acting on the section lying at the left support. As the distance x_1 increases, the moment $R_1 x_1$ also increases, and when $x_1 = a$ it will be

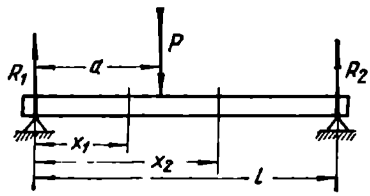

Fig. 323

$$M_a = R_1 a.$$

Inasmuch as the reaction $R_1 = P - R_2$, then

$$M_a = (P - R_2)a = Pa - R_2 a. \qquad (a)$$

Now let us take a section lying to the right of the point of application of force $\mathbf{P}$ at a distance $x_2 > a$.

The bending moment in this section

$$M_{x2} = R_1 x_2 - P(x_2 - a) = R_1 x_2 - Px_2 + Pa = Pa - (P - R_1)x_2 = Pa - R_2 x_2.$$

By comparing this equation with Eq. (a) we find that $M_{x_2} < M_a$. Thus, beginning with the section at which force **P** is applied, the bending moment diminishes, and at the right-hand support where $x_2 = l$, it will be $M_l = Pa - R_2 l = 0$; for when the beam is in equilibrium the algebraic sum of the moments in relation to the left support, just as at any other point, is zero.

Accordingly, the bending moment at the supports of the beam is equal to zero, while in the sections between the supports it increases up to the place where the external force **P** is acting; here the moment is the greatest. Hence this is the critical section where normal stresses are greatest.

If the beam is rectangular, the greatest tensile and compressive stresses — on the convex and concave surfaces respectively — are determined through Eq. (222):

$$\sigma = R_1 a : \frac{bh^2}{6} = \frac{6R_1 a}{bh^2} = \frac{6(P - R)a}{bh^2}. \qquad (226)$$

Note must be made that what has been said refers to bending caused by external forces applied to different sections of the beam without taking into account the weight of the beam itself.

Illustrative Problem 113. The dimension of a wooden beam of rectangular cross-section lying on two supports (Fig. 323) are $b = 140$ mm, and $h = 200$ mm throughout its length. Find the maximum stress at the critical section if the beam is loaded with a force $P = 1$ ton at a distance $a = 1.5$ m from the left support, and the distance between supports is $l = 4$ m.

Solution: first we calculate the reaction at support R_1. Since the algebraic sum of the moments of the external forces with respect to the right support is equal to zero, then

$$R_1 l - P(l - a) = 0, \text{ from which } R_1 = \frac{P(l-a)}{l}$$

$$= \frac{1{,}000 \times (400 - 150)}{400} = 625 \text{ kg}.$$

When this value enters Eq. (226) we obtain

$$\sigma = \frac{6 \times 625 \times 150}{14 \times 20^2} = 100.5 \text{ kg/cm}^2.$$

Illustrative Problem 114. In Illustrative Problem 13 (Fig. 38), work is done by a cutter fastened to a tool holder of rectangular cross-section whose dimensions $b = 30$ and $h = 40$ mm (Fig. 322). Find the maximum normal stress at the critical section of the tool holder, using the numerical values given in Illustrative Problem 13.

Solution: the maximum bending moment $M = P_z l = 5{,}400$ kg-cm. With Eq. (226) we obtain

$$\sigma = M : \frac{bh^2}{6} = \frac{5{,}400 \times 6}{3 \times 4^2} = 675 \text{ kg/cm}^2.$$

234. Questions for Review

1. What direction have the normal stresses in a bent beam with respect to a section perpendicular to its longitudinal axis?
2. What direction will the normal stresses in section *mn* have (Fig. 320*a*) if we consider the right half of the beam as a free body and figuratively discard the left?
3. Are the stresses the same in sections *mn* and m_1n_1 of the bent beam shown in Fig. 320*a*?
4. Define the terms "neutral plane" and "neutral axis". What is their position in relation to each other?
5. Assume that instead of force $\mathbf{P}_1$, indicated by a dotted line in Fig. 321*c*, a force equal and opposite to it is applied to the beam. What change would there be in the maximum stress at the critical section?
6. Under what circumstances will a rectangular beam (Fig. 322) best resist bending – when its broad side *h* or its narrow side *b* is in contact with the supports?
7. Why must the cutter on a lathe be set with the smallest possible distance between the cutting edge and the base of the tool?

