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THEORETICAL MECHANICS

Mir Publishers Moscow

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Theoretical Mechanics

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FROM THE AUTHORS

The new international system of units (SI) is used throughout the text. However, taking into account that the engineers' system (mkgfs) still finds wide use, the units of this system are also given. Moreover, wherever necessary the relations are indicated between the units of the international and engineers' systems. Normative and design data are given in both systems of units.

In the presentation of the material primary emphasis is placed on the practical significance of conclusions.

To gain a better understanding of theoretical propositions, the solutions of sample problems are given. All solutions are first carried out in algebraic form and then numerical data are substituted.

Although the book is mainly intended for full-time vocational schools, the large number of detailed and specially selected examples makes it handy for their evening and correspondence departments as well.

CONTENTS

NOTATION

A = area
A, B, C = points
a = acceleration, distance
a_a = absolute acceleration
a_b = base acceleration
a_{BA} = relative acceleration of B with respect to A
a_r = relative acceleration
b = distance, width
C = centroid or centre of gravity, instantaneous centre of rotation
$C_1, C_2, \ldots$ = constants of integration
c = distance, stiffness factor or spring constant
D = diameter
$D_1, D_2, \ldots$ = constants of integration
d = diameter, differential of (ds, dx), distance
E = kinetic energy
F = force, friction force
f = coefficient of friction
f_0 = coefficient of static friction
G = weight
g = acceleration of gravity
H = height, horizontal component of reaction
h = distance, height, lead, pitch
I = moment of inertia of mass
i = gear ratio
K = centre of oscillation
k = coefficient of rolling friction, radius of gyration, stability factor
k_a = acceleration scale
k_F = force scale
k_S = space scale
k_v = velocity scale
L = length

l = distance, length, span
M, m = mass, moment
M_{st} = stability moment
M_t = tipping moment
N = normal component of reaction, power
n = normal direction, revolutions per minute
O = origin of co-ordinates
P = force, load, weight
P_c = circumferential force
P_e = elastic force
p = pole of velocity diagram, unit pressure
Q = force, load, momentum of system, weight
q = load per unit length, momentum of particle
R = radius, reaction, resistance, resultant force
r = radius
S = displacement, distance, force, impulse of a force
T = tension, torque
t = pitch, tangential direction, time
U = work
$U_{m.f}$ = work of moving forces
$U_{p.f}$ = work of parasitic resistance forces
$U_{r.f}$ = work of resistance forces
$U_{u.f}$ = work of useful resistance forces
V = potential energy, vertical component of reaction, volume
v = velocity
v_a = absolute velocity
v_b = base velocity
v_{BA} = relative velocity of B with respect to A
v_r = relative velocity
W = inertia force
X, Y, Z = reactions at supports
x, y, z = rectangular co-ordinates
x_c, y_c, z_c = rectangular co-ordinates of centroid or centre of gravity
Z, z = number of gear teeth
α (alpha) = angle, angular acceleration
β (beta) = angle
γ (gamma) = specific weight
δ (delta) = coefficient of fluctuation
ε (epsilon) = slip ratio

η (eta) = efficiency
λ (lambda) = lead angle
π (pi) = pole of acceleration diagram
ρ (rho) = radius of curvature
φ (phi) = angle, angle of friction, angular co-ordinate
ω (omega) = angular velocity

INTRODUCTION

Theoretical mechanics treats of the general laws of motion and equilibrium of particles and bodies. Here, procedures and methods are developed for solving problems involving mechanical motion.

Mechanical motion is defined as displacement of bodies or particles in space with time. In a wider sense, motion implies any change affecting living organisms, social formations, etc., as well as material bodies.

Motion is the basic property of matter and a form of its existence. From this it follows that matter and motion cannot exist separately. There can be no motion without matter, just as there can be no matter without motion.

The various types of motion are studied by different sciences. Mechanical motion is the simplest form of motion. The others (thermal, chemical, electrical motion) generally involve mechanical displacements of microparticles (molecules, atoms, electrons). Therefore, the laws of mechanical motion find application in many sciences.

Hereinafter the term motion will imply exclusively mechanical motion.

A particular case of motion is the state of rest. Rest is always relative since a body at rest is considered as stationary with respect to some other body which in turn may be moving in space. There are and there can be no absolutely stationary bodies in nature. A mechanical rest is also relative in the sense that a stationary body may be affected by all kinds of processes involving other, non-mechanical, forms of motion. For instance, we say that the support of a machine or the foundation of a structure is at rest. They are, indeed, stationary with respect to the earth, but they execute a complex motion with it about the sun and they are also moving with the sun relative to the galaxy, etc.

On the other hand, the support or foundation may undergo thermal, chemical or other processes accompanied by mechanical displacements of molecules and atoms. Under certain conditions these intrinsic processes may result in wear or even structural damage.

Thus, what we call rest is only an element of motion.

It is not possible to gain a true understanding of motion without fully realizing the relativity of rest. That is why the motion of bodies can and must be studied as the motion of one body with respect to another.

Theoretical mechanics is divided into three parts: statics, kinematics, and dynamics.

Statics is concerned with the conditions of equilibrium of bodies. The laws of equilibrium are much simpler than the laws of motion, therefore statics is the simplest part of theoretical mechanics.

Kinematics deals with the geometric properties of motion of material bodies without regard to its physical causes.

Dynamics, the most general and complex part of theoretical mechanics, establishes the relationships between the nature of motion and its physical causes. It derives the fundamental laws of mechanical motion.

CHAPTER I

Basic Definitions and Axioms of Statics

1. Fundamentals

Theoretical mechanics, being a mathematical science, deals not with real physical objects but with abstract schematized conceptions. Naturally, any schematization is acceptable as long as there is no significant discrepancy between the scheme and the real object from the point of view of the problem in hand.

Let us turn to the fundamental concepts of theoretical mechanics which have found their way into the science as a result of the practical activity of man for many centuries. They have been proved by numerous experiments and observations of nature.

One of these fundamental concepts is the concept of a *material particle*. A body may be considered as a particle, i.e., it can be represented by a geometric point at which the entire mass of the body is concentrated, provided the dimensions of the body are immaterial in the particular case. Thus, in studying the motion of planets and satellites they are treated as particles since the dimensions of planets and satellites are negligibly small compared with their orbits. On the other hand, when studying the motion of a planet (for example, the earth) about its axis it can no longer be treated as a particle. A body may be considered as a particle if all its points execute a similar motion. For instance, a piston in an internal-combustion engine may be considered as a particle in which all its mass is concentrated.

By extending the concept of a material particle we arrive at the concept of a system of material particles.

A *system* is a set of material particles whose motions and positions are interdependent. From this definition it follows that any physical body is a system of material particles.

Bodies which are studied in theoretical mechanics are assumed perfectly rigid, i.e., it is taken that external effects produce no change in their dimensions or shape and the distance between any two points in the body remains constant. In

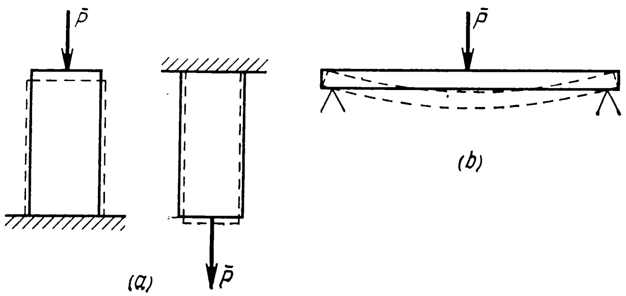


Fig. 1.

actual fact, all bodies encountered in engineering change their dimensions and shape under the action of forces exerted by other bodies. Thus, if a bar of some material employed in practice, say, steel or wood, is compressed, its length decreases, and when stretched its length increases (Fig. 1*a*).

Likewise, the shape of a bar resting on two supports changes under the action of a load perpendicular to its axis (Fig. 1*b*). The bar bends.

In most practical cases deformations of bodies, i.e., changes in their dimensions and shape are very small and can be neglected to a first approximation.

Thus, the concept of a perfectly rigid body is conventional (abstract). It is introduced to simplify the study of the action of forces on a body and its motion. Only after studying the mechanics of perfectly rigid bodies can one proceed to the equilibrium and motion of elastic and deformable bodies, fluids, etc. Thus, after going through theoretical mechanics we shall consider deformations of bodies in strength of materials. Deformations play an important role in problems treated in strength of materials and they cannot be neglected. In turn, other abstractions are introduced there and the properties of bodies which are of secondary importance are neglected.

In theoretical mechanics the concept of *force* is introduced which is extensively used in other engineering sciences. The physical essence of this concept is self-evident.

Let us turn to the definition of a force for perfectly rigid bodies considered in theoretical mechanics. These bodies may interact with a subsequent change in the nature of their motion. Force is a measure of this interaction.

For instance, the interaction of planets and the sun is determined by forces of attraction; the interaction of the earth and various bodies on its surface is determined by forces of gravity.

It should be noted that when real, not absolutely rigid, bodies interact the resulting forces may cause a change in their shape and dimensions without changing the nature of their motion. In other words, in real physical bodies forces may be the cause of deformations.

Theoretical mechanics considers and studies not the nature of acting forces but the effect produced by them.

The effect of a force on a perfectly rigid body is completely defined by two factors: the direction or line of action of the force and its numerical value (modulus).

In other words, *a force is a vector quantity.*

Besides forces, other vector quantities are frequently encountered in theoretical mechanics. Among these are velocity, acceleration.

A quantity which has no direction is called a *scalar* or a *scalar quantity.* Examples are time, temperature, volume.

A vector is represented by a segment with an arrowhead at the end. The arrowhead indicates the sense of the vector, the length of the segment represents the magnitude of the vector measured to an arbitrarily chosen scale.

For some vector quantities, say, velocity, it is necessary also to specify the point of application.

A vector with tail at B and tip at C (Fig. 2*a*) may be denoted as $\overline{BC}$ in this order. Sometimes vectors are represented by a single letter, $\overline{P}$, $\overline{Q}$, $\overline{a}$, etc. (Fig. 2*b*) or boldface type.

The line of action of a force is a straight line extending infinitely along the direction of the force (Fig. 2*c*).

The *modulus* or *magnitude of a force* is a quantitative characteristic of the measure of interaction of bodies. The magnitude of a force in the international system of units (SI) is measured in newtons (N).

Larger units of force are also used: 1 kilonewton = 1 kN = $=10^3$ N, 1 meganewton = 1 MN $= 10^6$ N.

Before the international system of units was introduced extensive use was made of the unit of the engineers' system,

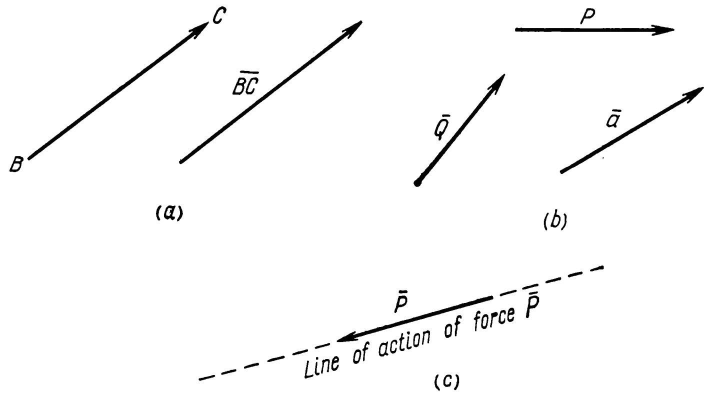


Fig. 2.

the kilogram force (kgf). This unit is still in use. The relationship between the units of force in the international and engineers' systems is derived in dynamics (Sec. 80). Here we give it without derivation: 1 kgf = 9.81 N, 1 N = 0.102 kgf. One can take that 1 kgf = 10 N, 1 N $\cong$ 0.1 kgf.

The simplest and most familiar example of a force is the force of gravity. The force of gravity or the weight of a body near the earth's surface is the force with which this body is attracted by the earth. The force of gravity is always directed vertically downward, i.e., toward the centre of the earth.

A set of forces applied to a body, a particle or a system of particles and bodies is called a *force system*.

Force systems are classified according to the relative position in space of the lines of action of the forces involved.

Thus, a system of forces whose lines of action lie in different planes is referred to as *three-dimensional*. If the lines of action lie in the same plane, the system is referred to as *two-dimensional*. A system of forces with the lines of action intersecting at a common point is called *concurrent*. A concurrent force system may be either three-dimensional or two-dimensional. Finally, a *system of parallel forces* is distinguished, with the lines of action located either in different planes or in a single plane.

Two force systems are said to be *equivalent* if, acting separately, they produce the same effect on a body. From this definition it follows that two force systems are equivalent if each of them is equivalent to a third one. Any complicated force system can always be replaced by a simpler force system equivalent to it.

A single force equivalent to a given force system is called the *resultant* of this system.

A force equal in magnitude to the resultant and directed along the same line of action but in the opposite sense is called the *equilibrant*. If to a force system is added the equilibrant, the new system so obtained, when applied to a body, will not change its mechanical state.

Forces acting on a system of particles are divided into two groups: *external forces* and *internal forces*.

External forces are exerted on the particles of a given system by other bodies not belonging to this system. Internal forces of a system are the forces of interaction of the particles involved in this system. Thus, for any body located on the surface of the earth its weight is an external force. External forces produce internal forces in bodies. These internal forces acting between the particles of solids are investigated in strength of materials and in the theory of elasticity. The study of internal forces in liquid and gaseous media is the realm of hydromechanics and aerodynamics, respectively.

The general methods for revealing and determining internal forces are largely based on statics.

In the introduction, we have already said that statics deals with the equilibrium of particles and bodies. Now that the fundamental concepts have been explained the subject of statics can be formulated in greater detail.

Statics deals with the equilibrium of particles and bodies under various force systems and, besides, treats the transformation of forces, i.e., the replacement of a system of forces by equivalent forces. To sum up, statics establishes relations for solving problems in equilibrium under any loading conditions.

2. Axioms of Statics

Mechanics is based on some axioms, self-evident truths derived from experience. The general principles of mechanics were first formulated by Sir Isaac Newton in his classical work *Philosophiae Naturalis Principia Mathematica* (1687).

These general principles form a basis for the development of the laws of statics and are given below as axioms.

The axioms of statics establish the basic properties of forces applied to a perfectly rigid body.

The conclusions and equations of statics based on these axioms are valid only for absolutely rigid bodies. However, no absolutely rigid bodies exist in nature. In the further study of technical mechanics (strength of materials and machine elements) statics will be applied to elastic bodies undergoing deformations under the action of applied forces. This is justified by the fact that the observed deformations are usually very small compared to the dimensions of the bodies and can be neglected.

The axioms of statics are:

Axiom I (law of inertia). A force system applied to a particle is balanced if the particle is in a state of relative rest or moves with constant speed in a straight line. This proposition was formulated in a somewhat different form by Galileo Galilei in 1638.

A particle moving along a straight line is said to be in rectilinear motion; the motion is uniform if the particle travels equal distances in equal intervals of time.

It can easily be deduced from the first axiom that a balanced force system is equivalent to zero. It is also evident that any force of a balanced system is the equilibrant in relation to the remaining forces of the system.

It should be noted that a body, unlike a particle, will not necessarily be at rest or move with constant speed in a straight line under the action of a balanced force system. Under certain conditions a balanced force system may cause a body to rotate uniformly about a fixed axis. This case is frequently met in engineering and therefore is of great importance.

Axiom II (condition of equilibrium of two forces). Two equal and opposite forces applied to a perfectly rigid body are mutually balanced (Fig. 3*a*).

Axiom III (principle of superposition). The equilibrium of a perfectly rigid body will not be disturbed if we add or subtract a balanced force system.

Let a body (Fig. 3*b*) be in a state of equilibrium. If we apply to it several mutually balanced forces $\bar{P}_1 = \bar{P}_2$, $\bar{Q}_1 = \bar{Q}_2$,

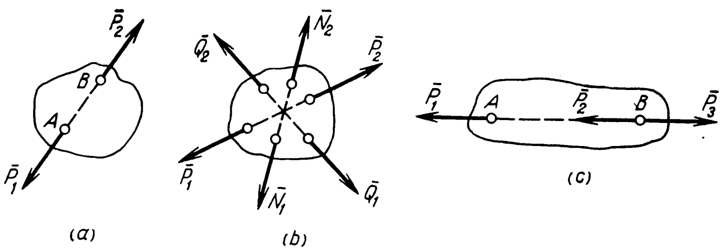


Fig. 3.

$\bar{N}_1 = \bar{N}_2$, the equilibrium of the body will not be disturbed. A similar effect is obtained when these balanced forces are subtracted.

The force systems shown in Fig. 3*a* and *b* are equivalent as they produce the same effect; the body is in equilibrium in either case.

Corollary. From Axiom III it follows that *any force acting on a perfectly rigid body may be moved along its line of action to any point of the body without disturbing its equilibrium.*

Indeed, let a force $\bar{P}_1$ act on the body at point A (Fig. 3*c*). At an arbitrary point B on the line of action of the force $\bar{P}_1$, we apply two forces, $\bar{P}_2$ and $\bar{P}_3$, equal in magnitude to the force $\bar{P}_1$ and having opposite sense. The equilibrium of the body is not upset; the removal of the forces $\bar{P}_1$ and $\bar{P}_3$, as being equal and opposite, does not upset the equilibrium either. Thus, we have replaced the force $\bar{P}_1$ by an equal force $\bar{P}_2$ moved from A to B along the line of action of the force $\bar{P}_1$.

Vectors that may be transmitted along their lines of action are called *sliding vectors*. As shown above, force is a sliding vector. It should be emphasized that the transmission of a force along its line of action is permissible only when the bodies are considered as perfectly rigid. Otherwise, it is not permissible.

Axiom IV (parallelogram rule). The resultant of two non-parallel forces applied at a common point is applied at the same point

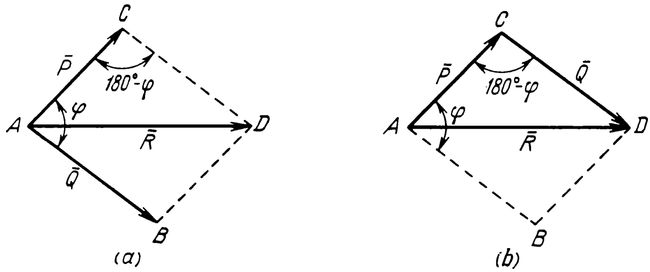


Fig. 4.

and represented in magnitude and direction by the diagonal of the parallelogram constructed on the given forces. Thus, the resultant of two forces $\bar{P}$ and $\bar{Q}$ applied at point A (Fig. 4*a*) is the force $\bar{R}$ representing the diagonal of the parallelogram $ACDB$ constructed on the given force vectors. The determination of the resultant of two forces according to the parallelogram rule is called vector or geometric addition and expressed by the vector equality

$$\bar{R} = \bar{P} + \bar{Q}. \tag{1}$$

In determining the resultant of two forces graphically use can be made of the triangle rule instead of the parallelogram rule. From an arbitrary point A of the body (Fig. 4*b*), we draw, preserving the scale and the given direction, a vector representing the first component force $\bar{P}$; from its tip, we draw a vector parallel and equal to the second component force $\bar{Q}$. The closing side AD of the triangle is the desired resultant $\bar{R}$.

It can also be represented as the diagonal of the parallelogram *ACDB* constructed on the given forces.

Axiom V (*law of action and reaction*). *When bodies interact, to each action corresponds an equal and opposite reaction.* Thus, if a body *B* (Fig. 5) is acted upon by a force $\bar{P}_1$ exerted by a material body *A*, then the body *A* is acted upon by an

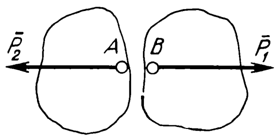


Fig. 5.

equal force $\bar{P}_2$ exerted by the body *B*. The two forces act along the same straight line and have opposite sense.

Action and reaction are always applied to different bodies and hence they cannot be balanced.

The fifth axiom states that there can be no unilateral action of a force in nature. The following example supports this statement.

Consider universal gravitation which represents the force of attraction between two material bodies. The sun and the earth are mutually attracted with equal forces; these forces act in opposite sense along a straight line joining the centres of the sun and the earth. The forces are applied to different bodies and, consequently, they cannot be balanced if we consider each body separately. If, however, we study the solar system as a whole, we shall reveal no forces of attraction between the planets. For the solar system, the forces of attraction between the planets involved are internal. Internal forces in any system, including solids, obey the fifth axiom.

3. Constraints and Their Reactions. Axioms of Constraints

Bodies considered in mechanics may be free or constrained. A *free body* is not prevented from moving in space in any direction. If, however, a body is connected to other bodies which restrict its motion in one or several directions, it is a *const-*

rained body. The bodies which restrict the motion of the body considered are called *constraints*. As a result of the interaction between the given body and its constraints, forces arise in

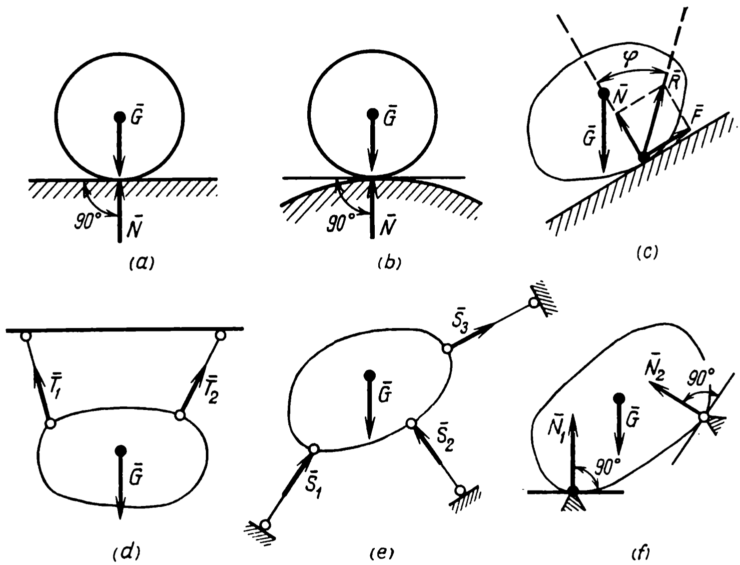


Fig. 6.

the latter which oppose possible motions of the body and are called *reactions of constraints.*

A constraining force always acts in a direction opposite to that in which the constraint prevents the body from moving.

The determination of reactions of constraints is one of the important problems of statics.

Below are listed the most common types of constraints encountered in mechanics.

1. A constraint in the form of a smooth (frictionless) plane or surface (Fig. 6*a* and *b*). In this case the reaction of the constraint is always directed along the normal to the supporting surface.

2. A constraint in the form of a rough plane (Fig. 6*c*). Here, two reaction components arise: a normal component of reaction $\bar{N}$ perpendicular to the plane and a tangential component

of reaction $\bar{F}$ parallel to the plane. The tangential reaction $\bar{F}$ is called the friction force; it is always opposite to the actual or possible motion of a body. The total reaction $\bar{R}$ which is equal to the geometric sum of the normal and tangential components, $\bar{R} = \bar{N} + \bar{F}$, deviates from the perpendicular to the supporting surface by an angle φ.

All real constraints involve forces of friction. In many cases, however, these friction forces are insignificant owing to some constructional features of the constraints. Hence friction forces are often ignored.

Frictionless constraints are called ideal constraints. The above constraint in the form of a smooth plane or surface falls into this category.

3. A flexible constraint effected by a rope, cable, chain, etc. (Fig. 6*d*). The reaction of a flexible constraint is directed along the constraint. A flexible constraint can act only in tension.

4. A constraint in the form of a rigid rod (with hinged ends, Fig. 6*e*). Here, as in a flexible constraint, the reactions are always directed along the axes of the rods but the rods can be either in tension or in compression.

5. A constraint realized by the edge of a dihedral angle or by a point support (Fig. 6*f*). The reaction of such a constraint is perpendicular to the supported surface of a body.

The existence of the reactions of constraints is consistent with the axiom of action and reaction. To determine reactions of constraints and also to extend the laws of motion of a free body, use is made of the following axiom concerning the isolation of a body from constraints.

Axiom VI. The equilibrium (or motion) of a body or of a system of bodies will remain unchanged if we remove any constraint imposed on the system and replace it by the reaction it produces.

Another important axiom finding application in special sections of mechanics is the axiom of imposition of new constraints.

Axiom VII. If a body or a system of bodies is in equilibrium, the imposition of new constraints will not change the state of equilibrium.

If, for example, a blackboard hanging on a wall is in equilibrium, addition of new fastenings will not upset its equilibrium.

CHAPTER II

Systems of Concurrent Forces in a Plane

4. Analytic Determination of the Resultant of Two Forces Applied at the Same Point

The simplest force system studied in statics is a system of concurrent forces. Forces are concurrent if their lines of action intersect at a common point.

A distinction is made between a two-dimensional system of concurrent forces where the lines of action of all forces lie in the same plane, and a three-dimensional system of concurrent forces where the lines of action of the forces lie in different planes.

On the basis of the corollary of Axiom III a force can be transmitted along its line of action. Thus, concurrent forces may always be moved until they are applied at the same point, the point of intersection of their lines of action.

In the present chapter we shall study two-dimensional systems of concurrent forces. Figure 7 shows a system of forces whose lines of action intersect at point A. By using the above corollary of Axiom III, we move all forces to point A. This transmission is necessary, as shown above, for the graphical determination of the resultant of the system.

If we have two concurrent forces, their resultant is determined by the parallelogram or triangle rule on the basis of Axiom IV.

The modulus of the resultant of two forces can be determined analytically from triangle ACD (Fig. 4*b*)

$$R^2 = P^2 + Q^2 - 2PQ\cos(180^\circ - \varphi), \qquad (2)$$

where φ is the angle between the directions of the forces $\overline{P}$ and $\overline{Q}$.

Equation (2) can be represented as

$$R^2 = P^2 + Q^2 + 2PQ\cos\varphi,$$

whence

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\varphi}. \qquad (2a)$$

Consider three specific cases.

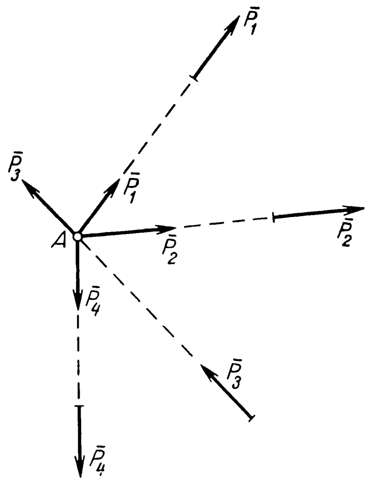


Fig. 7.

1. The angle between the directions of the forces $\bar{P}$ and $\bar{Q}$ is equal to zero, $\varphi = 0$. In this case $\cos\varphi = 1$, then

$$R = \sqrt{P^2 + Q^2 + 2PQ}$$

or

$$R = P + Q. \qquad (2b)$$

2. The angle between the directions of the forces $\bar{P}$ and $\bar{Q}$ is 180°. In this case $\cos\varphi = -1$, then

$$R = \sqrt{P^2 + Q^2 - 2PQ}$$

or

$$R = P - Q. \qquad (2c)$$

3. The angle between the directions of the forces $\bar{P}$ and $\bar{Q}$ is 90°. In this case $\cos\varphi = 0$, then

$$R = \sqrt{P^2 + Q^2}. \qquad (2d)$$

The first case corresponds to the determination of the modulus of the resultant of two forces applied at the same point and having the same sense.

The second case corresponds to the determination of the resultant of two forces applied at the same point and having opposite sense.

The third case corresponds to the determination of the resultant of two mutually perpendicular forces.

The above cases of the determination of the modulus of the resultant illustrate the marked difference between vector and scalar addition. The vector sum is identical with the scalar sum (or difference) only when the vectors are directed along the same straight line and have the same or opposite sense.

5. Resolution of a Force into Two Components Applied at the Same Point

We have discussed in detail the rule for the geometric addition of two concurrent forces. In some cases it is necessary to solve the inverse problem when a given force is replaced by an equivalent system of two or even more forces. Such an operation is called resolution of a force.

Consider a force $\bar{R}$ applied at a point A (Fig. 8*a*). This force can be represented as the resultant of two arbitrary component forces. If no conditions are specified, an infinite number of component forces can be chosen. The problem becomes indefinite in this case. Consequently, to resolve a force requires some additional information about its components.

We shall discuss cases when a force can be resolved into two components in a unique way.

1. The force $\bar{R}$ can be resolved into two components, $\bar{P}$ and $\bar{Q}$, whose directions are specified by angles α and β (Fig. 8*b*). From the tail and tip of the force $\bar{R}$ (points A and B) we draw components in the given directions, which intersect at C. The segment AC gives the magnitude of one component, $\bar{P}$, the segment CB gives the magnitude of the other component, $\bar{Q}$.

2. The force $\bar{R}$ can be resolved into two components, $\bar{P}$ and $\bar{Q}$, if their magnitudes are specified (Fig. 8*c*). From point A as a centre we draw an arc of radius P and from point B an

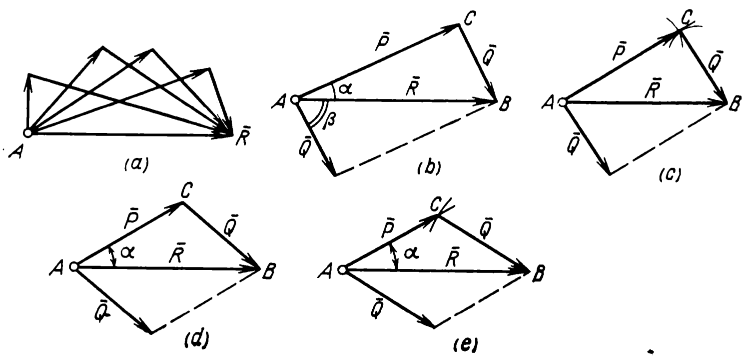


Fig. 8.

arc of radius Q. We obtain a force triangle ACB whose sides AC and CB represent the desired component forces $\bar{P}$ and $\bar{Q}$ in magnitude and direction.

By drawing from point A a segment equal and parallel to the side CB of the force triangle, we obtain two forces, $\bar{P}$ and $\bar{Q}$, concurrent at point A. As is known from geometry, this problem is solvable provided

$$\left.\begin{array}{l} P+Q>R, \\ P-Q<R. \end{array}\right\}$$

3. The force $\bar{R}$ can be resolved into two components, $\bar{P}$ and $\bar{Q}$, if the magnitude and direction of one of the components, say $\bar{P}$, are specified (Fig. 8*d*). Through point A we draw the force $\bar{R}$ and the force $\bar{P}$ at the given angle α. By joining points C and B by a straight line, we find the second component $\bar{Q}$. By drawing from point A a segment equal and parallel to CB, we obtain two forces, $\bar{P}$ and $\bar{Q}$, concurrent at point A.

4. The force $\bar{R}$ can be resolved into two components, $\bar{P}$ and $\bar{Q}$, when one of them is known only in magnitude and the other only in direction. Let the magnitude of the force $\bar{Q}$

and the direction of the force $\bar{P}$ be specified, the force $\bar{P}$ making an angle α with the direction of the force $\bar{R}$. We lay off the force $\bar{R}$ from a certain point A and draw from the same point the direction line of the force $\bar{P}$ at the angle α to the direction of $\bar{R}$ (Fig. 8*e*). From the tip of the force $\bar{R}$, point B, we swing, by means of a compass, a circular arc of radius equal to the magnitude of the force $\bar{Q}$ which crosses the direction line of the force $\bar{P}$ at a point C. The intercept AC on the direction line of the force $\bar{P}$ determines the magnitude of this force. The segment CB will determine the magnitude and direction of the force $\bar{Q}$. Laying off from point A a vector equal and parallel to the segment BC, we obtain two concurrent forces, $\bar{P}$ and $\bar{Q}$.

6. Addition of Concurrent Forces in a Plane. Force Polygon

Using the triangle rule we shall show how to add concurrent forces.

Let several concurrent forces, $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ and $\bar{P}_4$, be applied at a point A (Fig. 9*a*). To determine their resultant,

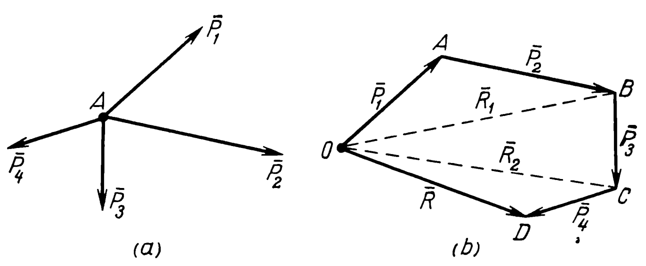


Fig. 9.

we add all the forces successively by using the triangle rule. We first add two forces, $\bar{P}_1$ and $\bar{P}_2$. From an arbitrary point O, we draw the force $\bar{P}_1$, preserving the scale and direction. From the tip of the force $\bar{P}_1$, we draw the second force $\bar{P}_2$. By joining point O to the tip of the force $\bar{P}_2$, we obtain a force

$\bar{R}_1$ equal to the sum of the forces $\bar{P}_1$ and $\bar{P}_2$

$$\bar{R}_1 = \bar{P}_1 + \bar{P}_2.$$

From the tip of the force $\bar{R}_1$, we draw the third force $\bar{P}_3$. By joining point O to the tip of the force $\bar{P}_3$, we obtain a force $\bar{R}_2$ equal to the sum of the forces $\bar{P}_3$ and $\bar{R}_1$. But $\bar{R}_1 = \bar{P}_1 + \bar{P}_2$, whence

$$\bar{R}_2 = \bar{P}_1 + \bar{P}_2 + \bar{P}_3.$$

From the tip of the force $\bar{R}_2$, we draw the fourth, and last, force $\bar{P}_4$. By joining point O to the tip of the force vector $\bar{P}_4$, we obtain a force $\bar{R}$ equal to the sum of the forces $\bar{R}_2$ and $\bar{P}_4$, i.e.,

$$\bar{R} = \bar{R}_2 + \bar{P}_4 = \bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \bar{P}_4.$$

The intermediate vectors $\bar{R}_1$ and $\bar{R}_2$ need not be constructed. The given forces are laid off in succession in the order indicated above and the tail of the first force is joined to the tip of the last one.

The figure $OABCD$ thus obtained is called the *force polygon*. The closing side of this polygon is the geometric sum of the given forces, i.e., their resultant. It should be noted that the resultant force $\bar{R}$ is always directed from the tail of the first component to the tip of the last one. In other words, the arrowhead of the resultant force always points opposite to the general trend of the component forces (Fig. 9*b*).

These forces could be added in a similar way by using the parallelogram rule but the constructions would be more complicated.

7. Projection of a Force on an Axis

The direct geometric addition of force vectors often requires involved and cumbersome constructions. In such cases an alternate method is used, replacing geometric construction by computation of scalar quantities. This is achieved by projecting the forces on an axis. In mechanics, it is common to use rectangular axes.

An *axis* is an infinite straight line which is assigned a certain sense.

The rectangular projection of a vector is a scalar quantity which is determined by the segment cut off on the axis by the perpendiculars dropped upon the axis from the tail and tip of the vector.

The projection of a vector is considered positive (+) if the vector has the same sense as the axis, and negative (—) if the vector has the opposite sense.

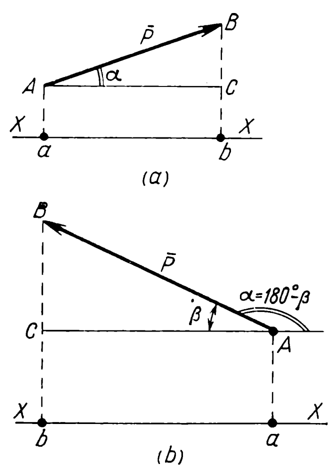


Fig. 10.

Consider some cases of projecting forces on an axis.

1\. A force $\bar{P}$ is given (Fig. 10*a*) which lies in the same plane as the x axis. The force has the same sense as the axis and makes an acute angle α with the positive x axis.

In order to find the magnitude of the projection, we drop perpendiculars upon the x axis from the tail and tip of the force vector; we obtain

$$P_x = ab = P\cos\alpha.$$

The projection of the vector is positive.

2\. A force $\bar{P}$ is given (Fig. 10*b*) which lies in the same plane as the x axis but makes an obtuse angle α with the positive direction of the axis. The projection of the force $\bar{P}$ on the x axis is

$$P_x = ab = P\cos\alpha,$$

but

$$\cos\alpha = -\cos\beta.$$

It is seen from the drawing that $\alpha > 90°$, hence $\cos\alpha$ is a negative quantity. Replacing $\cos\alpha$ by $\cos\beta$ (β is an acute angle), we obtain finally

$$P_x = -P\cos\beta.$$

In this case the projection of the force is negative.

Thus, *the projection of a force on a co-ordinate axis is equal to the product of the modulus of the force and the cosine of the angle between the vector and the positive direction of the axis.*

In determining the projection of a force vector on an axis, the cosine of an acute angle is usually used, irrespective of whether the angle is made with the positive or negative direction of the axis. The sign of the projection is more easily established by inspection.

For instance, in Fig. 10*b* the force $\bar{P}$ makes an acute angle β with the negative x axis and an obtuse angle α with the

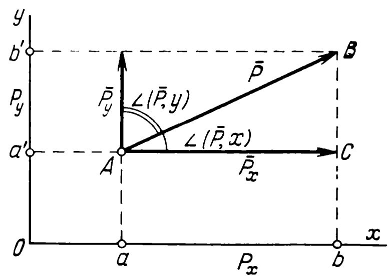


Fig. 11.

positive x axis. From the construction in Fig. 10*b* it is clear that the projection of the force $\bar{P}$ is negative and its magnitude is determined from the triangle ABC

$$P_x = -P\cos\beta.$$

A force located in the plane xOy can be projected on two co-ordinate axes Ox and Oy. Figure 11 represents a force $\bar{P}$ and its projections $\bar{P}_x$ and $\bar{P}_y$. Since the projections are perpendicular to each other, it follows from the right triangle ABC:

$$\left.\begin{aligned} &\text{(a)}\ \ P = \sqrt{P_x^2 + P_y^2},\\ &\text{(b)}\ \ \cos\angle(\bar{P}, x) = \frac{P_x}{P},\\ &\text{(c)}\ \ \cos\angle(\bar{P}, y) = \frac{P_y}{P}. \end{aligned}\right\} \qquad (3)$$

These formulas may be used to determine the magnitude and direction of a force when its projections on the co-ordinate axes are known. The same formulas may be used to determine the magnitude and direction of any vector in terms of its projections.

In a plane, a force $\bar{P}$ can be represented as the vector sum of two mutually perpendicular forces $\bar{P}_x$ and $\bar{P}_y$ which are equal in modulus to the absolute values of the corresponding projections

$$\bar{P} = \bar{P}_x + \bar{P}_y. \qquad (4)$$

Here we have the resolution of a force into two components along the co-ordinate axes. The difference between the projection of a force and its component is that the projection is a scalar while the component is a vector. This should be remembered when solving problems.

8. Projection of a Vector Sum on an Axis

Concurrent forces $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$, $\bar{P}_4$, $\bar{P}_5$, $\bar{P}_6$ are given (Fig. 12*a*). The geometric sum or the resultant of these forces

$$\bar{R} = \sum_{i=1}^{n} \bar{P}_i = \bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \bar{P}_4 + \bar{P}_5 + \bar{P}_6. \qquad (5)$$

is determined by the closing side of the force polygon $\overline{AL} = \bar{R}$ (Fig. 12*b*).

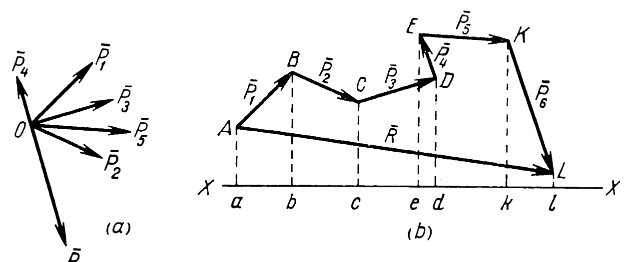


Fig. 12.

We project all the vertices of the force polygon *ABCDEKL* on the *x* axis and denote their projections by *a*, *b*, *c*, *d*, *e*, *k*, *l*, respectively. On the basis of the proposition concerning the projection of a force on an axis (see p. 34) we obtain

$$P_{1x}=ab,\ P_{2x}=bc,\ P_{3x}=cd,$$
$$P_{4x}=-de,\ P_{5x}=ek,\ P_{6x}=kl.$$

The sum of the projections can be represented as

$$\sum_{i=1}^{n} P_{ix}=P_{1x}+P_{2x}+P_{3x}+\ldots+P_{6x}=$$
$$=ab+bc+cd-de+ek+kl=al.$$

Since al is the projection of the resultant force $\bar{R}$ on the x axis, i.e., $al=R_x$, then

$$R_x=P_{1x}+P_{2x}+\ldots+P_{6x}$$

or

$$R_x=\sum_{i=1}^{n} P_{ix}, \tag{6}$$

where n is the number of component vectors.

Consequently, *the projection of a vector sum on any axis is equal to the algebraic sum of the projections of the component vectors on the same axis.*

In a plane, the geometric sum of forces can be projected on two co-ordinate axes.

9. Analytic Determination of the Resultant of a System of Concurrent Forces (Method of Projections)

The resultant of a system of concurrent forces can be found in terms of the projections of the components. By way of example let us consider the system of forces $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ represented in Fig. 13*a*. The resultant of these concurrent forces is constructed in Fig. 13*b*

$$\bar{R}=\bar{P}_1+\bar{P}_2+\bar{P}_3.$$

Projecting all forces on the axes Ox and Oy and using the theorem on the projection of a vector sum, we obtain

$$\left.\begin{aligned} R_x&=P_{1x}+P_{2x}+P_{3x}=\sum_{i=1}^{n} P_{ix},\\ R_y&=P_{1y}+P_{2y}+P_{3y}=\sum_{i=1}^{n} P_{iy}.\end{aligned}\right\} \tag{7}$$

The modulus of the resultant force R is defined in terms of its projections by the formula

$$R=\sqrt{R_x^2+R_y^2}. \tag{8}$$

Substituting the values of the projections R_x and R_y in Eq. (8), we find

$$R=\sqrt{\left(\sum_{i=1}^{n} P_{ix}\right)^2+\left(\sum_{i=1}^{n} P_{iy}\right)^2}. \tag{8a}$$

The direction of $\bar{R}$ is determined from the cosines of

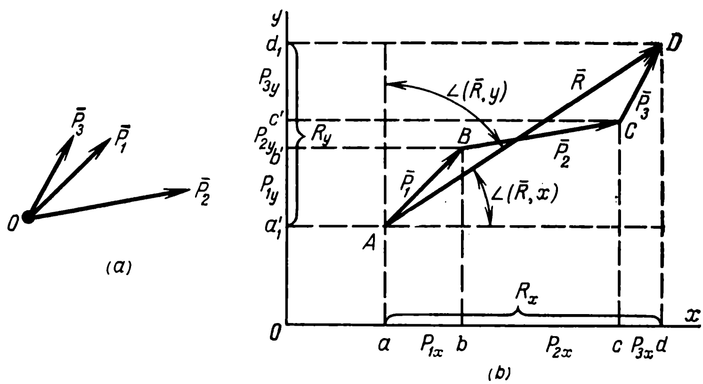


Fig. 13.

the angles which it makes with the co-ordinate axes

$$\left.\begin{aligned}\cos\angle(\bar{R},x)&=\frac{R_x}{R}=\frac{\sum_{i=1}^{n} P_{ix}}{R},\\ \cos\angle(\bar{R},y)&=\frac{R_y}{R}=\frac{\sum_{i=1}^{n} P_{iy}}{R}.\end{aligned}\right\} \tag{9}$$

With these formulas one can determine analytically the magnitude and direction of the resultant force from the known projections of the component forces.

10. Conditions and Equations of Equilibrium for a System of Concurrent Forces

If, in constructing the force polygon for a two-dimensional system of concurrent forces, the tip of the last component force coincides with the tail of the first one, the resultant $\bar{R}$ of all component forces $\bar{P}_1$, $\bar{P}_2$, etc. is zero. In this case the system of concurrent forces is in equilibrium.

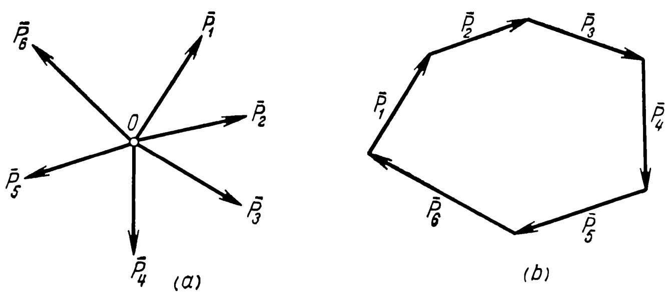


Fig. 14.

The geometric condition for the equilibrium of a given system of concurrent forces is that the force polygon be closed. This condition permits direct solution of many problems.

The physical meaning of this condition is that a given force system is balanced, the resultant force vector reducing to a point. Figure 14*a* shows a balanced system of concurrent forces; in Fig. 14*b* a closed force polygon is constructed for this system.

The projections of the resultant of a concurrent force system on the co-ordinate axes are equal to the sum of the projections of the component forces on the same axes, i.e.,

$$\left.\begin{aligned} R_x &= \sum_{i=1}^{n} P_{ix}, \\ R_y &= \sum_{i=1}^{n} P_{iy}. \end{aligned}\right\}$$

When the bar MN is compressed, its reactions are directed *toward* the points of support, i.e., outward (Fig. 15*b*).

Consequently, it may be said that in a stretched bar the reactions are directed away from the joints, and in a compressed bar toward the joints.

An analogy may be drawn here with a deformed spring (Fig. 15*c*, *d*, *e*).

In solving problems by the analytic method, it is sometimes difficult to determine the direction of reactions. In these cases bars will be assumed stretched with the reactions directed inward, away from the joints. If the solution yields a negative value for the reaction, it means that the bar is actually in compression and not in tension. Thus, in equilibrium equations the reactions of stretched bars will be positive and those of compressed bars negative.

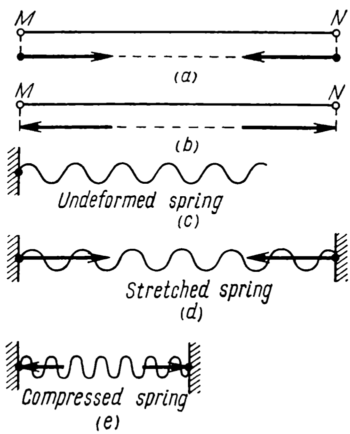


Fig. 15.

Example 1. A weight $Q = 400$ N is suspended from a bracket ABC at a point B by a cable which passes over a fixed pulley D (Fig. 16*a*).

Determine the forces in the bars AB and BC of the bracket if the cable by which the weight is suspended makes a right angle with the bar BC. The bars of the bracket are at 45° to each other; $\angle ABC = 45°$. Neglect the weight of the cable and the friction in the pulley D.