CHAPTER XXV

GENERAL PRINCIPLES OF COMBINED STRAIN

235. Simple and Combined Strain

We have investigated the chief kinds of simple strain – tension, compression, shear, torsion, and bending. But it must not be thought that the elements of machines and other engineering structures undergo only one kind of strain in each separate instance. Very frequently members are subjected to the action of forces applied in such a way that several strains occur simultaneously, accompanied by corresponding stresses which must be taken into account in calculating the dimensions required for strength. In such cases we must deal with *combined strain* as distinguished from simple strain. Let us examine a few examples of this kind.

Assume that a force **P** is applied to the centre of gravity of a bar (Fig. 321). We will take section mn_1 at a freely-chosen angle to the cross-section *mn*. By figuratively discarding the upper part of the bar, we obtain an internal force **P′** which is equal and opposite to force **P**. Resolving force **P′** into two components — $\mathbf{P}_n$ perpendicular to section mn_1 and $\mathbf{P}_t$ lying within the section — we find that, aside from elongation the bar is subject to shear strain in the sections not perpendicular to the axis of the bar.

Assume that a rectangular wooden beam is resting on two supports (Fig. 325*a*). Under the action of force **P** it bends and its butt ends *A* and *B* turn relative to each other. Now assume the beam to be sawn in width into three boards along its entire length. When these boards are placed on the same supports and the former force **P** applied (Fig. 325*b*) we shall find that bending is more pronounced and the ends of the three boards have not remained in the same plane but have formed steps. From this we conclude that the boards slide against each other when they are under a load, causing their resistance to bending to be less under the same load.

Now if we cut transverse channels into the boards and place tightly-fitting keys into them as shown in Fig. 325c, we shall see that deflection under the same force **P** is just as for the whole uncut beam and that the ends of all the boards remain in the same plane. From this we deduce that shearing stresses have been

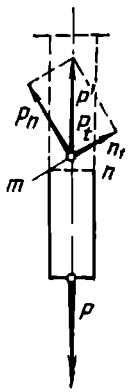

Fig. 324

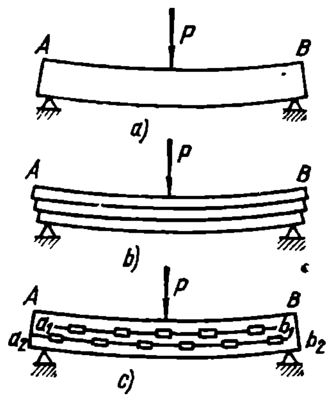

Fig. 325

set up in the bent beam, and if the keys are not strong enough they may shear along the lines a_1b_1 and a_2b_2. Moreover, if the beam consisted of several layers of plate steel rivetted together to form its height h, the rivets would be subject to shearing strain where the planes of their shanks coincide with the planes of the plate steel layers. Hence, bending is a combination of tension, compression, and shear.

236. Combined Tension, Compression, and Bending Strains

Fig. 326 shows a spiked-head bolt. If this bolt is used to tightly fasten a joint with a force **P**, its shank will undergo tension. On the other hand, the bolt-head will be subjected, by the surface of the jointed part, to a reaction **P′** equal and opposite to force **P**. The bolt will therefore be also subjected to bending under the action of the moment of the couple **P** and **P′** equal to **P′**e, in which e is the eccentricity (the distance to the point of application of the resultant **P′** of the elementary forces exerted on the head of the bolt and coming from the direction of the parts being fastened). At the same time the combined action of tensile and bending stresses will cause a considerable increase in tensile stresses in the shank of the bolt, which will increase as the arm of the moment e increases. Hence, when there is eccentric tension and one of the tensile forces does not coincide with the longitudinal axis passing through the centre of gravity of a straight bolt, the resulting tensile stress will be greater than if there were simple tension.

Now let us assume that the top of the square post of height h, represented in Fig. 327, is under the action of force **P** applied to its plane of symmetry. Under the action of force **P** the post will bend, the bending moment reaching its maximum equal to