Solution. *Analytic method.*

Isolate the joint B which is considered as an object of equilibrium in this problem. Apply to this joint the given force $\overline{Q}$ which is transmitted through the cable. Account is taken of the fact that the fixed pulley D changes the direction of the force but does not affect its magnitude. Separate the joint B from the constraints which are realized by the bars AB and BC. Apply instead the reactions of the bars $\overline{S}_1$ and $\overline{S}_2$ and direct them away from the joint, i.e., assume that tensile forces are induced in both bars. We choose co-ordinate axes *xy* (see Fig. 16*b*) and set up equilibrium equations

$$\sum_{i=1}^{n} P_{ix}=0,\quad S_1-Q\cos 45^\circ+S_2\cos 45^\circ=0,$$

$$\sum_{i=1}^{n} P_{iy}=0,\quad S_2\cos 45^\circ+Q\cos 45^\circ=0.$$

Solving the equilibrium equations, we find

$$S_2=-Q=-400\ \text{N},$$

$$S_1=Q\cos 45^\circ-S_2\cos 45^\circ=$$

$$=400\times 0.707-(-400)\times 0.707=565.6\ \text{N}.$$

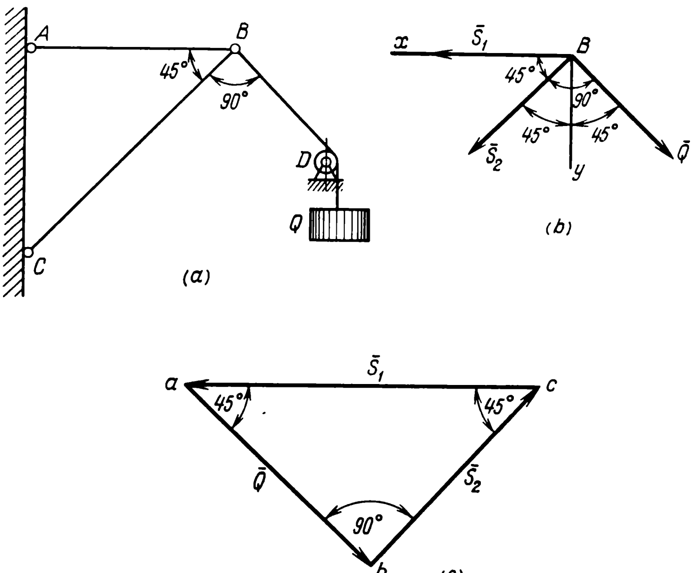


Fig. 16.

The minus sign in front of the numerical value of the reaction $\bar{S}_2$ indicates that the bar *AB* is in compression and not in tension as was assumed.

Semigraphical method.

This example can be conveniently solved by the semigraphical method, as it involves the equilibrium of three concurrent forces. Sketch a closed triangle for the given force $\bar{Q}$ and the reactions of constraints $\bar{S}_1$ and $\bar{S}_2$. Laying off the force $\bar{Q}$ as a segment *ab* (Fig. 16*c*), draw from its tail the direction line of the force $\bar{S}_1$ and from the tip that of the force $\bar{S}_2$. These lines intersect at a point *c*. The triangle *abc* so obtained enables one to find the sense and magnitude of the forces in the bars of the bracket. From Fig. 16*c* it follows that the force triangle *abc* is a right isosceles triangle. Hence

$$S_2 = Q = 400 \text{ N}, \quad S_1 = \frac{Q}{\cos 45^\circ} = \frac{400}{0.707} = 565.6 \text{ N}.$$

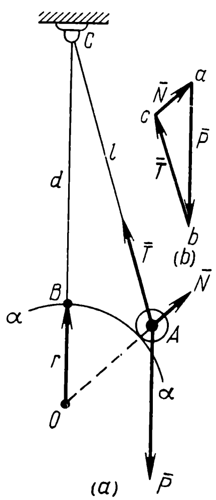


Fig. 17.

From the sense of the forces found it follows that the bar *BC* is in compression (force $\bar{S}_2$ is directed toward the joint), while the bar *AB* is in tension (force $\bar{S}_1$ is directed away from the joint).

The graphical method of solution is basically the same as the semigraphical method, except that the force triangle is not sketched but is drawn to a certain scale, care being taken to have the forces strictly parallel to the corresponding bars; the required forces are measured to scale instead of computed. Therefore, the accuracy of the graphical method depends on the accuracy of the construction.

Example 2. A ball is supported on a smooth spherical surface α — α by a flexible string attached to a fixed point *C* (Fig. 17*a*).

Determine the reaction of the spherical surface and the tension in the string if the following data are given: weight of the ball $P = 10$ N, dimensions: $AC = l = 60$ cm, $OB = r = 20$ cm, $CB = d = 55$ cm.

The ball radius can be neglected because of its smallness, i.e. we can take that $OB = OA = r$.

Solution. The ball is in equilibrium under the action of its weight and the reactions of the string and the spherical surface. Consider the equilibrium of the ball. The only active force exerted on the ball is its weight $\overline{P}$. Separate the ball from the constraints, i.e., the flexible string and the smooth spherical surface, and replace their action by unknown reactions. The reaction of the string $\overline{T}$ is directed along the line AC, the reaction of the spherical surface along the radius OA.

The three forces $\overline{P}$, $\overline{N}$ and $\overline{T}$ intersect at a common point A and are in equilibrium, therefore the force triangle *abc* (Fig. 17*b*) must be closed. From the similar triangles AOC and *abc*, which have respectively parallel sides, it follows that

$$\frac{ca}{ab}=\frac{AO}{CO} \quad \text{or} \quad \frac{N}{P}=\frac{r}{r+d}$$

and

$$\frac{cb}{ab}=\frac{AC}{OC} \quad \text{or} \quad \frac{T}{P}=\frac{l}{r+d},$$

whence we obtain

$$N=P\frac{r}{r+d}=10\frac{20}{20+55}=2.67 \text{ N},$$

$$T=P\frac{l}{r+d}=10\frac{60}{20+55}=8 \text{ N}.$$

Example 3. Two weights $P = 3$ kN and $Q = 1$ kN are suspended from a bracket ABC at a point B as shown in Fig. 18*a*. The bar AB of the bracket makes an angle $\alpha = 30°$ with the vertical wall, the bar BC an angle $\beta = 60°$. Neglecting the friction in the pulley D, determine the forces in the bars AB and BC.

Solution. Isolate the joint B as an object of equilibrium. Apply the given forces $\overline{P}$ and $\overline{Q}$ to it. The force $\overline{Q}$ is applied horizontally as it is transmitted through the fixed pulley D which changes the direction of the force but does not affect its magnitude. Separate the joint B from the constraints, i.e., remove the bars AB and BC and apply their reactions $\overline{S}_1$ and $\overline{S}_2$. As shown in Fig. 18*b*, these reactions are directed away from the joint. This corresponds to the assumption that both bars, AB and BC, are in tension. We choose co-ordinate axes xy, with the x axis directed from the point B to the left

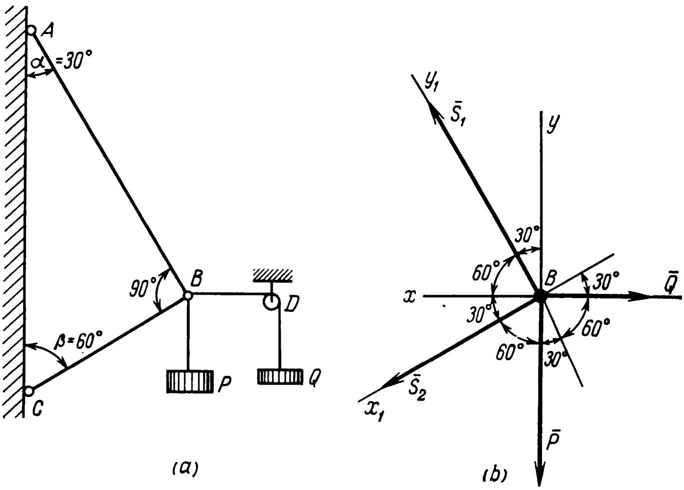


Fig. 18.

so that the greatest possible number of forces are projected on it with the positive sign. Set up equilibrium equations

$$\sum_{i=1}^{n} P_{ix}=0, \quad S_1 \cos 60° + S_2 \cos 30° - Q = 0,$$

$$\sum_{i=1}^{n} P_{iy}=0, \quad S_1 \cos 30° - S_2 \cos 60° - P = 0.$$

Solve the equilibrium equations. From the first equation

$$S_1 = \frac{Q - S_2 \cos 30°}{\cos 60°}.$$

Substitute into the second equation

$$\frac{Q - S_2 \cos 30°}{\cos 60°} \cos 30° - S_2 \cos 60° - P = 0.$$

After simple transformations we have

$$Q \cos 30° - S_2 \cos^2 30° - S_2 \cos^2 60° - P \cos 60° = 0,$$

whence

$$S_2 = \frac{Q \cos 30° - P \cos 60°}{\cos^2 30° + \cos^2 60°} = \frac{1 \times 0.866 - 3 \times 0.5}{0.866^2 + 0.5^2} = -0.633 \text{ kN}.$$

The minus sign in front of the numerical value of the reaction $\bar{S}_2$ indicates that the bar *BC* is actually in compression and not in tension.

Substituting the magnitude of $\bar{S}_2$ into the formula for $\bar{S}_1$, we obtain

$$S_1 = \frac{Q - S_2 \cos 30^\circ}{\cos 60^\circ} = \frac{1 - (-0.633) \times 0.866}{0.5} = 3.100 \text{ kN}.$$

Here we had to solve the equilibrium equations simultaneously. This can be avoided by a rational choice of co-ordinate axes. Taking into account that the bars *AB* and *BC* are mutually perpendicular, we choose co-ordinate axes x_1y_1 along the respective bars (Fig. 18*b*). Set up equilibrium equations

$$\sum_{i=1}^{n} P_{ix_1} = 0, \quad S_2 + P \cos 60^\circ - Q \cos 30^\circ = 0,$$

$$\sum_{i=1}^{n} P_{iy_1} = 0, \quad S_1 - P \cos 30^\circ - Q \cos 60^\circ = 0.$$

By solving the equations, we find

$$S_2 = Q \cos 30^\circ - P \cos 60^\circ = 1 \times 0.866 - 3 \times 0.5 = -0.633 \text{ kN},$$

$$S_1 = P \cos 30^\circ + Q \cos 60^\circ = 3 \times 0.866 + 1 \times 0.5 = 3.100 \text{ kN}.$$

In this case the solution of the set of equations is far easier as each equation involves only one unknown.

Example 4. A winch *D* (Fig. 19*a*) is used to lift a load $P = 2$ kN, the cable passes over a pulley fixed at a point *B* by means of bars *AB* and *BC*. The angles between the bars *AB* and *BC* and the cable *BD* are indicated in Fig. 19*a*. Neglecting the dimensions of the pulley *B*, find the forces in the bars *AB* and *BC*. The bars are pin-connected at points *A*, *B* and *C*.

Solution. Isolate the joint *B* and, neglecting the dimensions of the pulley, apply to the isolated joint the given forces: the load $\bar{P}$ and the tension in the cable $\bar{R}$ which is equal in magnitude to the load being lifted, $R = P$. Separate the joint *B* from the constraints, the bars *AB* and *BC*, and apply their reactions $\bar{T}_1$ and $\bar{T}_2$.

The required reactions $\bar{T}_1$ and $\bar{T}_2$ are directed away from the point B, assuming the bars to be in tension (Fig. 19b).

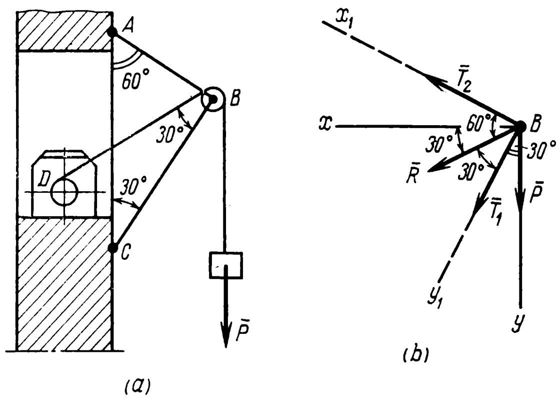


Fig. 19.

Choose co-ordinate axes and set up equilibrium equations

$$\sum P_{ix}=T_2\cos 30°+R\cos 30°+T_1\cos 60°=0,$$

$$\sum P_{iy}=-P-T_1\cos 30°-R\cos 60°+T_2\cos 60°=0.$$

From the first equation we find

$$T_1=-(R+T_2)\frac{\cos 30°}{\cos 60°},$$

but $R=P$, hence

$$T_1=-(P+T_2)\frac{\cos 30°}{\cos 60°}$$

and, substituting into the second equation, we obtain

$$-P+P\frac{\cos^2 30°}{\cos 60°}+T_2\frac{\cos^2 30°}{\cos 60°}-P\cos 60°+T_2\cos 60°=0.$$

Reduce to a common denominator and solve for T_2

$$-P\cos 60°+P\cos^2 30°-P\cos^2 60°+T_2\cos^2 30°+$$
$$+T_2\cos^2 60°=0.$$

$$P\left(-\frac{1}{2}+\frac{3}{4}-\frac{1}{4}\right)+T_2\left(\frac{3}{4}+\frac{1}{4}\right)=0,\ T_2=0,$$

then

$$T_1 = -P\frac{\cos 30^\circ}{\cos 60^\circ} = -2\ \frac{0.866}{0.500} = -3.47\ \text{kN}.$$

The minus sign indicates that a compressive force, and not a tensile force as was assumed, is induced in the bar BC.

We had to solve the two equations simultaneously. This can be avoided by choosing other directions of the co-ordinate axes.

Choose co-ordinate axes x_1 and y_1 with the x_1 axis along the bar BA and the y_1 axis along BC (Fig. 19*b*). The equilibrium equations become

$$\sum P_{ix_1} = T_2 + P\cos 60^\circ - P\cos 60^\circ = 0,$$
$$\sum P_{iy_1} = T_1 + P\cos 30^\circ + P\cos 30^\circ = 0,$$

whence

$$T_2 = 0,$$
$$T_1 = -2P\cos 30^\circ = -2\times 2\times 0.866 = -3.47\ \text{kN}.$$

As is evident the proper choice of co-ordinate axes considerably simplifies the solution of the problem.

12. Theorem of Concurrence of Three Mutually Balanced Non-Parallel Forces

In this section a theorem is proved which often helps in the solution of equilibrium problems. It is formulated as follows.

If three non-parallel forces lying in the same plane are balanced, their lines of action intersect at a common point.

Mutually balanced non-parallel forces $\overline{P}_1$, $\overline{P}_2$ and $\overline{P}_3$, lying in the same plane, are applied at points A_1, A_2 and A_3 (Fig. 20). The forces $\overline{P}_1$ and $\overline{P}_2$, as non-parallel, intersect at a certain point A. Transmitting the forces $\overline{P}_1$ and $\overline{P}_2$ along their lines of action to point A, we find their resultant $\overline{R}$, applied at the same point, as the diagonal of the parallelogram constructed on these forces.

The forces $\overline{P}_1$, $\overline{P}_2$ and $\overline{P}_3$ are mutually balanced. Clearly, the force $\overline{R}$ (the resultant of the forces $\overline{P}_1$ and $\overline{P}_2$) and the force $\bar{P}_3$ must also be balanced. This means that they have the same magnitude, same line of action and opposite sense, i.e., the forces $\bar{R}$ and $\bar{P}_3$ pass through point A, which proves the theorem.

Of course, if three forces intersect at a common point, this does not yet mean that they are in equilibrium. Equili-

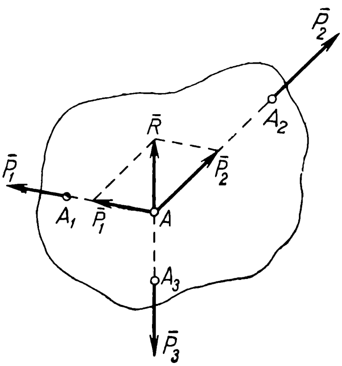


Fig. 20.

brium requires that their resultant be zero. Thus, the inverse theorem is not true.

Let us apply the theorem of equilibrium of three non-parallel forces to the solution of a problem.

Example 5. A homogeneous beam of weight $P = 200$ kN is suspended by three cables AB, CD and CE arranged as shown in Fig. 21*a*.

Determine the forces induced in the cables.

Solution. Isolate the beam AC and, taking into account its homogeneity, apply its weight at mid-point. Separate the beam from the constraints and apply instead reactions $\bar{S}_1$, $\bar{S}_2$ and $\bar{S}_3$. The force system obtained is not concurrent, it involves three unknowns. Replace two unknown reactions $\bar{S}_2$ and $\bar{S}_3$ by their resultant $\bar{R}_C$ (Fig. 21*a*). It may now be stated that the beam is subjected to three balanced forces $\bar{P}$, $\bar{S}_1$ and $\bar{R}_C$ which intersect at a common point K. In other words, having found the point of intersection of the forces $\bar{S}_1$ and $\bar{P}$

we determine the direction of the reaction $\overline{R}_C$. It is inclined to the horizon and vertical at an angle of 45°. We construct the

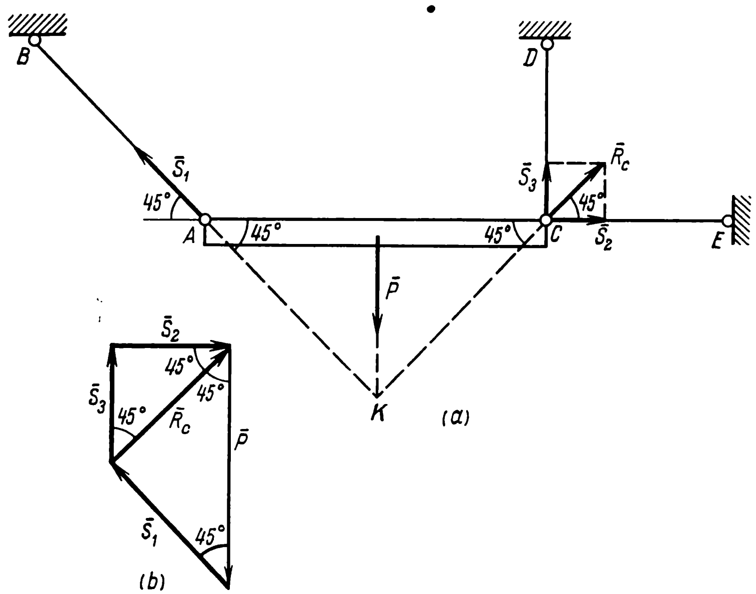


Fig. 21.

force triangle (Fig. 21*b*) from which the magnitudes of the forces $\overline{S}_1$ and $\overline{R}_C$ are easily determined

$$S_1 = R_C = P \sin 45° = 200 \times 0.707 = 141.4 \text{ kN}.$$

In the same figure the force $\overline{R}_C$ is resolved into two components, $\overline{S}_2$ and $\overline{S}_3$, whose magnitudes are also easy to compute

$$S_2 = S_3 = R_C \cos 45° = 141.4 \times 0.707 = 100 \text{ kN}.$$

The above example demonstrates the advantages of applying the theorem of concurrence of three balanced forces in the solution of problems.

The methods of determining the forces in structural elements based on the theorem of concurrence of three balanced forces are extensively used in some branches of mechanics.

CHAPTER III

Couple

13. Addition of Two Parallel Forces of the Same Sense

Let a force $\overline{P}$ be applied at a point A of a body (Fig. 22), and a force $\overline{Q}$ parallel to $\overline{P}$, at a point B. We propose to determine the magnitude, direction and point of application of the resultant force $\overline{R}$.

Apply equal and opposite forces $\overline{F}_1$ and $\overline{F}_2$ at points A and B respectively; according to Axiom III this will not disturb the mechanical state of the body.

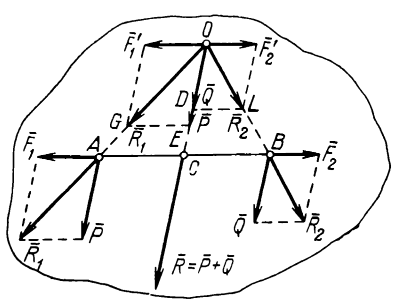


Fig. 22.

Add the forces $\overline{P}$ and $\overline{F}_1$. Their resultant is $\overline{R}_1$. Add the forces $\overline{Q}$ and $\overline{F}_2$. Their resultant is $\overline{R}_2$. Transmit the forces $\overline{R}_1$ and $\overline{R}_2$ to the point of intersection O of their lines of action. Resolve the force $\overline{R}_1$ at the point O into two components, one, $\bar{F}_1'$, equal and parallel to the force $\bar{F}_1$, the other equal and parallel to the force $\bar{P}$. Also, resolve the force $\bar{R}_2$ into two components, one, $\bar{F}_2'$, equal and parallel to the force $\bar{F}_2$, the other equal and parallel to the force $\bar{Q}$.

Since the forces $\bar{F}_1'$ and $\bar{F}_2'$ are equal in modulus and, being applied at the same point, are mutually balanced, they may be discarded according to Axiom III. Then only forces $\bar{P}$ and $\bar{Q}$ applied at point O are left.

The sum of these forces represents the resultant of the given forces, i.e., $\bar{R} = \bar{P} + \bar{Q}$; its line of action passes through C.

We now move the force $\bar{R}$ along its line of action to the point C located on the line AB.

From the similar triangles ACO and GOE, we have

$$\frac{AC}{OC} = \frac{GE}{OE}. \tag{a}$$

The triangles BCO and LOD are also similar, hence

$$\frac{BC}{OC} = \frac{DL}{OD}. \tag{b}$$

By dividing (a) by (b), we obtain

$$\frac{AC \times OC}{OC \times BC} = \frac{GE \times OD}{OE \times DL}.$$

From the construction it is clear that $GE = DL$; hence, after reduction the equation becomes

$$\frac{AC}{BC} = \frac{OD}{OE} = \frac{Q}{P}.$$

Thus, *the resultant of two parallel forces of the same sense is equal to the arithmetic sum of these forces, is parallel to them and has the same sense; the line of action of the resultant divides the straight line joining the points of application of the components into parts inversely proportional to these forces.*

14. Addition of Two Parallel Forces of Opposite Sense

Let a force $\bar{Q}$ be applied at a point A of a body (Fig. 23), and a force $\bar{P}$ parallel and opposite to $\bar{Q}$, at a point B. The forces $\bar{P}$ and $\bar{Q}$ are not equal in magnitude; suppose that

$$P > Q.$$

We resolve the larger force $\bar{P}$ into two parallel components so that one of the components, $\bar{Q}'$, applied at point A, is equal in magnitude but opposite in sense to the force $\bar{Q}$. Since the

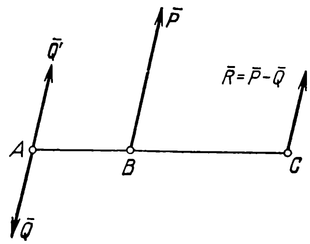


Fig. 23.

force $\bar{P}$ must be equal to the sum of the components, the second component force is determined as the difference between the force $\bar{P}$ and the component $\bar{Q}$

$$\bar{R} = \bar{P} - \bar{Q}.$$

The point of application C of this force can be determined by a proportion

$$\frac{AB}{BC} = \frac{P-Q}{Q}. \tag{11}$$

The forces $\bar{Q}$ and $\bar{Q}'$, being applied at the same point and being equal and opposite, are mutually balanced and so may be discarded. In this case the remaining force $\bar{R} = \bar{P} - \bar{Q}$ is the resultant.

On the basis of the property of derived proportions we obtain from Eq. (11)

$$\frac{AB+BC}{BC} = \frac{P-Q+Q}{Q}$$

or

$$\frac{AC}{BC} = \frac{P}{Q}. \tag{12}$$

Thus, we come to the conclusions:

1. *The resultant of two parallel forces of opposite sense is equal to the difference of these forces, parallel to them and has the same sense as the larger force. The line of action of the resultant divides the extended line segment AB, joining the points of application of the forces, on the side of the larger force in inverse proportion to the magnitudes of the forces.*

2. *If the forces $\bar{P}$ and $\bar{Q}$ are equal in magnitude, the resultant force is zero.*

It should be remembered that in the latter case the body will not be in a state of equilibrium under the action of this

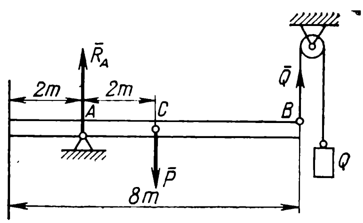


Fig. 24.

system as, according to Axiom II, two equal and oppose- forces are in equilibrium only if they have the same line of action.

Example 6. A rod of length $l = 8$ m (Fig. 24) and weight $P = 21$ kgf $= 206$ N is placed horizontally. It is supported at a point A distant 2 m from the end. The other end is attached to a rope passing over a pulley.

Determine the load $\bar{Q}$ required to hold the rod in equilibrium.

Solution. The rod is subjected to three parallel forces, $\bar{R}_A$, $\bar{P}$ and $\bar{Q}$. According to the condition of the problem, the force $\bar{Q}$ must balance two opposite parallel forces.

For the rod to be in equilibrium, the resultant of two parallel forces $\bar{P}$ and $\bar{Q}$ must pass through the point of support A (Fig. 24). The resultant will be directed downward, opposite to the reaction $\bar{R}_A$. Consequently,

$$\frac{P}{Q} = \frac{AB}{AC},$$

whence

$$Q=\frac{P\times AC}{AB}=\frac{21\times 2}{6}=7\text{ kgf}=68.7\text{ N}.$$

15. Moment of a Couple

The point of application of the resultant of two parallel forces $\bar{P}$ and $\bar{Q}$ of opposite sense is determined from the equality [see Sec. 14, formula (11)]

$$\frac{AB}{BC}=\frac{P-Q}{Q},$$

whence

$$BC=AB\frac{Q}{P-Q}.$$

A particular case of two parallel forces of opposite sense, when these forces, $\bar{P}$ and $\bar{Q}$, are equal, is of special interest

$$BC=AB\frac{Q}{0}=\infty.$$

Consequently, if two equal and parallel forces have opposite sense, then, first, the magnitude of the resultant force is zero ($R = P - Q = 0$) and, second, the point of application of the resultant force ($BC = \infty$) is at infinity.

A system of two equal and parallel forces which have opposite sense and different lines of action is called a *couple.*

This particular case of antiparallel forces is of great importance in practice. This is why the properties of a couple as a specific measure of mechanical interaction of bodies are studied separately.

A body acted upon by a couple is not in equilibrium. As experience shows, the action of a couple on a rigid body consists in tending to rotate the body.

The ability of a couple to produce a rotation is expressed quantitatively by the moment of the couple which is equal to the product of the force and the shortest distance (measured along the perpendicular to the forces) between the lines of action of the forces.

Denote the moment of a couple by M and the shortest distance between the forces by a, then the absolute value of the moment (Fig. 25) is

$$M = P_1 a = P_2 a.$$

Since $P_1 = P_2 = P$, then

$$M = Pa.$$

The shortest distance between the lines of action of the forces is called the *arm of a couple.* It may therefore be said that the *moment of a couple is numerically equal to one of the forces times the arm.*

In the international system of units (SI), the force P is measured in newtons and the arm in metres. Accordingly,

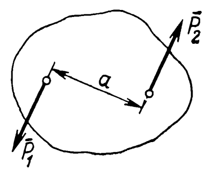


Fig. 25.

the moment of a couple in the SI system is measured in newton-metres (N-m) or in units derived from the newton-metre, kN-m, MN-m, etc. In the engineers' system of units, the force

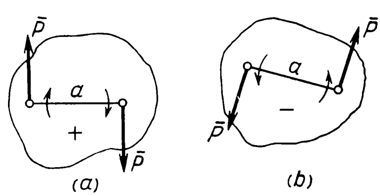


Fig. 26.

is measured in kilograms and the arm in metres, therefore the moment of a couple is measured in kilogram-metres (kgf-m) or in units derived from the kilogram-metre, ton-m, kgf-cm, etc.

Similarly to a force, a couple is assigned a sense. We agree to consider the moment of a couple positive if the couple tends to rotate a body clockwise (Fig. 26*a*) and negative if the couple tends to rotate a body counterclockwise (Fig. 26*b*).

Thus, the moment of a couple can be written as

$$M = \pm Pa.$$

The sign rule adopted for moments of couples is conventional. We might just as well have adopted opposite signs. To avoid confusion, one must always adhere to a definite sign rule in solving problems.

It is often found convenient or even necessary to represent the moment of a couple as a vector.

The moment vector of a given couple is a vector equal in magnitude to one of the forces multiplied by the arm and dire-

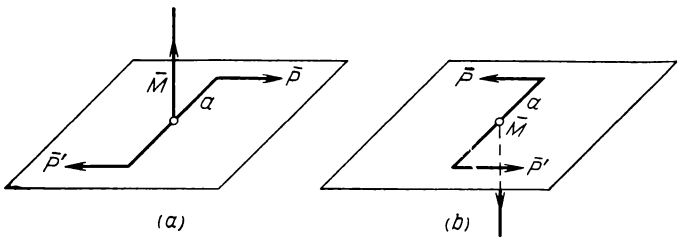


Fig. 27.

cted perpendicular to the plane of action of the couple so that the rotation produced by the couple is clockwise when viewed from the tip of the vector (Fig. 27*a* and *b*).

16. Equivalence of Couples. Translation of a Couple in Its Plane of Action

In accordance with the definition of equivalent force systems given in Sec. 1, *two couples are considered equivalent if, after one of the couples is replaced by the other, the mechanical state of the body remains unchanged, i.e., neither the motion of the body nor its equilibrium is affected.*

To establish the conditions of equivalence of couples, an important property of a couple is proved below:

The mechanical state of a body will not be disturbed if a given couple applied to it is translated to any other position in its plane.

Consider a couple consisting of equal and opposite forces $\overline{P}_1$ and $\overline{P}_2$ with an arm AB (Fig. 28).

Let us take an arbitrarily located segment $CD = AB$ and show that, without disturbing the mechanical state, we can

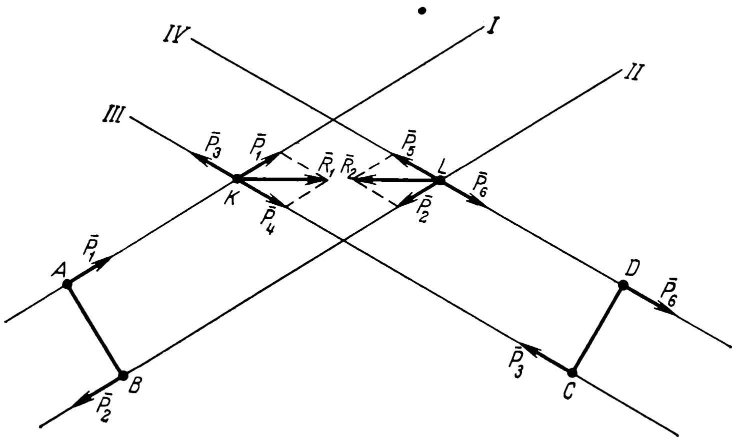


Fig. 28.

move the given couple so that its arm will coincide with the segment CD.

Denote the lines of action of the forces $\overline{P}_1$ and $\overline{P}_2$ by I and II and erect perpendiculars III and IV to the segment CD at points C and D.

Point K is the intersection of the lines I and III and point L the intersection of the lines II and IV.

Transmit the points of application of the forces $\overline{P}_1$ and $\overline{P}_2$ along their lines of action I and II to points K and L, respectively. Next, apply at K and L mutually balanced forces $\overline{P}_3$ and $\overline{P}_4$, $\overline{P}_5$ and $\overline{P}_6$ numerically equal to the forces $\overline{P}_1$ and $\overline{P}_2$ and directed, respectively, along the lines III and IV

$$P_1 = P_2 = P_3 = P_4 = P_5 = P_6.$$

Add the force $\overline{P}_1$ applied at point K and the force $\overline{P}_4$. By constructing a parallelogram (rhombus) on these forces we obtain their resultant $\overline{R}_1$.

Likewise, adding the force $\overline{P}_2$ applied at point L and the force $\overline{P}_5$ we obtain their resultant $\overline{R}_2$.

The parallelograms constructed at points K and L are equal. Consequently, $R_1 = R_2$.

The force $\bar{R}_1$, as the diagonal of the rhombus, divides the angle between the forces $\bar{P}_1$ and $\bar{P}_4$ in half.

On the other hand, the line KL, as the diagonal of a rhombus formed by the straight lines I, II, III, IV, also divides the same angle in half.

Hence, the force $\bar{R}_1$ is directed along the line KL.

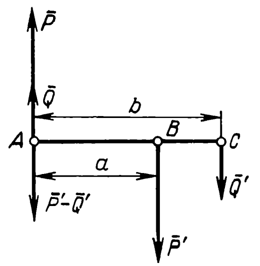
It can be proved in a similar manner that the force $\bar{R}_2$ is also directed along the line KL.

The conclusion is evident: the forces $\bar{R}_1$ and $\bar{R}_2$, being equal and opposite and having the same line of action, are mutually balanced. The remaining two forces, $\bar{P}_3$ and $\bar{P}_6$, act along the lines III and IV, respectively. Transmit the points of application of these forces along the lines III and IV to points C and D. Thus, the given couple has been moved to the new required position.

17. Theorem of Equivalent Couples

The theorem which is proved in this section forms the basis for the addition of couples. It states:

The mechanical state of a body will not be disturbed if the magnitude of the forces and the arm of a couple are arbitrarily



F g. 29.

changed provided the moment of the couple remains unchanged.

Consider a couple $\bar{P}\bar{P}'$ with an arm $AB = a$ (Fig. 29). Replace this couple by a new couple with an arm $AC = b$ so that the moment of the couple remains the same.

Resolve the force $\overline{P}'$ into two forces parallel to it with points of application at A and C. Denote the force applied at point C by $\overline{Q}'$, then the magnitude of the force applied at point A is $\overline{P}' - \overline{Q}'$.

Add the forces $\overline{P}$ and $\overline{P}' - \overline{Q}'$ applied at point A into their resultant; taking into account that $\overline{P}'$ is equal to $\overline{P}$, we have $Q = P - (P' - Q') = Q'$.

Thus, we obtain two forces, $\overline{Q}$ and $\overline{Q}'$, which form a couple of moment $M_1 = Qb$.

Let us prove that the moment of the given couple $M = Pa$ is equal to the moment M_1 of the newly obtained couple $(\overline{Q}\overline{Q}')$. Indeed, resolving the force $\overline{P}'$ into two forces $\overline{P}' - \overline{Q}'$ and $\overline{Q}'$, we obtain

$$\frac{P' - Q'}{Q'} = \frac{BC}{AB}.$$

Applying a derived proportion, we find

$$\frac{P' - Q' + Q'}{Q'} = \frac{BC + AB}{AB}$$

or

$$\frac{P'}{Q'} = \frac{b}{a},$$

whence $P'a = Q'b$, i.e., $M = M_1$, which was to be proved. Hence

$$Pa = Qb.$$

Thus, in place of the given couple $\overline{P}\overline{P}'$ with the arm a we now have an equivalent couple $\overline{Q}\overline{Q}'$ with the arm b.

18. Addition of Couples Acting in the Same Plane

Couples can be added as well as forces. A couple having the same effect on a body as a given set of couples is called the *resultant couple*. The resultant of several couples is equivalent to them.

The determination of the resultant from given couples is called *addition of couples*. The inverse problem, i.e., the replacement of a single given couple by several couples is called *resolution of a couple*.

Let us find the resultant of two couples acting in the same plane.

We have two couples $\overline{P}_1\overline{P}'_1$ and $\overline{P}_2\overline{P}'_2$ with arms a and b (Fig. 30), i.e.,

$$\left.\begin{aligned} M_1 &= -P_1 a, \\ M_2 &= P_2 b. \end{aligned}\right\} \quad (13)$$

Reduce the given couples to a common arm leaving the magnitude of the moment of each couple unaltered.

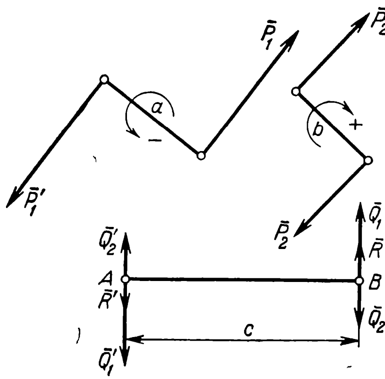


Fig. 30.

A certain segment $AB = c$ is taken as a common arm of the two given couples. Find couples equivalent to the couples of moments M_1 and M_2. Denote the forces of the equivalent couples by $\overline{Q}_1$ and $\overline{Q}_2$, then

$$\left.\begin{aligned} M_1 &= -P_1 a = -Q_1 c, \\ M_2 &= P_2 b = Q_2 c. \end{aligned}\right\} \quad (14)$$

Adding the forces applied at points A and B, we find their resultants

$$\left.\begin{aligned} R' &= Q'_1 - Q'_2, \\ R &= Q_1 - Q_2. \end{aligned}\right\} \quad (15)$$

The resultants $\bar{R}$ and $\bar{R}'$, being equal and opposite, form a couple $\bar{R}\bar{R}'$ whose moment is

$$M = -Rc. \tag{16}$$

The couple $\bar{R}\bar{R}'$ is the resultant couple.

Substituting the value of R from Eq. (15) into Eq. (16), we obtain

$$M = -Rc = -(Q_1 - Q_2)\,c = Q_2c - Q_1c,$$

and since

$$M_2 = Q_2c \text{ and } M_1 = -Q_1c,$$

then

$$M = M_1 + M_2. \tag{17}$$

Thus, we conclude that *the moment of the resultant couple is equal to the algebraic sum of the moments of the component couples.*

This statement applies to any number of couples lying in the same plane.

Therefore, for an arbitrary number of component couples lying in the same or parallel planes the moment of the resultant couple is determined by the formula

$$M = M_1 + M_2 + \ldots + M_n = \sum_{i=1}^{n} M_i. \tag{18}$$

On the basis of the foregoing rule for the addition of couples the following condition of equilibrium of a system of couples is established: *for several couples to be balanced, it is necessary and sufficient that the moment of the resultant couple be zero, or that the algebraic sum of the moments of the couples be zero*

$$M = \sum_{i=1}^{n} M_i = 0. \tag{19}$$

Example 7. Determine the moment of a resultant couple equivalent to a system of three couples lying in the same plane. The first couple is formed by forces $P_1 = P_1' = 2$ kN, has an arm $h_1 = 1.25$ m and acts clockwise; the second couple is formed by forces $P_2 = P'_2 = 3$ kN, has an arm $h_2 = 2$ m and acts counterclockwise; the third couple is formed by

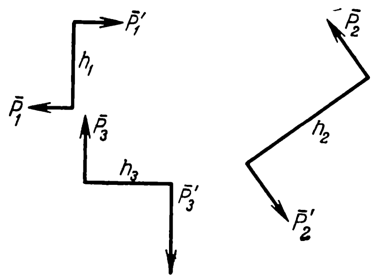


Fig. 31.

forces $P_3 = P'_3 = 4.5$ kN, has an arm $h_3 = 1.2$ m and acts clockwise (Fig. 31).

Solution. We compute the moments of the component couples

$$M_1 = P_1 h_1 = 2 \times 1.25 = 2.5 \text{ kN-m},$$

$$M_2 = -P_2 h_2 = -3 \times 2 = -6 \text{ kN-m},$$

$$M_3 = P_3 h_3 = 4.5 \times 1.2 = 5.4 \text{ kN-m}.$$

Since the couples lie in the same plane, the moment of the resultant couple is obtained by adding their moments algebraically

$$M = \sum_{i=1}^{n} M_i = M_1 + M_2 + M_3 = 2.5 - 6 + 5.4 = 1.9 \text{ kN-m}.$$

CHAPTER IV

Two-Dimensional Systems of Arbitrarily Located Forces

19. Moment of a Force About a Point

When the lines of action of a system of forces lie in the same plane, such a system is referred to as *two-dimensional.* The two-dimensional system of concurrent forces considered above is a particular case of two-dimensional systems. In the general case the lines of action of the forces of a two-dimensional system may not intersect at a common point. An important concept commonly used in the study of two-dimensional force systems is the concept of the moment of a force about a point.

The moment of a force about a point is defined as the product of the magnitude of the force and the perpendicular distance from the point to the line of action of the force (*Fig.* 32*a*).

If the body were fixed at the point O, the force $\bar{P}$ would tend to rotate the body about this point.

The point O about which the moment is taken is called the moment centre, and the perpendicular a the arm of the force with respect to the moment centre.

The moment of the force $\bar{P}$ about O is denoted as

$$M_O(\bar{P}) = Pa.$$

The moments of forces are measured in newton-metres (N-m) or kilogram-metres (kgf-m), as well as the moments of couples.

Join the tail and tip of the force $\bar{P}$ to the moment centre O; the resulting figure OAB is called the moment triangle.

The product Pa is equal to twice the area of this triangle (Fig. 32a). Consequently, we obtain an alternate expression for the moment of the force $\bar{P}$ about the point O

$$M_O(\bar{P}) = 2 \text{ area } \triangle AOB. \tag{20}$$

The moment of a force about a point is equal to twice the area of the moment triangle constructed on the force vector.

We distinguish positive and negative moments according to the sense of rotation. The moment is generally considered

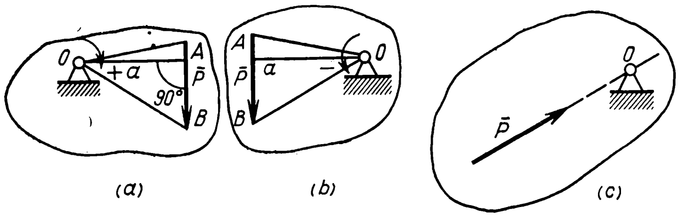


Fig. 32.

as positive if the force tends to rotate a body clockwise (Fig. 32a), otherwise negative (Fig. 32b). As in the case of moments of couples, the adopted sign rule for moments of forces is conventional.

When the line of action of a force passes through the moment centre, the moment of the force about this point is zero as the arm $a = 0$ (Fig. 32c).

There is a significant difference between the moment of a couple and the moment of a force. The magnitude and sense of the moment of a couple are the same at any point in space. By contrast, the magnitude and sense of the moment of a force depend on the position of the point about which the moment is being determined.

By analogy with the moment of a couple, the moment of a force about a point can be represented by a vector. As with the moment of a couple, the vector representation of the moment of a force is used in the study of three-dimensional force systems. The magnitude of the vector representing the moment of a force is equal to the product of the force times the arm with respect to the point. The vector representing the moment of a force is always applied at this point and is perpendicular to a plane passing through this point and the line of action of the force. The vector representing

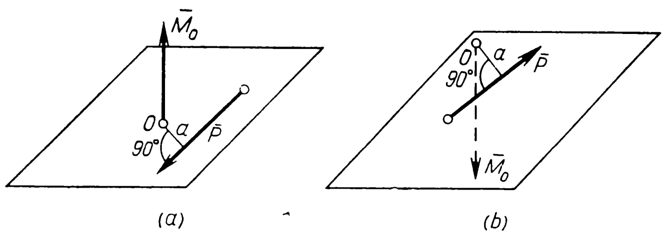


Fig. 33.

the moment of a force points in the direction from which the rotation produced by the force about the point considered appears clockwise (Fig. 33*a* and *b*).

20. Equilibrium of a Lever

Many problems of mechanics involve the equilibrium of a body hinged about a fixed axis. This body is called a *lever*.

A lever is capable of rotating about the fixed axis (Fig. 34*a*).

A lever is in equilibrium only if the sum of the moments of all forces about the fixed point of the lever is zero.

The fixed point of a lever about which we shall set up the moment equation is the point of intersection of the axis of rotation of the lever and the plane of the drawing (Fig. 34*a*)

$$\sum M_O(\bar{P}_i) = 0 \qquad (21)$$

or

$$P_1h_1 - P_2h_2 - P_3h_3 = 0.$$

A lever can be used to lift loads, to produce high pressures with relatively little effort, etc. Levers of two kinds are shown in Fig. 34*b* and *c*. These are respectively the levers of the first and second kind.

In the lever of the first kind (Fig. 34*b*), the fixed point O is located between the load $\bar{P}_2$ to be lifted and the point of application of the lifting force $\bar{P}_1$. In the lever of the second kind (Fig. 34*c*), the fixed point O lies to the same side of the load $\bar{P}_2$ to be lifted and of the lifting force $\bar{P}_1$.

Example 8. A horizontal homogeneous rod AB of weight $G = 2$ N $= 204$ gf and length $l = 50$ cm can rotate about the

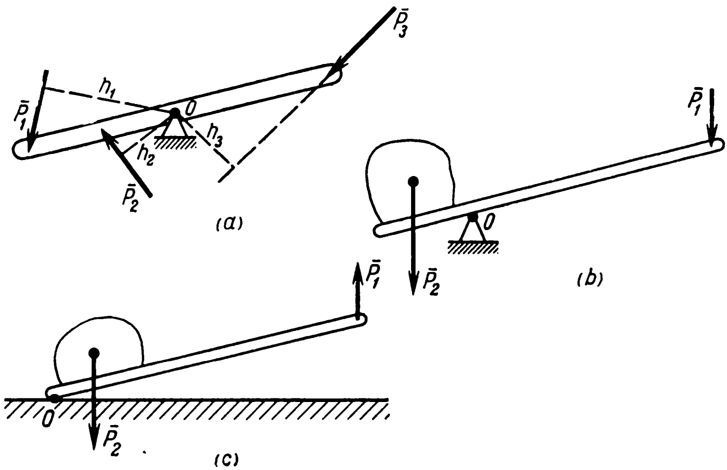


Fig. 34.

axis of a hinge A (Fig. 35). The end B of the rod is pulled up by means of a weight $P = 3$ N $= 306$ gf and a rope passing over a fixed pulley D.

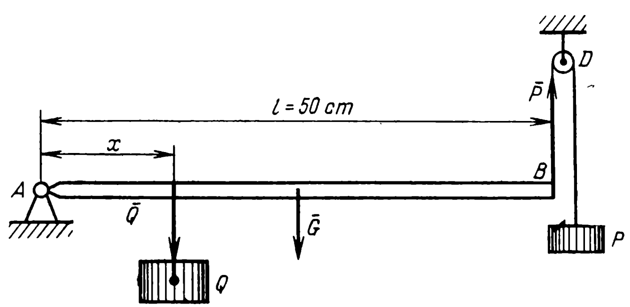


Fig. 35.

Determine the distance from the hinge A at which a weight $Q = 5$ N $= 510$ gf must be suspended if the rod AB is to be in equilibrium.

Solution. We apply all given forces to the rod AB. The rod acts as a lever of the second kind. We set the sum of moments about the hinge A equal to zero

$$\sum M_A = 0, \quad -Pl + G\frac{l}{2} + Qx = 0,$$

whence

$$x = \frac{Pl - G\frac{l}{2}}{Q} = \frac{3 \times 50 - 2 \times 25}{5} = 20 \text{ cm}.$$

Example 9. Considering the rod of Example 6 as a lever and using the same data (Fig. 24) determine the load Q required to maintain the rod in equilibrium.

Solution. We set up the equation of equilibrium of the rod as a lever

$$\sum M_A = 0, \quad -Q \cdot AB + P \cdot AC = 0,$$

whence

$$Q = P\frac{AC}{AB} = 21\,\frac{2}{6} = 7 \text{ kgf} = 68.7 \text{ N}.$$

The solution of the problem based on the equation of equilibrium of the lever proves more graphic and simple than the solution in Example 6 by using the rule for the addition of parallel forces.

21. Reduction of a Force to a Given Point

As is known, every force may be moved to any point on its line of action without changing the mechanical state of the body. Consider now transferring a force to an arbitrary point off its line of action (Fig. 36*a*).

Let a force $\bar{P}$ be applied at a point A. It is required to transfer this force parallel to itself to a point B. Apply at point B two opposite forces $\bar{P}'$ and $\bar{P}''$ equal in modulus and parallel to the given force $\bar{P}$, i.e., $P' = P'' = P$. These two forces do not change the state of the body as they are mutually balanced.

Drop a perpendicular a from point B upon the line of action of the force $\bar{P}$. The resulting system of three forces can be regarded as consisting of the force $\bar{P}'$ applied at point B and the couple $(\bar{P}\bar{P}'')$ of moment $M = Pa$. This couple is called an *associated couple.*

Thus, *the reduction of a force $\bar{P}$ to a point off the line of action of the force results in an equivalent system consisting of a force*

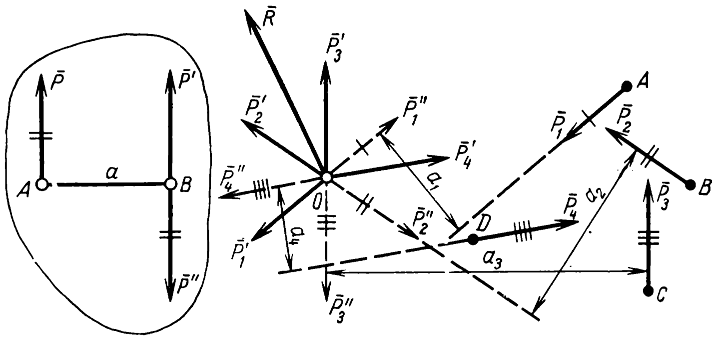


Fig. 36.

of the same modulus and direction as the force $\vec{P}$ and an associated couple whose moment is equal to the moment of the given force about the point of reduction, i.e.,

$$M = M_O(\bar{P}). \tag{22}$$

22. Reduction of a Two-Dimensional Force System to a Given Point

The above method of reducing a force to a point can be applied to any number of forces. Suppose that forces $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ and $\bar{P}_4$ are applied, respectively, at points A, B, C and D of a body (Fig. 36*b*). It is required to reduce these forces to a point O in the plane. Reduce first the force $\bar{P}_1$ applied at point A. Apply at point O two opposite forces $\bar{P}_1'$ and $\bar{P}_1''$, each equal in modulus to the given force $\bar{P}_1$ and parallel to it. As a result of the reduction of the force $\bar{P}_1$ we obtain a force $\bar{P}_1'$ applied at point O and a couple $\bar{P}_1\bar{P}_1''$

(forces forming a couple are marked with the same number of dashes) with an arm a_1; proceeding in the same way with the force $\overline{P}_2$ applied at point B, we obtain a force $\overline{P}'_2$ applied at point O and a couple $\overline{P}_2\overline{P}''_2$ with an arm a_2, etc.

We have replaced the system of forces applied at points A, B, C and D by the concurrent forces $\overline{P}'_1$, $\overline{P}'_2$, $\overline{P}'_3$ and $\overline{P}'_4$ applied at point O and the couples of moments equal to the moments of the given forces about point O

$$M_1 = +P_1a_1 = M_O(\overline{P}_1), \quad M_3 = -P_3a_3 = M_O(\overline{P}_3),$$
$$M_2 = -P_2a_2 = M_O(\overline{P}_2), \quad M_4 = -P_4a_4 = M_O(\overline{P}_4).$$

The forces concurrent at point O can be replaced by a single force $\overline{R}'$ equal to the geometric sum of the components

$$\overline{R}' = \overline{P}'_1 + \overline{P}'_2 + \overline{P}'_3 + \overline{P}'_4 = \overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \overline{P}_4 = \sum_{i=1}^{n=4} \overline{P}_i. \quad (23)$$

This force equal to the geometric sum of the given forces is called *the resultant force of the given force system.*

On the basis of the rule for the addition of couples their moments can be replaced by the moment of the resultant couple, which is equal to the algebraic sum of the moments of the components about point O

$$M_O = M_1 + M_2 + M_3 + M_4 = \sum_{i=1}^{n=4} M_i = \sum_{i=1}^{n} M_O(\overline{P}_i). \quad (24)$$

By analogy with the resultant force, the moment M_O of a couple representing the algebraic sum of the moments of all forces about the centre of reduction O is called *the resultant moment of the system about the given centre of reduction O.* Consequently, in the general case a two-dimensional force system is replaced, as a result of the reduction to a given point O, by an equivalent system consisting of a single force geometrically equal to the resultant force and a single couple of moment equal to the resultant moment of the given force system about the point O.

It should be noted that the resultant force $\overline{R}'$ is not the resultant of the given force system as this system is not equivalent to the force $\bar{R}'$ alone when the resultant moment is not zero, $M_O \neq 0$. Only when the resultant moment vanishes is the resultant force identical with the resultant.

Since the resultant force is equal to the geometric sum of the forces of a given system, neither its magnitude nor its direction depends on the choice of the centre of reduction.

Any point in the plane taken as a centre of reduction will always yield a system of concurrent forces whose components are equal to the given forces.

The magnitude of the resultant moment M_O depends on the position of the centre of reduction since the arms of the component couples depend on the position of the centre about which the moments are taken.

The following variants may be encountered when a force system is reduced to a point:

1. The general case $\bar{R}' \neq 0$, $M_O \neq 0$, the system reduces to the resultant force and the resultant couple.

2. $\bar{R}' \neq 0$, $M_O = 0$, the system reduces to the resultant force alone.

3. $\bar{R}' = 0$, $M_O \neq 0$, the system reduces to a couple of moment equal to the resultant moment.

4. $\bar{R}' = 0$, $M_O = 0$, the system is in equilibrium.

23. Resultant of a Two-Dimensional Force System

Let us prove that in the general case when $\bar{R}' \neq 0$ and $M_O \neq 0$, a point can be found about which the resultant moment of the force system is zero.

Consider a two-dimensional force system reduced to a point O; in general, we have the resultant force $\bar{R}'$ and the resultant moment M_O at point O (Fig. 37). For definiteness, we assume that the resultant moment is clockwise, i.e., $M_O > 0$. Represent this resultant moment by a pair of forces whose modulus is chosen equal to the modulus of the resultant force $\bar{R}'$. Apply one of the forces constituting the couple, $\bar{R}''$, at the centre of reduction O, and the other force, $\bar{R}$, at a certain point C defined by

$$M_O = OC \cdot R.$$

Consequently,

$$OC = \frac{M_O}{R}.$$

Locate the couple ($\bar{R}$, $\bar{R}''$) so that the force $\bar{R}''$ is opposite to the resultant force $\bar{R}'$. At point O, we have two equal and

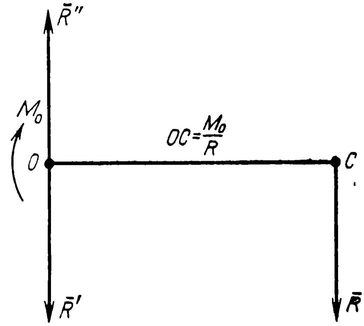


Fig. 37.

opposite forces $\bar{R}'$ and $\bar{R}''$ directed along the same straight line; they may be discarded according to Axiom III.

Consequently, the resultant moment of the given force system about point C is zero and the system reduces to the resultant $\bar{R}$.

24. Theorem of the Moment of a Resultant (Varignon's Theorem)

Consider again a two-dimensional force system (Fig. 36*b*). At an arbitrary point O, this force system reduces to a resultant force $\bar{R}'$ and a resultant moment M_O (Fig. 37), the resultant moment being equal to the algebraic sum of the moments of the given forces about point O

$$M_O = M_1 + M_2 + \ldots + M_n = \sum_{i=1}^{n} M_O(\bar{P}_i). \qquad \text{(a)}$$

At a point C, the system can be reduced to a single force $\bar{R}$ equal in magnitude to the resultant force, $R = R'$. Determine the moment of the resultant $\bar{R}$ about point O.

Taking into account that the arm of the force $\bar{R}$ is

$$OC = \frac{M_O}{R},$$

we obtain (see Fig. 37)

$$M_O(\bar{R}) = R \cdot OC = R\frac{M_O}{R} = M_O. \qquad \text{(b)}$$

Two quantities equal to a third one are equal to each other, hence, by comparing equations (a) and (b), we obtain

$$M_O(\bar{R}) = \sum_{i=1}^{n} M_O(\bar{P}_i). \qquad (25)$$

This equation expresses Varignon's theorem: *the moment of the resultant of a force system about an arbitrarily taken point*

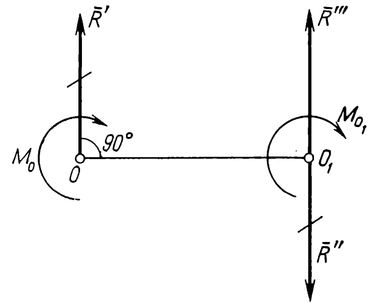


Fig. 38.

is equal to the algebraic sum of the moments of the component forces about the same point.

From Varignon's theorem it follows that the resultant moment of a two-dimensional force system about any point on the line of action of its resultant is zero.

As noted above, the resultant moment of a two-dimensional system depends on the choice of the centre of reduction. Let us see how the resultant moment varies when passing to a new centre of reduction. Suppose a two-dimensional force system is reduced to a point O (Fig. 38) and yields the resultant force $\bar{R}'$ and the resultant moment M_O at this centre. We propose to reduce this force system to a new centre O_1.

To do this, it is sufficient to reduce the resultant force and the resultant moment of the system to the new centre. From the construction (Fig. 38), the line OO_1 is perpendicular to the direction of the resultant force. The resultant moment of the system, being a couple, can be moved freely in its plane of action ($M_O = M_{O_1}$). The resultant force is reduced to the point O_1 in accordance with the reduction rule given in Sec. 21. Apply at the new centre of reduction O_1 two opposite forces $\bar{R}''$ and $\bar{R}'''$ equal in modulus and parallel to the resultant force $\bar{R}'$, i.e., $R'' = R''' = R'$. The forces $\bar{R}''$ and $\bar{R}'''$ are mutually balanced and can be applied according to Axiom III. The forces $\bar{R}'$ and $\bar{R}''$ form an associated couple in the resulting new system. The moment of this couple is equal to the moment of the resultant force about the new centre of reduction O_1

$$M'_{O_1} = R' \cdot OO_1 = M_{O_1}(\bar{R}').$$

In a plane, the moments of couples are added algebraically. Therefore, the resultant moment at the new centre of reduction O_1 is determined as the algebraic sum of the resultant moment at the original centre of reduction O and the additional moment of the associated couple, i.e.,

$$M_{O_1} = M_O + M'_{O_1} = M_O + M_{O_1}(\bar{R}'),$$

whence

$$M_{O_1} - M_O = M_{O_1}(\bar{R}'), \tag{26}$$

i.e., *the change in the resultant moment of a two-dimensional force system when the centre of reduction is changed is equal to the moment of the resultant force of this system applied at the original centre of reduction about the new centre.*

Below, the application of Varignon's theorem is illustrated by examples.

Example 10. Determine the line of action of the resultant of two parallel forces $\bar{P}$ and $\bar{Q}$ of the same sense by using Varignon's theorem (Fig. 39).

Solution. We choose as the moment centre a point C on the line of action of the resultant. Write Varignon's theorem with respect to this point

$$M_C(\bar{R}) = M_C(\bar{P}) + M_C(\bar{Q}).$$

The moment of the resultant about point C on its line of action is zero, $M_C(\overline{R})=0$. Varignon's theorem is now written as

$$M_C(\overline{P})+M_C(\overline{Q})=0.$$

Find the arms of the forces $\overline{P}$ and $\overline{Q}$ with respect to point C. For this purpose, we drop perpendiculars $A'C$ and $B'C$ from this point upon the lines of action of the forces $\overline{P}$ and $\overline{Q}$

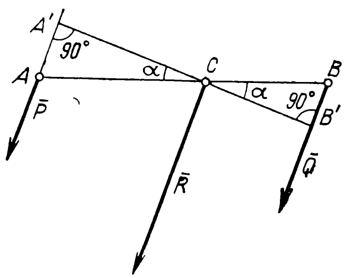


Fig. 39.

(Fig. 39). The magnitudes of these arms are determined in terms of the angle α between the line AB and the perpendicular to the lines of action of the forces

$$A'C=AC\cos\alpha,$$
$$B'C=BC\cos\alpha.$$

The moments of the forces $\overline{P}$ and $\overline{Q}$ about point C are

$$M_C(\overline{P})=-P\cdot A'C=-P\cdot AC\cos\alpha,$$
$$M_C(\overline{Q})=Q\cdot B'C=Q\cdot BC\cos\alpha.$$

Substituting these moments into the expression for Varignon's theorem, we obtain

$$-P\cdot AC\cos\alpha+Q\cdot BC\cos\alpha=0 \text{ or } P\cdot AC=Q\cdot BC,$$

whence

$$\frac{Q}{P}=\frac{AC}{BC},$$

i.e., the line of action of the resultant of two parallel forces divides the distance between the component forces in inverse proportion to their magnitudes.

The same result was obtained in a different and more complicated way in Sec. 13.

Example 11. The resultant moments of a two-dimensional force system about three points A, B, C are, respectively,

$$M_A = 0, \quad M_B = 0, \quad M_C = 2.6 \text{ kN-m}.$$

The points A, B and C form an equilateral triangle of side $a = 3$ m (Fig. 40).

Determine the magnitude and direction of the resultant of this force system.

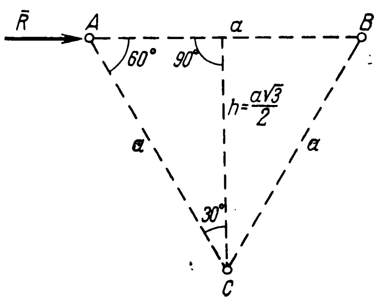


Fig. 40.

Solution. When proving Varignon's theorem, we established that the moment of the resultant about any point is equal to the corresponding resultant moment, i.e.,

$$M_A(\bar{R}) = M_A = 0,$$

$$M_B(\bar{R}) = M_B = 0,$$

$$M_C(\bar{R}) = M_C = Rh = 2.6 \text{ kN-m},$$

where $h = a\frac{\sqrt{3}}{2} = \frac{3\times 1.73}{2} = 2.6$ m is the arm of the resultant with respect to point C.

From the first two equalities it follows that the line of action of the resultant passes through points A and B. Consequently, this force is directed along the straight line AB. Since its moment about point C is positive, i.e., clockwise, the force $\bar{R}$ should accordingly be directed from left to right. The magnitude of the force is determined from the third equality

$$M_C(\bar{R}) = M_C = Rh,$$

whence

$$R = \frac{M_C}{h} = \frac{M_C}{\frac{a\sqrt{3}}{2}} = \frac{2.6}{2.6} = 1 \text{ kN} = 1{,}000 \text{ N}.$$

Example 12. Determine the moment of a force $\bar{P}$ about a point O chosen as the origin of co-ordinates (Fig. 41). The

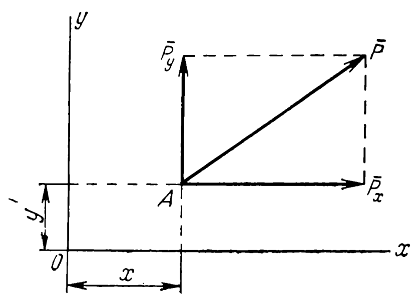


Fig. 41.

following data are given: the projections of the force on the co-ordinate axes, $P_x = 8$ N, $P_y = 6$ N and the co-ordinates of the point of application A, $x = 6$ cm, $y = 3$ cm.

Solution. Resolve the force vector $\bar{P}$ into two components parallel to Ox and Oy. Denote these components by $\bar{P}_x$ and $\bar{P}_y$. The magnitudes of the components are equal to the corresponding projections. The given force $\bar{P}$ is the resultant of the forces $\bar{P}_x$ and $\bar{P}_y$. According to Varignon's theorem we can write

$$M_O(\bar{P}) = M_O(\bar{P}_x) + M_O(\bar{P}_y).$$

From Fig. 41 we find

$$M_O(\bar{P}_x) = P_x y,$$

$$M_O(\bar{P}_y) = -P_y x,$$

therefore

$$M_O(\bar{P}) = P_x y - P_y x. \qquad \text{(a)}$$

This is an analytic expression for the moment of a force about the origin of co-ordinates. Substituting the numerical values, we obtain

$$M_O(\bar{P}) = P_x y - P_y x = 8 \times 6 - 6 \times 3 = 30 \text{ N cm}.$$

If n forces were given in place of a single force, then, resolving each of the given forces into two components parallel to the co-ordinate axes, we would obtain the components

$$\bar{P}'_x, \ \bar{P}''_x, \ \bar{P}'''_x, \ \ldots, \ \bar{P}^n_x;$$

$$\bar{P}'_y, \ \bar{P}''_y, \ \bar{P}'''_y, \ \ldots, \ \bar{P}^n_y.$$

We can write the expression (a) for the moment of a force about the origin for each pair of components $\bar{P}_x$ and $\bar{P}_y$.

The algebraic sum of the moments determines the moment of the resultant of n forces. The analytic expression for the moment of the resultant is

$$M_O(\bar{R}') = \sum_{i=1}^{n} [P_{ix} y_i - P_{iy} x_i].$$

25. A Case of the Reduction of a Two-Dimensional Force System to a Couple

Consider in greater detail the case of reduction of a two-dimensional force system when the resultant force is zero and the resultant moment is different from zero, $\bar{R}' = 0$, $M_O \neq 0$ [see Sec. 22, case (3)]. The resultant force of a system vanishes when the geometric sum of the given forces is zero, in other words, when the force polygon of the system is closed. It is evident that, after reducing this system to a centre O, we obtain a single couple of moment equal to the resultant moment of the system

$$M_O = \sum M_O(\bar{P}_i).$$

There is no resultant force in this case and the resultant moment remains unchanged whatever the centre to which we reduce the force system. Indeed, since the resultant force is zero, it follows from equality (26) that

$$M_{O_1}-M_O=M_{O_1}(\overline{R}')=0 \text{ or } M_{O_1}=M_O,$$

where M_{O_1} is the resultant moment of the system about an arbitrary new centre O_1.

Thus, after reducing the given force system to any two centres O and O_1, we obtain respectively two couples of equal moments M_O and M_{O_1}.

Since each of the couples is equivalent to the given force system, they must naturally be equivalent to each other. Thus, *if the resultant force of a given two-dimensional force system is zero, but its resultant moment about any centre is not zero, this system is equivalent to a couple whose moment is equal to the resultant moment of the system and independent of the choice of the centre of reduction.*

26. Conditions and Equations of Equilibrium for a Two-Dimensional Force System

Every system of forces arbitrarily located in a plane can be reduced to a resultant force and a resultant moment. Therefore, as shown above, the necessary conditions for the equilibrium of forces in a plane are

$$\left.\begin{aligned} &\overline{R}'=0,\\ &M_O=\sum_{i=1}^{n} M_O(\overline{P}_i)=0. \end{aligned}\right\} \qquad (27)$$

The same conditions (27) are sufficient for the equilibrium of a body.

Indeed, from the equality $\overline{R}'=0$ it follows that all forces $\overline{P}'_1$, $\overline{P}'_2$, ..., $\overline{P}'_n$ equal to the given forces $\overline{P}_1$, $\overline{P}_2$, ..., $\overline{P}_n$ and applied at the centre of reduction O are balanced. Likewise, from the equality $M_O=0$ it follows that the sum of the moments of the associated couples $(\overline{P}_1\overline{P}'_1)$, $(\overline{P}_2\overline{P}'_2)$, ..., $(\overline{P}_n\overline{P}'_n)$ is zero, i.e., these couples are also balanced.

Thus, *for a system of forces arbitrarily located in a plane to be in equilibrium, it is necessary and sufficient that the resultant* *force and the resultant moment of these forces about any centre should each be zero.*

The resultant force $\overline{R}'$ represents the geometric sum of all forces of the system transferred to the centre of reduction.

The magnitude of the resultant force can be determined in terms of the projections of all acting forces on the co-ordinate axes.

If the sums of the projections of all forces on the x and y axes are respectively

$$\sum_{i=1}^{n} P_{ix} \quad \text{and} \quad \sum_{i=1}^{n} P_{iy},$$

then the resultant force is

$$R' = \sqrt{[\sum_{i=1}^{n} P_{ix}]^2 + [\sum_{i=1}^{n} P_{iy}]^2}.$$

Equilibrium requires that the resultant force be zero; if this condition is fulfilled, we obtain

$$\left.\begin{aligned} \sum_{i=1}^{n} P_{ix} = 0, \\ \sum_{i=1}^{n} P_{iy} = 0. \end{aligned}\right\} \tag{28a}$$

Besides, equilibrium requires that the resultant moment be zero as well, i.e.,

$$\sum_{i=1}^{n} M_O(\overline{P}_i) = 0. \tag{28b}$$

Henceforth, we shall use a more compact form of writing equilibrium equations: instead of $\sum_{i=1}^{n} P_{ix} = 0$ we shall write $\sum P_{ix} = 0$; instead of $\sum_{i=1}^{n} M_O(\overline{P}_i) = 0$, $\sum M_O = 0$.

Equations (28a) and (28b) are the equations of equilibrium for a body subjected to a system of forces arbitrarily located in a plane.

Two of these equations (28a) express that the algebraic sum of the projections of all forces of the system on two arbitrary co-ordinate axes is zero; equation (28b) expresses that the sum of the moments of all forces about any chosen point is zero.

27. Three Forms of Equilibrium Equations

The equations of equilibrium of an arbitrary two-dimensional force system can be represented in three forms.

The *first*, fundamental, form of these equations

$$\left.\begin{aligned} \sum P_{ix} &= 0, \\ \sum P_{iy} &= 0, \\ \sum M_O &= 0 \end{aligned}\right\} \qquad (29)$$

has been derived above. Other forms can also be used for solving problems.

Since, when a rigid body is in equilibrium, the sum of the moments of all forces applied to it about any point is zero, the following three equations of equilibrium can be obtained by choosing three arbitrary points A, B, C and equating to zero the sum of the moments about each of these points

$$\left.\begin{aligned} \sum M_A &= 0, \\ \sum M_B &= 0, \\ \sum M_C &= 0. \end{aligned}\right\} \qquad (30)$$

This is the *second* form of equilibrium equations.

The three points A, B and C should not be in a straight line. If this condition is not fulfilled, one of the three equations of equilibrium is a consequence of the other two and we obtain only two independent equations.

Indeed, from the equalities $\sum M_A = 0$ and $\sum M_B = 0$ it only follows that the resultant of a given force system is directed along the line AB (Fig. 42*a*). Clearly, this resultant will give a zero moment about any point on the line AB through which it will pass. If, however, the point C lies off the line AB, the equation $\sum M_C = 0$ indicates that the resultant is zero and, together with the equations $\sum M_A = 0$ and $\sum M_B = 0$, determines the equilibrium of the force system.

The *third* form of equilibrium equations is obtained by writing that the sum of the moments about two arbitrary points

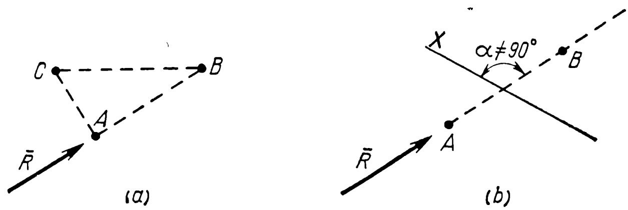


Fig. 42.

A and *B* and the sum of the projections on some axis are zero

$$\left.\begin{aligned} \sum M_A = 0, \\ \sum M_B = 0, \\ \sum P_{ix} = 0. \end{aligned}\right\} \quad (31)$$

When using this form of equilibrium equations, it is essential that the *x* axis is not perpendicular to the line joining points *A* and *B* (Fig. 42*b*). Otherwise, the force system may have a non-zero resultant directed along *AB*. This resultant gives zero moments about points *A* and *B* and ensures the fulfilment of the equations $\sum M_A = 0$ and $\sum M_B = 0$. The projection of the resultant on the *x* axis, perpendicular to the line *AB*, is a point and the third equation $\sum P_{ix} = 0$ will also be satisfied in the absence of equilibrium. If, however, the *x* axis is not perpendicular to the line *AB*, the equation $\sum P_{ix} = 0$ indicates that the resultant of the system (along line *AB*) is zero. Consequently, when the above condition is fulfilled the third form of equations determines the equilibrium of a two-dimensional force system.

Consider a particular case of equilibrium of a body subjected to a two-dimensional system of parallel forces. Let a balanced system of parallel forces $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$, $\bar{P}_4$, $\bar{P}_5$ be applied to a given body (Fig. 43). Through an arbitrary point O of the body we draw an axis Ox perpendicular to the forces and an axis Oy parallel to these forces. We write the first form of equilibrium equations for this force system

$$\left.\begin{aligned}\sum P_{ix}=0,\\ \sum P_{iy}=0,\\ \sum M_O=0.\end{aligned}\right\}$$

The first equation reduces to the identity $0 = 0$ which is fulfilled, whether the forces are balanced or not. Each

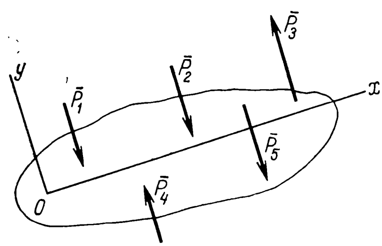


Fig. 43.

force is perpendicular to the axis Ox and its projection on this axis is zero. Consequently, only two equilibrium equations remain for the two-dimensional system of parallel forces, the forces being projected on the y axis full size since the axis Oy is parallel to them.

The first form of equilibrium equations for a two-dimensional system of parallel forces becomes

$$\left.\begin{aligned}\sum P_{iy}=0,\\ \sum M_O=0.\end{aligned}\right\} \tag{32}$$

The second and third forms of equilibrium equations for a two-dimensional system of parallel forces are the same

$$\left.\begin{aligned}\sum M_A=0,\\ \sum M_B=0.\end{aligned}\right\} \tag{33}$$

Thus, for an arbitrary two-dimensional force system we have three equilibrium equations while for a two-dimensional system of parallel forces we have only two. Accordingly, three unknowns can be found when solving equilibrium problems for an arbitrary two-dimensional force system and no more than two when dealing with a two-dimensional system of parallel forces.

If the number of unknowns exceeds the number of equations of statics, the problem becomes statically indeterminate. Statically indeterminate problems can be solved if the elastic properties of a body and the deformations produced in it are taken into account. Methods of solving such problems are considered in strength of materials.

28. Supporting Devices of Beam Systems

So-called beam systems are often found in machines and structures. These systems are primarily designed to carry loads perpendicular to their axis. Beam systems are provided with special supporting devices to transmit forces and connect them to other elements.

The following types of support are distinguished.

A *movable hinge support* (Fig. 44*a*). This support permits a rotation about the axis of the hinge and a linear displacement through a small distance parallel to the supporting plane. The point of application of the reaction exerted by the support is known to be at the centre of the hinge and its direction is along the normal to the supporting surface (if the friction of the rollers is neglected).

Thus, we have only one unknown, the magnitude of the reaction $\bar{R}_A$ at the support. Diagrammatic sketches of movable hinge supports are shown in Fig. 44*b*, *c*, *d*. It should be noted that a movable hinge support may have an inclined supporting surface (Fig. 44*d*). The reaction $\bar{R}_A$ will also be inclined as it is perpendicular to the supporting surface.

An *immovable hinge support* (Fig. 45*a*). This support permits a rotation about the axis of the hinge but prevents any linear displacements. Only the point of application of the reaction at the support is known, being at the centre of the hinge; the direction and magnitude of the reaction at the support are not known. Thus, an immovable hinge support involves two unknowns—the horizontal and vertical compo-

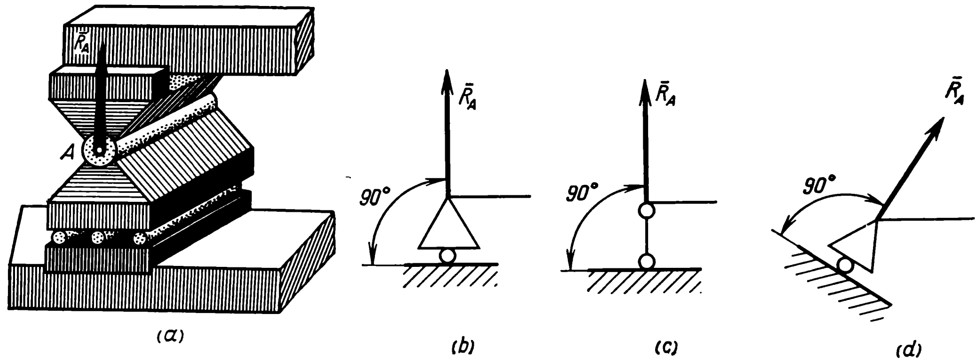

$\bar{R}_A$
A
(a)
$\bar{R}_A$
90°
(b)
$\bar{R}_A$
90°
(c)
$\bar{R}_A$
90°
(d)

Fig. 44.

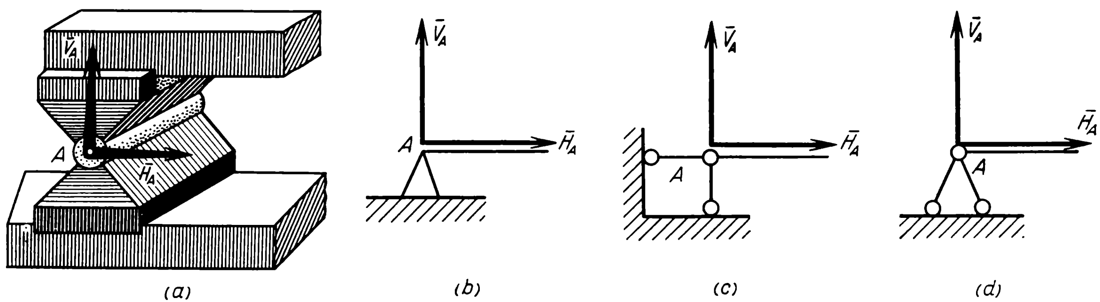

$\bar{V}_A$
A
$\bar{H}_A$
(a)
$\bar{V}_A$
A
$\bar{H}_A$
(b)
$\bar{V}_A$
$\bar{H}_A$
A
(c)
$\bar{V}_A$
$\bar{H}_A$
A
(d)

Fig. 45.

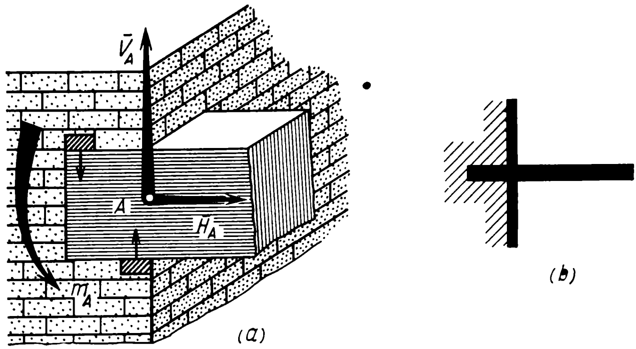


Fig. 46.

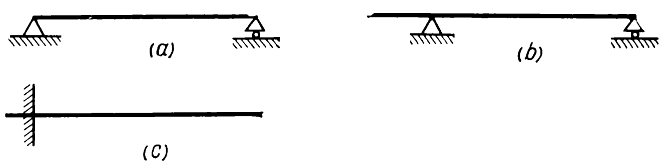


Fig. 47.

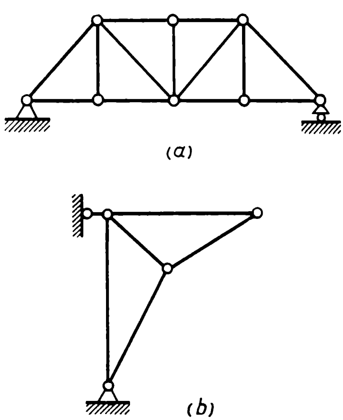


Fig. 48.

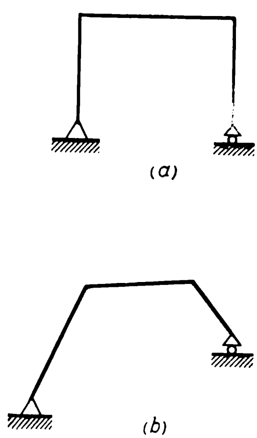


Fig. 49.

nents $\bar{H}_A$ and $\bar{V}_A$ of the reaction at the support. Diagrammatic sketches of immovable hinge supports are shown in Fig. 45*b*, *c*, *d*.

A *fixed* (*restrained*) *support*. This support (Fig. 46*a*) permits neither linear displacements nor rotation. The unknowns are not only the magnitude and direction of the reaction but also its point of application. Thus, to determine the reaction at the support it is necessary to find three unknowns: the components $\bar{H}_A$ and $\bar{V}_A$ of the reaction at the support along the co-ordinate axes and the reactive moment m_A about the centroid of the section at the support. A diagrammatic sketch of a fixed support is shown in Fig. 46*b*.

Reactions at supports can also be designated by letters corresponding to co-ordinate axes along which they are directed with a subscript appropriate to the support. For example, $\bar{Y}_A$ and $\bar{X}_A$, etc.

Beam systems employed in engineering are divided into three groups: beams (Fig. 47*a*, *b*, *c*), trusses (Fig. 48*a*, *b*), frames (Fig. 49*a*, *b*).

29. Classification of Loads

Loads are external forces acting on elements of machines and structures. The determination of the nature of the load, its magnitude and point of application is a prerequisite to design. Loads are classified according to various characteristics:

1. as *distributed* and *concentrated*, depending upon the manner in which they are applied.

(a) distributed loads are applied to an element in a continuous manner. They may be uniformly distributed or non-uniformly distributed. If loads are distributed along a length, they are measured in units of force divided by units of length (N/m, kN/m, kgf/m, ton/m, etc.). If loads are distributed over an area or a volume, they are measured, respectively, in units of force divided by units of area (N/m^2, kN/m^2, kgf/m^2, ton/m^2, etc.) or in units of force divided by units of volume (N/m^3, kN/m^3, kgf/m^3, ton/m^3, etc.). An example of a load distributed over a volume is the weight of a body. Snow or wind loads are distributed over an area.

(b) concentrated loads are measured in units of force (N, kN, kgf, ton, etc.). They are assumed to be applied at a single point. Examples are pressure exerted by elements on each other, pressure exerted by walls and columns on the foundation. Actually, concentrated loads are distributed over a relatively small area whose dimensions can be neglected.

2. as *static* and *dynamic*, depending upon the way they act.

(a) static loads increase gradually from zero to their final value.

(b) dynamic loads are applied rapidly, instantaneously. Therefore, they are sometimes called impact loads.

3. as *dead* and *live*, depending upon the duration of action.

(a) dead loads act at all times (e.g. the weight of a body).

(b) live loads act over a limited period of time (wind or snow).

30. Practical Solution of Equilibrium Problems for Two-Dimensional Force Systems

When solving equilibrium problems for two-dimensional force systems, we can use any form of equilibrium equations given in Sec. 27. It will be useful to remember, in setting up equilibrium equations, that we have a complete freedom of choice of co-ordinate axes and moment points. This will often serve to simplify computations.

In dealing with statics problems, it is advisable to set up equations so that they can be solved as easily and rapidly as possible. A system of equilibrium equations each of which involves only one of the unknown forces is simplest to solve. Such a system can be obtained by suitably choosing the direction of the co-ordinate axes and the location of the moment centre. It is well to place the moment centre at the intersection of two unknown forces; the moment equation with respect to this point will contain only one unknown.

The direction of the co-ordinate axes x and y should be chosen so that they are perpendicular to some unknown forces. When setting up projection equations, the unknowns perpendicular to the corresponding axes will not enter into these equations.

It is good practice to begin the computation of unknowns with moment equations and then proceed to projection

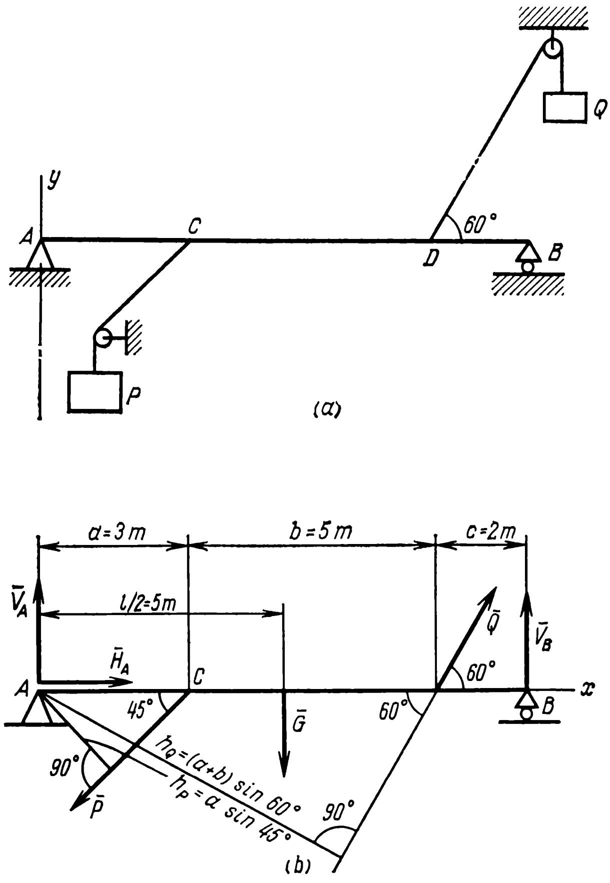


Fig. 50.

equations, thus avoiding simultaneous solution of equations and reducing the probability of errors.

One more important consideration. We can choose any number of projection axes and moment points for a two-dimensional force system. Projecting the forces of a given two-dimensional system on different axes and setting up moment equations with respect to any points, we can write as many equilibrium equations as we like but only three of these will be independent. The remaining equations follow from these three equations and can serve to check the solution.

Example 13. A uniform beam AB of weight $G = 10$ kN and span $l = 10$ m carries two loads $P = 100$ kN and $Q = 20$ kN applied at points C and D, which are supported by fixed pulleys as shown in Fig. 50*a*. $AC = a = 3$ m, $CD = b = 5$ m, $DB = c = 2$ m.

Determine the reactions at the supports of the beam.

Solution. Isolate the beam AB and apply to it the given forces ($\overline{G}$ at mid-span, $\overline{P}$ at point C and $\overline{Q}$ at point D) and the unknown reactions of the constraints $\overline{V}_A$, $\overline{H}_A$ and $\overline{V}_B$ (Fig. 50*b*). Choose co-ordinate axes xy and set up equilibrium equations

$$\sum P_{ix} = 0, \quad H_A - P\cos 45^\circ + Q\cos 60^\circ = 0,$$

$$\sum P_{iy} = 0, \quad V_A - P\sin 45^\circ - G + Q\sin 60^\circ + V_B = 0,$$

$$\sum M_A = 0, \quad Ph_P + G\frac{l}{2} - Qh_Q - V_B l = 0,$$

where the arms of the forces are given by the formulas (see Fig. 50*b*)

$$h_P = a\sin 45^\circ,$$

$$h_Q = (a+b)\sin 60^\circ.$$

Substituting the values of the arms into the third equation. we obtain

$$Pa\sin 45^\circ + G\frac{l}{2} - Q(a+b)\sin 60^\circ - V_B l = 0,$$

whence

$$V_B = \frac{Pa\sin 45^\circ + G\frac{l}{2} - Q(a+b)\sin 60^\circ}{l} =$$

$$= \frac{100\times 3\times 0.707 + 10\times 5 - 20\times 8\times 0.866}{10} = 12.3 \text{ kN}.$$

Solving the remaining equations, we find

$$H_A = P\cos 45° - Q\cos 60° = 100 \times 0.707 - 20 \times 0.5 = 60.7 \text{ kN},$$

$$V_A = P\sin 45° + G - Q\sin 60° - V_B =$$
$$= 100 \times 0.707 + 10 - 20 \times 0.866 - 12.3 =$$
$$= 80.7 - 29.6 = 51.1 \text{ kN}.$$

The solution obtained should be checked with the aid of an additional equilibrium equation not used in the solution. For instance,

$$\sum M_B = 0,$$

$$V_A l - P(b+c)\sin 45° - G\frac{l}{2} + Qc\sin 60° = 0.$$

Substituting the values of the reactions and distances, we obtain

$$51.1 \times 10 - 100 \times 7 \times 0.707 - 10 \times 5 + 20 \times 2 \times 0.866 = 0,$$
$$511 - 495 - 50 + 34 = 0,$$
$$545 - 545 = 0.$$

Example 14. A uniform beam of weight 60 kN and length 4 m rests against a smooth floor with one of its ends and

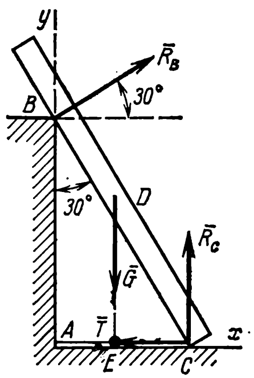


Fig. 51.

against a column of height 3 m at a point B, making an angle of 30° with the vertical (Fig. 51). The beam is held in position by a rope AC stretched on the floor.

Neglecting friction, determine the tension $\bar{T}$ in the rope and the reactions at points B and C.

Solution. Consider the beam under the action of the applied forces and the reactions of the constraints.

The beam is subjected to the following forces: the weight of the beam $\bar{G}$ (applied at the centre of gravity D), the reaction of the rope $\bar{T}$, the reaction of the column $\bar{R}_B$, the reaction of the floor $\bar{R}_C$. The beam is in equilibrium under the action of this force system.

It is seen from Fig. 51 that we have a system of forces arbitrarily located in a plane, therefore, to solve the problem, we must apply all three equilibrium equations, say, of the first form

$$\sum P_{ix}=0,$$
$$\sum P_{iy}=0,$$
$$\sum M_C=0.$$

Choose a system of rectangular co-ordinate axes. Set up equilibrium equations

$$\sum P_{ix}=0, \quad R_B\cos 30°-T=0,$$
$$\sum P_{iy}=0, \quad R_B\cos 60°-G+R_C=0,$$
$$\sum M_C=0, \quad R_B\,BC-G\cdot DC\cos 60°=0.$$

It is well to take the sum of moments about point C, thus eliminating the magnitudes of two unknown reactions, $\bar{R}_C$ and $\bar{T}$.

Substituting $BC=\frac{AB}{\cos 30°}$ into the third equation, we obtain

$$R_B\frac{AB}{\cos 30°}-G\cdot DC\cos 60°=0,$$

whence

$$R_B=\frac{G\cdot DC\cos 60°\cos 30°}{AB}.$$

Substituting the given values, we find

$$R_B=\frac{60\times 2\times 0.5\times 0.866}{3}=17.3\text{ kN}.$$

From the equation $R_B \cos 30° - T = 0$ we obtain

$$T = R_B \cos 30° = 17.3 \times 0.866 = 15 \text{ kN}.$$

From the equation $R_B \cos 60° - G + R_C = 0$ we have

$$R_C = G - R_B \cos 60° = 60 - 17.3 \times 0.5 = 51.3 \text{ kN}.$$

This solution can be checked by applying an additional equilibrium equation not used in the solution. For instance,

$$\sum M_B = 0,$$

$$G(AC - EC) + T \cdot AB - R_C AC = 0, \quad EC = 1 \text{ m},$$

but

$$AC = AB \tan 30° = 3 \times 0.578 = 1.73 \text{ m},$$

then

$$60(1.73 - 1) + 15 \times 3 - 51.3 \times 1.73 = 0,$$

$$43.8 + 45 - 88.8 = 0, \quad 88.8 - 88.8 = 0.$$

Example 15. A homogeneous beam of length 10 m and weight G = 200 kN rests on two supports A and B (Fig. 52*a*).

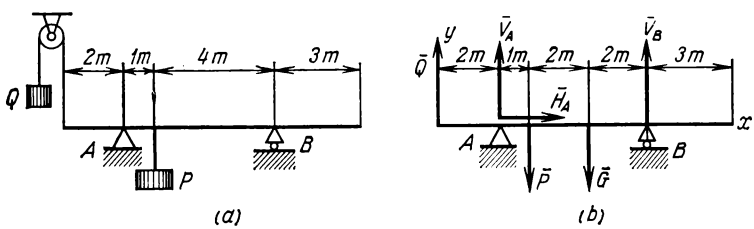


Fig. 52.

The support A is 2 m from the left end of the beam, the support B is 3 m from the right one. The left end of the beam is pulled vertically upward by means of a load Q = 300 kN and a rope passing over a pulley. A load P = 800 kN is suspended from the beam at a distance of 3 m from the left end.

Neglecting friction in the pulley, determine the reactions at the supports.

Solution. Apply to the beam the given forces. Taking into account that a fixed pulley changes only the direction of a force, we apply the force $\bar{Q}$ vertically upward. The weight of the beam $\bar{G}$ is applied at mid-length as the beam is homogeneous according to the condition of the problem. The application of the force $\bar{P}$ presents no difficulty. Separate the beam from the constraints at points A and B and apply the reactions. There are two reaction components, a vertical component $\bar{V}_A$ and a horizontal component $\bar{H}_A$, at the immovable hinge support A and a reaction $\bar{V}_B$ perpendicular to the supporting surface at the movable hinge support B.

Figure 52*b* shows all forces acting on the beam and the chosen co-ordinate axes x and y. Set up three equilibrium equations

$$\sum P_{ix} = 0, \quad H_A = 0,$$

$$\sum P_{iy} = 0, \quad Q + V_A - P - G + V_B = 0,$$

$$\sum M_A = 0, \quad Q \times 2 + P \times 1 + G \times 3 - V_B \times 5 = 0.$$

From the first equation, we find at once that the horizontal component of the reaction at the support A is zero. This is due to the fact that all acting forces (except for the reaction $\bar{H}_A$) are vertical and as such are not projected on the x axis.

Solving the other two equations, we obtain

$$V_B = \frac{P \times 1 + G \times 3 + Q \times 2}{5} =$$

$$= \frac{800 \times 1 + 200 \times 3 + 300 \times 2}{5} = 400 \text{ kN},$$

$$V_A = P + G - Q - V_B = 800 + 200 - 300 - 400 = 300 \text{ kN}.$$

Example 16. Determine the reaction at the fixed end of a cantilever beam (Fig. 53). A load $P = 1$ kN is suspended from the end of the beam, the length of the beam is $l = 4$ m; its weight $Q = 0.4$ kN is applied at mid-length.

Solution. Consider the equilibrium of the beam AB and apply to it the active forces: its own weight $Q = 0.4$ kN and the load $P = 1$ kN.

Separate the beam AB from the constraints, remove the fixed support at A and replace the effect of fixing by reactions.