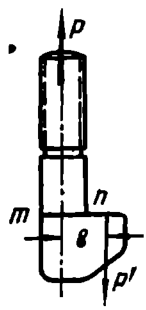

Fig. 326

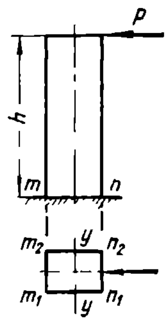

Fig. 327

Ph in section mn where there are tensile stresses between the neutral axis yy and edge n_1n_2, and compressive stresses between yy and m_1m_2. Furthermore, compressive stresses are being caused by the weight of the post equally distributed over the cross-section. Here we have an instance of combined compression and bending strain; in that part of the section between yy and edge m_1m_2 the two kinds of stresses will combine, whereas between the neutral axis and edge n_1n_2, the stresses at various points will be equal to their difference; whether the tensile or compressive stress prevails will depend on which is the greater. To preclude the possibility of tensile stresses occurring where they are undesirable (e.g., in brick construction, which offers poor resistance to tensile stresses) the cross-sectional dimensions of the post must be calculated so that the tensile stresses from bending along the edge n_1n_2 will not be greater than the compressive stress due to the post's own weight.

237. Combined Torsion and Bending Strains

Combined torsion and bending strains are frequently met with: when transmitting a definite torque, a shaft is also subjected to bending from its own weight, the weight of its sheaves or gears, and the pull of the belt or the peripheral force of the gears.

By way of illustration, let us investigate the work of shaft *II* in Fig. 328*a* on which are mounted gears *2* and *3*. Gear *2* receives rotation from gear *1* on shaft *I*, and gear *3* transmits rotation to gear *4* on shaft *III*. Let vector $\mathbf{P}_1$ represent the effective pull acting on gear *2*, and vector $\mathbf{P}_4$ be the effective pull acting on the driving gear *3* from the driven gear *4*.

We apply opposite forces $\mathbf{P}_2$ and $\mathbf{P}_2'$, which are each equal in magnitude to force $\mathbf{P}_1$ and parallel to it, to the centre O_2 of gear *2*. As a result we obtain three forces $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_2'$, of which the first two result in a couple with the arm of the couple equal to the radius r_2 of the pitch circle of gear *2*. The moment of this couple is equal to $P_1 r_2$, i. e., to the torque transmitted to shaft *II*. As concerns the third force $\mathbf{P}_2'$, it acts in the axial plane of the shaft.

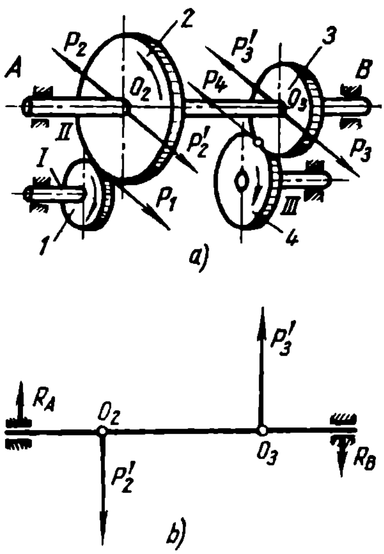

Fig. 328

Let us now consider gear *3*. A force $\mathbf{P}_4$ equal in magnitude to the torque on shaft *II* divided by the radius r_3 of gear *3*, i.e., $\frac{P_1 r_2}{r_3}$, is acting on it from gear *4*. By applying to centre O_3 of gear *3* two equal and opposite forces $\mathbf{P}_3$ and $\mathbf{P}_3'$, each equal to force $\mathbf{P}_4$ and parallel to it, the result is again a couple $\mathbf{P}_4$ and $\mathbf{P}_3$ with its moment equal and opposite to the moment $P_1 r_2$. Force $\mathbf{P}_3'$ is applied to the shaft at section O_3.

Thus we have found that shaft *II* is under the action of torque $Mt = P_1 r_2 - P_4 r_3$ along the part between sections O_2 and O_3, and of two forces $\mathbf{P}_2'$ and $\mathbf{P}_3'$ applied to these sections. In the present case these two forces are parallel. Fig. 328*b* contains a diagram of the system of forces applied to the shaft: forces $\mathbf{P}_2'$ and $\mathbf{P}_3'$ and the reactions $\mathbf{R}_A$ and $\mathbf{R}_B$ at the bearings. Under the action of this system of forces the shaft will bend throughout its length between the two bearings and also twist along length O_2O_3.

In detailed courses of Strength of Materials and machine parts, methods are given for calculating the dimensions of heavily loaded shafts. These methods of calculation take into account the stresses arising from combined torque and bending. But when the bending moment is small, as compared to torque (as for example in transmission shafts), bending is ignored and, by incorporating the smallest allowable stresses, calculations are based only on twist strain.