The fixed support involves three unknowns: the reactive moment m_A and two force components, $\bar{H}_A$ and $\bar{V}_A$. Choose co-ordinate axes as shown in Fig. 53 and set up equilibrium equations

$$\sum P_{ix}=0, \quad H_A=0,$$
$$\sum P_{iy}=0, \quad V_A-Q-P=0,$$
$$\sum M_A=0, \quad m_A+Q\frac{l}{2}+Pl=0.$$

Solving the equations, we obtain

$$V_A=Q+P=0.4+1=1.4 \text{ kN},$$
$$m_A=-\left(Q\frac{l}{2}+Pl\right)=-(0.4\times 2+1\times 4)=-4.8 \text{ kN}.$$

The minus sign in front of the numerical value of the moment indicates that the moment acts counterclockwise

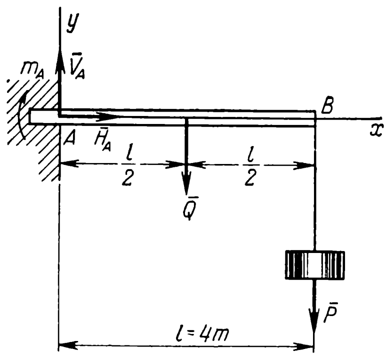


Fig. 53.

and not as in Fig. 53. From the first equation we have $H_A = 0$, hence the vertical load does not produce any horizontal component of the reaction at the support.

Thus, the fixed support A of the beam involves only two reaction components, the third one vanishes.

Example 17. Determine the reactions at the supports of a beam (Fig. 54) subjected to a uniformly distributed vertical load $q = 0.5$ kN/m; the span of the beam is $l = 8$ m. Neglect the weight of the beam.

Solution. Consider the equilibrium of the beam. Replace the uniformly distributed load by its resultant which is equal to the intensity of the load multiplied by the length on which the load acts (ql), and is applied at mid-span of the beam.

The reactions at the supports A and B are vertical. There is no horizontal component of the reaction at the immovable

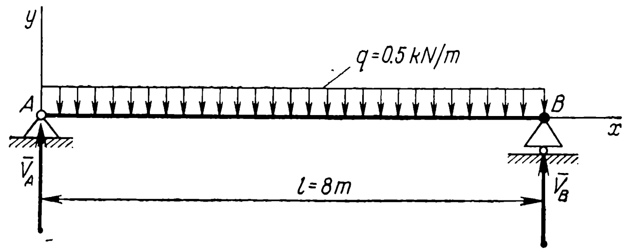


Fig. 54.

hinge support A, as the active load and the reaction at the support B are vertical. We have a system of parallel forces in a plane.

Choose co-ordinate axes and set up equilibrium equations.

The first equation $\sum P_{ix} = 0$, the sum of projections on the x axis, reduces to the identity $0 = 0$ as the x axis is perpendicular to the direction of the forces.

The remaining equations are

$$\sum P_{iy} = 0, \quad V_A - ql + V_B = 0,$$

$$\sum M_A = 0, \quad -V_B l + ql\frac{l}{2} = 0.$$

Solving them simultaneously, we obtain

$$V_A = V_B = \frac{ql}{2} = \frac{0.5 \times 8}{2} = 2 \text{ kN}.$$

Example 18. A homogeneous beam of length $l = 12$ m and weight $G = 600$ kN (Fig. 55) rests on an immovable hinge support A and a movable hinge support B with the supporting surface inclined at an angle $\alpha = 30°$ to the horizon. A load $Q = 1{,}500$ kN is suspended from the beam at a distance of 4 m ($a = 4$ m) from the support A.

Determine the reactions at the supports of the beam.

Solution. Apply to the beam the given forces and the reactions of the constraints (Fig. 55). Set up equilibrium equations

$$\sum P_{ix}=0, \quad H_A - R_B \cos(90-\alpha)=0,$$
$$\sum P_{iy}=0, \quad V_A - Q - G + R_B \cos\alpha = 0,$$
$$\sum M_A = 0, \quad Qa + G\frac{l}{2} - R_B l \cos\alpha = 0.$$

The arm of the reaction $\overline{R}_B$ is determined by dropping a perpendicular from the point A upon its line of action.

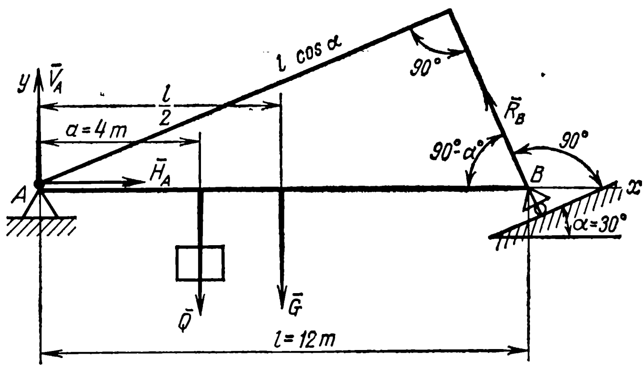


Fig. 55.

Solving this system of equations, we find the unknown reactions at the supports

$$R_B = \frac{Qa + G\frac{l}{2}}{l\cos\alpha} = \frac{1{,}500\times 4 + 600\times 6}{12\times 0.866} = 923 \text{ kN},$$
$$V_A = Q + G - R_B\cos\alpha =$$
$$= 1{,}500 + 600 - 923\times 0.866 = 1{,}300 \text{ kN},$$
$$H_A = R_B\cos(90-\alpha) = 923\times 0.5 = 461.5 \text{ kN}.$$

In this example the horizontal component of the reaction at the support A is not zero, even though the given forces are vertical. This is due to the fact that the supporting surface at B is inclined to the horizon and therefore the reaction at the support $\overline{R}_B$ is not vertical.

Example 19. A symmetrical roof truss ABC (Fig. 56) rests on an immovable hinge support at A and a movable hinge support at B. The weight of the truss is $P = 10$ kN. The side AC is subjected to a uniformly distributed wind

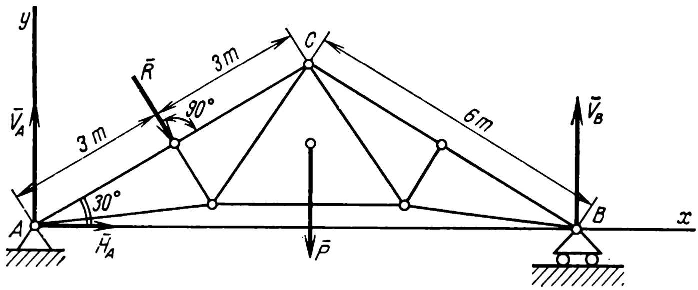


Fig. 56.

pressure perpendicular to it; the resultant of the wind pressure $R = 0.8$ kN is applied at the mid-point of the side AC. $AC = 6$ m, $\angle CAB = \angle CBA = 30°$.

Determine the reactions at the supports.

Solution. The reaction $\bar{V}_B$ at the movable support B is vertical and the reaction at the immovable support A is unknown in magnitude and direction; its components are denoted by $\bar{V}_A$ and $\bar{H}_A$; we have altogether three unknown forces

$$\bar{V}_B, \ \bar{V}_A \text{ and } \bar{H}_A.$$

The directions of the co-ordinate axes are indicated in Fig. 56.

Set up equilibrium equations

$$\sum M_A = -V_B AB + P\frac{AB}{2} + R\frac{AC}{2} = 0,$$

where

$$AB = 2AC\cos 30°,$$

$$\sum P_{ix} = H_A + R\cos 60° = 0,$$

$$\sum P_{iy} = V_A + V_B - P - R\cos 30° = 0.$$

Substituting the value of AB into the first equation and dividing through by AC, we obtain

$$V_B = \frac{P \cdot AC \cos 30^\circ + R\frac{AC}{2}}{2AC \cos 30^\circ} = \frac{10 \times 0.866 + 0.8 \times 0.5}{2 \times 0.866} = 5.23 \text{ kN}.$$

Solving the remaining equations, we find

$$H_A = -R\cos 60^\circ = -0.8 \times 0.5 = -0.4 \text{ kN},$$
$$V_A = -V_B + P + R\cos 30^\circ =$$
$$= -5.23 + 10 + 0.8 \times 0.866 \cong 5.46 \text{ kN}.$$

Example 20. Determine the reactions at the supports of a beam (Fig. 57) subjected to a couple of moment $m =$

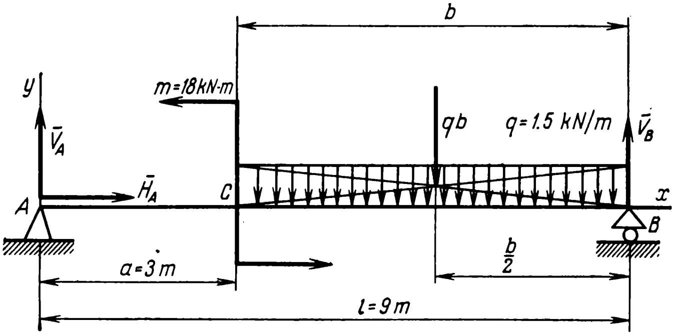


Fig. 57.

$= 18$ kN-m at C and a distributed load $q = 1.5$ kN/m over a portion CB; $AC = a = 3$ m, $CB = b = 6$ m.

Solution. Replace the distributed load in the portion CB by its resultant which is equal to qb and is applied at the middle of the loaded portion. Separate the beam from the supports and apply the reactions $\overline{V}_A$, $\overline{H}_A$ and $\overline{V}_B$. Choose co-ordinate axes x and y and set up equilibrium equations

$$\sum P_{ix} = 0, \qquad H_A = 0,$$
$$\sum P_{iy} = 0, \qquad V_A + V_B - qb = 0$$

(the couple is not projected on any of the co-ordinate axes),

$$\sum M_A = 0, \qquad -m + qb\left(l - \frac{b}{2}\right) - V_B l = 0.$$

Solving the equations, we obtain

$$H_A = 0,$$

$$V_B = \frac{-m + qb\left(l - \frac{b}{2}\right)}{l} = \frac{-18 + 1.5 \times 6(9-3)}{9} = 4 \text{ kN},$$

$$V_A = qb - V_B = 1.5 \times 6 - 4 = 5 \text{ kN}.$$

It should be noted that no horizontal component of the reaction arises at an immovable hinge support under the action of a couple, provided, of course, the supporting surface of a movable hinge support is horizontal.

31. Equilibrium of a System of Connected Bodies

Statics problems often deal not with a single body but with several connected bodies forming an unchangeable system. Forces exerted on such a system by other bodies not included in it are called *external forces*. Forces of interaction between the bodies of the system are called *internal forces*. For a two-dimensional force system the number of unknowns to be determined may be greater than three in this case. The number of equations which can be set up increases accordingly. Three equilibrium equations can be written for each body forming part of the system if the force system acting on it is two-dimensional. Each body or a group of bodies of the system can be separated and considered in the state of equilibrium under the action of the external and internal forces applied to this part of the system. Such a procedure of solving equilibrium problems for a system of bodies is known as *the method of separation*. When considering the equilibrium of a system of connected bodies it is sometimes convenient to set up equilibrium equations not only for individual parts of the system but also for the system as a whole.

Below are examples illustrating the application of the method of separation.

Example 21. A homogeneous inclined rod *CD* of weight $P_2 = 0.8$ kN rests against a homogeneous beam *AB* of weight $P_1 = 1$ kN at a point *C* located at a quarter of the span. The other end of the rod *CD* is hinged at *D*. The span of the beam *AB* is $l = 8$ m.

Determine the reactions at the supports A and B (Fig. 58a).

Solution. Here we have a system of two connected bodies. To determine the reactions at the supports A and B, it is necessary to know all the forces acting on the beam AB. The

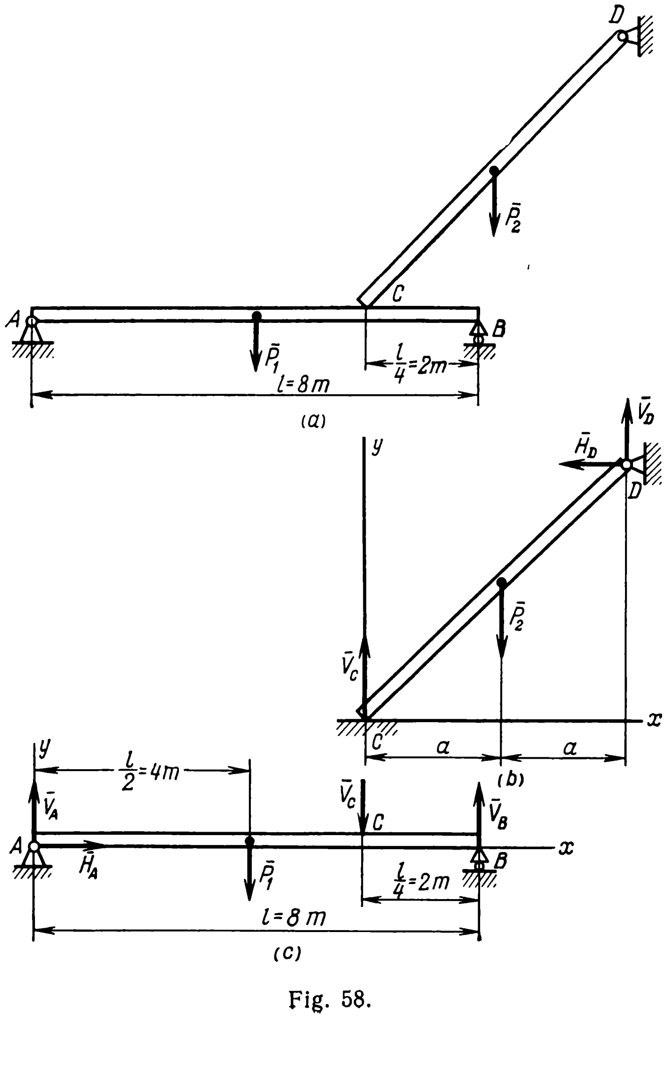


Fig. 58.

pressure exerted by the rod *CD* on this beam at point *C* is an internal force which is not known. It is to be determined by considering the equilibrium of the isolated rod *CD* (Fig. 58*b*). According to the condition of the problem, the rod is homogeneous, hence its weight is applied at mid-length, halfway between the points *C* and *D*. Since the reactions arising at the hinge *D* are of no interest to us, we set up only one equilibrium equation for the rod *CD*, viz. the sum of moments about the hinge *D*

$$\sum M_D = 0, \quad V_C 2a - P_2 a = 0,$$

whence

$$V_C = \frac{P_2}{2} = \frac{0.8}{2} = 0.4 \text{ kN}.$$

On the basis of the axiom of action and reaction we conclude that the force exerted by the rod *CD* on the beam *AB* is equal in magnitude to the reaction $\bar{V}_C$ but is directed downward. Consider now the equilibrium of the beam *AB* (Fig. 58*c*) and set up three equilibrium equations for it

$$\sum P_{ix} = 0, \qquad H_A = 0,$$

$$\sum P_{iy} = 0, \qquad V_A - P_1 - V_C + V_B = 0,$$

$$\sum M_A = 0, \qquad P_1 \frac{l}{2} + V_C \frac{3}{4} l - V_B l = 0.$$

Solving these equations, we obtain

$$H_A = 0,$$

$$V_B = \frac{P_1}{2} + \frac{3}{4} V_C = \frac{1}{2} + \frac{3}{4} 0.4 = 0.8 \text{ kN},$$

$$V_A = P_1 + V_C - V_B = 1 + 0.4 - 0.8 = 0.6 \text{ kN}.$$

Example 22. A bridge consists of two half-arches connected by a hinge *C* and attached to the land abutments by hinges *B* and *A*. The weight of each half of the bridge is $P_1 = P_2 = 30$ kN; their centres of gravity are at points *D* and *E*, respectively. The right half-arch supports a load $Q = 40$ kN. The relevant dimensions are indicated in the drawing.

Determine the reactions at the hinges *A*, *B* and *C* (Fig. 59*a*).

Solution. Consider first the equilibrium of the arch as a whole. Denote the components of the reactions at the

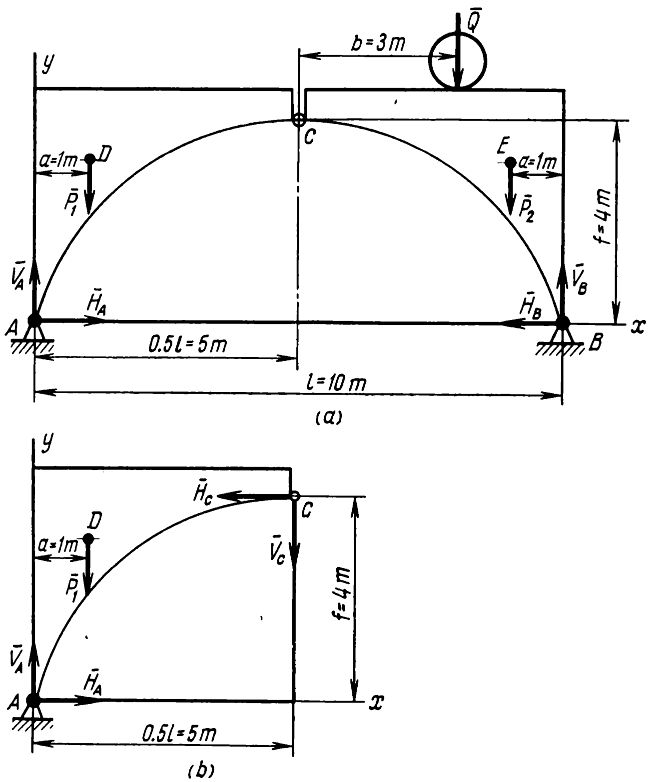


Fig. 59.

hinges A and B by $\bar{V}_A$, $\bar{H}_A$ and $\bar{V}_B$, $\bar{H}_B$ (Fig. 59*a*); set up equilibrium equations

$$\sum P_{ix}=0,\quad H_A-H_B=0,$$

$$\sum M_A=0,\quad P_1a+Q\left(\frac{l}{2}+b\right)+P_2(l-a)-V_Bl=0,$$

$$\sum M_B=0,\quad V_Al-P_1(l-a)-Q\left(\frac{l}{2}-b\right)-P_2a=0,$$

These three equations involve four unknown reaction components.

Consider the equilibrium of one of the half-arches. Note that it is always simpler to consider the equilibrium of that part of a connected system to which fewer external forces are applied. This simplifies equilibrium equations. Therefore we shall consider the equilibrium of the left half-arch. The reactions arising at the hinge C are denoted by $\bar{V}_C$ and $\bar{H}_C$ (Fig. 59*b*). Set up equilibrium equations

$$\sum P_{ix}=0, \quad H_A-H_C=0,$$

$$\sum M_A=0, \quad P_1a+V_C\frac{l}{2}-H_Cf=0,$$

$$\sum M_C=0, \quad V_A\frac{l}{2}-H_Af-P_1\left(\frac{l}{2}-a\right)=0.$$

Thus, we have set up altogether six equilibrium equations which involve six unknown reaction components

$$\bar{H}_A, \bar{V}_A, \bar{H}_B, \bar{V}_B, \bar{H}_C, \bar{V}_C.$$

Solving these equations, we find

$$H_A=H_B=H_C=17.5 \text{ kN},$$

$$V_B=62 \text{ kN}, \ V_A=38 \text{ kN}, \ V_C=8 \text{ kN}.$$

When solving problems concerning the equilibrium of connected systems, one should remember the axiom of action and reaction (Axiom V). Reactions or internal forces revealed at the section made when dismembering the system should be directed so that the forces exerted on each of the cut-off portions of the system are opposite. These opposite reactions are always equal in magnitude.

CHAPTER V

Friction

32. Types of Friction

A friction force arises when one body moves on another and is always opposite to the motion. Friction plays an important role in machines. For instance, in many transmission mechanisms, such as belt, friction, rope drives, the motion is transmitted with the aid of friction. In all machines friction opposes the motion and leads to a useless expenditure of work.

Two types of friction are distinguished, depending upon the form of motion: *sliding friction*, or friction of the first kind, and *rolling friction*, or friction of the second kind. The friction of flexible bodies is a special type of friction.

In some cases we deal with a combination of sliding and rolling.

As experiments show, friction is a complex phenomenon. Here is a simplified explanation of sliding friction. (The physical side of rolling friction will be discussed in Sec. 36.)

The surfaces of any contacting bodies have irregularities (Fig. 60*a*).

When one body moves on another the asperities of one surface will interlock with those of the other, causing their deformation. As a consequence, tangential as well as normal forces will develop at the surfaces in contact, as shown at one of the points of contact in Fig. 60*a*. The friction force is the resultant of these tangential forces. If the asperities of the surfaces are in direct contact (Fig. 60*a*), we have *dry friction*. When the surfaces are lubricated, it is *fluid friction* (Fig. 60*b*).

Fluid friction is always much lower than dry friction. Besides, there is semidry and semifluid friction.

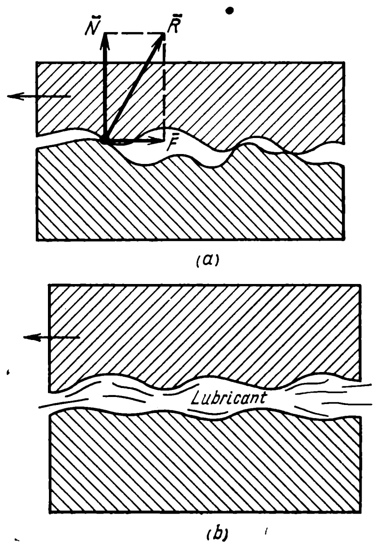


Fig. 60.

In *semidry friction* less than half the surface of contact is lubricated, in *semifluid friction*, more than half.

33. The Laws of Sliding Friction

Friction depends on a series of complex mechanical, chemical and other phenomena. The laws of sliding friction are the result of generalization of a great body of experimental data. The basic laws of sliding friction are presently formulated as follows:

1. The friction force is proportional to the normal pressure.

2. The coefficient of friction depends on the nature of the bodies in contact and the physical condition of the surfaces in contact.

3. Friction between similar bodies is generally larger than between dissimilar bodies.

4. The friction force does not depend upon the area of contact, except at high unit pressures.

5. The static-friction force is larger than the kinetic-friction force for most bodies.

6. The friction force depends on the relative velocity of the bodies in contact. In practice the friction force is often assumed to be independent of the velocity in the range of velocities encountered in engineering.

The basic relationship for the magnitude of a sliding-friction force can be approximated by the formula

$$F = fN, \tag{34}$$

where F = sliding-friction force opposing the relative motion of two bodies in contact,

N = normal reaction,

f = coefficient of proportionality, or coefficient of sliding friction.

From (34) we find the value for the coefficient of friction

$$f = \frac{F}{N} = \frac{F}{G}, \tag{35}$$

where G is the normal pressure.

The coefficient of sliding friction is a dimensionless quantity.

Since the surfaces of bodies in contact gradually change during operation, the coefficients of friction in reference tables are average values based on a great body of experimental observations.

Friction occurs not only when bodies are in motion but also when they are just in contact. The necessary condition for friction to manifest itself is the presence of a moving force. Hence static friction is defined as the resistance to any tendency of one body to move relative to another.

It becomes clear, therefore, why we understand by the friction force the tangential reaction directed opposite to the moving force.

For the limiting static friction, when motion is impending, we have

$$\left.\begin{aligned} F_0 &= f_0 N \\ \text{or} \quad F_0 &= f_0 G \end{aligned}\right\} \tag{36}$$

and

$$f_0 = \frac{F_0}{G} = \frac{F_0}{N}, \qquad (37)$$

where N = normal reaction,
f_0 = coefficient of static friction (it is always larger than coefficient of kinetic friction),
F_0 = static-friction force.

34. Angle and Cone of Friction

Let a body of weight $\bar{Q}$ rest on a horizontal plane (Fig. 61). The normal reaction $N = Q$. A moving force $\bar{P}$ gives rise

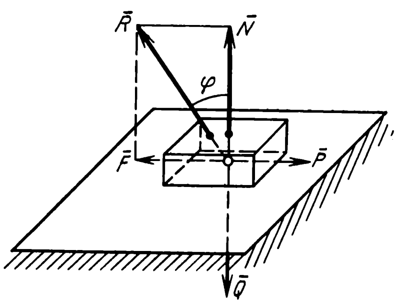


Fig. 61.

to the tangential reaction, the sliding-friction force

$$F = fN = fQ.$$

From Fig. 61 it follows that

$$\tan\varphi = \frac{F}{N} = \frac{fN}{N} = f,$$

i.e.,

$$f = \tan\varphi. \qquad (38)$$

The angle φ that the direction of the total reaction $\bar{R}$ makes with the direction of the normal reaction $\bar{N}$ is known as the angle of friction.

From equation (38) it may be concluded that *the coefficient of friction f is equal to the tangent of the angle of friction.* This conclusion is valid both for static friction, or friction at limiting equilibrium, and for kinetic friction. If the surface of a circular cone (Fig. 62) is generated by revolving the vector representing the total reaction about an axis

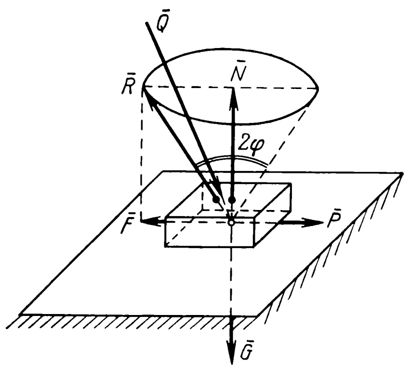


Fig. 62.

perpendicular to the plane along which the body is moving, we obtain a so-called *cone of friction* with an angle at the apex equal to twice the angle of friction.

If the line of action of the resultant of all forces applied to a body makes with the normal an angle smaller than the angle of friction and hence is located inside the cone of friction, it cannot displace the body from its equilibrium position, no matter how large this force may be. The body can be displaced from its equilibrium position only if the line of action of the resultant of all forces is located outside the cone of friction, i.e., if it makes with the normal an angle larger than the angle of friction.

Example 23. A steam boiler of weight $G = 15{,}000$ kN is moved with constant speed up a wooden floor of slope $\alpha = 30°$ by means of a winch (Fig. 63).

Determine the tension $\bar{P}$ in the cable if the coefficient of friction between the cylinder and the wooden floor is $f = 0.25$. The cable is parallel to the floor.

Solution. Isolate the boiler and apply to its centre of gravity all acting forces: the weight $\bar{G}$, the tension $\bar{P}$ in the cable, the normal reaction $\bar{N}$ and the friction force $\bar{F}$. As the boiler moves up at constant speed all these forces must

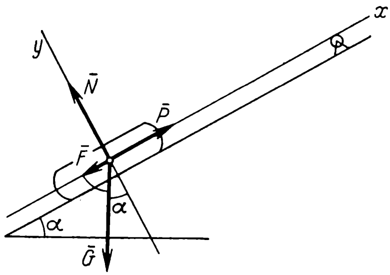


Fig. 63.

be in equilibrium. Set up equilibrium equations. Since all forces are applied at the same point, we have two equations of equilibrium (Fig. 63)

$$\sum P_{ix}=0, \qquad P-F-G\sin\alpha=0,$$

$$\sum P_{iy}=0, \qquad -G\cos\alpha+N=0.$$

According to the first law for sliding friction

$$F=fN.$$

Substituting for the friction force F, we obtain the following system of equations

$$P-fN-G\sin\alpha=0,$$

$$N-G\cos\alpha=0.$$

Solving, we find

$$N=G\cos\alpha,$$

$$P=fN+G\sin\alpha=fG\cos\alpha+G\sin\alpha=G(f\cos\alpha+$$
$$+\sin\alpha)=15{,}000\,(0.25\times 0.866+0.5)=10{,}740 \text{ kN}.$$

Example 24. A wedge A of slope $\tan\alpha = 0.05$ is driven into a recess BB_1 by a force $Q = 6$ kN. Determine the normal pressure exerted on the sides of the wedge if the coefficient of friction $f = 0.1$ (Fig. 64).

Solution. Denote the required pressure forces exerted on the sides of the wedge by $\bar{N}$ and $\bar{N}_1$; these sides are acted

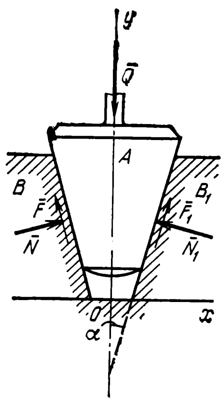


Fig. 64.

upon by friction forces denoted by $\bar{F}$ and $\bar{F}_1$. The friction forces are proportional to the normal pressure

$$F = fN,$$
$$F_1 = fN_1.$$

Choose co-ordinate axes as shown in Fig. 64 and project all forces on them

$$\sum P_{ix} = 0, \quad N\cos\alpha - N_1\cos\alpha - F\sin\alpha + F_1\sin\alpha = 0,$$
$$\sum P_{iy} = 0, \quad N\sin\alpha + N_1\sin\alpha + F\cos\alpha + F_1\cos\alpha - Q = 0.$$

Substitute the values of F and F_1 and group similar terms together. The equations of equilibrium become then

$$N(\cos\alpha - f\sin\alpha) - N_1(\cos\alpha - f\sin\alpha) = 0,$$
$$N(\sin\alpha + f\cos\alpha) + N_1(\sin\alpha + f\cos\alpha) - Q = 0.$$

From the first equation we find

$$N = N_1.$$

Substituting this result into the second equation, we obtain

$$N=\frac{Q}{2(\sin\alpha+f\cos\alpha)}\,.$$

Since the angle α is small, $\tan\alpha\cong\sin\alpha$ and $\cos\alpha=1$. Then

$$N=\frac{6}{2(0.05+0.1\times1)}=\frac{6}{0.3}=20\ \text{kN}.$$

Example 25. A couple $\overline{P}_1\overline{P}_2$ of moment $M=100$ N-m is applied to a shaft (Fig. 65). A brake wheel of radius $r=$ $=25$ cm is mounted on the shaft.

Find the force $\overline{Q}$ with which the brake shoes must be pressed against the wheel to assure that it remains at rest if the coefficient of friction between the wheel and the shoes is $f=0.25$.

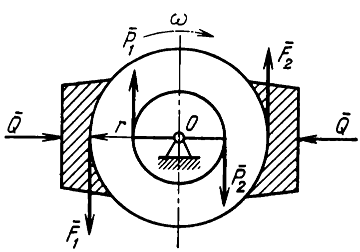


Fig. 65.

Solution. The couple $\overline{P}_1\overline{P}_2$ of moment $M=100$ N-m rotates the shaft (Fig. 65) clockwise. Friction forces $\overline{F}_1$ and $\overline{F}_2$ develop between the shoes and the brake wheel, which form a counterclockwise braking moment. Thus, the shaft is subjected to two couples. From theory it is known that for equilibrium of couples acting in the same plane, it is necessary and sufficient that the algebraic sum of the moments of the couples be zero.

The equilibrium equation is

$$\sum m_O=0,\ M-F\,2r=0,$$

whence

$$F=F_1=F_2=\frac{M}{2r}=\frac{100}{2\times0.25}=200\ \text{N}.$$

According to the laws of friction

$$F = fQ,$$

whence

$$Q = \frac{F}{f} = \frac{200}{0.25} = 800 \text{ N}.$$

35. Experimental Determination of Coefficients of Friction

It is established that the coefficient of friction is affected by the area of contact, the rate of application of load, the time the bodies are in contact at rest, the rate of sliding,

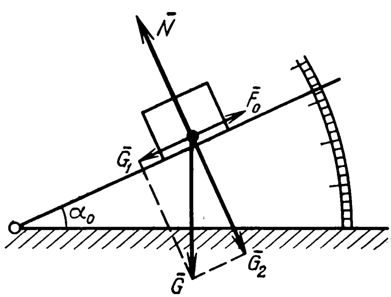


Fig. 66.

the nature and physical condition of the rubbing bodies, etc. Hence it is impossible to use tabulated coefficients of friction for any pairs of materials unless the conditions under which these coefficients were obtained are specified.

It is known that one and the same pair of materials may have widely differing coefficients of friction under different conditions.

That is why it is advisable to consider the simplest experimental methods of determining coefficients of static friction.

The coefficients of friction thus obtained will correspond to the actual operating conditions for the rubbing pair.

Let a heavy body of weight $\overline{G}$ rest on an inclined plane of slope α_0 (Fig. 66).

Resolve $\overline{G}$ into two components:
(1) in a direction parallel to the inclined plane

$$G_1 = G \sin \alpha_0;$$

(2) along the normal to the surfaces in contact

$$G_2 = G \cos \alpha_0.$$

The component $\overline{G}_2$ produces the normal reaction $\overline{N}$ of the plane and the component $\overline{G}_1$, the friction force

$$F_0 = f_0 N.$$

But

$$N = G_2,$$

consequently,

$$F_0 = f_0 G_2,$$

where f_0 is the coefficient of sliding friction.

The component $\overline{G}_1$, which is parallel to the inclined plane, tends to move the body down the inclined plane.

If $G_1 > F_0$, the body moves with constant acceleration; if $G_1 < F_0$, the body is at rest; if $G_1 = F_0$, the body is in a state of limiting equilibrium (motion is impending).

Determine the slope of the inclined plane at which the body is in a state of limiting equilibrium characterized by the equality $F_0 = G_1$.

The friction force is

$$F_0 = G_1 = f_0 G_2 = f_0 G \cos \alpha_0,$$

but

$$G_1 = G \sin \alpha_0,$$

consequently,

$$G \sin \alpha_0 = f_0 G \cos \alpha_0,$$

whence

$$f_0 = \frac{G \sin \alpha_0}{G \cos \alpha_0} = \tan \alpha_0. \tag{39}$$

By using this method the coefficient of static friction is determined as the slope of the plane at which the body starts sliding down it.

36. The Laws of Rolling Friction

When one curved surface rolls on another, the resistance to rolling is called friction of the second kind, or rolling friction. Rolling friction, as well as sliding friction, is a complex phenomenon.

Rolling friction can be primitively explained as follows (Fig. 67).

If a cylinder of radius R rolls over an irregularity, it is necessary, in order to overcome resisting forces, to apply

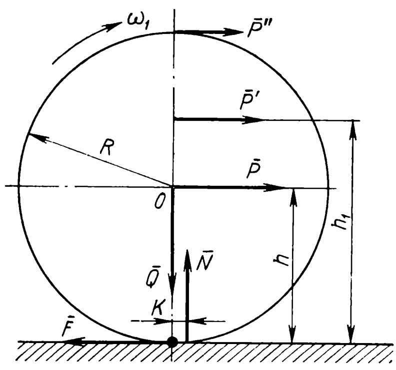


Fig. 67.

at the centre of the cylinder a moving force $\bar{P}$ of moment equal to the moment of the resisting force, i.e.,

$$Ph = Nk,$$

but

$$N = Q,$$

then

$$Ph = Qk,$$

whence

$$P = \frac{Qk}{h}, \tag{40}$$

$$k = \frac{Ph}{Q}, \tag{41}$$

where h = arm of force $\overline{P}$ with respect to point A over which the body is to roll (if the force is applied at centre of cylinder, it can be taken with sufficient accuracy for practical purposes that $h = R$),

$\overline{N}$ = normal reaction,

$\overline{Q}$ = normal pressure,

k = coefficient of proportionality or coefficient of friction of second kind depending on material and physical properties of rolling surfaces.

Analysing formula (40), we conclude that the magnitude of the moving force $\overline{P}$ depends on where it is applied to the rolling cylinder. We can imagine a friction force $F = fQ$, which is called a *cohesive force* in this case, to be applied at the point of contact. If $P < fQ$, the roller is "held up" at the point of contact and rolling takes place. If, however, the magnitude of the force $\overline{P}$, as determined from formula (40), is larger than the magnitude of $\overline{F}$, no rolling is possible. The roller will slide along the plane since motion occurs in the direction of the lower resistance.

Thus, the condition of rolling is

$$P < F \tag{42}$$

or

$$f > \frac{k}{h}. \tag{43}$$

In contrast to the coefficient of sliding friction, the coefficient of rolling friction has a linear dimension.

If the moving force is applied not at the centre of the cylinder but at a distance h_1 from point A (Fig. 67), then

$$P_1 = k\frac{Q}{h_1}. \tag{44}$$

If the moving force is directed along the tangent, then

$$P_2 = k\frac{Q}{2R}. \tag{45}$$

The following points will aid in understanding the physical nature of rolling friction between elastic bodies.

1. If a perfectly rigid cylinder resting on a perfectly rigid flat surface (Fig. 68*a*) is acted upon by a force $\overline{Q}$, the normal reaction $\overline{N}$ is equal and opposite to this force.

In this case no external moment would be required to roll the cylinder, i.e.

$$M_{mov} = 0.$$

2. If the cylinder undergoes local crushing deformation in the static condition under the action of the external force

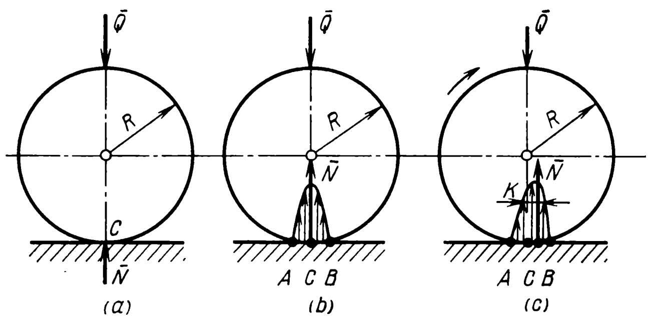


Fig. 68.

$\overline{Q}$ (Fig. 68*b*), stresses occur on the area AB which add up into the normal reaction $\overline{N}$, which in turn is equal in magnitude to the force $\overline{Q}$ from the condition of equilibrium, i.e., $N = Q$.

By Hertz's theory, the stresses are distributed over the area AB according to an elliptic law.

As a result, the stress resultant equal to $\overline{N}$ is equal and opposite to the external force $\overline{Q}$.

Consequently, in this case, too, there is no reason for rolling resistance to occur.

3. If the cylinder is not in the static condition (Fig. 68*c*), the elliptic law of stress distribution is violated: the stresses in the portion CB of the contact area are higher than in the portion AC.

The portion CB is called the zone of increasing deformations and the portion AC the zone of decreasing deformations.

The zone of decreasing deformations is attributed to the fact that part of the deformation remains for some time after the cause is removed.

That is why the stresses in the zone of decreasing deformations are less than in the zone of increasing deformations.

The stress distribution over the area can be seen in Fig. 68*c*.

The resultant of the stresses on the entire area of contact, which is equal to the external force $\overline{Q}$, is shifted beyond the vertical axis of symmetry of the cylinder through a distance k.

The distance k is called the *arm of rolling friction*. The arm of friction is numerically equal to the coefficient of rolling friction. This arm comprises part of the width of the contact area AB

$$k = xb, \tag{46}$$

where b is the width of the contact area.

The factor x depends on the elastic properties of the materials of bodies in rolling contact. It is always less than unity.

CHAPTER VI

Three-Dimensional Force Systems

37. Force Parallelepiped

If the lines of action of forces applied to a body do not lie in the same plane, the force system is referred to as *three-dimensional*. In space, forces have to be projected on three co-ordinate axes.

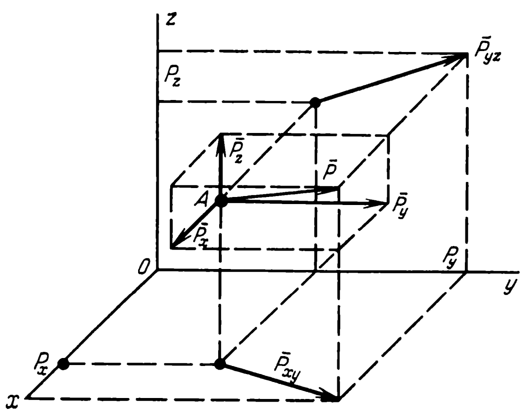


Fig. 69.

Let us see how the projections of an arbitrary force $\bar{P}$ (Fig. 69) applied at a point A are determined in the general case.

The force $\overline{P}$ is first projected on a co-ordinate plane, say, xOy or xOz. These projections will be denoted by $\overline{P}_{xy}$ or $\overline{P}_{xz}$, respectively.

The projections of the force on the co-ordinate planes are vector quantities. Each of the projections on the co-ordinate planes, $\overline{P}_{xy}$ and $\overline{P}_{xz}$, can now be easily projected on two co-ordinate axes in the plane of which it lies. We obtain three projections of the force $\overline{P}$

$$P_x, \ P_y \text{ and } P_z.$$

The force $\overline{P}$ can be represented by the diagonal of a rectangular parallelepiped constructed on the components $\overline{P}_x$, $\overline{P}_y$, $\overline{P}_z$ (Fig. 69) which are equal in magnitude to the projections. Consequently, the magnitude and direction of the force $\overline{P}$ are defined by the formulas

$$\left.\begin{aligned} P &= \sqrt{P_x^2 + P_y^2 + P_z^2}, \\ \cos \angle (P, x) &= \frac{P_x}{P}, \\ \cos \angle (P, y) &= \frac{P_y}{P}, \\ \cos \angle (P, z) &= \frac{P_z}{P}. \end{aligned}\right\} \qquad (47)$$

38. Equilibrium of a System of Concurrent Forces in Space

If the lines of action of all forces of a three-dimensional system intersect at a common point, the forces are said to be concurrent. The resultant of a three-dimensional system of concurrent forces can be determined in exactly the same manner as for a two-dimensional force system by using the polygon rule. But here the force polygon will not be two-dimensional.

The geometric condition of equilibrium of a three-dimensional force system is that the force polygon is closed. The analytic condition of equilibrium expresses that the resultant vector is zero

$$\overline{R} = 0. \qquad (48)$$

The absolute value of the resultant can be determined in terms of the projections of the component forces on three axes of a co-ordinate system in space (on the basis of the

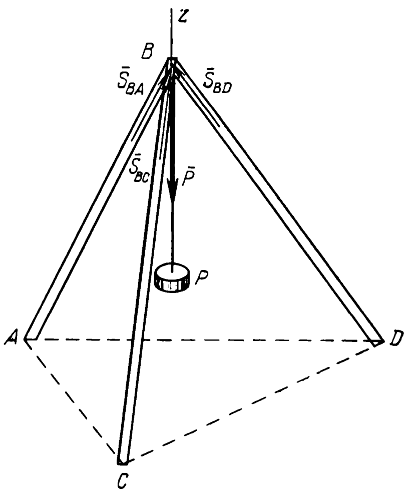


Fig. 70.

theorem on the projection of a vector sum, see Sec. 8)

$$R=\sqrt{\left(\sum_{i=1}^{n} P_{ix}\right)^2+\left(\sum_{i=1}^{n} P_{iy}\right)^2+\left(\sum_{i=1}^{n} P_{iz}\right)^2}, \qquad (49)$$

where $\sum_{i=1}^{n} P_{ix}$ = sum of projections of all forces on x axis,

$\sum_{i=1}^{n} P_{iy}$ = sum of projections of all forces on y axis,

$\sum_{i=1}^{n} P_{iz}$ = sum of projections of all forces on z axis.

This analytic condition of equilibrium of a three-dimensional force system provides the following three equations of equilibrium

$$\left.\begin{aligned}\sum_{i=1}^{n} P_{ix} = 0;\\ \sum_{i=1}^{n} P_{iy} = 0,\\ \sum_{i=1}^{n} P_{iz} = 0.\end{aligned}\right\} \qquad (50)$$

Consequently, the equilibrium equations for concurrent forces in space permit determination of no more than three unknowns.

Example 26. A weight $P = 10$ kN is suspended from a tripod *ABCD* (Fig. 70) at point *B*. The legs are of equal length. They are hinged to a horizontal floor and make equal angles with each other.

Determine the force in each of the legs if they make an angle of 30° with the vertical.

Solution. We have a three-dimensional system of concurrent forces. Because of the symmetrical arrangement of the legs the compressive forces induced in them should be the same, i.e.,

$$S_{BA} = S_{BC} = S_{BD} = S.$$

Set up an equilibrium equation

$$\sum P_{iz} = 0,$$

$$3S \cos 30^\circ - P = 0,$$

whence

$$S = \frac{P}{3 \cos 30^\circ} = \frac{10}{3 \times 0.866} = 3.9 \text{ kN}.$$

39. Moment of a Force About an Axis

Consider the moment of a force about an axis. The tendency of a force to rotate a body about a fixed axis depends on the magnitude of the force, its inclination and distance from the axis.

It is known from experience that forces passing through or parallel to an axis produce no rotation about that axis.

Neither the force $\bar{P}_1$ (Fig. 71) intersecting the axis nor the force $\bar{P}_2$ parallel to the axis can cause the body to rotate about the axis Ox.

To assess the tendency of forces to produce rotation about an axis, the concept of the moment of a force about an axis

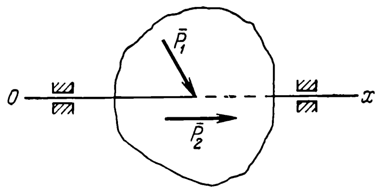


Fig. 71.

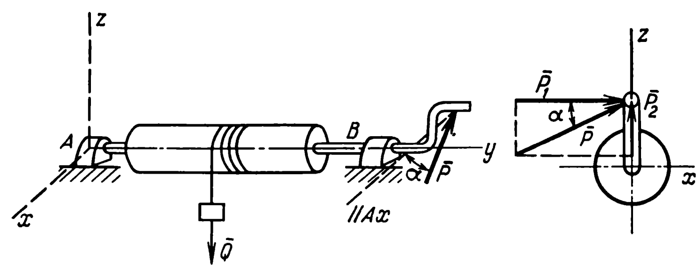


Fig. 72.

$M_x(\bar{P})$ is introduced. The rotational effect of a force about an axis is expressed by its moment. For instance, to start a well windlass, one must exert a force $\bar{P}$ on the handle (Fig. 72).

Resolve the force $\bar{P}$ (Fig. 72) into two components, one, $\bar{P}_1$, directed horizontally and perpendicular to the handle and the other, $\bar{P}_2$, directed vertically. Clearly, only $\bar{P}_1$ produces a moment about the axis AB. The other component, $\bar{P}_2$, gives no moment. If a force parallel to the axis AB is applied on the handle, it will give no moment either. The foregoing can easily be verified by considering the rotation of a door or a window frame about the vertical axis.

Let an arbitrary force $\overline{P}$, which is neither parallel to the axis of rotation Oz nor intersects it, act on a body at a point A (Fig. 73a).

Resolve the force $\overline{P}$ into two components: $\overline{P}_z$, parallel to the axis of rotation Oz, and $\overline{P}_{xy}$, located in the xy plane perpendicular to the axis Oz.

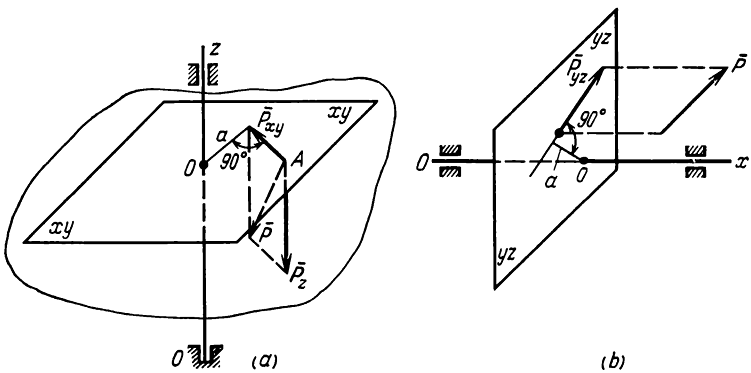


Fig. 73.

The component $\overline{P}_z$, which is parallel to the z axis, produces no moment about this axis.