Coefficient of Sliding Friction *f* (for dry bodies)

Materials	*f*	Materials	*f*
Mild steel on mild steel	0.14-0.19	Leather on cast iron .	0.56
Cast iron on cast iron	0.16	Leather on oak .	0.37-0.48
Bronze on bronze .. .	0.20	Steel on ice (skates) ..	0.02-0.03
Mild steel on bronze .	0.18	Steel runners on smooth wooden or stone floor	0.4
Cast iron on bronze .	0.21	Wooden runners on snow and ice ...	0.035
Cast iron on oak .. .	0.49	The same but runners faced with steel ..	0.02
Wood on wood	0.32-0.60		
Oak on oak (along the grain of both bodies)	0.48		
Oak on oak (one body along the grain, the other across the grain)	0.34		

Coefficient of Rolling Friction *k* (in centimetres)

Materials	*r*	Materials	*r*
Wood on wood . .	0.05-0.08	Steel railway-car wheels on rails	0.05
Steel on steel	0.005		
Steel ball on steel ...	0.0005-0.001		

Modules of Gears

0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 1; 1.25; 1.5; 1.75; 2; 2.25; 2.5; (2.75); 3; (3.25); 3.5; (3.75); 4; (4.25); 4.5; 5.5; 6; 6.5; 7; 8; 9; 10; 11; 12; 13; 14; 15; 16; 18; 20; 22; 26; 28; 30; 33; 36; 39; 42; 45; 50.

Modules in parenthesis should not be used if possible.

For bevel gears the module refers to the external diameter.

Supplement IV

The Greek Alphabet

Letter	Name of Letter	Letter	Name of Letter
A α	Alpha	N ν	Nu
B β	Beta	Ξ ξ	Xi
Γ γ	Gamma	O o	Omicron
Δ δ	Delta	Π π	Pi
E ε	Epsilon	P ϱ	Rho
Z ζ	Zeta	Σ σ	Sigma
H η	Eta	T τ	Tau
Θ θ ϑ	Theta	Υ υ	Upsilon
I ι	Iota	Φ φ	Phi
K $\varkappa$	Kappa	X χ	Chi
Λ λ	Lambda	Ψ ψ	Psi
M μ	Mu	Ω ω	Omega

ANSWERS TO EXERCISES

Statics

4. 141 kg; **5.** $P = 429$ kg; **6.** $P_x = 245$ kg, $P_y = 350$ kg; **7.** Support AB is compressed with a force of 600 kg, support BC is stretched with a force of 1,082 kg; **8.** Bar is stretched by a force of 150 kg, and bar is stretched by a force of 250 kg; **9.** The system is in equilibrium; **10.** At a distance of 90 mm from the line of force P_1; **11.** 60 mm to the right of the line of force P_2; **12.** $R = 300$ kg (downwards), its line of action is 1,583 mm from the extreme left-hand force; **13.** A couple with a moment of 120 kg-m; **14.** $R_A = 190$ kg and $R_B = 800$ kg, both reactions being directed upwards; **15.** $P_1 = 34.5$ kg, $R = 214.5$ kg; **22.** 2.515 kg; **23.** Tipping moment = 727.4 kg m, coefficient of stability = 1.73; **25.** 187.5 times; **26.** $P = 5.2$ and 7.5 kg; **27.** $P = 12.1$ kg; **28.** $P = 11$ kg; **29.** $P = 7.65$ kg; **30.** $\tan \alpha = \frac{k}{R}$.

Kinematics

35. $v_p = 15$ m/min, $v_r = 30$ m/min; **36.** $a = 0.25$ m/sec², $v = 54$ km/hr; **37.** $a = 0.125$ m/sec², $T = 10$ min 50 sec, $v_{av} = 55.8$ km/hr; **38.** $h = -78.48$ m, $t = 8$ sec; **41.** 18 km; **42.** $v = 100$ mm/min, $v_1 = 608$ mm/min; **43.** $v_n = 1.2$ m/sec, $a_t = 0.21$ m/sec², $a_n = 0.72$ m/sec², $a_t = 0.76$ m/sec²; **45.** $\approx 1{,}000$ rpm; **46.** $D = 280$ mm; **47.** $v = 215$ m/min; **48.** $\approx 2{,}740$ rpm; **49.** $\varepsilon = 1.64$ deg/sec², $a_t = 0.02$ m/sec², $\omega = 881$ deg/sec, $v = 10.8$ m/sec; **50.** $\varepsilon = 21.33$ deg/sec², $a_t = 0.238$ m/sec², $v = 16.7$ m/sec; **51.** $\varepsilon = 4.8$ deg/sec², $n \approx 460$ rpm.