The component $\overline{P}_{xy}$, acting in the xy plane, produces a moment about the axis Oz.

The moment of the force equals the product of the force $\overline{P}_{xy}$ and the perpendicular distance a from the point O to the direction of this force, i.e.,

$$M_z(\overline{P}) = -P_{xy}a. \tag{51a}$$

Consider the moment of the force $\overline{P}$ about the axis Ox (Fig. 73b). The force $\overline{P}$ is projected on the yz plane perpendicular to the axis of rotation Ox. The projection of this force on the yz plane is denoted by $\overline{P}_{yz}$.

From the point of intersection O of the axis Ox and the yz plane on which the force $\overline{P}$ was projected, drop a perpendicular upon the direction of the force $\overline{P}_{yz}$. The moment of the force $\overline{P}$ about the axis Ox is then

$$M_x(\overline{P}) = +P_{yz}a. \tag{51b}$$

The expression for the moment of the force involves only its component lying in a plane perpendicular to the axis of rotation and not the total force.

The sign of the moment is determined according to the general rule by the sense of rotation of a body: (+) for a clockwise motion, (—) for a counterclockwise motion. When

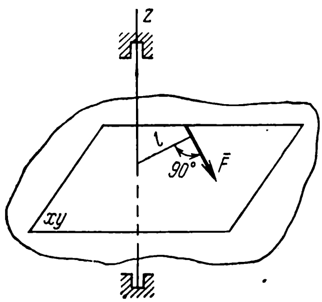


Fig. 74.

determining the sign of the moment the observer must look from the positive direction of the axis. Thus, in Fig. 73*a* the moment of the force $\overline{P}$ about the axis Oz is negative as the force tends to rotate the body counterclockwise when viewed from the positive direction of the axis Oz (from above).

In Fig. 73*b* the moment of the force about the axis Ox is positive.

In a particular case the moment of a force $\overline{F}$ (Fig. 74) contained in the xy plane about the z axis perpendicular to the xy plane is equal to the product of the total force F and its arm l with respect to the point of intersection of the z axis and the xy plane

$$M_z(\overline{F}) = +Fl.$$

Thus, *to determine the moment of a force about an axis, one must project the force on a plane perpendicular to the axis and* *find the moment of the projection of the force on the plane about the point of intersection of the axis with that plane.*

The moment of a force about an axis can be represented as a vector directed along the axis, the vector having the same sense as the axis when the moment is positive (Fig. 75*a*) and the opposite sense when the moment is negative (Fig. 75*b*).

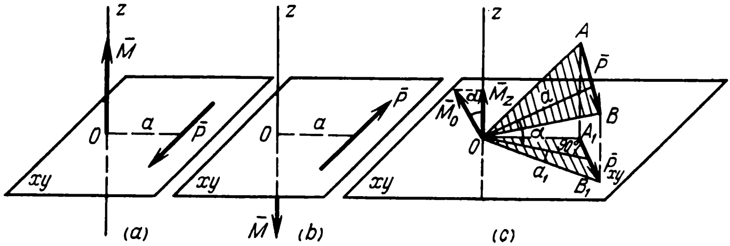


Fig. 75.

It can easily be shown that the projection of the vector representing the moment of a force $\bar{P}$ about a point O on an axis Oz passing through this point is equal to the moment of the force $\bar{P}$ about the axis Oz.

The magnitude of the moment of the force $\bar{P}$ about the point O (Fig. 75*c*) is

$$M_O(\bar{P}) = Pa = 2 \text{ area } \triangle\, OAB.$$

The magnitude of the moment of the force $\bar{P}$ about the z axis is

$$M_z(\bar{P}) = P_{xy}a_1 = 2 \text{ area } \triangle\, OA_1B_1.$$

The triangle OA_1B_1 is the projection of the triangle OAB on the xy plane; from geometry we know that

$$\text{area } \triangle\, OA_1B_1 = \text{area } \triangle\, OAB \cos\alpha,$$

where α is the angle between the planes of the triangles.

The angle between the moment vector $\bar{M}_O$ and the z axis is equal to the angle α as $\bar{M}_O$ and the z axis are respectively perpendicular to the planes of the moment triangles.

Consequently,

$$M_z(\bar{P}) = 2 \text{ area } \triangle\, OA_1B_1 = 2 \text{ area } \triangle\, OAB \cos\alpha = M_O(\bar{P}) \cos\alpha.$$

40. Equilibrium of an Arbitrary Three-Dimensional Force System

A three-dimensional force system, just as a two-dimensional force system, can be reduced to any point in space. The procedure of reduction is the same as for a two-dimensional force system, a force and a couple being obtained at the centre of reduction from each force.

The geometric sum of all forces of a given three-dimensional system is called the resultant force. The magnitude of the resultant force is determined from the projections of all forces of the system on the co-ordinate axes in space in the same way as for a three-dimensional system of concurrent forces

$$R = \sqrt{\left(\sum_{i=1}^{n} P_{ix}\right)^2 + \left(\sum_{i=1}^{n} P_{iy}\right)^2 + \left(\sum_{i=1}^{n} P_{iz}\right)^2}. \qquad (52)$$

In contrast to a two-dimensional force system the moments of the forces of a three-dimensional system about the point of reduction act in different planes. Therefore, *the resultant moment of a three-dimensional force system is determined as the geometric sum of the moment vectors of all forces about the point of reduction.*

The magnitude of the resultant moment of a given force system about a point O is given by the formula

$$M = \sqrt{\left[\sum_{i=1}^{n} M_{ix}(\bar{P})\right]^2 + \left[\sum_{i=1}^{n} M_{iy}(\bar{P})\right]^2 + \left[\sum_{i=1}^{n} M_{iz}(\bar{P})\right]^2}, \qquad (53)$$

where $\sum_{i=1}^{n} M_{ix}(\bar{P})$ = sum of projections of all moment vectors on x axis, or, which is the same thing, the algebraic sum of moments of all forces of the system about x axis,

$\sum_{i=1}^{n} M_{iy}(\bar{P})$ = sum of projections of all moment vectors on y axis, or the algebraic sum of moments of all forces of the system about y axis,

$\sum_{i=1}^{n} M_{iz}(\overline{P})$ = sum of projections of all moment vectors on z axis, or the algebraic sum of moments of all forces of the system about z axis.

For equilibrium of a three-dimensional force system, it is necessary and sufficient that the resultant force and the resultant moment be zero, i.e.,

$$\left.\begin{aligned} \overline{R} &= 0, \\ \overline{M} &= 0. \end{aligned}\right\} \tag{54}$$

Thus we can write six equations of equilibrium for a body subjected to a system of forces arbitrarily located in space.

1. The algebraic sums of the projections of all forces of the system on three co-ordinate axes in space must be zero

$$\left.\begin{aligned} \sum_{i=1}^{n} P_{ix} &= 0, \\ \sum_{i=1}^{n} P_{iy} &= 0, \\ \sum_{i=1}^{n} P_{iz} &= 0. \end{aligned}\right\} \tag{55a}$$

2. The algebraic sums of the moments of all forces about three co-ordinate axes must be zero

$$\left.\begin{aligned} \sum_{i=1}^{n} M_{ix}(\overline{P}) &= 0, \\ \sum_{i=1}^{n} M_{iy}(\overline{P}) &= 0, \\ \sum_{i=1}^{n} M_{iz}(\overline{P}) &= 0. \end{aligned}\right\} \tag{55b}$$

Example 27. A gear wheel C of radius $R = 1$ m and a pinion D of radius $r = 10$ cm are mounted on a horizontal shaft AB (Fig. 76). The other dimensions are indicated in the figure. A horizontal force $P = 10$ N is applied to the wheel C along a tangent and a vertical force $\bar{Q}$ is applied to the pinion D also along a tangent.

Determine the force $\bar{Q}$ and the reactions at the bearings A and B in the position of equilibrium.

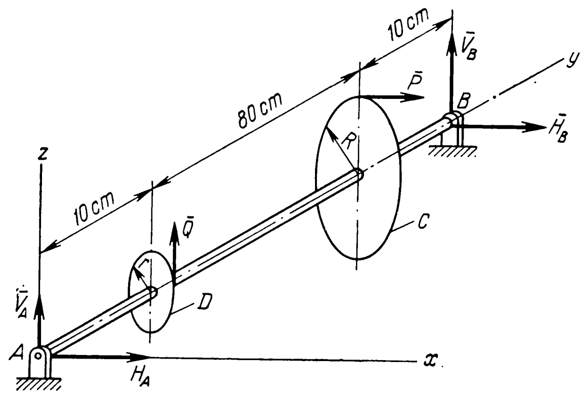


Fig. 76.

Solution. The hinges A and B each involve two unknown components, $\bar{H}_A$, $\bar{V}_A$ and $\bar{H}_B$, $\bar{V}_B$ (Fig. 76); besides, the magnitude of $\bar{Q}$ is unknown. There are altogether five unknowns.

Set up equilibrium equations

$$\sum M_{iy} = PR - Qr = 0,$$

hence

$$Q = P\frac{R}{r} = 10\,\frac{100}{10} = 100 \text{ N};$$

$$\sum M_{iz} = P \cdot AC + H_B \cdot AB = 0,$$

hence

$$H_B = -P\frac{AC}{AB} = -10\,\frac{90}{100} = -9 \text{ N};$$

$$\sum M_{ix} = -Q \cdot AD - V_B \cdot AB = 0,$$

hence

$$V_B = -Q\frac{AD}{AB} = -100\frac{10}{100} = -10 \text{ N};$$

$$\sum P_{ix} = H_A + H_B + P = 0,$$

hence

$$H_A = -P - H_B = -10 - (-9) = -1 \text{ N};$$

$$\sum P_{iz} = V_A + V_B + Q = 0,$$

hence

$$V_A = -Q - V_B = -100 - (-10) = -90 \text{ N}.$$

The minus sign in front of the values of the reactions $\bar{H}_A$, $\bar{V}_A$, $\bar{H}_B$ and $\bar{V}_B$ indicates that these reactions have senses opposite to those shown in the drawing.

The sixth equation $\sum P_{iy} = 0$ reduces to the identity $0 = 0$ in this problem as none of the forces is projected on the y axis.

The problem involved five unknowns and accordingly only five equations were required to determine them. That is why we could have guessed at the very outset that one of the six equations of equilibrium should reduce to an identity. If a problem dealing with the equilibrium of an arbitrary three-dimensional force system involves four unknowns, two equations reduce to identities, etc.

Example 28. A horizontal transmission shaft carrying two pulleys E and D of a belt drive revolves in bearings A and B. The radii of the pulleys are $r_1 = 20$ cm, $r_2 = 25$ cm; the distances of the pulleys from the bearings $a = b = 50$ cm; the distance between the pulleys $c = 100$ cm. The tensions in both parts of the belt passing over the pulley E are horizontal and have magnitudes T_1 and t_1, with $T_1 = 2t_1 = 500$ N; the tensions in the two parts of the belt passing over the pulley D make an angle $\alpha = 30°$ with the vertical and have magnitudes T_2 and t_2, with $T_2 = 2t_2$.

Determine the tensions $\bar{T}_2$ and $\bar{t}_2$ under equilibrium conditions and the reactions at the bearings due to the tension in the belts (Fig. 77a).

Solution. Consider the equilibrium of the shaft subjected to the forces $\bar{T}_1$, $\bar{t}_1$, $\bar{T}_2$, $\bar{t}_2$ and the reactions at the bearings A and B they produce.

The components of these reactions along the co-ordinate axes x and z are $\bar{X}_A$, $\bar{Z}_A$, $\bar{X}_B$ and $\bar{Z}_B$, with $\bar{X}_A$ and $\bar{X}_B$ opposite to the x axis.

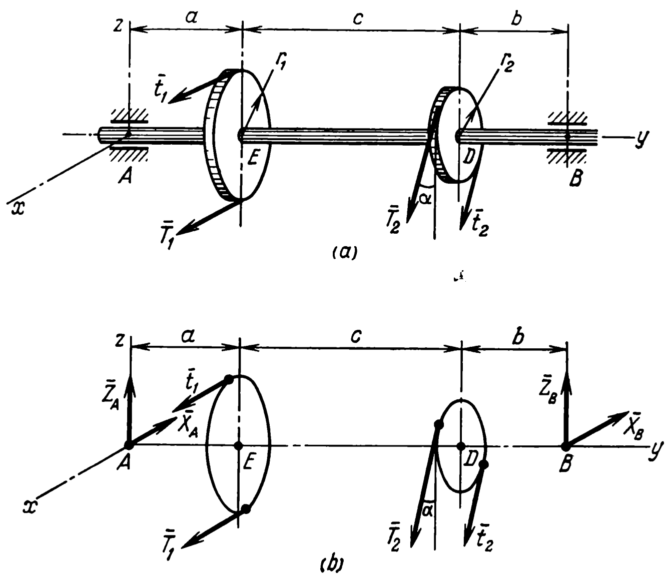


Fig. 77.

No reaction will be exerted along the y axis as the tensions in the belts and all the other forces act in planes perpendicular to that axis (Fig. 77*b*). Write the equilibrium equations for this system

$$\sum P_{ix}=0,\quad T_1+t_1+(T_2+t_2)\sin\alpha-X_A-X_B=0,$$

$$\sum P_{iy}=0,\quad 0=0,$$

$$\sum P_{iz}=0,\quad Z_A+Z_B-(T_2+t_2)\cos\alpha=0,$$

$$\sum M_x=0,\quad (T_2+t_2)(a+c)\cos\alpha-Z_B(a+c+b)=0,$$

$$\sum M_y=0,\quad -t_1r_1-T_2r_2+T_1r_1+t_2r_2=0,$$

$$\sum M_z = 0,\ (t_1 + T_1)\,a + (T_2 + t_2)\,(a + c)\sin\alpha - \\ - X_B\,(a + c + b) = 0.$$

Besides, according to the condition of the problem we have $T_2 = 2t_2$.

The above equations involve six unknowns

$$\overline{t}_2,\ \overline{T}_2,\ \overline{X}_A,\ \overline{Z}_A,\ \overline{X}_B \text{ and } \overline{Z}_B.$$

The equation of statics $\sum P_{iy} = 0$ reduces to an identity in this problem as all forces lie in planes perpendicular to the y axis.

Substituting $T_1 = 2t_1$ and $T_2 = 2t_2$ and solving the system of equations obtained, we find

$$t_2 = \frac{T_1 r_1 - t_1 r_1}{r_2} = \frac{(2t_1 - t_1)\,r_1}{r_2} = \frac{t_1 r_1}{r_2} = \frac{250 \times 20}{25} = 200 \text{ N},$$

$$T_2 = 2t_2 = 2 \times 200 = 400 \text{ N},$$

$$Z_B = \frac{(T_2 + t_2)\,(a + c)\cos\alpha}{a + c + b} = \frac{(400 + 200)\,(50 + 100)\cos 30^\circ}{200} = 389.7 \text{ N},$$

$$X_B = \frac{(t_1 + T_1)\,a + (T_2 + t_2)\,(a + c)\sin\alpha}{a + c + b} =$$

$$= \frac{(250 + 500) \times 50 + (400 + 200)\,(50 + 100)\,\frac{1}{2}}{200} =$$

$$= \frac{750 \times 50 + 600 \times 75}{200} = 412.5 \text{ N},$$

$$X_A = T_1 + t_1 + (T_2 + t_2)\sin\alpha - X_B =$$

$$= 500 + 250 + (400 + 200)\,\frac{1}{2} - 412.5 = 637.5 \text{ N},$$

$$Z_A = (T_2 + t_2)\cos\alpha - Z_B = (400 + 200)\,\frac{\sqrt{3}}{2} - 389.7 = 129.9 \text{ N}.$$

The values obtained for the reactions $\overline{X}_A$ and $\overline{X}_B$ are positive, this indicating that the directions chosen for them in Fig. 77 are correct.

CHAPTER VII

Centroids and Centres of Gravity

41. Centre of Parallel Forces

Every body can be regarded as consisting of a large number of small particles acted upon by forces of gravity. All these forces are directed toward the centre of the earth along the radius. Since the dimensions of bodies dealt with in engineering are negligibly small compared to the radius of the earth (about 6,371 km), it may be assumed that the forces of gravity applied to the particles are parallel and vertical.

Consequently, the forces of gravity acting on individual particles of a body form a system of parallel forces. The resultant of these forces represents the weight of the body.

Recall an important property of the point of application of the resultant of two parallel forces.

Let parallel forces $\bar{Q}$ and $\bar{P}$ act on a body at points A and B (Fig. 78), respectively. The resultant of these forces is equal to their sum, is parallel to them and has the same sense; its line of action divides the straight line AB into parts inversely proportional to these forces, i.e., $\frac{AC}{BC}=\frac{P}{Q}$ (see Sec. 13).

Rotate the forces $\bar{P}$ and $\bar{Q}$ about their points of application through an arbitrary angle α, i.e., change their directions maintaining parallelism. The resultant remains equal to their sum, parallel to them and of the same sense. Again, its line of action divides the straight line AB into parts inversely proportional to the given forces.

Thus, the point of intersection of the line of action of the resultant and the line AB joining the points of application of the component forces (point C in Fig. 78) is independent of their directions, depending only on the moduli of the compo-

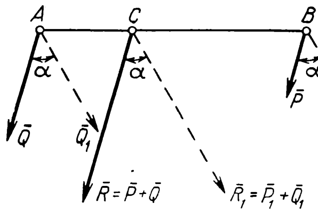


Fig. 78.

nent parallel forces and on the position of their points of application. This point is called *the centre of parallel forces.*

The above conclusion is valid for any number of parallel forces.

The centre of parallel forces of gravity acting on all particles of a body is called *the centre of gravity of the body.*

Since the centre of parallel forces remains the same regardless of the direction of the forces, the centre of gravity of a body does not change its position with the body rotation.

42. Co-ordinates of the Centre of Parallel Forces

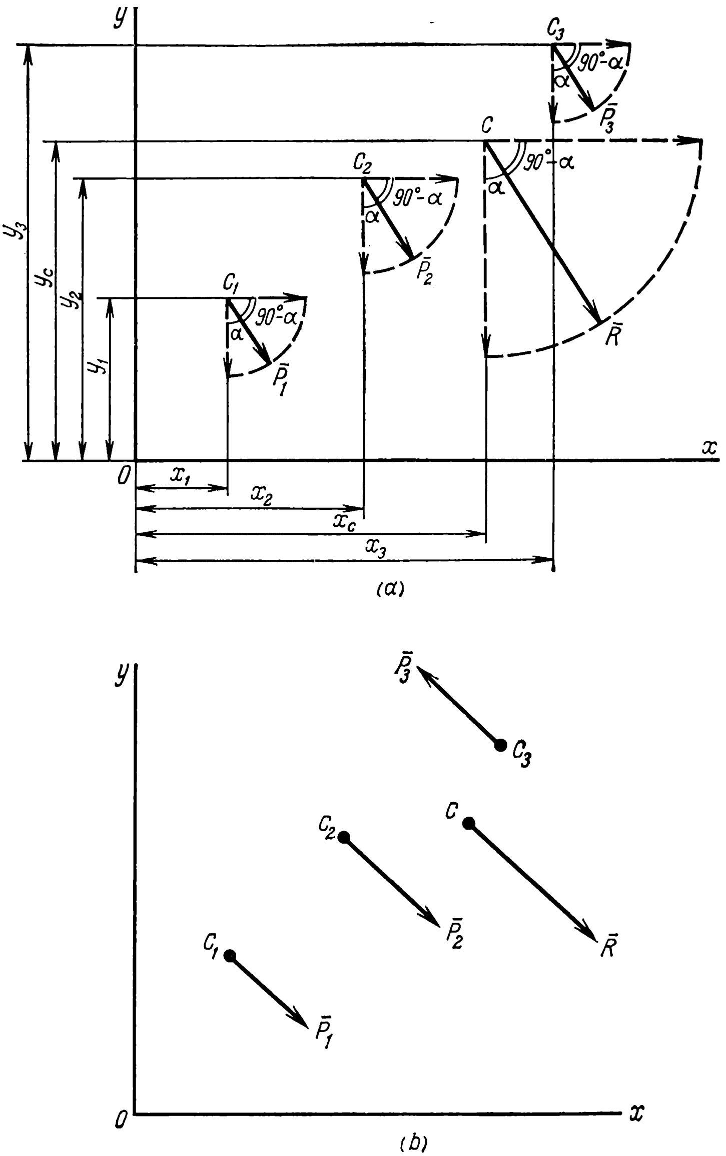
We shall derive formulas defining the position of the centre of any parallel forces. Let a system of parallel forces $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$, . . ., $\bar{P}_n$ be given; the co-ordinates of the points C_1, C_2, C_3, . . ., C_n of application of these forces are known (Fig. 79*a*).

Denote the point of application of the resultant $\bar{R}$ by C and the co-ordinates of this point, which is the centre of the given parallel forces, by x_c and y_c.

Clearly

$$R = P_1 + P_2 + P_3 + \ldots + P_n = \sum_{i=1}^{n} P_i.$$

If among the given parallel forces are forces of opposite sense, they should be taken with different signs (Fig. 79*b*).



y
x
0
C_1
C_2
C_3
C
$\bar{P}_1$
$\bar{P}_2$
$\bar{P}_3$
$\bar{R}$
90°-α
α
x_1
x_2
x_c
x_3
y_1
y_2
y_c
y_3
(a)
(b)

Fig. 79.

Therefore, the formula

$$R=\sum_{i=1}^{n} P_i$$

can be used for calculating the resultant in all cases if P_i implies the algebraic values of the forces.

Since the centre of parallel forces is independent of their direction, we rotate all the given component forces clockwise through an angle α so that they become parallel to the y axis (Fig. 79*a*). The resultant will also rotate clockwise through the angle α.

Apply the theorem of the moment of the resultant (Varignon's theorem) about the origin of co-ordinates O

$$Rx_c=P_1x_1+P_2x_2+P_3x_3\ldots=\sum_{i=1}^{n} P_ix_i,$$

hence

$$x_c=\frac{P_1x_1+P_2x_2+P_3x_3}{R},$$

but since

$$R=P_1+P_2+P_3,$$

then

$$x_c=\frac{P_1x_1+P_2x_2+P_3x_3}{P_1+P_2+P_3}=\frac{\sum_{i=1}^{n} P_ix_i}{\sum_{i=1}^{n} P_i}. \qquad (56)$$

Likewise, rotating the given forces counterclockwise through an angle (90° — α) so that they become parallel to the x axis and using the theorem of the moment of the resultant, we obtain the formula for the other co-ordinate of the centre of parallel forces

$$y_c=\frac{P_1y_1+P_2y_2+P_3y_3}{P_1+P_2+P_3}=\frac{\sum_{i=1}^{n} P_iy_i}{\sum_{i=1}^{n} P_i}. \qquad (57)$$

If a given system of parallel forces involves forces of opposite sense, forces of one sense in formulas (56) and (57) should be recorded with a plus sign and those of the opposite sense with a minus sign.

If, for example, the force $\overline{P}_3$ has a sense opposite to that of the forces $\overline{P}_1$ and $\overline{P}_2$ (Fig. 79*b*), formulas (56) and (57) are then

$$\left.\begin{aligned} x_c &= \frac{P_1x_1+P_2x_2-P_3x_3}{P_1+P_2-P_3}, \\ y_c &= \frac{P_1y_1+P_2y_2-P_3y_3}{P_1+P_2-P_3}. \end{aligned}\right\} \tag{58}$$

To determine the centre of a system of parallel forces in space, the following three formulas are available

$$x_c = \frac{\sum_{i=1}^{n} P_ix_i}{\sum_{i=1}^{n} P_i}, \quad y_c = \frac{\sum_{i=1}^{n} P_iy_i}{\sum_{i=1}^{n} P_i}, \quad z_c = \frac{\sum_{i=1}^{n} P_iz_i}{\sum_{i=1}^{n} P_i}. \tag{59}$$

These formulas serve to calculate the co-ordinates of the centre of gravity of a body, P_i implying the weight of individual parts of the body and x_i, y_i, z_i the co-ordinates of their centres of gravity.

The centre of gravity of a symmetrical body lies in a plane of symmetry. A *plane of symmetry* is a plane such that to every particle located on one side of it corresponds a particle of equal mass and weight located on the other side, the line joining the particles being perpendicular to the plane of symmetry and divided in half by that plane. Since two equal and parallel forces of gravity are applied at two points A and B equidistant from the plane of symmetry, the resultant of these forces is contained in the same plane. Clearly, the above arguments apply to any pair of symmetrical points. Therefore, we come to the conclusion that the point of application of the resultant of all elementary forces of gravity, i.e., the centre of gravity of a homogeneous body lies in a plane of symmetry.

Hence the centre of gravity of a line segment is at its middle.

The centre of gravity of a plane symmetrical figure, a thin homogeneous plate, lies on an axis of symmetry, i.e., on a line dividing the figure into two equal parts.

43. Centroid of a Volume

If a body is homogeneous, the weight of its individual parts is proportional to their volume

$$P_i = \gamma V_i,$$

where γ is the specific weight (a constant for a homogeneous body).

Putting γ before the summation sign in the numerator and denominator in the general formulas (59) and cancelling it,

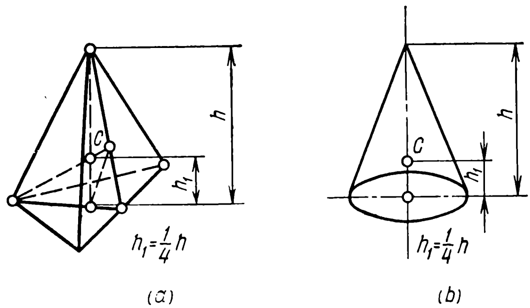


Fig. 80.

we obtain formulas for determining the co-ordinates of the centroid of a homogeneous volume

$$x_c = \frac{\sum_{i=1}^{n} V_i x_i}{\sum_{i=1}^{n} V_i}, \quad y_c = \frac{\sum_{i=1}^{n} V_i y_i}{\sum_{i=1}^{n} V_i}, \quad z_c = \frac{\sum_{i=1}^{n} V_i z_i}{\sum_{i=1}^{n} V_i}, \qquad (60)$$

where V_i are the volumes of individual parts of the body, x_i, y_i, z_i the co-ordinates of their centroids.

The centroid always lies in a plane of symmetry of a body. For example, the centroid of a prism or a cylinder is at the middle of the line joining the centroids of the bases. The centroid of a sphere coincides with its geometric centre. The centroid of a pyramid lies on the line segment joining the centroid of the base and the opposite vertex at one-fourth of its length from the base (Fig. 80*a*).

The centroid of a cone lies on the line segment joining the centroid of the base and the vertex at one-fourth of its length from the base (Fig. 80*b*).

Example 29. Find the co-ordinates of the centroid of a wooden hammer consisting of a head—rectangular parallelepiped—and a handle of square section (Fig. 81).

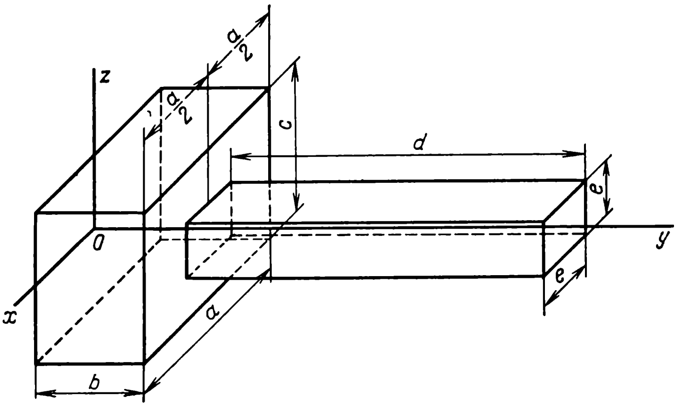


Fig. 81.

The dimensions of the hammer are: $a = 10$ cm, $b = 8$ cm, $c = 18$ cm, $d = 40$ cm, $e = 3$ cm.

Solution. Consider two parallelepipeds: (1) the handle and (2) the head. The arrangement of the chosen co-ordinate axes is shown in Fig. 81. The co-ordinates of the centroids are: for the handle

$$x_1 = 0, \ y_1 = b + \frac{d}{2} = 28 \text{ cm}, \ z_1 = 0;$$

for the head

$$x_2 = 0, \ y_2 = \frac{b}{2} = 4 \text{ cm}, \ z_2 = 0.$$

The volume of the handle is $V_1 = de^2 = 360$ cm^3, the volume of the head $V_2 = abc = 1{,}440$ cm^3. By using the formulas for determining the co-ordinates of the centroid of a homogeneous body, we obtain

$$x_c = \frac{V_1x_1 + V_2x_2}{V_1 + V_2} = \frac{360 \times 0 + 1,440 \times 0}{360 + 1,440} = 0,$$

$$y_c = \frac{V_1y_1 + V_2y_2}{V_1 + V_2} = \frac{360 \times 28 + 1,440 \times 4}{360 + 1,440} = 8.8 \text{ cm},$$

$$z_c = \frac{V_1z_1 + V_2z_2}{V_1 + V_2} = \frac{360 \times 0 + 1,440 \times 0}{360 + 1,440} = 0.$$

44. Centroid of an Area. Static Moments of an Area

In engineering problems it is often necessary to determine the centroids of various sections of bodies which represent geometric plane figures, sometimes of rather complex shape. Consider a two-dimensional homogeneous body. The weight of each of its parts is proportional to the area. Denote by γ' the weight of one square metre, then $P_i = \gamma' A_i$.

Dividing the numerator and denominator in formulas (59) by γ', we obtain formulas for determining the co-ordinates of the centroid of a plane figure in its plane. Therefore, we have only two such formulas and not three

$$x_c = \frac{\sum_{i=1}^{n} A_i x_i}{\sum_{i=1}^{n} A_i}, \quad y_c = \frac{\sum_{i=1}^{n} A_i y_i}{\sum_{i=1}^{n} A_i}, \tag{61}$$

where A_i are the areas of individual parts of the figure, x_i, y_i the co-ordinates of their centroids.

The product of a part of the area A_i of a figure (Fig. 82*a*) *and the distance of its centroid to any axis is called the "static moment" of this part of the area with respect to the given axis.*

Thus, the static moment of the area A_i with respect to the axis Ox is $S_{ix} = A_i y_i$, and with respect to the axis Oy, $S_{iy} = A_i x_i$.

The sum of the static moments of all parts of a figure is called the "static moment" of the area of the figure with respect to a given axis.

The static moments of the entire area with respect to the axes Ox and Oy are defined by the formulas

$$\left.\begin{aligned} S_x &= \sum_{i=1}^{n} A_i y_i, \\ S_y &= \sum_{i=1}^{n} A_i x_i. \end{aligned}\right\} \qquad (62)$$

If a complex figure is divided into infinitesimal parts, the

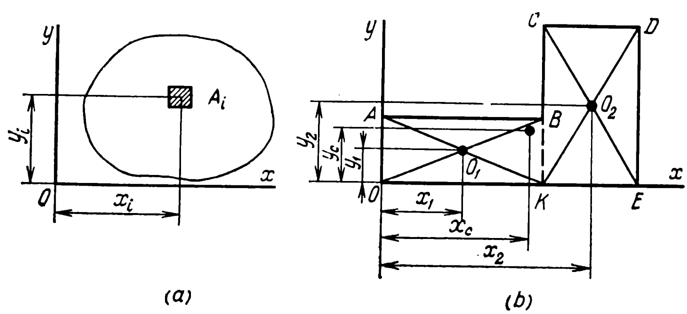


Fig. 82.

static moments of its area can be represented as integrals taken over the entire area

$$\left.\begin{aligned} S_x &= \int_A y\,dA, \\ S_y &= \int_A x\,dA. \end{aligned}\right\} \qquad (63)$$

The static moment of an area is expressed in units of length to the third power, for instance, in cubic centimetres.

The concept of the static moment of an area facilitates the determination of the co-ordinates of the centroids of complex figures.

Let us determine the centroid of an area $OABCDEO$ (Fig. 82*b*). Break this complex figure into simple figures whose centroids are easy to determine. Thus, extending the side CB until it intersects the axis Ox at a point K, we obtain two rectangles whose centroids O_1 and O_2 coincide with the geometric centres of the figures, the points of intersection of their diagonals.

Let the co-ordinates of the centroid of the entire area be x_c and y_c, and those for the two rectangles x_1, y_1 and x_2, y_2, respectively. Denote the area of the rectangle $OABK$ by A_1 and that of $KCDE$ by A_2. The entire area of the figure is

$$A = A_1 + A_2.$$

Based on the definition of the static moments of an area with respect to the co-ordinate axes, we find

$$S_x = Ay_c = A_1y_1 + A_2y_2,$$
$$S_y = Ax_c = A_1x_1 + A_2x_2.$$

From these equations we obtain

$$y_c = \frac{A_1y_1 + A_2y_2}{A},$$
$$x_c = \frac{A_1x_1 + A_2x_2}{A}.$$

In general, if any complex figure of area A is divided into n simple parts, then

$$y_c = \frac{\sum\limits_{i=1}^{n} A_iy_i}{A} = \frac{S_x}{A}, \quad x_c = \frac{\sum\limits_{i=1}^{n} A_ix_i}{A} = \frac{S_y}{A}. \tag{64}$$

When a given figure cannot be broken into a finite number of simple shapes, the static moments of the area are determined by integration according to formulas (63). In this general case the co-ordinates of the centroid are calculated by the formulas

$$y_c = \frac{\int\limits_A y\,dA}{A}, \quad x_c = \frac{\int\limits_A x\,dA}{A}. \tag{65}$$

If the origin of co-ordinates is placed at the centroid of an area, the static moments of the area with respect to the axes Ox and Oy passing through the centroid are zero since

$$y_c = 0 \text{ and } x_c = 0.$$

To conclude this section we give data (without derivation) on the centroids of some simple figures that may be encountered in the solution of problems.

The centroid of a parallelogram, rectangle or square coincides with the point of intersection of the diagonals (Fig. 83*a*).

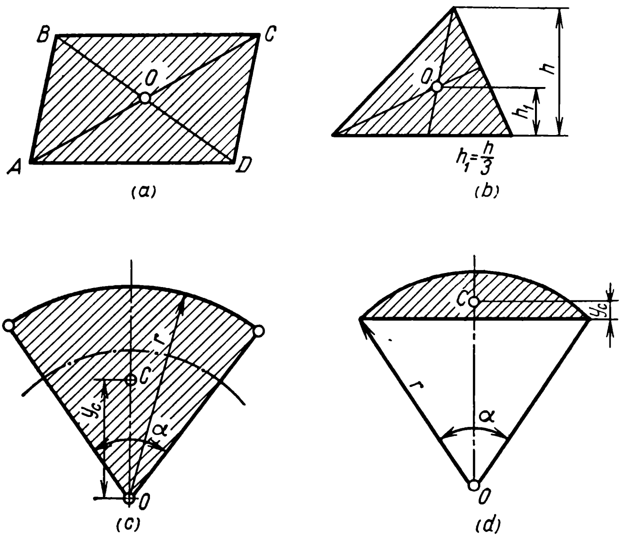


Fig. 83.

The centroid of a triangle lies at the intersection of its medians (Fig. 83*b*). The centroid of a circular sector is defined by the formula (Fig. 83*c*)

$$y_c = \frac{2}{3} r \frac{\sin \alpha}{\alpha},$$

where α is the central angle of the sector in radians.

The centroid of a segment of a circle is defined by the formula (Fig. 83*d*)

$$y_c = \frac{4}{3} \frac{r \sin^3 \frac{\alpha}{2}}{(\alpha - \sin \alpha)};$$

where α is the central angle of the segment in radians.

Example 30. Determine the co-ordinates of the centroid of the plane figure with a circular hole shown in Fig. 84. The dimensions are indicated in the drawing.

Solution. Break the figure into three simple parts: two rectangles I and II and the circular hole III. Calculate the co-

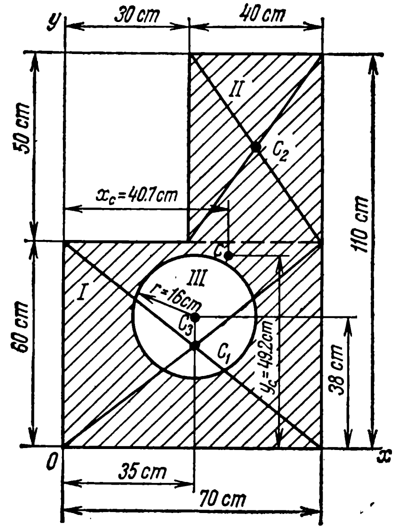


Fig. 84.

ordinates of the centroids and the areas of these parts (see Fig. 84)

$$x_1 = 35 \text{ cm}, \quad y_1 = 30 \text{ cm}, \quad A_1 = 60 \times 70 = 4{,}200 \text{ cm}^2,$$

$$x_2 = 50 \text{ cm}, \quad y_2 = 85 \text{ cm}, \quad A_2 = 40 \times 50 = 2{,}000 \text{ cm}^2,$$

$$x_3 = 35 \text{ cm}, \quad y_3 = 38 \text{ cm}, \quad A_3 = -3.14 \times 16^2 = -804 \text{ cm}^2.$$

Calculate the co-ordinates of the centroid of the entire figure

$$x_c = \frac{A_1x_1 + A_2x_2 + A_3x_3}{A_1 + A_2 + A_3} = \frac{4{,}200 \times 35 + 2{,}000 \times 50 - 804 \times 35}{4{,}200 + 2{,}000 - 804} =$$

$$= \frac{147{,}000 + 100{,}000 - 28{,}140}{5{,}396} = \frac{218{,}860}{5{,}396} = 40.6 \text{ cm},$$

$$y_c = \frac{A_1y_1 + A_2y_2 + A_3y_3}{A_1 + A_2 + A_3} = \frac{4{,}200 \times 30 + 2{,}000 \times 85 - 804 \times 38}{4{,}200 + 2{,}000 - 804} =$$

$$= \frac{126{,}000 + 170{,}000 - 30{,}552}{5{,}396} = \frac{265{,}448}{5{,}396} = 49.2 \text{ cm}.$$

From the above example it follows that the determination of the centroid of a plane figure with holes is considerably simplified by assuming the areas of the holes to be negative. A similar procedure should be adopted when determining the centroids of three-dimensional bodies.

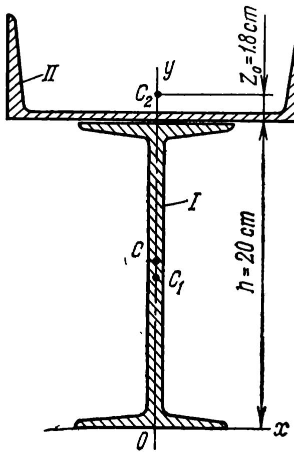


Fig. 85.

Example 31. Determine the centroid of the section of a beam (Fig. 85) made up of a No. 20 I-beam ($A_1 = 26.4$ cm^2) and a No. 16 channel ($A_2 = 18.1$ cm^2, $z_0 =$ $= 1.8$ cm).

Solution. The section is symmetrical about the vertical axis. The centroids of the two figures forming the section (I—I-beam and II—channel) lie on the vertical axis of symmetry of the section which is chosen as the y axis. The x axis is directed horizontally along the bottom of the I-beam.

Determine the co-ordinates of the centroids of the component parts of the section. The areas of the two parts and the distance of the centroid of the channel from its web z_0 are taken from the USSR State Standard GOST 8239-56 and GOST 8240-56. We have for the I-beam $A_1 = 26.4$ cm^2, $x_1 = 0$, $y_1 = 10$ cm; for the channel $A_2 = 18.1$ cm^2, $x_2 = 0$, $y_2 = 20 + 1.8 = 21.8$ cm.

Calculate the co-ordinates of the centroid of the entire section

$$x_c = \frac{A_1x_1 + A_2x_2}{A_1 + A_2} = \frac{26.4 \times 0 + 18.1 \times 0}{26.4 + 18.1} = 0,$$

$$y_c = \frac{A_1y_1 + A_2y_2}{A_1 + A_2} = \frac{26.4 \times 10 + 18.1 \times 21.8}{26.4 + 18.1} = 14.8 \text{ cm}.$$

45. Centroid of a Line

The centroid of a homogeneous body of uniform section made up of several parts of certain extent is determined in the same way as for lines. An example of such a body is a segment of wire. Clearly, the weight of each portion of line is proportional to its length. Denoting by γ'' the weight per unit length, we obtain $P_i = \gamma'' L_i$.

The constant factor γ'' in formulas (59) can be put before the summation sign in the numerator and denominator and cancelled. We obtain formulas for determining the centroid of a line. The centroid of a plane line is completely defined by two co-ordinates

$$x_c = \frac{\sum_{i=1}^{n} L_i x_i}{\sum_{i=1}^{n} L_i}, \quad y_c = \frac{\sum_{i=1}^{n} L_i y_i}{\sum_{i=1}^{n} L_i}, \qquad (66)$$

where L_i are the lengths of individual portions, x_i, y_i the co-ordinates of their centroids.

By using these formulas one can find the centroid of an arbitrary homogeneous line.

46. Stability of Equilibrium

Problems involving stability of equilibrium are encountered in nearly all domains of applied mechanics.

Theoretical mechanics deals with the stability of equilibrium of a perfectly rigid body.

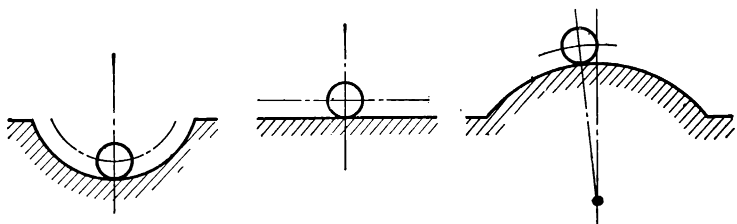
Every rigid body may be in a state of stable, neutral or unstable equilibrium under the action of external forces.

A body is in a position of stable equilibrium if, being displaced slightly from this position, it always returns to it. Displacements of a body from its equilibrium position are assumed to be small.

Stable equilibrium is characterized by such a position of a body where its centre of gravity occupies the lowest of all positions as compared to any other possible close positions.

For instance, a ball resting on a concave spherical surface at its lowest point is in a state of stable equilibrium under the action of its own weight. Any displacement of the ball from this position results in an increase in the height of its centre of gravity and as soon as the forces which have caused the displacement are removed the ball returns to the position of stable equilibrium (Fig. 86*a*).

Neutral equilibrium is characterized by such a position of a body where its possible displacements do not change the height of the centre of gravity, i.e., all close positions of the body are



(*a*) (*b*) (*c*)

Fig. 86.

equivalent positions of equilibrium. For instance, a ball resting on a horizontal surface is in a state of neutral equilibrium (Fig. 86*b*).

Unstable equilibrium is characterized by such a position of a body where its centre of gravity is lowered with any possible displacements of the body.

Thus, a ball resting on a convex spherical surface at its highest point is in a state of unstable equilibrium (Fig. 86*c*). When displaced slightly from this position the ball rolls down the surface.

The stability of equilibrium depends on the forces acting on the body. In the above examples the ball is acted upon only by its own weight and the reaction at the support. In equilibrium positions, the weight of the ball is perpendicular to the supporting surface, it passes through the point of contact and is balanced by the reaction at the support.

The forces acting in structures and machines include not only forces of weight but also moving forces, resistance forces, etc. For example, a dam is subjected to water pressure, a machine tool is acted upon by forces transmitted from an electric motor, etc.

Under the action of all applied forces structures and machines must always be in a state of stable equilibrium.

The conditions of stable equilibrium of a body are provided by appropriate relations between the forces acting on the

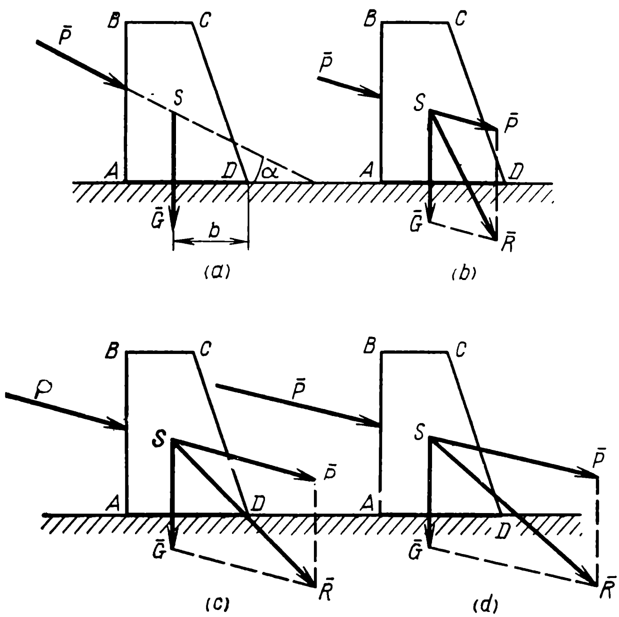


Fig. 87.

body and its geometric dimensions. When acting forces are specified, all structures and machines must be so dimensioned as to assure stable equilibrium; when the dimensions of a structure are given, the acting forces must not exceed certain values.

In practice it is often necessary to solve problems concerning stability of a solid resting on a plane, as in the stability analysis of a retaining wall, dam, crane, machine tool.

Consider an arbitrary body resting on a horizontal plane; let the figure *ABCD* (Fig. 87*a*) be the section of this body by a vertical plane passing through the centre of gravity S.

The body is acted upon by a force $\overline{P}$ and its own weight $\overline{G}$. Assume that the body cannot slide on the supporting plane. Clearly, the force $\overline{P}$ tends to rotate the body about the edge D and the weight $\overline{G}$ opposes the tipping action of the force $\overline{P}$.

The moment of the force $\overline{P}$ about the edge D is called the tipping moment. Its magnitude is equal to the product of the force $\overline{P}$ and its arm b with respect to the edge D

$$M_t = Pb.$$

The moment of the force $\overline{G}$ about the edge D is called the stability moment or the moment against tipping. By multiplying the weight $\overline{G}$ by its arm b with respect to the edge D we obtain the magnitude of this moment

$$M_{st} = Gb.$$

It should be noted that when determining the tipping moment and the moment against tipping we calculate their absolute values without regard to sign.

The condition of stable equilibrium is

$$M_{st} > M_t, \tag{67}$$

i.e., in order to provide stable equilibrium for a body resting on a plane it is necessary and sufficient that the stability moment be larger than the tipping moment.

The ratio

$$k = \frac{M_{st}}{M_t} \tag{68}$$

is called the *stability factor*. This factor must obviously be greater than unity to provide stable equilibrium

$$k = \frac{M_{st}}{M_t} > 1. \tag{69}$$

In this case the line of action of the resultant of the tipping forces and of the forces against tipping passes within the limits of the base AD to the left of the edge D and the body $ABCD$ is in stable equilibrium (Fig. 87*b*).

When $k = 1$ and $M_{st} = M_t$ we obtain the position of limiting equilibrium, the line of action of the resultant of the forces $\overline{G}$ and $\overline{P}$ passes through the edge D (Fig. 87*c*).

Finally, when $k < 1$ and $M_{st} < M_t$ the line of action of the resultant of the forces $\bar{G}$ and $\bar{P}$ is outside the limits of the base AD and the body $ABCD$ will tip about the edge D, i.e., in this case the position of the body is unstable (Fig. 87*d*).

The stability factor k is generally taken to be no less than 1.3 to 1.5 to allow for possible variations in the magnitudes of acting forces.

Stability analysis is of particular importance for tall structures (chimneys, masts, high walls, cranes).

In all cases the stability factor is assigned by design standards.

Example 32. A crane, minus a counterweight, has a weight $G = 500$ kN which acts at a distance $c_1 = 1.5$ m from a

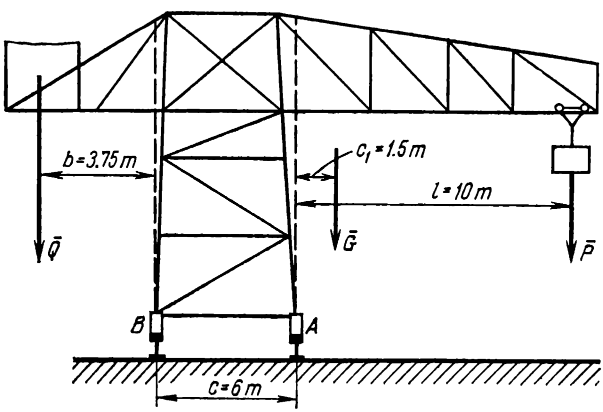


Fig. 88.

vertical through the right-hand rail (Fig. 88), the lifting capacity of the crane trolley is $P = 250$ kN, its boom $l = 10$ m, the rail gauge $c = 6$ m. The counterweight is at a distance $b = 3.75$ m from a vertical through the left-hand rail.

Find the magnitude of the counterweight Q required to provide a stable position of the crane when lifting the maximum load if the stability factor $k = 1.5$. Check the stability of the crane under no-load conditions.

Solution. As the load is lifted the crane may tip about the rail A

$$M_t = Pl + Gc_1 = 250 \times 10 + 500 \times 1.5 = 3{,}250 \text{ kN-m},$$

$$M_{st} = Q(b+c) = Q(3.75+6) = 9.75Q \text{ kN-m}.$$

The stability factor is

$$k = \frac{M_{st}}{M_t},$$

whence

$$M_{st} = kM_t.$$

Substituting the numerical values, we obtain

$$9.75Q = 1.5 \times 3{,}250,$$

whence

$$Q = \frac{1.5 \times 3{,}250}{9.75} = 500 \text{ kN}.$$

In the unloaded position the crane may tip about the rail B

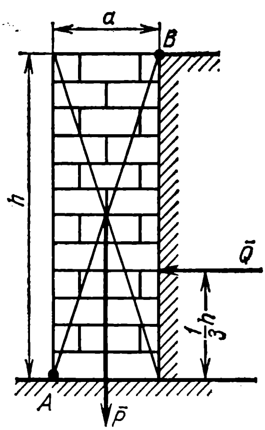


Fig. 89.

$$M_t = Qb = 500 \times 3.75 = 1{,}875 \text{ kN-m},$$

$$M_{st} = G(c_1 + c) = 500 \times 7.5 = 3{,}750 \text{ kN-m}.$$

The stability factor is

$$k=\frac{M_{st}}{M_t}=\frac{3,750}{1,875}=2.0>1.5.$$

The stable position of the crane is ensured.

Example 33. An earth embankment is supported by a vertical brick wall AB.

Find the thickness of the wall a required to provide stability (Fig. 89) assuming that the earth pressure on the wall is horizontal, applied at one-third of its height and equal to 60 kN/m (per metre of the wall length); the specific weight of the masonry is $\gamma = 20$ kN/m³.

Solution. Check the wall for tipping about the edge A. The calculation is carried out for 1 m of the wall length.

The tipping moment is produced by the force Q

$$M_t=Q\frac{h}{3}.$$

The moment against tipping is due to the action of the weight of the wall P

$$P=ha\times 1\times\gamma,$$

$$M_{st}=P\frac{a}{2}=ha\gamma\frac{a}{2}=\frac{ha^2\gamma}{2}.$$

The condition of stability of the wall is

$$M_{st}\geqslant M_t.$$

Substituting, we obtain

$$\frac{ha^2\gamma}{2}\geqslant\frac{Qh}{3}.$$

Cancelling h, we solve for a

$$a\geqslant\sqrt{\frac{2Q}{3\gamma}}=\sqrt{\frac{2\times 60}{3\times 20}}=1.41 \text{ m}.$$

Consequently, the stable position of the wall is ensured when its thickness is not less than 1.41 m.

PART 2. KINEMATICS

CHAPTER VIII

Fundamentals

47. Subject of Kinematics

Kinematics is the part of theoretical mechanics in which the mechanical motion of particles and rigid bodies is studied without regard to the forces producing this motion.

It will be recalled that mechanical motion is the displacement of particles and bodies in space with time. Mechanical motion is the most common type of motion. When studying non-mechanical forms of motion, such as thermal and electric motion, use is often made of the laws established in kinematics for mechanical motion, because thermal, electric and other forms of motion are associated with mechanical displacement of atoms, molecules, electrons, etc. We have already mentioned this in the introduction. Here we again draw attention to the generality of the laws of mechanical motion in order to stress their importance.

Kinematics is subdivided into two parts: the kinematics of a particle and the kinematics of a rigid body.

The two parts are taken up in independent subdivisions of the text.

The motion of many bodies and other objects can be reduced to the motion of a particle. Therefore, the kinematics of a particle is of great importance by itself.

Besides, the laws of motion established for a particle form the basis for the derivation of the laws of motion of rigid bodies which are treated as systems of particles. Consequently, both parts of kinematics are intimately connected. Kinematics is often called the geometry of motion, it is largely based on the axioms of mathematics.

48. Space and Time

As follows from the foregoing, every mechanical motion takes place in space and in time. Space and time are forms of the existence of matter. Space and time are closely interlinked, their unity manifests itself in motion.

In theoretical mechanics, the space in which bodies move is considered as a three-dimensional space all properties of which are governed by the system of axioms and theorems of Euclidean geometry. Time is assumed not to be connected with anything and flowing uniformly. These conceptions of space and time were introduced by the founder of classical mechanics Sir Isaac Newton at the end of the XVII century in his famous *Philosophiae Naturalis Principia Mathematica* (1687).

The development of physics has led to different conceptions of space and time. The theory of relativity developed by one of the greatest scientists of our times Albert Einstein shows that space and time cannot be separated from motion. Space and time are not absolute concepts as is assumed in mechanics. Actually, the geometric properties of space and the properties of time are inherently related to the properties of moving matter. This interrelation stands out most prominently at high velocities close to that of light (300,000 km/sec). At ordinary speeds encountered in engineering space and time only slightly depend on the properties of moving matter. Therefore, this dependence can be ignored in theoretical mechanics and space and time conventionally considered as absolute categories independent of the nature of motion.

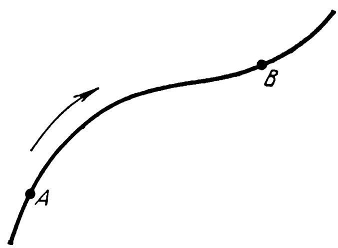


Fig. 90.

The international system and the engineers' system use the same units for measuring space: the fundamental unit, metre (m), and derived units (cm, mm, dm, km). To measure time, all systems use the second (sec) as the fundamental unit, and derived units (minute, hour, etc.).

Kinematics distinguishes an interval or period of time and the initial instant. A *time interval* is the time elapsed between any two fixed events. For instance, when a particle moves (Fig. 90) its coincidence with a point A and subsequently with a point B may be considered such fixed events. The time interval is then the time corresponding to the displacement of the particle from position A to position B.

The initial instant is the time from which the measurement is made. By a given instant is meant the boundary between any two adjacent time intervals.

49. Basic Definitions

Examining the motion of a rigid body it can readily be seen that, in general, its different particles execute different motions. Hence the need to study first of all the motion of individual particles of a body. In order to define the position of a particle in space, it is necessary to have a fixed body or a system of co-ordinate axes attached to it which is called a reference system.

By a reference system is meant an absolutely rigid body or a co-ordinate system invariably attached to it with respect to which a given motion is considered.

The motion of a given body is revealed only by comparison with a reference system.

In some cases a moving reference system which executes motion with respect to the basic reference system is considered in kinematics.

In nature, no fixed bodies exist and consequently there can be no fixed reference systems. A fixed reference system is usually assumed to be a system of co-ordinate axes attached to the earth.

As an example, consider the motion of a particle in a presumably fixed system of co-ordinates xyz (Fig. 91). The position of a particle in space is defined by three co-ordinates.

These co-ordinates change when the particle moves to another position.

The curve described by a particle as it moves in space with respect to a chosen reference system is called its path.

Paths are classed as *rectilinear* (the motion of points of the piston of an engine) and *curved* (circular—the motion of points of a pulley, of a circular saw; parabolic—the motion of the fluid issuing from an orifice in the lateral wall of a vessel, etc.).

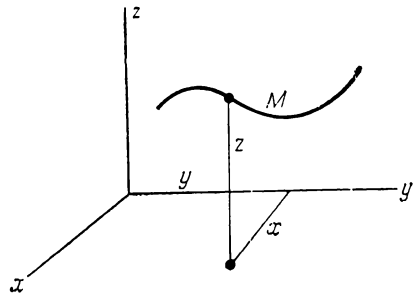


Fig. 91.

The paths of one and the same particle may be different depending upon the adopted reference (co-ordinate) system. For instance, the path of the hook of a bridge crane with respect to its frame is a vertical straight line. If the crane moves, however, the path of the hook with respect to a fixed reference system, i.e., with respect to an immovable observer, will be different.

In the case of uniform rectilinear motion of the bridge crane the hook moves along a sloping straight line with respect to a fixed reference system.

The motion of a particle in space is defined above all by its velocity. *Velocity is a quantity characterizing the speed and direction of motion of a particle at a given instant.* The motion of a body may be *uniform* or *non-uniform* depending upon its velocity.

In uniform motion the velocity is constant in magnitude, in non-uniform motion it is variable. *The change of velocity in time is characterized by acceleration.*

When studying the motion of a particle two important concepts are distinguished, the *distance*, or the *position co-ordinate*, and the *total distance travelled.* The distance defines the position of a particle on its path and is measured from a certain reference point or origin. The position co-ordinate is an algebraic quantity as it may be positive or negative according to the position of the particle relative to the origin and the adopted sense of the axis of distances. As contrasted to the position, the distance travelled by the particle is always expressed by a positive number. The distance travelled equals the absolute value of the position co-ordinate only when the motion of the particle starts from the origin and continues in one and the same direction.

In the general case, the motion of a particle may start from any of its positions on the path. The concept of the *initial distance* is then introduced to define this original position of the particle. If a particle starting from a given initial position other than the origin moves along its path in one and the same direction, the distance travelled is equal to the absolute value of the difference of the distances in the initial and terminal positions. A particle may execute an oscillatory motion along the path, as in the motion of a piston of an internal-combustion engine, the motion of a clock pendulum. Here, when determining the relationship between the position and the distance travelled one should take into account the number of oscillations executed before the instant considered. We denote distances by S but we shall not introduce any special designations for the distance travelled for reasons just explained. The position co-ordinate and the distance travelled may differ only by a constant depending on the location of the origin. This can be expressed as

$$S_p = S_{tr} + S_0, \tag{70}$$

where S_p = distance of particle from origin,
S_{tr} = distance travelled,
S_0 = distance from origin to initial position of particle (constant).

Later on we will have to calculate derivatives of the position co-ordinate and of the distance travelled with respect to time. By differentiating formula (70) with respect to time, we obtain

$$\frac{dS_p}{dt} = \frac{dS_{tr}}{dt} + \frac{dS_0}{dt}. \tag{71}$$

Since S_0 is a constant, its derivative is zero, $\frac{dS_0}{dt} = 0$, and we obtain finally

$$\frac{dS_p}{dt} = \frac{dS_{tr}}{dt}. \tag{72}$$

Thus, *the time derivatives of the position co-ordinate and of the distance travelled are identically equal.*

CHAPTER IX

Kinematics of Particles

50. Methods of Specifying the Motion of a Particle

The most general case of motion of a particle is motion along a curved path.

To study the curvilinear motion of a particle, one should know how to determine the position of the particle in a prescribed reference (co-ordinate) system at any time.

Equations expressing the position of a moving particle as a function of time are called the *equations of motion*. The motion of a particle can be prescribed in two ways.

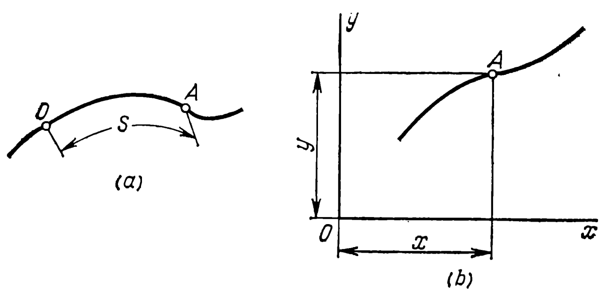


Fig. 92.

1. *Natural* or *geometric method*. In this method the path of a particle is given (graphically or analytically) as well as the law of motion of the particle along its path. Let an arbitrary particle A move along a given path (Fig. 92*a*). Choosing a point O as the origin of the distance travelled by the particle

A along its path, the equation of motion can be written as

$$S = f(t), \tag{73}$$

where S is the distance travelled by the particle A from the origin, t is the time.

This equation determines the position of the particle A at any given time.

2. *Co-ordinate* or *analytic method*. The position of a particle moving in a plane (Fig. 92*b*) can be determined if its co-ordinates x and y with respect to a system of two mutually perpendicular co-ordinate axes Ox and Oy are known. When the particle moves its co-ordinates vary with time, consequently x and y are functions of time

$$\left.\begin{aligned} x &= f_1(t), \\ y &= f_2(t). \end{aligned}\right\} \tag{74}$$

The motion of a particle in a plane is completely defined by these equations as we can calculate the co-ordinates of the particle and consequently indicate its position at any given time t.

Equations (74) are called the equations of motion of a particle in rectangular co-ordinates. With these equations we can find the equation of the path followed by the particle. For this purpose it is necessary to eliminate the parameter t (time) from Eqs. (74) and find the relation between the co-ordinates of the particle

$$y = f(x). \tag{75}$$

The motion of a particle in space is accordingly defined by three equations

$$\left.\begin{aligned} x &= f_1(t), \\ y &= f_2(t), \\ z &= f_3(t). \end{aligned}\right\} \tag{76}$$

Example 34. Given the equations of motion of a particle

$$x = at^2, \tag{a}$$

$$y = bt, \tag{b}$$

find the equation of its path.

Solution. Eliminate the parameter t from the equations. From equation (b) we have

$$t=\frac{y}{b},$$

then

$$x=a\frac{y^2}{b^2},$$

whence

$$ay^2-b^2x=0.$$

This equation defines the path as a parabola.

51. Velocity of a Particle

The determination of the velocity of a particle in curvilinear motion depends on whether the natural or co-ordinate method is used to specify the motion.

Below we give some basic definitions which will be used in the later discussion. *If a particle travels equal distances in equal time intervals, its motion is said to be uniform.*

The velocity of a particle in uniform motion is defined as the quotient of the distance travelled by the particle in a certain time interval and this time interval

$$v=\frac{S}{t}, \tag{77}$$

where v is the velocity, S the distance travelled, t the time.

Formula (77) shows that velocity is measured in units of length divided by units of time—m/sec, cm/sec, km/hr, etc.

Sometimes it is necessary to transform one unit into another, mostly km/hr into m/sec or vice versa. This can easily be done noting that 1 km = 1,000 m, 1 hr = 3,600 sec and consequently 1 km/hr = 1,000 m/hr = $\frac{1,000}{3,600}$ m/sec = = $\frac{10}{36}$ m/sec = 0.278 m/sec. When transforming m/sec into km/hr it should be remembered that 1 m/sec = 3,600 m/hr = = 3.6 km/hr.

When deriving formulas and getting acquainted with new physical quantities one should always examine their dimensions.

From Eq. (77) the distance travelled is

$$S = vt, \tag{78}$$

and the time

$$t = \frac{S}{v}. \tag{79}$$

If a particle travels unequal distances in equal time intervals, its motion is said to be non-uniform.

From this definition it is clear that the velocity of a particle in non-uniform motion is a variable quantity and is a function of time

$$v = f(t). \tag{80}$$

It is often necessary to determine the *average velocity* of a particle in non-uniform motion over a certain time interval, i.e., *the velocity of the particle in such uniform motion in which the distance travelled by the particle during a definite time interval is the same as in the non-uniform motion.*

Let S be the distance travelled by the particle in non-uniform motion and t the time in which the particle travels this distance. The average velocity is determined from formula (77)

$$v_{av} = \frac{S}{t}.$$

Let us now see how the velocity is determined when the motion of a particle is prescribed by the natural method.

Let a particle A move along a given path according to a certain law $S = f(t)$ (Fig. 93*a*). Over an interval of time Δt, the particle A will have moved to a position A_1 along an arc AA_1 whose length is denoted by ΔS.

If the arc AA_1 is replaced by its chord, the average velocity of the particle can be found to a first approximation from the formula

$$v_{av} = \frac{AA_1}{\Delta t} = \frac{\Delta S}{\Delta t}.$$

This velocity is directed along the chord from point A to point A_1. The true velocity is found by passing to the limit as $\Delta t \to 0$; taking into account that the chord AA_1 approaches the arc, we obtain

$$v=\lim_{\Delta t\to 0}\frac{AA_1}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta S}{\Delta t}=\frac{dS}{dt}. \qquad (81)$$

When $\Delta t\to 0$, the direction of the chord coincides in the limit with the direction of the tangent to the path at point A.

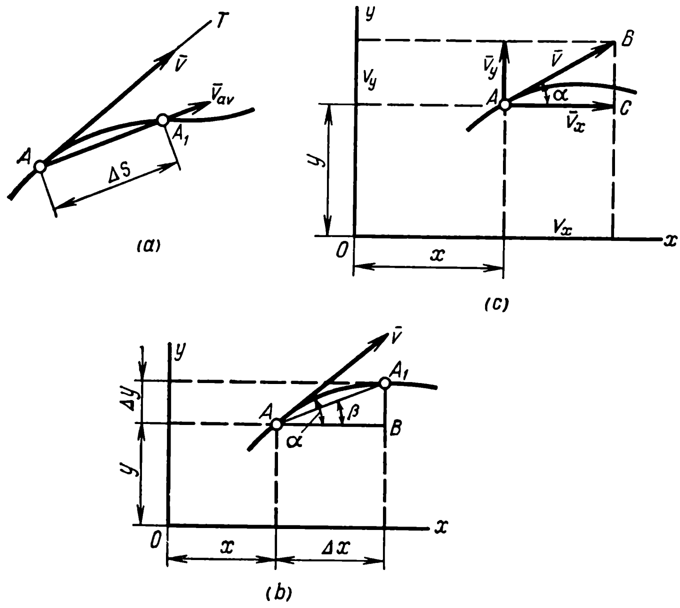


Fig. 93.

Thus, *the magnitude of the velocity of a particle is determined as the first derivative of the distance with respect to time and its direction coincides with the tangent to the path at a given point.*

When the motion of a particle is prescribed by the co-ordinate method, its velocity can be determined in terms of the projections on the co-ordinate axes.

The distance travelled by the particle during the time interval Δt is the arc AA_1 whose length is denoted by ΔS. We project the chord of this arc on the co-ordinate axes

(Fig. 93*b*)

$$\Delta x = AB = AA_1 \cos\beta,$$
$$\Delta y = BA_1 = AA_1 \sin\beta,$$

where AA_1 is the chord.

Divide both sides of these equalities by the time, Δt

$$\frac{\Delta x}{\Delta t} = \frac{AA_1}{\Delta t}\cos\beta,$$
$$\frac{\Delta y}{\Delta t} = \frac{AA_1}{\Delta t}\sin\beta.$$

Passing to the limit as Δt tends to zero and taking into account that in the limit the chord AA_1 approaches the arc AA_1 and the angle β approaches the angle α, we obtain

$$\left.\begin{aligned} \frac{dx}{dt} &= \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0}\left[\frac{AA_1}{\Delta t}\cos\beta\right] = \\ &= \frac{dS}{dt}\cos\alpha = v\cos\alpha, \\ \frac{dy}{dt} &= \lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0}\left[\frac{AA_1}{\Delta t}\sin\beta\right] = \\ &= \frac{dS}{dt}\sin\alpha = v\sin\alpha. \end{aligned}\right\} \tag{82}$$

From Fig. 93*c* it follows that $v\cos\alpha$ is the projection of the velocity on the axis of abscissas and $v\sin\alpha$ on the axis of ordinates; therefore we have finally

$$\left.\begin{aligned} \frac{dx}{dt} &= v\cos\alpha = v_x, \\ \frac{dy}{dt} &= v\sin\alpha = v_y. \end{aligned}\right\} \tag{83}$$

Consequently, *when the motion of a particle is prescribed by the co-ordinate method the projections of its velocity on the rectangular co-ordinate axes are determined as the first derivatives of the corresponding co-ordinates of the particle with respect to time.*

The magnitude and direction of the velocity are determined from the right triangle ABC (Fig. 93c)

$$\left.\begin{aligned} v &= \sqrt{v_x^2+v_y^2}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}, \\ \cos\angle(\bar{v}, x) &= \frac{v_x}{v}=\frac{v_x}{\sqrt{v_x^2+v_y^2}}, \\ \cos\angle(\bar{v}, y) &= \frac{v_y}{v}=\frac{v_y}{\sqrt{v_x^2+v_y^2}}. \end{aligned}\right\} \qquad (84)$$

The velocity of a particle can be resolved in a plane into two components in the direction of the co-ordinate axes, $\bar{v}_x$ and $\bar{v}_y$ (Fig. 93c). The magnitudes of the components are numerically equal to the projections of the velocity on the co-ordinate axes.

Using the relations derived above it is possible to pass over from the co-ordinate to the natural method of prescribing the motion of a particle. The value of the velocity can be found from the formula

$$v=\sqrt{v_x^2+v_y^2}.$$

Substituting the value of v in formula (81) $v=\frac{dS}{dt}$ and separating the variables, we have

$$dS = v\,dt. \qquad (85)$$

By integrating this differential equation, we obtain the natural equation of motion

$$S=f(t)+S_0, \qquad (86)$$

where S_0 is a constant of integration which characterizes the initial position of the particle on its path, i.e., the distance of the particle from the reference point at the beginning of the motion.

Example 35. The motion of a particle is defined by the equations

$$x = 5\cos 2t,$$

$$y = 5\sin 2t$$

(x and y in cm, t in sec).

Determine the path of the particle and its law of motion along this path measuring the distance travelled from the initial position of the particle.

Solution. To determine the path, eliminate the time t from the given equations. Solve the equations for the trigonometric functions

$$\cos 2t = \frac{x}{5},$$

$$\sin 2t = \frac{y}{5}.$$

Square and add these expressions

$$\cos^2 2t + \sin^2 2t = \frac{x^2}{25} + \frac{y^2}{25} = 1.$$

The equation of the path followed by the particle defines a circle of radius $r = 5$ cm.

Determine the projections of the velocity

$$v_x = -10 \sin 2t,$$

$$v_y = 10 \cos 2t.$$

Calculate the magnitude of the total velocity

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-10 \sin 2t)^2 + (10 \cos 2t)^2} =$$

$$= 10 \sqrt{\sin^2 2t + \cos^2 2t} = 10 \text{ cm/sec}.$$

Substitute the value of the velocity in formula (85)

$$dS = v\, dt = 10 dt.$$

Integrating this expression, we obtain

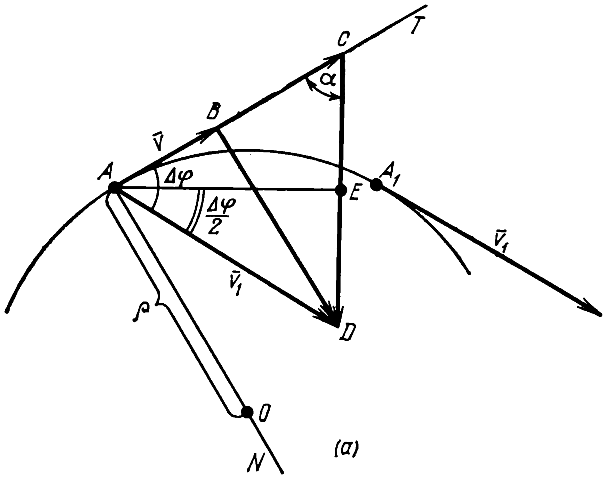
$$S = 10t + S_0;$$

$S_0 = 0$ since the distance travelled is measured from the initial position of the particle according to the condition of the problem.

52. Acceleration of a Particle

In the general case of motion of a particle along a curved path its velocity changes both in direction and magnitude. *The change of velocity per unit time is characterized by acceleration.*

Let a particle A move along a given curved path from position A to position A_1 in a time Δt. The distance travelled by the particle is the arc AA_1, whose length is denoted by ΔS



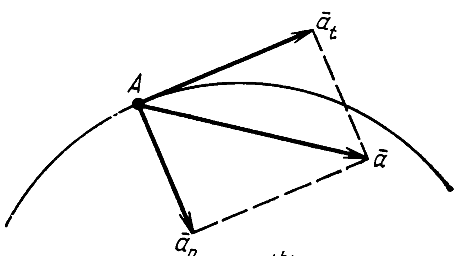


Fig. 94.

(Fig. 94a). In the position A the particle had a velocity $\bar{v}$ and in the position A_1, $\bar{v}_1$. The geometric difference of the velocities is found by constructing the vector $\bar{v}_1$ from point A. In Fig. 94 the increment of the velocity is represented by the vector $\overline{BD}$. Lay off in the direction of the velocity $\bar{v}$ a vector $\overline{AC}$ equal in magnitude to the vector $\bar{v}_1$, i.e., $AC =$

$= v_1$. Join point C and point D by a vector $\overline{CD}$. It is convenient to represent the velocity increment vector $\overline{BD}$ as the geometric sum of the vectors $\overline{BC}$ and $\overline{CD}$, i.e., $\overline{BD} = \overline{BC} + \overline{CD}$.

The average acceleration is found by dividing the velocity increment vector by Δt

$$\bar{a}_{av} = \frac{\overline{BD}}{\Delta t} = \frac{\overline{BC}}{\Delta t} + \frac{\overline{CD}}{\Delta t}. \tag{87}$$

The true acceleration is determined by passing to the limit as $\Delta t \to 0$

$$\bar{a} = \lim_{\Delta t \to 0} \frac{\overline{BD}}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\overline{BC}}{\Delta t} + \frac{\overline{CD}}{\Delta t} \right) = \lim_{\Delta t \to 0} \frac{\overline{BC}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\overline{CD}}{\Delta t}. \tag{88}$$

As follows from the construction, the vector $\overline{BC}$ is equal in magnitude to the algebraic difference between the initial and final velocities, i.e.,

$$BC = v_1 - v = \Delta v.$$

Denoting the angle between the velocity vectors at point A by $\Delta\varphi$, we find the side CD from the isosceles triangle ACD. For this purpose we drop an altitude AE (from the vertex A upon CD), which is also a median and bisectrix in an isosceles triangle. Taking this into account, we obtain

$$CD = 2ED = 2AD \sin \frac{\Delta\varphi}{2} = 2v_1 \sin \frac{\Delta\varphi}{2},$$

since $AD = v_1$.

Consider each of the two terms appearing in the expression for the total acceleration. Determine the magnitude of the first term

$$\lim_{\Delta t \to 0} \frac{BC}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}. \tag{89}$$

Thus, the first term is equal in magnitude to the first derivative of the velocity with respect to time. As regards the direction of the vector $\overline{BC}$, it coincides with the direction of the tangent. Thus, the first term of the total acceleration characterizes the change in magnitude of the velocity and is directed along the tangent to the path. It is called the *tangential component of acceleration*

$$a_t = \frac{dv}{dt}. \tag{90}$$

Determine the magnitude of the second term

$$\lim_{\Delta t \to 0} \frac{CD}{\Delta t} = \lim_{\Delta t \to 0} \frac{2v_1 \sin \frac{\Delta\varphi}{2}}{\Delta t}. \tag{91}$$

Multiply and divide the expression under the limit sign by ΔS and $\Delta\varphi$ (ΔS is the distance travelled during the time Δt, $\Delta\varphi$ the angle between the velocity vectors) and arrange the terms as shown below

$$\lim_{\Delta t \to 0} \frac{CD}{\Delta t} = \lim_{\Delta t \to 0} \frac{2v_1 \sin \frac{\Delta\varphi}{2} \Delta S \, \Delta\varphi}{\Delta t \, \Delta S \, \Delta\varphi} = \lim_{\Delta t \to 0} v_1 \frac{\sin \frac{\Delta\varphi}{2}}{\frac{\Delta\varphi}{2}} \frac{\Delta S \, \Delta\varphi}{\Delta t \, \Delta S} =$$

$$= \lim_{\Delta t \to 0} v_1 \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} \lim_{\Delta t \to 0} \frac{\sin \frac{\Delta\varphi}{2}}{\frac{\Delta\varphi}{2}} \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta S}.$$

Find the values of the four limits involved in the last formula.

The first limit

$$\lim_{\Delta t \to 0} v_1 = v.$$

The second limit

$$\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \frac{dS}{dt} = v.$$

The third limit

$$\lim_{\Delta t \to 0} \frac{\sin \frac{\Delta\varphi}{2}}{\frac{\Delta\varphi}{2}} = 1.$$

This is the limit of equivalent infinitesimals known from mathematics.

The fourth limit determines the curvature of the path

$$\lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta S} = \frac{1}{\rho},$$

where ρ is the radius of curvature of the path at a given point.

Substituting the values of the four limits, we obtain

$$\lim_{\Delta t \to 0} \frac{CD}{\Delta t} = v \times v \times 1 \times \frac{1}{\rho} = \frac{v^2}{\rho}. \qquad (92)$$

Determine the limiting direction of the vector $\overline{CD}$ as $\Delta t \to 0$. This vector makes an angle α with the tangent or the velocity (see Fig. 94*a*).

From the right triangle *ACE*

$$\alpha = 90° - \frac{\Delta\varphi}{2}.$$

As $\Delta t \to 0$, $\Delta\varphi \to 0$, consequently, $\alpha \to 90°$. Thus, the limiting position (as $\Delta t \to 0$) of the vector $\overline{CD}$ is perpendicular to the velocity, i.e., it is coincident with the normal and directed toward the inside of the path. The second term of the total acceleration, $\lim_{\Delta t \to 0} \frac{\overline{CD}}{\Delta t}$, determines the change in direction of the velocity and is called the *normal component of acceleration*

$$a_n = \frac{v^2}{\rho}. \qquad (93)$$

The total acceleration of the particle is determined as the geometric sum of the tangential and normal components (Fig. 94*b*)

$$\overline{a} = \overline{a}_t + \overline{a}_n. \qquad (94)$$

Since the components $\overline{a}_n$ and $\overline{a}_t$ are mutually perpendicular, the magnitude of the total acceleration can be found by applying the Pythagorean theorem

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}. \qquad (95)$$

If the motion of a particle is prescribed by the co-ordinate method, its total acceleration can be determined in terms of the projections on the co-ordinate axes. Without repeating the theoretical arguments used in the determination of the velocity in terms of its projections, we arrive at the following.

The projections of the acceleration of a particle on the co-ordinate axes are equal to the second derivatives of the corresponding co-ordinates with respect to time

$$a_x = \frac{d^2x}{dt^2}, \quad a_y = \frac{d^2y}{dt^2}. \tag{96}$$

The total acceleration can be found from its given pro-

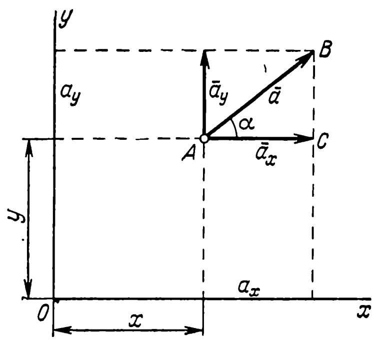


Fig. 95.

jections. From the right triangle *ABC* (Fig. 95) we have

$$a = \sqrt{a_x^2 + a_y^2}. \tag{97}$$

The total acceleration vector can be resolved into components along the co-ordinate axes

$$\bar{a} = \bar{a}_x + \bar{a}_y. \tag{98}$$

The direction of the total acceleration vector can be determined from the cosine of the angle that it makes with some axis, say, the x axis

$$\cos\alpha = \cos\angle(\bar{a}, \bar{x}) = \frac{a_x}{a}. \tag{99}$$

Example 36. The motion of a particle is defined by the equations

$$x = 2t - \sin 2t,$$

$$y = 1 - \cos 2t$$

(x and y in m, t in sec).

Determine the tangential and normal components of acceleration of the particle at $t=\frac{\pi}{2}$ sec.

Solution. Find the projections of the velocity on the co-ordinate axes

$$v_x=\frac{dx}{dt}=2-2\cos 2t,$$

$$v_y=\frac{dy}{dt}=+2\sin 2t.$$

Calculate the velocity

$$v=\sqrt{v_x^2+v_y^2}=\sqrt{(2-2\cos 2t)^2+(2\sin 2t)^2}=$$
$$=\sqrt{4-8\cos 2t+4(\sin^2 2t+\cos^2 2t)}=2\sqrt{2(1-\cos 2t)}=$$
$$=2\sqrt{2\times 2\sin^2 t}=4\sin t.$$

Find the projections of the acceleration on the co-ordinate axes

$$a_x=\frac{d^2x}{dt^2}=4\sin 2t,$$

$$a_y=\frac{d^2y}{dt^2}=4\cos 2t.$$

The total acceleration is

$$a=\sqrt{a_x^2+a_y^2}=4\sqrt{\sin^2 2t+\cos^2 2t}=4 \text{ m/sec}^2.$$

The tangential component of acceleration is determined as the first derivative of the velocity

$$a_t=\frac{dv}{dt}=4\cos t.$$

The total acceleration of the particle is determined in terms of the normal and tangential components by the formula

$$a=\sqrt{a_n^2+a_t^2},$$

whence we can find the normal component of acceleration

$$a_n=\sqrt{a^2-a_t^2}=\sqrt{4^2-(4\cos t)^2}=4\sqrt{1-\cos^2 t}=4\sin t.$$

At $t=\frac{\pi}{2}$

$$a_t = 4\cos\frac{\pi}{2} = 0,$$

$$a_n = 4\sin\frac{\pi}{2} = 4 \text{ m/sec}^2.$$

53. Types of Motion of a Particle as Related to Acceleration

The nature of motion of a particle and the shape of its path are eventually defined by its acceleration. Consider all possible cases of motion of a particle and analyse the formulas derived above for the tangential and normal components of acceleration.

Uniform Rectilinear Motion. A uniform rectilinear motion is characterized by the fact that the velocity of a particle *A* is constant and the radius of curvature of the path followed by the particle *A* is infinite (Fig. 96*a*).

In this case the tangential component of acceleration is

$$a_t = \frac{dv}{dt} = 0 \ \ (v = \text{constant}),$$

the normal component of acceleration

$$a_n = \frac{v^2}{r} = 0 \ \ (r = \infty).$$

Hence the total acceleration of the particle is also zero

$$\bar{a} = 0.$$

Uniform Curvilinear Motion. A uniform curvilinear motion is characterized by the fact that the magnitude of the velocity of a particle *A* is constant, the velocity changes only in direction and the radii of curvature of the path followed by the particle have finite values.

The velocity of the particle *A* is directed along the tangent to the path at any given instant (Fig. 96*b*). In this case the tangential component of acceleration is

$$a_t = \frac{dv}{dt} = 0 \ \ (v = \text{constant}),$$

the normal component of acceleration

$$a_n = \frac{v^2}{r} \neq 0$$

(r is a finite quantity).

Thus, *the total acceleration of a particle in uniform curvilinear motion consists only of the normal component of acceleration*, i.e.,

$$\bar{a} = \bar{a}_n.$$

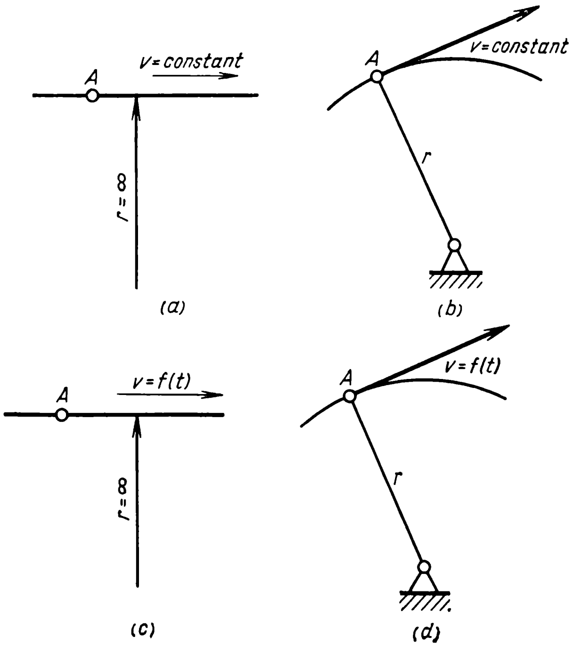


Fig. 96.

Non-Uniform Rectilinear Motion. A non-uniform rectilinear motion is characterized by the fact that the velocity of a particle A is a function of time (Fig. 96c) and the radius of curvature of the path followed by the particle A is infinite. Therefore the tangential component of acceleration is

$$a_t = \frac{dv}{dt} \neq 0,$$

the normal component of acceleration

$$a_n = \frac{v^2}{r} = 0 \quad (r = \infty).$$

Thus, *the total acceleration of a particle in non-uniform rectilinear motion consists only of the tangential component of acceleration*, i.e.,

$$\bar{a} = \bar{a}_t.$$

Non-Uniform Curvilinear Motion. A non-uniform curvilinear motion (Fig. 96*d*) is characterized by the fact that the velocity of a particle A is a function of time and the radius of curvature of its path is a finite quantity. In this case the tangential component of acceleration is

$$a_t = \frac{dv}{dt} \neq 0,$$

the normal component of acceleration

$$a_n = \frac{v^2}{r} \neq 0$$

(r is a finite quantity).

Thus, *the total acceleration of a particle in non-uniform curvilinear motion is geometrically composed of the tangential and normal components of acceleration*, i.e.,

$$\bar{a} = \bar{a}_n + \bar{a}_t.$$

54. Uniformly Variable Motion of a Particle

When the magnitude of the tangential component of acceleration is constant, a_t = constant, the motion of a particle is referred to as *uniformly variable*. A uniformly variable motion can be either uniformly accelerated or uniformly decelerated (retarded) according as the absolute value of the velocity increases or decreases.

Write the formula for determining the tangential component of acceleration

$$a_t = \frac{dv}{dt}.$$

Separate the variables and write this formula as the differential equation

$$dv = a_t\,dt.$$

Integrate this equation

$$v = \int a_t\,dt.$$

Since the tangential component of the acceleration is constant, $a_t = \text{constant}$, we put it before the integral sign and, carrying out the integration, we obtain

$$v = a_t \int dt = a_t t + v_0, \tag{100}$$

where v_0 is a constant of integration which characterizes the initial velocity of the particle.

In uniformly accelerated motion the acceleration a_t is considered as positive and in uniformly decelerated motion as negative.

Determine the tangential component of the acceleration from Eq. (100)

$$a_t = \frac{v - v_0}{t}.$$

When the initial velocity is zero, $v_0 = 0$, the formula becomes

$$a_t = \frac{v - 0}{t} = \frac{v}{t}.$$

Taking into account that the velocity is the first time derivative of the distance, Eq. (81) can be represented as

$$v = \frac{dS}{dt} = a_t t + v_0.$$

Separate the variables

$$dS = (a_t t + v_0)\,dt.$$

Integrating this equation and noting that $a_t = \text{constant}$ and $v_0 = \text{constant}$, we obtain a formula for determining the distance of a particle from the reference point in uniformly variable motion for any time t

$$S = \frac{a_t t^2}{2} + v_0 t + S_0, \tag{101}$$

where S_0 is a constant of integration which characterizes the initial position of the particle.

Choosing the initial position of the particle as the reference point, it is generally taken that $S_0 = 0$ and $S = v_0t + \frac{a_tt^2}{2}$.

After simple algebraic transformations of formula (101) we obtain

$$S = \frac{t}{2}[v_0 + (v_0 + a_tt)].$$

Taking into account that $v_0 + a_tt = v$ [see formula (100)], we reduce the expression for determining the distance travelled by the particle to

$$S = \frac{v_0 + v}{2}t. \qquad (102)$$

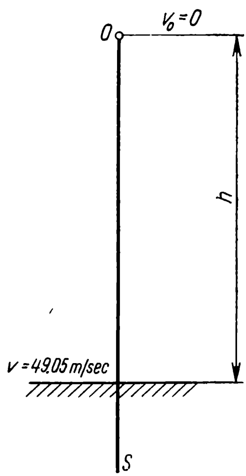


Fig. 97.

An example of uniformly accelerated motion is provided by a freely falling body. The acceleration of a freely falling body is denoted by g. Experiments show that the average value of this acceleration is 9.81 m/sec^2 at the surface of the earth.

Example 37. Determine the height h from which a heavy body should be dropped with no initial velocity if its velocity is to be $v = 49.05$ m/sec when it strikes the ground. Neglect air resistance.

Solution. We choose as a reference point the point coincident with the highest position of the body and direct the axis of distances downward (Fig. 97). Since the initial velocity is zero, $v_0 = 0$, the equations of uniformly accelerated motion are in this case

$$S = h = \frac{gt^2}{2}$$

and

$$v = gt.$$

From the second equation we find the time required for the falling body to attain the velocity $v = 49.05$ m/sec

$$t = \frac{v}{g} = \frac{49.05}{9.81} = 5 \text{ sec}.$$

Substituting the time in the first equation, we obtain.

$$h = \frac{gt^2}{2} = \frac{9.81 \times 5^2}{2} = 122.6 \text{ m}.$$

Example 38. Moving along a circular arc of radius $R = 1{,}000$ m, a train is slowing down at a constant rate. At the beginning of the motion along the arc its speed was $v_0 = 54$ km/hr.

After the train had travelled a distance $S = 500$ m, its speed decreased to $v = 36$ km/hr. Determine the total acceleration at the beginning and at the end of the motion.

Solution. The motion is prescribed by the natural method. From the equations for the distance travelled and the speed determine the magnitude of the tangential component of acceleration.

It is known that

$$v = v_0 + a_t t,$$

$$S = v_0 t - \frac{a_t t^2}{2} = \frac{v_0 + v}{2} t.$$

From the second equation

$$t = \frac{2S}{v_0 + v} = \frac{2 \times 500}{\frac{54{,}000}{3{,}600} + \frac{36{,}000}{3{,}600}} = \frac{1{,}000}{15 + 10} = 40 \text{ sec}.$$

From the first equation

$$a_t = \frac{v - v_0}{t} = \frac{10 - 15}{40} = -\frac{5}{40} = -0.125 \text{ m/sec}^2.$$

The tangential component of the acceleration is constant throughout the motion and negative as the motion is uniformly decelerated.

Calculate the normal component of acceleration:
at the beginning of the motion

$$a_n = \frac{v_0^2}{R} = \frac{15^2}{1{,}000} = 0.225 \text{ m/sec}^2,$$

at the end of the motion

$$a_n = \frac{v^2}{R} = \frac{10^2}{1{,}000} = 0.1 \text{ m/sec}^2.$$

Calculate the magnitude of the total acceleration:
at the beginning of the motion

$$a=\sqrt{a_t^2+a_n^2}=\sqrt{0.125^2+0.225^2}=0.258 \text{ m/sec}^2,$$

at the end of the motion

$$a=\sqrt{a_t^2+a_n^2}=\sqrt{0.125^2+0.1^2}=0.16 \text{ m/sec}^2.$$

55. Kinematic Graphs and Relationship Between Them

To visualize the motion of a particle use is often made of graphs showing the distance travelled, velocity and acceleration as functions of time in a system of rectangular axes.

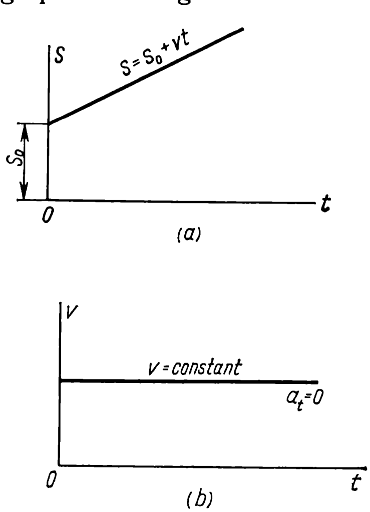


Fig. 98.

Consider kinematic graphs for uniform motion. Whether it is rectilinear or curvilinear, we have the following equations

$$S=S_0+vt, \quad v=\text{constant}.$$

From these equations it follows that the graph showing the distance travelled in uniform motion is a straight line cutting off S_0 on the axis of ordinates (axis of distance travelled), i.e., the distance of the particle from the reference point at the beginning of the motion (Fig. 98*a*).

The velocity graph is represented by a straight line parallel to the axis of abscissas since the velocity of a particle in uniform motion is a constant, $v=$ constant (Fig. 98*b*).

Consider kinematic graphs for uniformly variable motion. Whether it is rectilinear or curvilinear, the following equations derived in Sec. 54 are valid

$$v=v_0+a_t t,$$

$$S=S_0+v_0 t+\frac{a_t t^2}{2}.$$

The graph showing the distance travelled in uniformly variable motion is curved (parabolic) as it corresponds to the equation of a parabola (Fig. 99*a* and *b*). These graphs cut off S_0 on the axis of ordinates (S axis) at $t = 0$, i.e., the

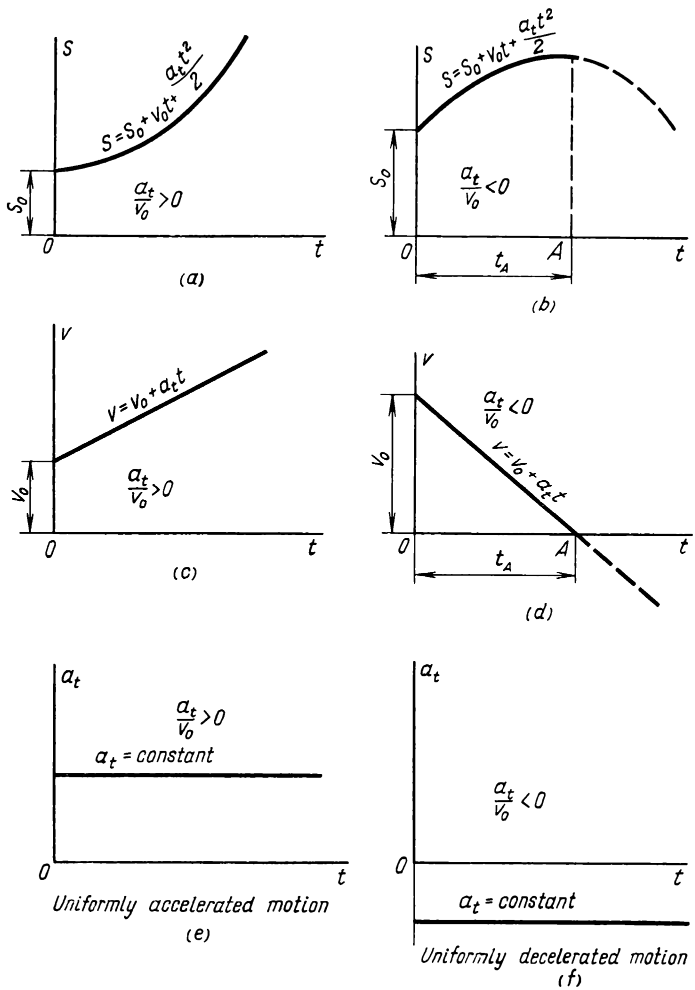


Fig. 99.

distance of the particle from the reference point at the beginning of the motion.