Dynamics

52. 200 kg-m⁻¹ sec²; **53.** $G = 1{,}177.2$ tons, $v = 6.75$ m/sec; **54.** $P = 20{,}194$ kg; **55.** $P = 13{,}551$ kg; **56.** $S = 26.5$ m; **57.** $N = 0.102$ kg; **58.** $\approx 20{,}680$ kg; **59.** At its highest position 1.67 kg, at its lowest position 3.17 kg; **60.** $T_u = 0.245$ kg, $N_u = 112$ kg; **61.** $\alpha = 7°20'$; **62.** $W = 2$ PS; **63.** 2.67 hp; **64.** 30 kg-m/sec; **65.** $N = 1{,}778$ hp; **66.** $G = 750$ tons, $P = 5{,}625$ kg; **67.** $N = 30.2$ hp, $M_2 = 108$ kg-m; **68.** $M_t = 3.56$ kg-m, $P = 39{,}5$ kg; **69.** $n = 240$ rpm; **70.** $N = 13.3$ hp, $P_z = 800$ kg; **71.** $v \approx 0.34$ m/sec; **72.** 2,648,700 kg-m, 5,297,400 kg-m; **73.** ≈ 4 min, $S = 2{,}119$ m; **74.** $P = 7{,}097$ kg, $N = 916$ hp, $F = 4{,}078$ kg; **75.** $\eta = 0.78$.

Elements of the Theory of Machines

76. $P = 117.3$ kg; **77.** $\eta = 0.85$; **78.** 19°28′; **79.** $G_1 = 1{,}414G_2$; **80.** $x = 200$ mm; **83.** $P = 1$ kg; **85.** 18 times; **86.** $P \approx 21.9$ kg; **87.** $n_2 = n_1 \frac{D_1}{D_3}$; **88.** $n_4 = 200$ rpm, $M_4 = 109.5$ kg-m; **89.** $M_2 = 21.92$ kg-m, $n_2 = 1{,}000$ rpm; **90.** $R_1 = 200$ mm; **91.** $\frac{z_0}{z_1}$, $\frac{z}{z_2}$, $\frac{z}{z_3} \ldots \frac{z}{z_7}$; **92.** $n_2 = 36$, $n_3 = 450$, $n_4 = 100$ rpm; **93.** $M_4 = 10.74$ kg-m; **94.** $n_2 = 6$, $n_3 = 50$, $n_4 = 450$ rpm; **95.** 327 and 467 rpm; **96.** $n_2 = 3{,}200$ rpm; **97.** 0.094 m/sec; **98.** 20 rpm; **99.** 60 rpm; **100.** $n_3 = n_1 \frac{z_1 z_3}{z_2 z_6}$ or $n_1 \frac{z_1 z_4}{z_2 z_7}$ or $n_1 \frac{z_1 z_5}{z_2 z_8}$, $n_1 = \frac{z_1 z_3 z_{11} z_{13}}{z_2 z_{10} z_{12} z_{14}}$; **101.** $P \approx 6$ kg, $N = 0.036$ hp; **102.** $P = 324$ kg, $v = 141.8$ m/min; **103.** $v_a = 31.4$ m/min, $v_c = 0.5$ m/min, $n_6 = 15$ rpm.

Stress and Strain

105. $\sigma = 497$ kg/cm^2; **106.** $\Delta l = 4.24$ mm; **107.** $\sigma = 1{,}000$ kg/cm^2; **108.** $\sigma = 584.5$ kg/cm^2, $P = 45{,}900$ kg; **109.** 5 km 96 m; **110.** $\approx 1{,}000{,}000$ kg/cm^2; **111.** $\sigma = 495$ kg/cm^2; **112.** $d_1 = 19$ mm, $d_2 = 13.5$ mm, $d_3 = 17.8$ mm; **113.** $\sigma = 675$ kg/cm^2, $\tau = 502$ kg/cm^2; **114.** $\tau = 62.5$ kg/cm^2; **115.** $\tau = 81.5$ kg/cm^2; **116.** $\tau = 393$ kg/cm^2; **117.** $d_1 = 43 \approx 45$ mm, $d_2 = 30$ mm; the first variant is more advantageous; **118.** $d_1 = 43 \approx 45$ mm, $d_2 = 38 \approx 40$ mm, the least advantageous variant.

TO THE READER

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