The velocity graph is represented by a straight line inclined to the axis of abscissas (Fig. 99*c* and *d*) and cuts off the initial velocity v_0 on the axis of ordinates (at $t = 0$).

The graph of acceleration in uniformly variable motion is represented by a straight line parallel to the axis of abscissas (time axis) (Fig. 99*e* and *f*).

In uniformly accelerated motion the acceleration graph is placed above the axis of abscissas, in uniformly decelerated motion, below it (Fig. 99*e* and *f*).

From the foregoing graphs it follows that in uniformly accelerated motion the magnitude of the velocity increases, i.e., the tangential component of acceleration $\bar{a}_t$ has the same sense as the initial velocity $\bar{v}_0$, the algebraic signs of v_0 and a_t are the same. Consequently, the ratio of the tangential component of acceleration to the initial velocity in uniformly accelerated motion is always positive, $\frac{a_t}{v_0} > 0$.

In uniformly decelerated motion the tangential component of acceleration is opposite in sign to the initial velocity, i.e., the ratio of the tangential component of acceleration to the initial velocity is always negative, $\frac{a_t}{v_0} < 0$. In uniformly decelerated motion the magnitude of the velocity diminishes. This is evident from Fig. 99*d*. The velocity may, in some cases, decrease to a zero value (point A in Fig. 99*d*). Subsequently the velocity changes sign and begins to increase in absolute value. Here a uniformly decelerated motion is actually transformed into a uniformly accelerated motion. Precisely this takes place in the case represented in Fig. 99*b* and *d* at $t = t_A$, i.e., when the velocity changes sign.

We note a definite relationship between kinematic graphs. Thus, for uniform motion the velocity graph is represented by a straight line parallel to the axis of abscissas and the distance graph is represented by a sloping straight line. For uniformly variable motion the acceleration graph is a straight line parallel to the axis of abscissas, the velocity graph a sloping straight line and the distance graph a parabolic curve. This correlation of the graphs follows directly from the differential expressions relating acceleration, velocity and distance

$$v = \frac{dS}{dt}, \quad a_t = \frac{dv}{dt} = \frac{d^2S}{dt^2}.$$

Example 39. Two buses started simultaneously toward each other from towns A and B which are 480 km apart.

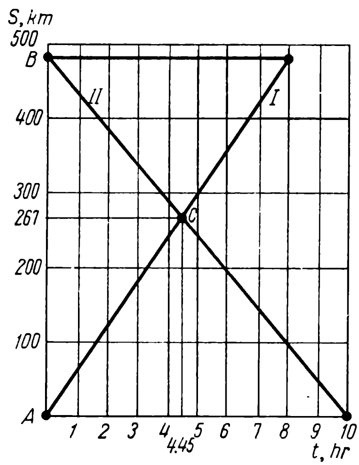


Fig. 100.

It took the first bus travelling from A to B eight hours to cover the distance, the second bus travelling from B to A was on the way for ten hours.

Determine how soon the buses met after starting and at what distance from town A.

Solution. This problem can most easily be solved by plotting motion graphs for the two buses (Fig. 100). The reference point is chosen at A. Assuming that the buses travel with constant speed, we plot distance graphs as follows.

For the first bus

$$\text{at } t=0, \quad S_1=0;$$

$$\text{at } t=8 \text{ hr}, \quad S_1=480 \text{ km}.$$

For the second bus

$$\text{at } t = 0, \quad S_2 = 480 \text{ km};$$

$$\text{at } t = 10 \text{ hr}, \quad S_2 = 0.$$

With two points we can easily plot straight lines *I* and *II* (Fig. 100).

The co-ordinates of the point of intersection C of these lines define the location and time of meeting. Find these co-ordinates from the graph

$$t_C = 4.45 \text{ hr}, \quad S_C = 267 \text{ km}.$$

Consequently, the buses will meet 4 hr 27 min after starting, at a distance of 267 km from town A.

CHAPTER X

Simple Motions of Rigid Bodies

56. Translation of a Rigid Body

The simplest motions of a rigid body are translation and rotation about a fixed axis.

Translation is a motion of a rigid body in which any straight line drawn on the body remains parallel to its initial position.

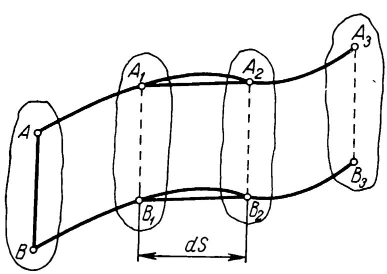


Fig. 101.

Consider a body (Fig. 101) in translation. A straight line AB drawn on the body moves parallel to its initial position during the motion. Consider the displacement of the body during a short interval of time dt. It may be assumed that the points A and B move along parallel straight lines. They travel the same distance dS in the time dt. Consequently, the velocities of these points are equal in magnitude

$$v_A = v_B = \ldots = \frac{dS}{dt} = v$$

and have the same sense, i.e.,

$$\overline{v}_A = \overline{v}_B = \ldots = \overline{v}. \qquad (103)$$

It may similarly be proved that the accelerations of the points of a body in translation are equal

$$\overline{a}_A = \overline{a}_B = \ldots = \overline{a}. \qquad (104)$$

Therefore, translation of a body is completely defined by the motion of any one of its points which may be prescribed either by the co-ordinate or the natural method. However, only a rigid body, and not a single particle, can have a motion of translation.

Examples of translation are the motion of a piston of a steam engine, the motion of a car along a straight section of track, etc.

Translation may be either rectilinear or curvilinear. In rectilinear translation, all points of a body describe equal and parallel straight lines. In curvilinear translation, all points of a body describe similar curves.

The identity of the paths of all points of a body in translation follows directly from its definition.

57. Rotation of a Rigid Body About a Fixed Axis

A rotary motion (or rotation) of a body about a fixed axis is a motion in which all points of the body located on the axis of rotation remain fixed. All the other points of the rotating body describe circles about the fixed axis in planes perpendicular to the axis with centres on the axis, the velocity of each point being proportional to its distance from the centre of rotation.

The motion of rotation is of great practical importance. Its laws govern the motion of pulleys, flywheels, gear wheels, circular saws, etc.

Consider a body rotating about an axis Oy (Fig. 102). Suppose that the body has rotated through an angle φ and occupied a position A_1A_1 in a time interval t.

A plane of the rotating body passing through the axis Oy and initially coincident with the fixed plane A_0A_0 (from which the angles of rotation are measured) makes the angle φ with it after the time t. The angle φ is called the angle of rotation or the angular co-ordinate of a body, it may be positive or negative according to the sense of rotation. The angle of rotation is measured in radians or in dimensionless units as the radian = 1. To a given angle of rotation φ corresponds a definite position of the body. Therefore, the equation defining the position of a rotating body at any given instant is an equation expressing the angle of rotation as a function of time

$$\varphi = f(t). \qquad (105)$$

This equation shows that we can determine the angle of rotation φ and consequently the corresponding position of a rotating body at any instant.

As noted above, to the instant t corresponds the position A_1A_1 of the body. Over a time interval $(t + \Delta t)$, the body will have rotated through an angle $(\varphi + \Delta\varphi)$, as measured from the original position A_0A_0, into a position A_2A_2. This means that the angle of rotation of the body has received an increment $\Delta\varphi$ in the time interval Δt.

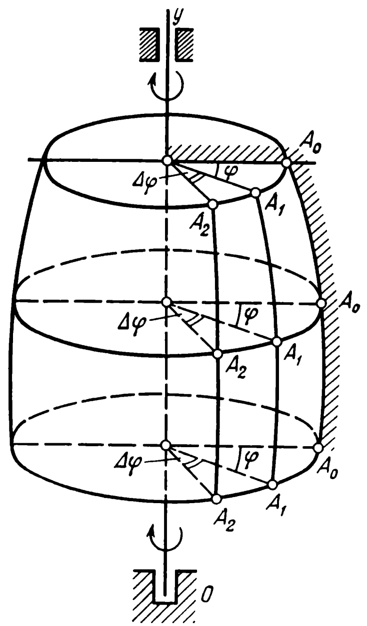


Fig. 102.

The average angular velocity of a rotating body over a time interval Δt is defined as the quotient of the increment of the angle of rotation $\Delta\varphi$ and the time Δt in which this increment occurs

$$\omega_{av} = \frac{\Delta\varphi}{\Delta t}. \qquad (106)$$

In the limit, as Δt approaches zero we obtain the angular velocity at a given instant

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{d\varphi}{dt}. \qquad (107)$$

Angular velocity is measured in radians per second

$$[\omega] = \left[\frac{\text{radians}}{\text{sec}}\right] = \left[\frac{1}{\text{sec}}\right].$$

The angular velocity of a body is equal to the first derivative of the angle of rotation with respect to time.

It is common practice to assign the number of revolutions per minute n rather than the angular velocity of a body. The angular velocity ω can be determined in terms of the number of revolutions per minute as follows.

The angle of rotation of a body during one revolution is

$$\varphi_1 = 2\pi.$$

If the rotating body makes n revolutions per minute, the angle of rotation per minute is

$$\varphi_2 = 2\pi n.$$

The angular velocity expressed in terms of the angle of rotation per unit time is given by the formula

$$\omega_{av} = \frac{\varphi_2}{60} = \frac{2\pi n}{60} = \frac{\pi n}{30}\left[\frac{\text{radians}}{\text{sec}}\right]. \quad (108)$$

When the angular velocity of a body is constant, ω = constant, we have a uniform rotation. Separate the variables in Eq. (107) $d\varphi = \omega\, dt$. Integrating this expression, we obtain the equation of uniform rotation

$$\varphi = \omega t + \varphi_0, \quad (109)$$

where φ_0 is a constant of integration which defines the initial position of the body.

When the angular velocity of a body is variable, $\omega = f(t)$, we have a non-uniform rotation. The change in angular velocity per unit time is characterized by angular acceleration. Suppose to a time t corresponds an angular velocity ω. During a time interval Δt, the angular velocity receives an increment $\Delta\omega$ and its value becomes $(\omega + \Delta\omega)$.

The average angular acceleration of a rotating body over a time interval Δt is defined as the quotient of the increment of the angular velocity $\Delta\omega$ and the time Δt in which this incre-

ment occurs

$$\alpha_{av} = \frac{\Delta\omega}{\Delta t}. \tag{110}$$

In the limit, as Δt approaches zero we obtain the angular acceleration at a given instant

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2}. \tag{111}$$

Angular acceleration is measured in radians per second per second

$$[\alpha] = \left[\frac{\text{radians}}{\text{sec}^2}\right] = \left[\frac{1}{\text{sec}^2}\right].$$

Thus, *the angular acceleration is equal to the first derivative of the angular velocity with respect to time or to the second derivative of the angle of rotation with respect to time.*

If a body rotates about an axis with a constant angular acceleration, α = constant, we have a *uniformly variable rotation.*

The equations of uniformly variable rotation are similar to those of uniformly variable motion of a particle, the only difference being that they involve angular instead of linear quantities. These equations are derived by integrating the corresponding differential relations.

The angle of rotation φ is given by the formula

$$\varphi = \frac{\alpha t^2}{2} + \omega_0 t + \varphi_0, \tag{112}$$

and the angular velocity at any instant is equal to

$$\omega = \omega_0 + \alpha t, \tag{113}$$

where φ_0 and ω_0 are constants of integration which define the angle of rotation and the angular velocity at the initial instant (at $t=0$). It is generally taken that $\varphi_0 = 0$, i.e., the body is assumed to start rotating from its initial position.

Angular acceleration α is an algebraic quantity. In uniformly accelerated rotation, it coincides in direction with the initial angular velocity ω_0. Therefore, the absolute value of the angular velocity always increases. In uniformly decelerated motion, the angular acceleration is opposite to the angular velocity and hence the absolute value of the latter decreases.

Example 40. A body starts rotating from rest with a constant angular acceleration and makes 7,200 revolutions in the first two minutes.

Determine the angular acceleration α.

Solution. We use the equation of uniformly variable rotation

$$\varphi = \omega_0 t + \frac{\alpha t^2}{2}.$$

Since the body starts rotating from rest, $\omega_0 = 0$ and $\varphi = \frac{\alpha t^2}{2}$, whence

$$\alpha = \frac{2\varphi}{t^2}.$$

Express the angle of rotation φ in radians (1 revolution $= 2\pi$ radians) and the time t in seconds, then

$$\alpha = \frac{2\pi \times 2 \times 7{,}200}{120^2} = 2\pi \frac{1}{\sec^2}.$$

Example 41. A flywheel rotates with an angular velocity corresponding to $n_1 = 300$ rpm. Then it slows down and its revolutions decrease to $n_2 = 120$ rpm in $t_1 = 3$ sec.

Determine the time required for the flywheel to come to rest assuming its motion to be uniformly decelerated. Find the number of revolutions it will make from the beginning of deceleration to a complete stop.

Solution. Write the equation of uniformly decelerated rotation for the angular velocity

$$\omega_2 = \omega_1 + \alpha t_1,$$

whence

$$\alpha = \frac{\omega_2 - \omega_1}{t_1}.$$

The angular velocities are

$$\omega_1 = \frac{\pi n_1}{30} = \frac{\pi \times 300}{30} = 10\pi \frac{1}{\sec},$$

$$\omega_2 = \frac{\pi n_2}{30} = \frac{\pi \times 120}{30} = 4\pi \frac{1}{\sec}.$$

Substituting the values of the angular velocities in the expression for angular acceleration, we obtain

$$\alpha = \frac{\omega_2 - \omega_1}{t_1} = \frac{4\pi - 10\pi}{3} = -2\pi \frac{1}{\sec^2}.$$

Knowing the angular acceleration (deceleration) α, we can calculate the time of rotation of the flywheel to a complete stop ($\omega = 0$)

$$\omega = \omega_1 + \alpha t_2,$$

whence

$$t_2 = \frac{\omega - \omega_1}{\alpha} = \frac{0 - 10\pi}{-2\pi} = 5 \text{ sec.}$$

The angle of rotation of the flywheel as it comes to rest is given by the formula

$$\varphi = \omega_1 t_2 + \frac{\alpha t_2^2}{2} = 10\pi \times 5 - \frac{2\pi \times 5^2}{2} = 25\pi \text{ radians.}$$

This corresponds to $\frac{25\pi}{2\pi} = 12.5$ revolutions of the flywheel.

58. Velocities and Accelerations of Points of a Rotating Body

The points of a body rotating about an axis move in circles (Fig. 103).

Consider a point A which travels a distance AA_1 in a time t. The arc AA_1 can here be determined as the product of

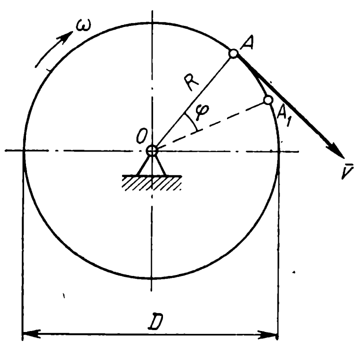


Fig. 103.

the angle of rotation and the radius of the circle, i.e.,

$$\text{arc } AA_1 = S = \varphi R.$$

As has been established above, the linear velocity of a point is equal to the first derivative of the displacement with respect to time

$$v = \frac{dS}{dt} = \frac{d(\varphi R)}{dt} = \frac{d\varphi}{dt} R,$$

where R is the distance of the point from the axis of rotation (a constant).

It is known that $\frac{d\varphi}{dt} = \omega$, hence

$$v = \omega R. \tag{114}$$

Substituting the angular velocity expressed in terms of the number of revolutions per minute in the formula for the linear velocity of the points of a body rotating about a fixed axis, we obtain

$$v = \frac{\pi n}{30} R. \tag{115a}$$

Taking into account that the radius of rotation R can be represented as half the diameter $R = \frac{D}{2}$ (Fig. 103), the formula for velocity can be written as

$$v = \frac{\pi n D}{60}. \tag{115b}$$

The tangential component of the acceleration of the point A moving in a circle is equal to the first derivative of the velocity with respect to time

$$a_t = \frac{dv}{dt} = \frac{d(\omega R)}{dt} = \frac{d\omega}{dt} R.$$

It is known that the angular acceleration is $\alpha = \frac{d\omega}{dt}$, then

$$a_t = \alpha R. \tag{116}$$

The normal component of the acceleration of the point A moving in a circle is equal to the ratio of the square of the velocity to the radius of the circle

$$a_n = \frac{v^2}{R} = \frac{(\omega R)^2}{R} = \frac{\omega^2 R^2}{R} = \omega^2 R. \tag{117}$$

The total acceleration of a point in non-uniform rotation about an axis (Fig. 104) is composed geometrically of the tangential and normal components of acceleration

$$\bar{a} = \bar{a}_n + \bar{a}_t.$$

Its magnitude is calculated as the diagonal of the rectangle

$$a = \sqrt{a_n^2 + a_t^2}.$$

Substituting the tangential and normal components of acceleration, we obtain

$$a = \sqrt{(\omega^2 R)^2 + (\alpha R)^2}$$

or

$$a = R\sqrt{\omega^4 + \alpha^2}. \tag{118}$$

The direction of the total acceleration vector of a point of a rotating body can be determined from the angle β that the vector makes with the radius.

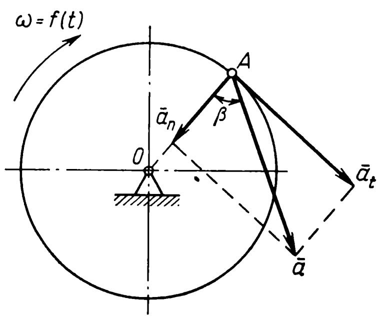


Fig. 104.

This angle is easily determined from its tangent

$$\tan \beta = \frac{a_t}{a_n} = \frac{\alpha R}{\omega^2 R} = \frac{\alpha}{\omega^2}. \tag{119}$$

Example 42. A point A (Fig. 105) located on the rim of a pulley moves with a velocity of 50 cm/sec and a point B moves with a velocity of 10 cm/sec; the distance $AB =$ $= 20$ cm.

Determine the angular velocity ω and the diameter of the pulley.

Solution. The angular velocity ω is the same for all points of the rotating pulley, consequently, the circumferential

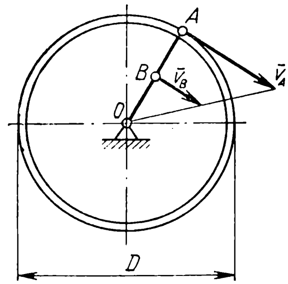


Fig. 105.

velocities $\bar{v}_A$ and $\bar{v}_B$ of points A and B are, respectively,

$$v_A = \omega\, OA = \omega\,(AB + OB) = \omega\,(20 + OB),$$

$$v_B = \omega\, OB,$$

then

$$\omega = \frac{v_A}{20 + OB} = \frac{v_B}{OB}.$$

Substituting the numerical values of v_A and v_B, we find OB

$$\frac{50}{20 + OB} = \frac{10}{OB},$$

$$OB = \frac{200}{40} = 5 \text{ cm}.$$

The diameter of the pulley is

$$D = 2\,AB + 2\,OB = 2 \times 20 + 2 \times 5 = 50 \text{ cm}.$$

The angular velocity is determined from the equation

$$\omega = \frac{v_B}{OB} = \frac{10}{5} = 2\,\frac{1}{\text{sec}}.$$

Example 43. The angle of inclination of the total acceleration of a point on the rim of a flywheel to its radius is 60° (Fig. 106). Its tangential component at the instant considered is $a_t = 10\sqrt{3}$ m/sec².

Find the normal component of the acceleration of a point B at a distance $r = 0.5$ m from the axis of rotation. The radius of the flywheel is $R = 1$ m.

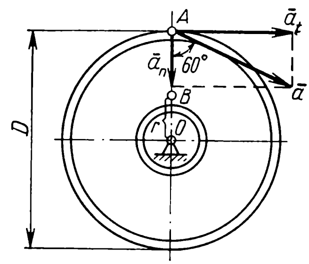


Fig. 106.

Solution. The normal component a_n of the acceleration of point A on the rim of the flywheel is

$$(a_n)_A = (a_t)_A \tan 30° = 10\sqrt{3}\,\frac{\sqrt{3}}{3} = 10 \text{ m/sec}^2.$$

The normal components of the accelerations of points A and B are defined by the formulas

$$(a_n)_A = \omega^2 R,$$

$$(a_n)_B = \omega^2 r.$$

Dividing the equations member by member and solving for $(a_n)_B$, we obtain

$$\frac{(a_n)_A}{(a_n)_B} = \frac{\omega^2 R}{\omega^2 r},$$

$$(a_n)_B = \frac{(a_n)_A\, r}{R} = \frac{10 \times 0.5}{1} = 5 \text{ m/sec}^2.$$

Example 44. A shaft A (Fig. 107) of radius $R = 10$ cm is set in rotation by a weight P suspended from it by a string. The weight is released with no initial velocity and moves with a constant acceleration $a = 200$ cm/sec^2.

Determine the angular velocity ω and the angular acceleration α of the shaft, also the total acceleration a of a point on the surface of the shaft at time t.

Solution. The velocity with which the weight moves down is equal to the velocity of a point on the periphery of the shaft

$$v = v_0 + at = 0 + at = 200t \text{ cm/sec}.$$

The angular velocity of the shaft is

$$\omega = \frac{v}{R} = \frac{200t}{10} = 20t \ \frac{1}{\text{sec}} .$$

The shaft starts rotating from rest, i.e., $\omega_0 = 0$, and acce-

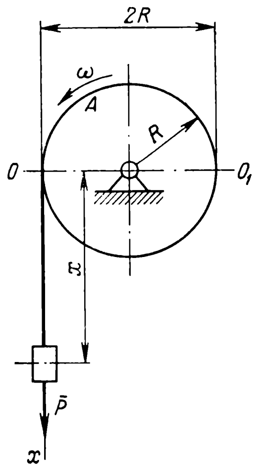


Fig. 107.

lerates uniformly as the weight moves down

$$\omega = \omega_0 + \alpha t = \alpha t,$$

whence

$$\alpha = \frac{\omega}{t} = \frac{20t}{t} = 20 \frac{1}{\text{sec}^2} .$$

The total acceleration of a point on the surface of the shaft is

$$a = R\sqrt{\omega^4 + \alpha^2} = 10\sqrt{(20t)^4 + (20)^2} =$$
$$= 200\sqrt{400t^4 + 1} \text{ cm/sec}^2.$$

The angle that the total acceleration of a point on the surface of the shaft makes with its radius can readily be determined by using formula (119)

$$\tan\beta = \frac{\alpha}{\omega^2}.$$

Substituting the values of the angular acceleration $\alpha = 20\,\frac{1}{\sec^2}$ and the angular velocity $\omega = 20t\,\frac{1}{\sec}$, we obtain

$$\tan\beta = \frac{\alpha}{\omega^2} = \frac{20}{400t^2}.$$

The angle β can easily be found from the known tangent for any time t.

CHAPTER XI

Methods of Transmission of Rotary Motion

59. Classification of Transmission Mechanisms

The transmission of energy from one machine to another or from one member to another in a machine is accomplished by means of various transmission mechanisms.

The types of drives may be classified according to various characteristics.

Classification of drives according to *the means for transmitting motion.*

1. Drives in which motion is effected by using frictional forces (friction drive, belt drive, rope drive).

2. Drives in which motion is accomplished by direct geometric engagement of one member of a kinematic pair with the other (gear drive, screw drive, worm drive, chain drive).

Classification of drives according to *the nature of transmission members between the driving and driven members.*

1. Motion is transmitted through direct contact between the driving and driven members (friction drive, gear drive, screw drive, worm drive).

2. Motion is transmitted through an intermediate link connecting the driving and driven members (belt drive, rope drive, chain drive, band transmission).

60. Gear Ratio

A shaft and transmission members (wheels or pulleys) mounted on it and transmitting a given torque are referred to as *driving*, while members receiving the torque and the motion from the driving members are termed *driven*.

Symbols pertaining to driving members of a direct drive (Fig. 108*a*) and a continuous drive (Fig. 108*b*) are assigned the subscript *1*, and for driven members, the subscript *2*.

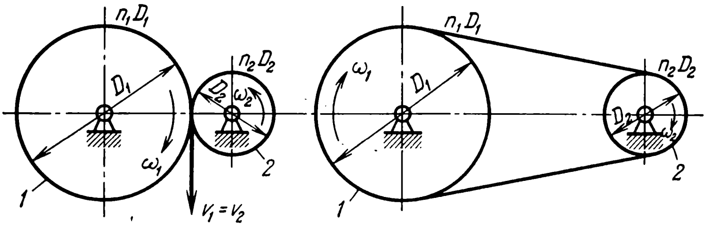


Fig. 108.

The peripheral speed of a driving member can be found from the formula

$$v_1 = \frac{\pi D_1 n_1}{60} \text{ m/sec.}$$

The peripheral speed of a driven member is

$$v_2 = \frac{\pi D_2 n_2}{60} \text{ m/sec.}$$

For transmission without slipping the peripheral speeds of the two members must be equal

$$v_1 = v_2$$

or

$$\frac{\pi D_1 n_1}{60} = \frac{\pi D_2 n_2}{60}.$$

Cancelling, we obtain

$$D_1 n_1 = D_2 n_2,$$

whence

$$\frac{n_1}{n_2} = \frac{D_2}{D_1}.$$

Here D_1 and D_2 are the diameters of the driving and driven members, n_1 and n_2 their numbers of revolutions, and v_1 and v_2 their peripheral speeds.

Since the ratio of the number of revolutions of the driving member to that of the driven member can be replaced by the ratio of their angular velocities, i.e.,

$$\frac{n_1}{n_2}=\frac{\omega_1}{\omega_2},$$

we have

$$\frac{\omega_1}{\omega_2}=\frac{n_1}{n_2}=\frac{D_2}{D_1}=\frac{R_2}{R_1}.$$

The ratio of the number of revolutions or the angular velocity of the driving shaft to the number of revolutions or the angular velocity of the driven shaft is called the *gear* (or *transmission*) *ratio* and is denoted by i. For the case under consideration

$$i=\frac{\omega_1}{\omega_2}=\frac{n_1}{n_2}=\frac{D_2}{D_1}, \tag{120}$$

where ω_1 = angular velocity of driving disk,
ω_2 = angular velocity of driven disk.

The gear ratio is assigned a sign according to the sense of the angular velocities. Thus, in the mechanism represented in Fig. 108*a* the angular velocities are opposite and so have different sign. Therefore the gear ratio i_{1-2} has a minus sign, i.e.,

$$i_{1-2}=-\frac{\omega_1}{\omega_2}=-\frac{n_1}{n_2}. \tag{121a}$$

For the mechanism shown in Fig. 108*b* the angular velocities ω_1 and ω_2 have the same sense and therefore the ratio i_{1-2} is assigned a plus sign

$$i_{1-2}=\frac{\omega_1}{\omega_2}=\frac{n_1}{n_2}. \tag{121b}$$

The sign rule adopted above for the gear ratio is conventional.

61. Cylinder Friction Drives

The operation of friction drives is based on the utilization of frictional forces due to the pressure of one friction roller on another. Friction drives are used to multiply rotary motion in cases where the distance between the axes of the driving and driven shafts is small and also where the angular velocity of the driven shaft must be changed during operation.

Friction drives with cylindrical or conical disks, with plain or grooved rollers are distinguished by simplicity of design and smooth, uniform running. When the external loading changes abruptly the rollers do not break but slip on each other.

The serious disadvantages of friction mechanisms are: small amount of power that can be transmitted, large loads on shafts and supports, variable angular velocity ratio, relatively low efficiency.

Consider two smooth cylindrical friction disks (Fig. 109). Due to the pressure exerted by normal forces (perpendicular

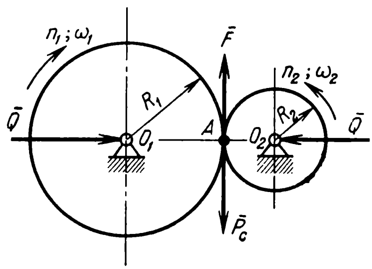


Fig. 109.

to the contact surfaces of the disks) a frictional force $F = fQ$ is developed by means of which the motion is transmitted (provided the circumferential force $\bar{P}_c \leqslant F$).

By the circumferential force $\bar{P}_c$ is meant the force transmitted from the driving to the driven member.

As noted above, if no slipping occurs between the driving and driven members the transmission ratio is

$$i_{1-2} = \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \frac{D_2}{D_1}. \tag{122}$$

However, slippage always occurs in a friction drive, which accounts for a reduction in the velocity of the driven roller as compared to the n_2 obtained from relation (122). The amount of slippage depends on the design of the drive, the loading conditions and other factors.

With the slip ratio ε taken into account, the transmission ratio of a friction drive is

$$i_{1-2} = \frac{D_2}{(1-\varepsilon) D_1}. \tag{123}$$

Consequently,

$$n_2 = n_1 \frac{D_1}{D_2} (1 - \varepsilon). \qquad (124)$$

The values of the slip ratio ε range from 0.005 to 0.05.

62. Face Friction Drives

In some types of friction drive it is possible to change the transmission ratio gradually. Disk A (Fig. 110a) rotating about its axis and pressed against disk B sets the latter in rotary motion.

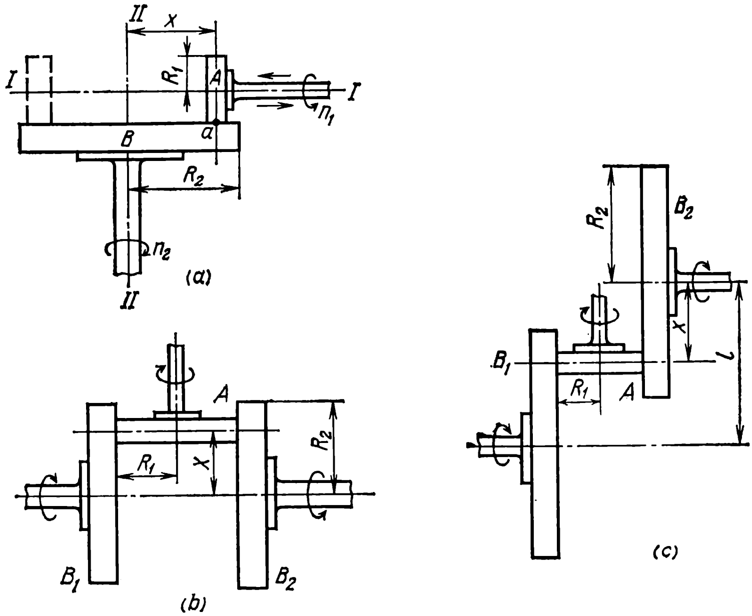


Fig. 110.

When disk A is moved along its axis of rotation the distance x between the disk and the axis II-II is changed and so is the transmission ratio.

The peripheral speed of the point a of the disk A is

$$(v_a)_1 = \frac{\pi n_1 R_1}{30}.$$

The peripheral speed of the point a of the disk B is

$$(v_a)_2 = \frac{\pi n_2 x}{30}.$$

The speed $(v_a)_1$ is equal to the speed $(v_a)_2$, consequently,

$$\frac{\pi n_1 R_1}{30} = \frac{\pi n_2 x}{30}$$

or

$$n_1 R_1 = n_2 x,$$

whence

$$\frac{n_1}{n_2} = \frac{x}{R_1} = i_{1-2}.$$

It is known that

$$\frac{n_1}{n_2} = \frac{\omega_1}{\omega_2},$$

then

$$i_{1-2} = \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \frac{x}{R_1}. \tag{125}$$

This drive with a variable transmission ratio is a face friction drive, one of the speed variators.

If the driving disk A is displaced beyond the axis *II-II* (along the axis *I-I*), the driven disk B will rotate in the opposite direction.

There exist more complex drives with a variable transmission ratio. Figure 110*b* shows a friction drive where the driven rollers B_1 and B_2 rotate in opposite directions. The friction drive in Fig. 110*c* has rollers B_1 and B_2 rotating in the same direction.

63. Cone Friction Drives

In cases where the axes of driving and driven members intersect at an angle friction drives with conical rollers are employed (Fig. 111). The peripheral speed of the roller A is

$$v_1 = \frac{\pi R_1 n_1}{30},$$

the peripheral speed of the roller B is

$$v_2 = \frac{\pi R_2 n}{30},$$

$v_1 = v_2$, hence $R_1 n_1 = R_2 n_2$, or

$$i_{1-2} = \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \frac{R_2}{R_1}.$$

For cone friction drives in which the axes of the rollers intersect at right angles (Fig. 111*a*), $\alpha_1 + \alpha_2 = 90°$, we find

$$\frac{R_1}{R_2} = \tan \alpha_1, \quad \frac{R_2}{R_1} = \tan \alpha_2.$$

It is seen from the formula that the angles α_1 and α_2 depend on the transmission ratio and cannot be chosen arbitrarily.

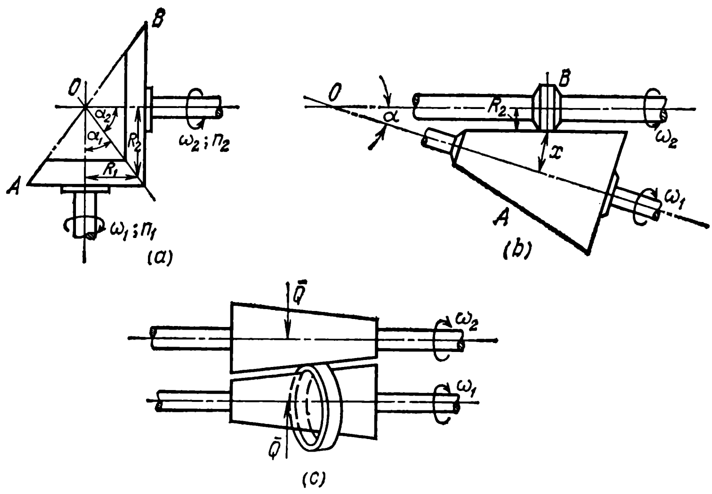


Fig. 111.

For this case the transmission ratio is expressed as

$$i = \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \frac{R_2}{R_1} = \tan \alpha_2 = \frac{1}{\tan \alpha_1}. \qquad (126)$$

There are friction drives in which the axes of the driving and driven rollers intersect at an arbitrary angle (Fig. 111*b*). When the roller *B* is moved along its axis the transmission ratio changes with the distance *x*. Figure 111*c* shows a sketch of a cone friction drive with a flexible ring and a variable transmission ratio. The transmission ratio is changed by moving the ring along the axis of the driving disk.

64. Belt Drives: Fundamental Concepts

If the driving shaft is at a considerable distance (up to 15 m) from the driven shaft, the moment of forces can be transmitted by a flexible connector using the frictional forces developed between the surface of the pulley and the flexible body.

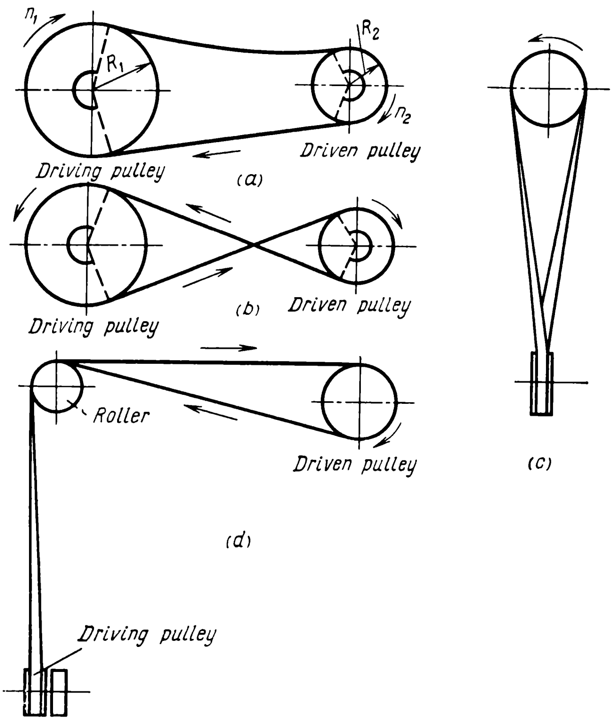


Fig. 112.

Belts and ropes are used as a flexible connector for this kind of drives (Fig. 112*a* through *d*).

The peripheral speed on the driving pulley (Fig. 112*a*) is

$$v_1 = \frac{\pi D_1 n_1}{60}.$$

The peripheral speed on the driven pulley is

$$v_2 = \frac{\pi D_2 n_2}{60}.$$

If no account is taken of slipping between the belt and the pulley, the speeds v_1 and v_2 must be equal

$$v_1 = v_2$$

and

$$\frac{\pi D_1 n_1}{60} = \frac{\pi D_2 n_2}{60}.$$

Cancelling, we obtain

$$D_1 n_1 = D_2 n_2 \text{ or } \frac{n_1}{n_2} = \frac{D_2}{D_1}.$$

In practice $v_1 \neq v_2$ since some elastic slippage always occurs between the belt and the pulley rim. Usually 1 to 3 per cent of the peripheral speed is lost in slipping. Introducing this correction, we obtain

$$v_2 = (1 - \varepsilon) v_1,$$

where ε is the slip ratio ($\varepsilon = 0.01$ to 0.03). Then

$$i_{1-2} = \frac{n_1 (1 - \varepsilon)}{n_2} = \frac{D_2}{D_1}. \qquad (127)$$

Elastic slippage is unavoidable in the operation of a belt drive under loading. This is not to be confused with parasitic slippage due to overloading or poor design.

65. Gear Drives: General Considerations

A gear drive differs from a friction drive in that both wheels (rollers) are provided with teeth of definite shape. When one of the wheels rotates its teeth engage the teeth of the mating wheel, setting it in rotation. A gear drive has a constant gear ratio.

Gear drives are the most common mechanisms for transmission of rotary motion and are capable of transmitting considerable power.

Gear drives are simple in design and highly efficient.

The following types of gear drives are distinguished according to the relative position of the geometric axes of the driving and driven shafts:

(a) spur gearing employed when the shaft axes are parallel,
(b) bevel gearing employed when the shaft axes intersect,
(c) helical gearing and worm gearing employed when the shaft axes cross in space.

66. Gear Ratio of a Spur Gear Drive

Figure 113 shows a sketch of spur gearing. Let ω_1 denote the angular velocity of the driving wheel and ω_2 the angular velocity of the driven wheel

$$\omega_1 = \frac{\pi n_1}{30}, \quad \omega_2 = \frac{\pi n_2}{30}.$$

The *gear ratio* is the angular velocity or the number of revolutions of the driving wheel divided by the angular

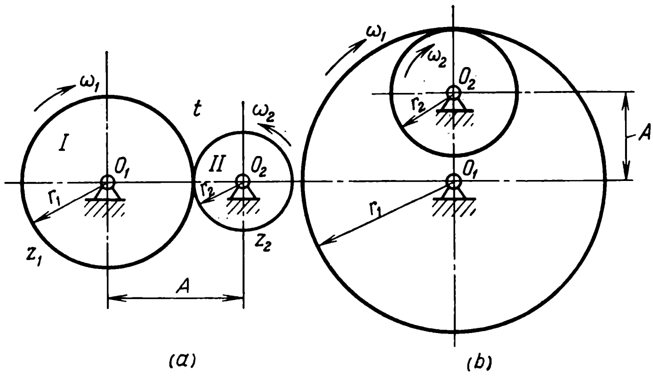


Fig. 113.

velocity or the number of revolutions of the driven wheel

$$i_{1-2} = \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2}.$$

But the numbers of revolutions are inversely proportional to the radii of the wheels, therefore

$$i_{1-2} = \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \frac{r_2}{r_1}.$$

Let z_1 denote the number of teeth of the driving wheel, z_2 the number of teeth of the driven wheel, t the tooth pitch.

The length of the circumference of the driving wheel is $2\pi r_1 = z_1 t$.

The length of the circumference of the driven wheel is $2\pi r_2 = z_2 t$.

Taking the ratio of the length of the circumference of the driven wheel to the length of the circumference of the driving wheel, we obtain

$$\frac{2\pi r_2}{2\pi r_1} = \frac{z_2 t}{z_1 t}$$

or

$$\frac{r_2}{r_1} = \frac{z_2}{z_1}.$$

The gear ratio for spur gears is then

$$i_{1-2} = \frac{\omega_1}{\omega_2} = \frac{n_1}{n_2} = \frac{r_2}{r_1} = \frac{z_2}{z_1}. \qquad (128)$$

Consequently, the gear ratio for a gear drive can also be expressed as the number of teeth of the driven wheel divided by the number of teeth of the driving wheel.

Besides external gearing (Fig. 113*a*), mechanical engineering employs internal gearing (Fig. 113*b*). In contrast to a pair of external wheels, internal wheels rotate in the same direction.

67. Gear Trains

Two cases are distinguished in series-connected gearing:

(a) a train with idler wheels—in-line connection;

(b) a train with multiple engagement—multistep, or compound, gearing.

In-Line Connection. If the driving shaft is at a considerable distance from the driven shaft, intermediate gears have to be inserted which do not affect the overall gear ratio of the drive (Fig. 114).

For instance, we have the gear ratio

$$i_{1-4} = \frac{\omega_1}{\omega_4} = \frac{r_4}{r_1}.$$

A wheel of radius r_1 is fitted on the driving shaft, and a wheel of radius r_4 on the driven shaft.

If an arbitrary number of gears are mounted between the driving and driven wheels, then in order to preserve the gear ratio the intermediate gears (called idler gears) should have the same tooth pitch as the driving and driven wheels.

We determine the overall gear ratio of the drive.

The gear ratio of the first pair of engaging gears is

$$i_{1-2}=\frac{\omega_1}{\omega_2}=\frac{r_2}{r_1}.$$

For the second pair

$$i_{2-3}=\frac{\omega_2}{\omega_3}=\frac{r_3}{r_2}.$$

For the third pair

$$i_{3-4}=\frac{\omega_3}{\omega_4}=\frac{r_4}{r_3}.$$

By multiplying the several gear ratios, we find

$$i_{1-2}i_{2-3}i_{3-4}=\frac{\omega_1}{\omega_2}\frac{\omega_2}{\omega_3}\frac{\omega_3}{\omega_4}=$$

$$=\frac{r_2}{r_1}\frac{r_3}{r_2}\frac{r_4}{r_3}.$$

After cancelling we obtain

$$i_{1-2}i_{2-3}i_{3-4}=\frac{\omega_1}{\omega_4}=\frac{r_4}{r_1}.$$

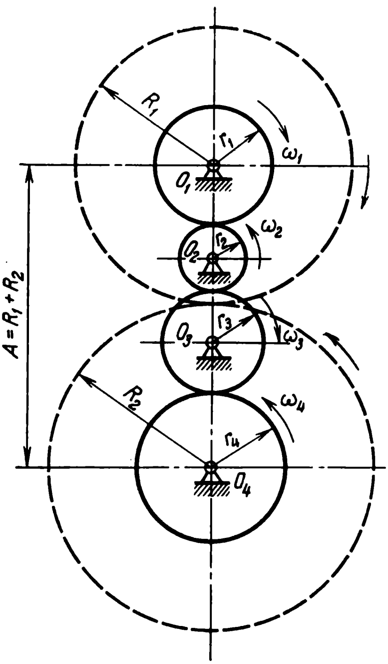


Fig. 114.

The result obtained is the assigned overall gear ratio of the drive, consequently,

$$i_{1-2}i_{2-3}i_{3-4}=\frac{\omega_1}{\omega_4}=\frac{r_4}{r_1}=i_{1-4}. \tag{129}$$

We arrive at the conclusions:

1. *The overall gear ratio of an in-line connection of engaging gears is equal to the product of the individual gear ratios of the drives forming the series.*

2. *The overall gear ratio is equal to the angular velocity of the first wheel divided by the angular velocity of the last wheel or to the inverse ratio of their radii.*

3. *Intermediate (idler) wheels do not change the overall gear ratio, but they affect the direction of rotation of the driven shaft: (a) with an even number of idler wheels the driving and driven shafts rotate in opposite directions; (b) with an odd number of idler wheels the direction of rotation is the same.*

The incorporation of a set of small gears in place of the two gears shown dashed in Fig. 114 does not change the assigned

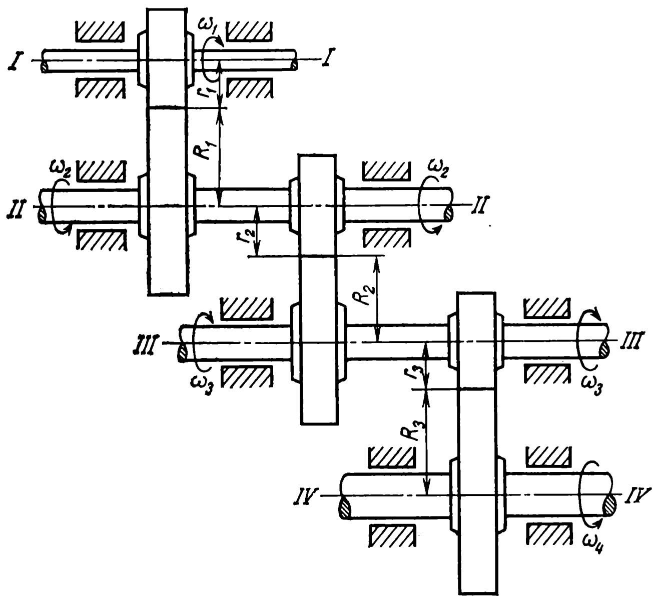


Fig. 115.

gear ratio, facilitates transmission, reduces the dimensions (this being very important in machines) and makes it possible to change the direction of rotation of the driven wheel.

Multistep Gearing. A large gear ratio cannot be obtained with just one pair of gears. To increase it, use is made of a set of series-connected gears called a multistep gear (Fig. 115). A multistep gear is characterized by the fact that two gears are mounted on each intermediate shaft.

The first wheel of the drive is fitted on the driving shaft, and the last on the driven one.

The gear ratio of the first engaging pair of gears is

$$i_{1-2}=\frac{\omega_1}{\omega_2}=\frac{R_1}{r_1}=\frac{Z_1}{z_1}.$$

For the second pair

$$i_{2-3}=\frac{\omega_2}{\omega_3}=\frac{R_2}{r_2}=\frac{Z_2}{z_2}.$$

For the last pair

$$i_{3-4}=\frac{\omega_3}{\omega_4}=\frac{R_3}{r_3}=\frac{Z_3}{z_3}.$$

By multiplying the individual gear ratios, we find

$$i_{1-2}i_{2-3}i_{3-4}=\frac{\omega_1}{\omega_2}\frac{\omega_2}{\omega_3}\frac{\omega_3}{\omega_4}=\frac{R_1}{r_1}\frac{R_2}{r_2}\frac{R_3}{r_3}=\frac{Z_1}{z_1}\frac{Z_2}{z_2}\frac{Z_3}{z_3}$$

or

$$i_{1-2}i_{2-3}i_{3-4}=\frac{\omega_1}{\omega_4}=\frac{R_1R_2R_3}{r_1r_2r_3}=\frac{Z_1Z_2Z_3}{z_1z_2z_3}.$$

The ratio of the angular velocity of the first driving wheel to the angular velocity of the last driven wheel is the gear ratio of the whole set of gears, i.e.,

$$i_{1-2}i_{2-3}i_{3-4}=\frac{\omega_1}{\omega_4}=i_{1-4}=\frac{R_1R_2R_3}{r_1r_2r_3}=\frac{Z_1Z_2Z_3}{z_1z_2z_3}. \quad (130)$$

Consequently, the gear ratio of the whole series-connected set of gears can be found by dividing the product of the radii of the driven wheels by the product of the radii of the driving wheels.

This arrangement of gears makes it possible to greatly increase (or reduce—less frequently) the overall gear ratio of the drive.

In the particular case when the radii of the driving wheels are equal and so are the radii of the driven wheels, i.e.,

$$r_1=r_2=r_3=r,$$

$$R_1=R_2=R_3=R,$$

the gear ratio of the drive is

$$i=\left(\frac{R}{r}\right)^3 \text{ or } i=\left(\frac{Z}{z}\right)^3.$$

In the general case this formula becomes

$$i = \left(\frac{R}{r}\right)^k = \left(\frac{Z}{z}\right)^k,$$

where k = number of pairs of gears forming the drive,
Z = number of teeth of driven wheel,
z = number of teeth of driving wheel.

68. Worm Gearing

General Considerations. A worm and worm wheel are used to transmit motion with a constant gear ratio between shafts crossing in space. We shall consider the most common worm drives with shafts crossing at right angles.

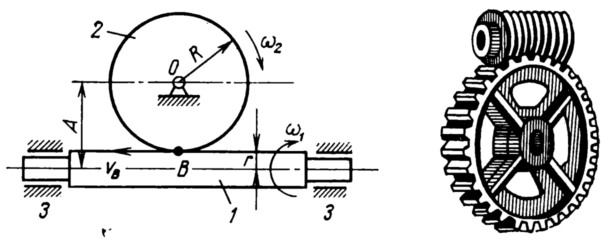


Fig. 116.

A worm drive (Fig. 116) consists of a worm (screw) *1* fitted on a shaft or made integral with the shaft, a worm wheel *2* mounted on another shaft, and bearings *3* supporting the shafts of the worm and worm gear.

Let us first see how a screw is generated.

The unwrapped circumference of the base of a right circular cylinder represents a straight line $AB = 2\pi r$ (Fig. 117*a*). Drawing from point A a sloping straight line at an angle λ to the base, we cut off a segment $BC = h$ on the side of the rectangle. Wrapping the sloping straight line AC around the cylinder, we obtain a continuous helical line.

The height h is called the *lead of the helix*, i.e., the axial distance that a point on the helix will move in one revolution, and the angle λ is termed the *lead angle*. From Fig. 117*a* it is seen that

$$h = 2\pi r \tan \lambda,$$

$$\tan \lambda = \frac{h}{2\pi r}.$$

Let us take a triangle the plane of which contains the axis of the cylinder and the base coincides with one of the generators. Displace the triangle so that its vertex moves along a helical line on the lateral surface of the cylinder. Then the sides of the triangle which include the vertex will describe helical surfaces which form a screw with a triangular thread.

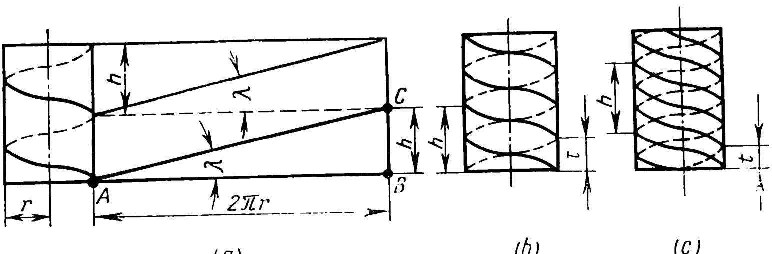


Fig. 117.

If the same motion is imparted to a rectangle, a screw with a square thread is formed. If there are two or three threads along one lead of a helix, the screw is referred to as double-threaded (Fig. 117*b*) or triple-threaded (Fig. 117*c*), respectively.

A body surrounding a screw the top of whose thread coincides with the thread groove of the screw is called a nut.

Gear Ratio. The gear ratio for a worm and worm wheel drive is defined by the same formula as for gear drives

$$i_{1-2} = \frac{z_2}{z_1},$$

where z_2 is the number of teeth in the wheel, z_1 the number of threads (or teeth) wrapped on the worm (in practice $z_1 \leqslant 5$ since when the number of threads is increased the lead angle becomes larger and this intensifies the wear of the worm).

In terms of the geometric elements of a worm and worm wheel the gear ratio is determined as follows. The linear velocity of a point of the thread on the base cylinder along the axis of the worm is

$$v_1 = \frac{hn_1}{60},$$

where h is the axial distance that a point on the helix of the worm will move in one revolution of the worm, n_1 the number of revolutions of the worm per minute.

The velocity of the teeth on the base circle of the worm gear is

$$v_2 = \frac{\pi D n_2}{60},$$

where D is the diameter of the worm wheel, n_2 the number of revolutions of the wheel per minute.

Since the velocities must be equal, we have

$$\frac{hn_1}{60} = \frac{\pi D n_2}{60}$$

or

$$\frac{n_1}{n_2} = \frac{\pi D}{h} = i_{1-2}.$$

It is known that

$$h = 2\pi r \tan \lambda = \pi d_1 \tan \lambda,$$

consequently,

$$i_{1-2} = \frac{n_1}{n_2} = \frac{\omega_1}{\omega_2} = \frac{\pi D}{h} = \frac{\pi 2R}{2\pi r_1 \tan \lambda} = \frac{R}{r_1 \tan \lambda},$$

where r_1 and d_1 are the radius and diameter of the worm, respectively.

It is also known that

$$\pi D = z_2 t,$$

where z_2 is the number of teeth in the worm wheel, t the pitch.

$$h = z_1 t,$$

where z_1 is the number of threads on the worm.

In the general form the gear ratio is

$$i = \frac{n_1}{n_2} = \frac{\omega_1}{\omega_2} = \frac{\pi D}{h} = \frac{R}{r_1 \tan \lambda} = \frac{z_2 t}{z_1 t} = \frac{z_2}{z_1}. \qquad (131)$$

Consider two examples of the kinematic analysis of transmission mechanisms.

1. *Spur Gear*. The gear ratio is $i = 2$. The number of revolutions of the driving shaft is $n_1 = 800$ rpm, the number of teeth of the driving wheel $z_1 = 30$.

We determine the number of revolutions and the number of teeth of the driven wheel

$$n_2 = \frac{n_1}{i} = \frac{800}{2} = 400 \text{ rpm},$$

$$z_2 = z_1 i = 30 \times 2 = 60.$$

2. *Worm and Worm Wheel*. The worm is double-threaded, i.e., $z_1 = 2$, the number of revolutions of the worm shaft is $n_1 = 2{,}000$ rpm, the gear ratio $i = 40$.

We determine the number of teeth of the worm wheel

$$i = \frac{z_2}{z_1}, \quad z_2 = iz_1 = 40 \times 2 = 80.$$

The number of revolutions of the driven shaft per minute is

$$n_2 = \frac{n_1}{i} = \frac{2{,}000}{40} = 50 \text{ rpm}.$$

CHAPTER XII

Complex Motion of Particles

69. Base, Relative and Absolute Motions

In many cases the motion of a particle has to be considered with respect to a moving reference system.

Imagine a body moving in translation with a velocity $\bar{v}_b$ (Fig. 118). All points of this body have the same velocity

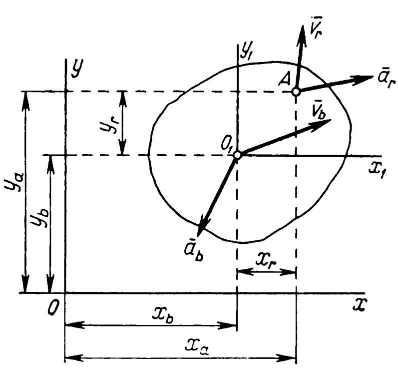


Fig. 118.

and acceleration at any given instant. The motion of the body with respect to a fixed co-ordinate system xOy is called a *base motion*.

Consider a particle which moves relative to the body. The motion of this material particle A with respect to the body or a moving co-ordinate system $x_1O_1y_1$ attached to it is a *relative motion*, i.e., a motion with respect to a point O_1 of the body chosen as the origin of a moving co-ordinate system.

The *absolute motion* of the particle A is its motion with respect to the fixed co-ordinate system xOy which is conventionally attached to the earth.

For instance, a man moves in a car of a moving train. The motion of the man with respect to the car is the relative motion, the motion of the car is the base motion, and the motion

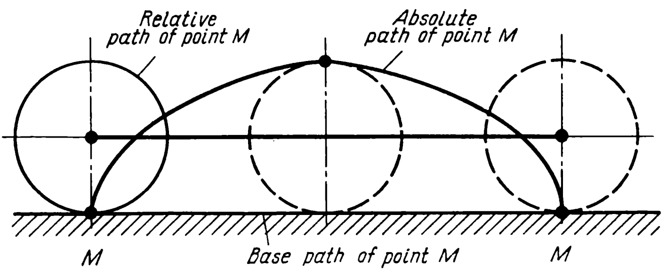


Fig. 119.

of the man with respect to a fixed point on the earth is the absolute motion.

Likewise, a piston of a steam locomotive has a relative motion with respect to the locomotive and an absolute motion with respect to the earth. The motion of the locomotive is the base motion in this case.

If the motion of a particle is composed of two or more independent motions, it is termed *complex* or *compound motion*. The motions involved are called *component motions*.

It is common to distinguish relative, base and absolute paths for a particle in complex motion. These paths may differ greatly according to the nature of the component motions.

Thus, the absolute path of a point on the rim of a wheel of an automobile moving in a straight line is a cycloid for an immovable observer. For an observer driving in the automobile the relative path of point M is a circle (Fig. 119).

The base path of point M is that which would be described by the point if it had the base motion with the automobile and remained fixed with respect to it. The base path of point M is a straight line.

By analogy with paths, we distinguish absolute, relative and base velocities and accelerations.

Sometimes it is necessary to determine the absolute motion of a point from its given relative and base motions.

Absolute motion is represented as composed of base and relative motions. The process of solving this problem is called *addition*, or *composition*, *of motions*.

It is often necessary to add two rectilinear motions. This case is of great importance as the addition of curvilinear

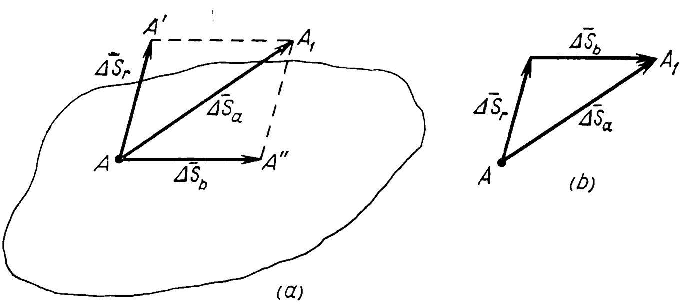


Fig. 120.

motions is virtually reduced to it by passing to infinitesimal displacements assumed to be directed along the corresponding velocities.

Let a particle A have a complex motion (Fig. 120): a base motion with the body and a relative motion with respect to it. Assume both motions to be rectilinear. During a time interval Δt the particle A undergoes a base displacement $\Delta \overline{S}_b$ with the body and a displacement $\Delta \overline{S}_r$ relative to the body. These displacements are represented in Fig. 120a as vectors in the direction of the corresponding motions.

In order to find the absolute displacement of the body imagine that the relative and base motions occur in succession and not simultaneously. If we suppose that the relative displacement precedes the base displacement, the particle A will move along a broken line $AA'A_1$ to a point A_1.

Let us see how the absolute displacement of the particle is determined from its known component displacements—base and relative.

If we now suppose that the base motion precedes the relative motion, the particle A will move along a broken line $AA''A_1$ to the same point A_1. In both cases the absolute displacement $\Delta\overline{S}_a$ is a diagonal of the parallelogram constructed on the component base and relative displacements (Fig. 120*a*). The absolute displacement can also be found by the triangle rule (Fig. 120*b*).

Thus, *in adding two rectilinear displacements*, base and relative, the absolute displacement is determined as their geometric sum, i.e.,

$$\Delta\overline{S}_a = \Delta\overline{S}_b + \Delta\overline{S}_r. \qquad (132)$$

Sometimes one has to solve the inverse problem, i.e., to determine one of the component motions from the known complex motion of a particle and the other component motion.

70. Theorems on Addition of the Velocities and Accelerations of a Particle in Complex Motion

When solving problems involving addition or resolution of motions it is necessary to have the relation between absolute, base and relative velocities and accelerations. This relation can be derived by using the rule for adding rectilinear displacements given in the preceding section.

Rewrite equation (132)

$$\Delta\overline{S}_a = \Delta\overline{S}_b + \Delta\overline{S}_r.$$

Divide all terms of the equation by Δt and pass to the limit as $\Delta t \to 0$

$$\lim_{\Delta t\to 0} \frac{\Delta\overline{S}_a}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta\overline{S}_b}{\Delta t} + \lim_{\Delta t\to 0} \frac{\Delta\overline{S}_r}{\Delta t}.$$

Clearly, the limits in the last equation represent respectively the absolute, base and relative velocity, i.e.,

$$\lim_{\Delta t\to 0} \frac{\Delta\overline{S}_a}{\Delta t} = \overline{v}_a,$$

$$\lim_{\Delta t\to 0} \frac{\Delta\overline{S}_b}{\Delta t} = \overline{v}_b,$$

$$\lim_{\Delta t\to 0} \frac{\Delta\overline{S}_r}{\Delta t} = \overline{v}_r.$$

Substituting the velocities, we obtain

$$\bar{v}_a = \bar{v}_b + \bar{v}_r. \tag{133}$$

Consequently, *the absolute velocity of a particle at a given instant is determined as the geometric sum of the relative and base velocities by the parallelogram rule.*

The same conclusion can be drawn with the co-ordinate method of prescribing base and relative motions. Let us take the simplest case of complex motion when the base motion is a translation. It will be recalled that the translation of a body is completely determined by the motion of a single point. Consequently, to specify the translation of the body in Fig. 118 it is sufficient to know the law of variation of the co-ordinates of point O_1 with respect to a fixed system. These co-ordinates $x_b = f_1(t)$ and $y_b = f_2(t)$ define the base motion.

The relative motion of a particle A in a moving system $x_1O_1y_1$ is defined by specifying the co-ordinates

$$x_r = f_3(t) \text{ and } y_r = f_4(t).$$

From Fig. 118 it follows that the co-ordinates x_a and y_a of the particle A in the absolute motion are equal to the algebraic sum of the like co-ordinates in the relative and base motions

$$x_a = x_r + x_b,$$

$$y_a = y_r + y_b.$$

Differentiate each of the above equations with respect to time taking into account that the derivative of the corresponding co-ordinate with respect to time is equal to the projection of the velocity on the co-ordinate axis, i.e.,

$$\frac{dx}{dt} = v_x, \quad \frac{dy}{dt} = v_y.$$

We obtain

$$\left.\begin{aligned}(v_x)_a &= (v_x)_r + (v_x)_b,\\ (v_y)_a &= (v_y)_r + (v_y)_b.\end{aligned}\right\} \tag{134}$$

It is known that the projection of a vector sum is equal to the sum of the projections of the components (Sec. 8); thus we conclude that two equations in projections (134) are equivalent to one vector equation

$$\bar{v}_a = \bar{v}_r + \bar{v}_b.$$

Figure 121a shows the construction of the parallelogram for the determination of the absolute velocity of the particle A from given relative and base velocities. The absolute velocity can also be constructed by the triangle rule (Fig. 121b).

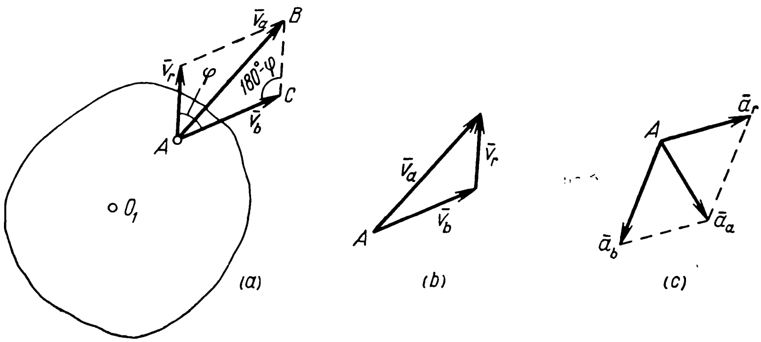


Fig. 121.

Consider the determination of the modulus, or magnitude, of the absolute velocity in terms of the magnitudes of the component velocities.

From the triangle ABC (Fig. 121a) we have

$$v_a^2 = v_b^2 + v_r^2 - 2v_b v_r \cos(180 - \varphi), \tag{135}$$

where φ is the angle between the directions of the relative and base velocities, $\bar{v}_r$ and $\bar{v}_b$.

Equation (135) can be represented as

$$v_a^2 = v_b^2 + v_r^2 + 2v_b v_r \cos \varphi,$$

whence

$$v_a = \sqrt{v_b^2 + v_r^2 + 2v_b v_r \cos \varphi}. \tag{136}$$

Consider three special cases.

1. The angle between the directions of the velocities $\bar{v}_b$ and $\bar{v}_r$ is zero ($\varphi = 0$). In this case $\cos \varphi = 1$ and

$$v_a = \sqrt{v_b^2 + v_r^2 + 2v_b v_r} = v_b + v_r.$$

2. The angle between the directions of the velocities $\bar{v}_b$ and $\bar{v}_r$ is 180° ($\varphi = 180°$). In this case $\cos\varphi = -1$ and

$$v_a = \sqrt{v_b^2 + v_r^2 - 2v_b v_r} = v_b - v_r.$$

3. The angle between the directions of the velocities $\bar{v}_b$ and $\bar{v}_r$ is 90° ($\varphi = 90°$). In this case $\cos\varphi = 0$ and

$$v_a = \sqrt{v_b^2 + v_r^2}.$$

In the first case, we have addition of two motions in the same direction; in the second case addition of two opposite motions; in the third case, of two motions at right angles to each other.

Using similar arguments for accelerations, we can write (Fig. 121*c*)

$$\bar{a}_a = \bar{a}_b + \bar{a}_r. \tag{137}$$

Equation (137) is valid only when the base motion is a translation. If the base motion is a rotation, a complementary

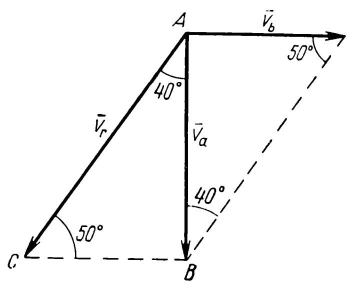


Fig. 122.

(rotational) acceleration, or Coriolis acceleration, will develop. This question is beyond the scope of the present course.

Thus, *if the base motion is a translation, the absolute acceleration of a particle at a given instant is determined as the geometric sum of the relative and base accelerations by the parallelogram or triangle rule.*

Example 45. Vertically falling raindrops leave paths at an angle of 40° to the vertical on the side windows of an automobile travelling on a horizontal road (Fig. 122); the speed of the automobile is v_b = 72 km/hr.

Determine the absolute velocity of the raindrops.

Solution. The relative velocity $\bar{v}_r$ of the raindrops with respect to the automobile is at an angle of 40° to the vertical, the base velocity $\bar{v}_b$ is horizontal.

According to the condition of the problem the absolute velocity $\bar{v}_a$ is vertical (Fig. 122).

The base velocity is

$$v_b = \frac{72 \times 1{,}000}{3{,}600} = 20 \text{ m/sec.}$$

From the right triangle ABC we have

$$AB = CB \tan 50^\circ$$

or

$$v_a = v_b \tan 50^\circ = 20 \times 1.19 = 23.8 \text{ m/sec.}$$

Example 46. A screw is entering a fixed nut at $n = 6$ rpm. The screw pitch is $h = 5$ mm. Find the velocity $\bar{v}_r$ of the

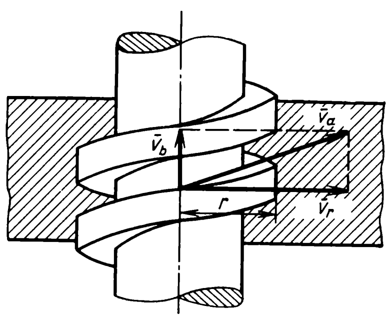


Fig. 123.

screw thread relative to the nut thread if the outer radius of the screw thread is $r = 1.5$ cm (Fig. 123).

Solution. Denote by $\bar{v}_b$ the velocity of a point of the screw thread in its translation and by $\bar{v}_r$ in rotation.

Determine the time of one turn of the screw

$$t = \frac{60}{n} = \frac{60}{6} = 10 \text{ sec.}$$

Determine the base velocity of the screw thread in the translation

$$v_b = \frac{h}{t} = \frac{5}{10} = 0.5 \text{ mm/sec.}$$

Determine the relative velocity

$$v_r = \omega r = \frac{\pi n}{30} r = \frac{3.14 \times 6}{30} 15 = 9.42 \text{ mm/sec.}$$

Determine the absolute velocity of the point of the screw thread

$$v_a = \sqrt{v_b^2 + v_r^2} = \sqrt{0.5^2 + 9.42^2} = 9.43 \text{ mm/sec.}$$

Here, the magnitude of the absolute velocity is calculated from the base and relative velocities by using the Pythagorean theorem. This is due to the fact that the component velocity vectors are at right angles to each other (Fig. 123).

If, however, the angle between the component velocity vectors is not 90°, the magnitude of the absolute velocity is determined by formula (136).

CHAPTER XIII

Plane Motion

71. Concept of Plane Motion of a Rigid Body

Plane motion of a rigid body is a motion in which a plane figure obtained at the intersection of the body and a fixed plane remains in this plane throughout the motion.

Consequently, in plane motion each point of the body describes a plane path located in a plane parallel to the given

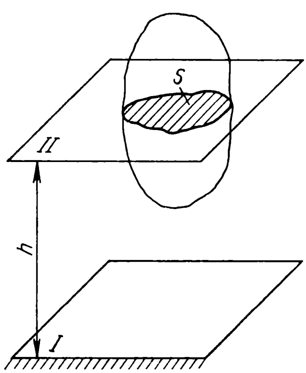


Fig. 124.

fixed plane. All points of the body located on a straight line perpendicular to the fixed plane move identically. Therefore, the study of the plane motion of a rigid body can be reduced to the study of the motion of a plane figure formed by cutting the body through a plane *II* parallel to a given fixed plane *I* provided the distance between the planes *I* and *II* is constant (h = constant) (Fig. 124).

The position of the plane figure in its plane can be determined at any instant if the co-ordinates of any one point

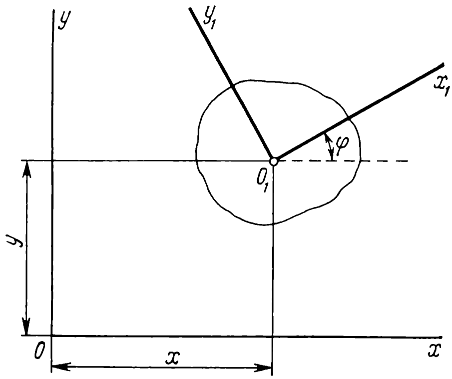


Fig. 125.

of the figure and the angle of rotation of the body are given as functions of time (Fig. 125)

$$\left.\begin{aligned} x &= f_1(t), \\ y &= f_2(t), \\ \varphi &= f_3(t). \end{aligned}\right\} \qquad (138)$$

These equations are called the *equations of plane motion.*

Examples of plane motion are the motion of the connecting rod of a slider-crank mechanism, the motion of a car wheel on a straight section of track, etc.

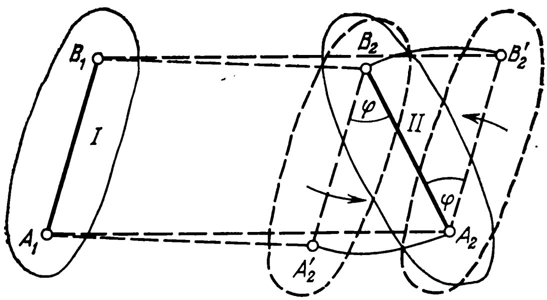
Clearly, the rotation of a rigid body about a fixed axis is a special case of plane motion when x = constant and y = = constant while $\varphi = f(t)$. Translation can also be regarded as a special case of plane motion when φ = constant.

72. Determination of the Velocity of Any Point of a Body in Plane Motion

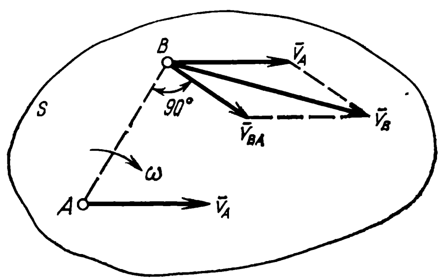
The discussion which follows will be concerned with the motion of a plane figure in its plane.

Let us show that any displacement of a plane figure may be accomplished by two simple motions: a translation and a rotation.

Consider an example. The position of the plane figure in Fig. 126*a* is completely determined by the segment A_1B_1.



(*a*)



(*b*)

Fig. 126.

This segment can be displaced from position *I* to position *II* as follows: transfer it parallel to itself to a position A_2B_2' (the figure executes a translation) and then rotate the segment about point A_2 counterclockwise through an angle φ (the figure executes a rotation and occupies position *II*).

We can proceed in a different way: first give a translation to the figure until the segment occupies a position B_2A_2' and then rotate it about point B_2 counterclockwise again through the angle φ.

The point about which the figure is rotated is called the *pole*. In the first case the pole is A_2 and in the second, B_2.

Clearly, any point of the figure can be chosen as the pole. Hence an important property of plane motion: *the angle of rotation is independent of the choice of the pole, i.e., when the pole is changed the translation changes but the rotation remains the same.*

Since the plane motion of a figure can be represented as the sum of two motions (translation and rotation), the velocity of any point B of the body (Fig. 126*b*) can be represented as the geometric sum of two velocities: the velocity $\bar{v}_A$ of the pole A and the velocity $\bar{v}_{BA}$ in the rotation about the pole A

$$\bar{v}_B = \bar{v}_A + \bar{v}_{BA}. \quad (139)$$

The magnitude of the velocity $\bar{v}_{BA}$ is given by the formula

$$v_{BA} = \omega AB,$$

where ω is the angular velocity, AB the radius of rotation of point B with respect to the pole A.

The velocity $\bar{v}_{BA}$ is perpendicular to the radius of rotation AB.

Since the rotational part of the motion is independent of the choice of the pole, the angular velocity ω which is the same for all poles is called *the angular velocity of a plane figure.*

73. Instantaneous Centre of Zero Velocity

It may be shown that in a plane figure there is always a point whose velocity is zero at a given instant. Indeed, choosing an arbitrary point O (Fig. 127*a*) as the pole, erect from it a perpendicular to the translational velocity vector $\bar{v}_O$ so that the velocity vector coincides with this perpendicular when turned through 90° in the direction of rotation. Lay off a segment $OC = \frac{v_O}{\omega}$ on this perpendicular, then the absolute velocity of point C is determined as the geometric sum of two equal and opposite vectors: the translational velocity $\bar{v}_O$ and the rotational velocity $v_{CO} = OC\,\omega = \frac{v_O}{\omega}\,\omega = v_O$, i.e., the velocity of point C is zero, $\bar{v}_C = 0$.

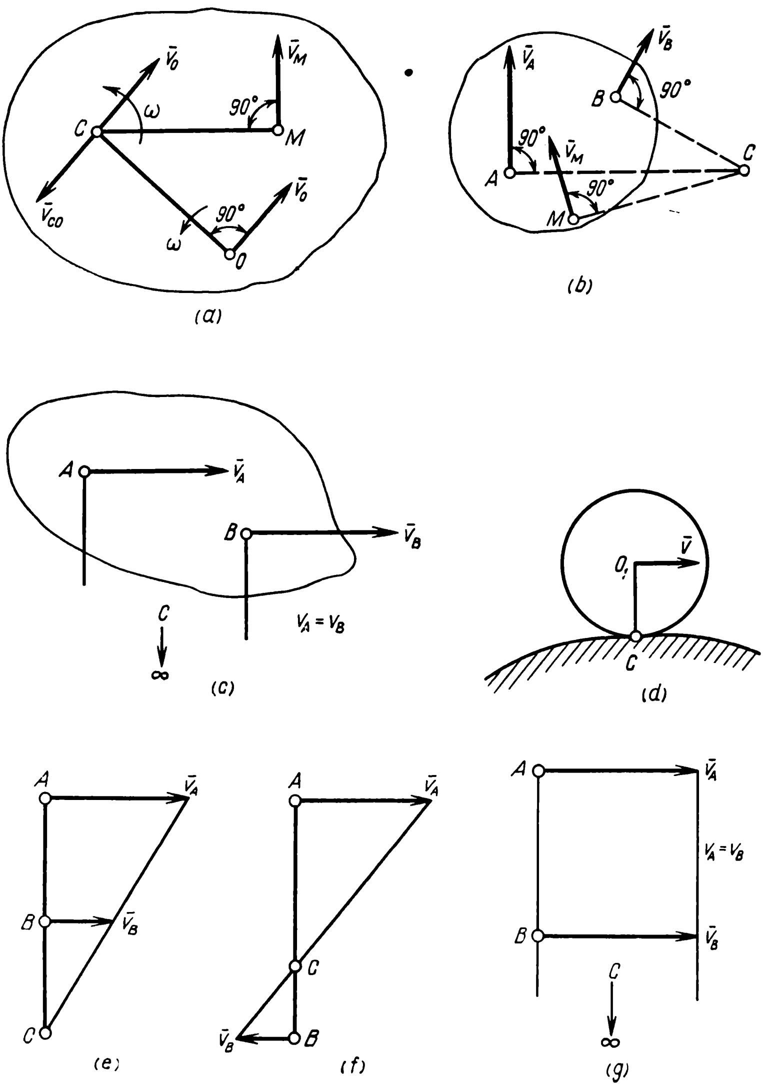

V̄O
V̄M
ω
C
90°
M
V̄CO
90°
V̄O
ω
O
(a)
V̄B
V̄A
B
90°
90°
V̄M
C
A
90°
M
(b)
A
V̄A
B
V̄B
C
∞
VA = VB
(c)
V̄
O1
C
(d)
A
V̄A
B
V̄B
C
(e)
A
V̄A
C
V̄B
B
(f)
A
V̄A
VA = VB
B
V̄B
C
∞
(g)

Fig. 127.

If point C is chosen as the pole (Fig. 127*a*), the velocity of an arbitrary point M is defined by the formula

$$\overline{v}_M = \overline{v}_C + \overline{v}_{MC},$$

but $\overline{v}_C = 0$, then $\overline{v}_M = \overline{v}_{MC}$ or

$$v_M = \omega\, CM, \tag{a}$$

i.e., the velocity of any point at a given instant is equal to the velocity in the rotation about a point C which is called *the instantaneous centre of zero velocity* or *the instantaneous centre of rotation.*

The position of the instantaneous centre of zero velocity varies with time. Consequently, plane motion can be represented as successive rotations about centres occupying different positions. This representation considerably simplifies the study of plane motion.

The instantaneous centre of zero velocity does not necessarily lie within the boundaries of the plane figure considered. Naturally, it may be located outside the figure but must always be contained in its plane.

The position of the instantaneous centre of zero velocity is easily determined in the following cases:

Case 1. If the directions of the velocities of two points A and B of a figure are known, the instantaneous centre of zero velocity is found at the intersection of perpendiculars to the velocity vectors $\overline{v}_A$ and $\overline{v}_B$ erected at points A and B (Fig. 127*b*).

In order to determine the magnitude of the velocity of any point of the figure it is necessary in this case to know, besides the directions of the velocities of two points, the magnitude of the velocity of one of these points. Thus, if the magnitude of the velocity of point A is known, we can find (having first determined the position of the instantaneous centre of zero velocity) the angular velocity of the plane figure with respect to this centre

$$v_A = \omega\, AC,$$

whence

$$\omega = \frac{v_A}{AC}\,.$$

Knowing the angular velocity of the plane figure and the position of the instantaneous centre of zero velocity, it is easy to find the magnitude of the velocity of any point of the figure. For instance (Fig. 127*b*),

$$v_B = \omega BC = \frac{BC}{AC} v_A,$$

$$v_M = \omega MC = \frac{MC}{AC} v_A, \text{ etc.}$$

It should be noted that the velocity vector of any point of a plane figure is always perpendicular to the line joining this point and the instantaneous centre of zero velocity (Fig. 127*b*). Besides, as follows from formula (a), the velocity of any point is proportional to the distance to the instantaneous centre of rotation.

Consequently, the farther the point of a plane figure from the instantaneous centre of zero velocity, the greater is the velocity of this point.

If the known directions of the velocities of two points of a body *A* and *B* are parallel (Fig. 127*c*), the perpendiculars to the velocity vectors will also be parallel. They will not intersect. Consequently, there is no instantaneous centre of zero velocity or, in other words, it is at infinity (Fig. 127*c*). The body is in translation at the instant considered and the velocities of the two points are equal

$$\bar{v}_A = \bar{v}_B.$$

Case 2. If a plane figure rolls without slipping along a fixed surface, the instantaneous centre of zero velocity is at the point of contact *C* of the rolling figure with the surface (Fig. 127*d*).

The velocity of this point is momentarily zero. The body is rotating, as it were, about the point of contact with the fixed surface at the instant considered. Likewise, for a wheel rolling without slipping on a straight track the instantaneous centre of zero velocity is at the point of contact of the wheel with the track surface.

Case 3. If the velocities of two points *A* and *B* of a plane figure are parallel to each other and at the same time are perpendicular to a line joining these points, then, to determine the instantaneous centre of zero velocity, it is necessary to join the extremities of the velocities and find the point of intersection of this line and the straight line AB.

Two situations may arise here: either the velocities of points A and B have the same sense (Fig. 127*e*) or they have opposite senses (Fig. 127*f*).

In the first instance (the velocities of points A and B have the same sense) the foregoing arguments hold if the velocities are unequal in magnitude, i.e., $v_A \neq v_B$. If the velocities

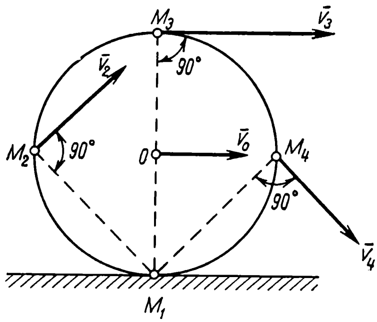


Fig. 128.

are equal, $v_A = v_B$ (Fig. 127*g*), there is no instantaneous centre of zero velocity (the instantaneous centre is at infinity). The body is then in translation.

Example 47. A wheel of radius $R = 0.5$ m rolls without slipping on a horizontal rail. The centre of the wheel moves with a constant velocity $v_O = 2$ m/sec.

Determine the velocities of the ends of the vertical and horizontal diameters of the wheel, M_1, M_2, M_3 and M_4 (Fig. 128).

Solution. The instantaneous centre of zero velocity of the wheel coincides with the point M_1 which is the point of contact with the fixed surface. Consequently, $v_1 = 0$. The velocity of any point can be regarded as the velocity in the rotation about the instantaneous centre

$$v_O = \omega \, M_1O,$$

whence

$$\omega = \frac{v_O}{M_1O} = \frac{2}{0.5} = 4 \, \frac{1}{\text{sec}} \, .$$

Knowing the instantaneous angular velocity and taking into account that $M_1M_2 = M_1M_4 = OM_1\sqrt{2}$, we can easily determine the velocities of the remaining points

$$v_2 = M_1M_2\,\omega = 0.5\sqrt{2} \times 4 = 2.82 \text{ m/sec},$$

$$v_3 = M_1M_3\,\omega = 1.0 \times 4 = 4 \text{ m/sec},$$

$$v_4 = M_1M_4\,\omega = 0.5\sqrt{2} \times 4 = 2.82 \text{ m/sec}.$$

The directions of these velocities are shown in Fig. 128.

Example 48. A rod AB moves in the plane of the drawing

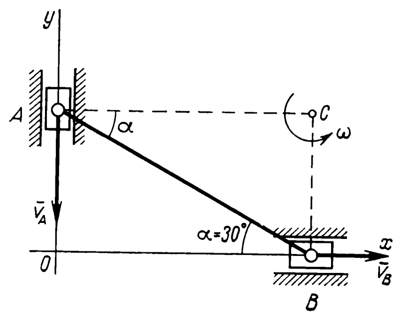


Fig. 129.

so that point A slides in vertical guides Oy and point B in horizontal guides Ox (Fig. 129).

Determine the velocity of point B if point A moves vertically down with a velocity of 1.5 m/sec; the angle that the line AB makes with the horizon is $\alpha = 30°$.

Solution. To determine the instantaneous centre of zero velocity erect at points A and B perpendiculars to the directions of the velocities. They intersect at a point C.

The instantaneous angular velocity is determined from the relation

$$v_A = \omega\, CA,$$

whence

$$\omega = \frac{v_A}{CA}.$$

Calculate the velocity of point B

$$v_B = \omega\, CB = \frac{v_A\, CB}{CA},$$

but

$$\frac{CB}{CA} = \tan\alpha = \tan 30^\circ$$

and

$$v_B = v_A \tan\alpha = 1.5 \times 0.577 = 0.865 \text{ m/sec.}$$

Example 49. The motion of an automobile along a straight horizontal track is defined by the equation

$$S = 5 + 2t + 2t^2 \quad (S \text{ in m, } t \text{ in sec}).$$

Determine the velocity of points A and B at the ends of the horizontal diameter of the wheel (Fig. 130) at time $t = 2$ sec.

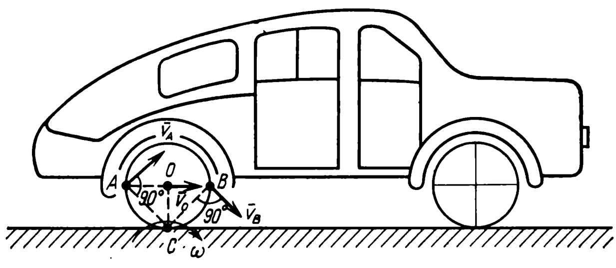


Fig. 130.

Solution. The centre of the wheel O moves according to the same law as the automobile. Determine the velocity of this point at the instant considered

$$v_O = \frac{dS}{dt} = 2 + 4t \text{ m/sec.}$$

At $t = 2$ sec

$$v_O = 2 + 4 \times 2 = 10 \text{ m/sec.}$$

When rolling on the straight track the wheel is in plane motion. The instantaneous centre of zero velocity or of rotation is at the point C where the wheel makes contact with the fixed road surface (see **Case 2**, Sec. 73).

The velocities of the points of a plane figure are proportional to their distance from the instantaneous centre of rotation, i.e.,

$$\frac{v_O}{OC} = \frac{v_A}{AC} = \frac{v_B}{BC}.$$

Since $AC = BC$, the velocities of points A and B are equal in magnitude, $v_A = v_B$.

By the above proportion we have

$$v_A = v_B = v_O \frac{AC}{OC}.$$

From a consideration of the isosceles right triangle AOC, we find

$$\frac{AC}{OC} = \frac{1}{\cos 45^\circ} = 1.41.$$

Substituting the value of this ratio in the expression for the velocities, we obtain

$$v_A = v_B = v_O \frac{AC}{OC} = 10 \times 1.41 = 14.1 \text{ m/sec.}$$

The velocity vectors of points A and B are perpendicular to straight lines joining these points to the instantaneous centre of rotation (Fig. 130).

Example 50. A wheel of radius $r_2 = 9$ cm rolls without slipping along the inside of a fixed cylinder of radius $r_1 =$ 24 cm (Fig. 131). The small wheel is driven by a crank OA rotating at 120 rpm ($n_O = 120$ rpm).

Determine the number of revolutions per minute for the small wheel.

Solution. Determine the velocity of the centre A of the small wheel

$$v_A = \omega_O \, OA,$$

where $\omega_O = \frac{\pi n_O}{30}$ is the angular velocity of the crenk, $OA = r_1 - r_2$ is the length of the crank.

Substituting the values of ω_O and OA, we obtain

$$v_A = \frac{\pi n_O}{30} (r_1 - r_2) = \frac{3.14 \times 120}{30} (24 - 9) = 188 \text{ cm/sec.}$$

The instantaneous centre of zero velocity C of the wheel coincides with the point of contact with the cylinder (**Case 2**). The velocity of point A can be expressed in terms of the instantaneous angular velocity

$$v_A = \omega r_2,$$

whence

$$\omega = \frac{v_A}{r_2} = \frac{188}{9} = 20.9 \frac{1}{\text{sec}}.$$

Taking into account that $\omega = \frac{\pi n}{30}$, we find the corresponding number of revolutions per minute for the wheel

$$n = \frac{30\,\omega}{\pi} = \frac{30 \times 20.9}{3.14} = 200 \text{ rpm}.$$

Example 51. Two parallel racks AB and DE move in opposite directions with velocities $v_1 = 60$ cm/sec and $v_2 = 20$ cm/sec, respectively. A disk of radius $r = 12$ cm is

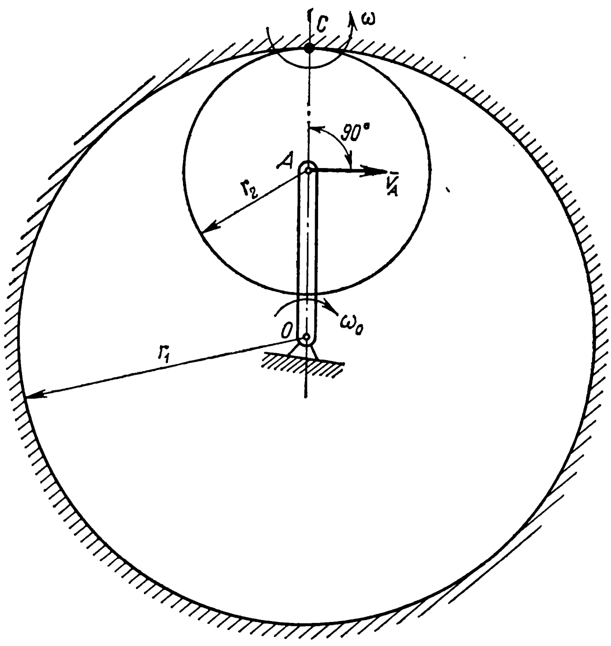


Fig. 131.

clamped between the racks. The points of the disk which are in contact with the racks move with the same velocities as the racks (Fig. 132a).

Determine the velocity of the centre O of the disk and its angular velocity.

Solution. The disk is in plane motion. The determination of the instantaneous centre of zero velocity reduces to **Case 3.** The instantaneous centre of zero velocity C is at the intersection of a line joining points M and N of the disk and a line joining the extremities of the velocities of these points

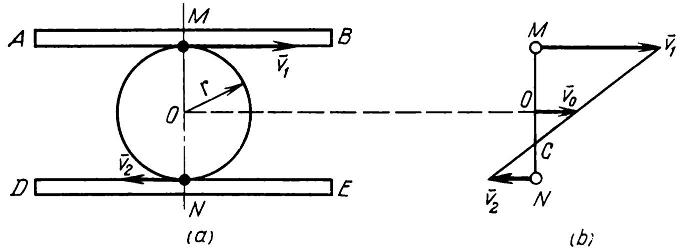


Fig. 132.

(Fig. 132*b*). The instantaneous centre of zero velocity C divides the diameter MN of the disk in proportion to the velocities, i.e.,

$$\frac{MC}{CN}=\frac{MN-CN}{CN}=\frac{v_1}{v_2},$$

whence

$$CN=\frac{MN}{1+\frac{v_1}{v_2}}=\frac{2r}{1+\frac{v_1}{v_2}}=\frac{2\times 12}{1+\frac{60}{20}}=6 \text{ cm}$$

and accordingly $CO=r-CN=12-6=6$ cm. The angular velocity of the disk is

$$\omega=\frac{v_2}{CN}=\frac{20}{6}=3.33\,\frac{1}{\text{sec}}\,.$$

The velocity of the centre O of the disk is

$$v_O=OC\,\omega=6\times 3.33=20 \text{ cm/sec}.$$

Example 52. The crank of a slider-crank mechanism rotates uniformly at $n = 180$ rpm (Fig. 133*a*). The length of the crank is $OA = r = 40$ cm, the length of the connecting rod $AB = l = 2$m.

Determine the angular velocity of the connecting rod and the velocity of its mid-point M in four positions of the crank for which the angle AOB is 0, $\pi/2$, π, $3\pi/2$, respectively.

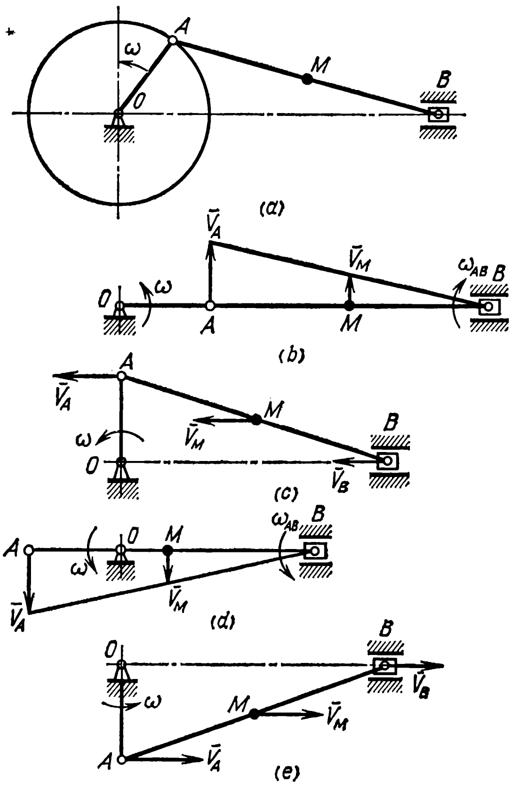


Fig. 133.

Solution. Calculate the velocity of point A

$$v_A = \frac{\pi n}{30} OA = \frac{\pi \times 180}{30} 0.4 = 2.4\,\pi \text{ m/sec.}$$

Consider now the four positions of the mechanism in succession.

Position I (Fig. 133*b*). Determine the instantaneous centre of zero velocity of the connecting rod AB. We know the directions of the velocities of two points of the rod. The velocity $\bar{v}_A$ of point A is always perpendicular to the crank OA, in the position considered it is vertical. The velocity of point B is always parallel to the guides, i.e., always horizontal. The determination of the instantaneous centre of zero velocity reduces to **Case 1** (see Sec. 73). The instantaneous centre of zero velocity is at the intersection of perpendiculars to the directions of the velocities. In the position under consideration, this point of intersection coincides with the point B which is therefore the instantaneous centre of zero velocity.

The instantaneous angular velocity of the connecting rod is

$$\omega_{BA} = \frac{v_A}{AB} = \frac{2.4\pi}{2} = 1.2\pi \frac{1}{\text{sec}} .$$

The velocity of point M is

$$v_M = \omega_{BA} MB = \frac{v_A}{AB} MB = \frac{v_A}{2} = 1.2\pi \text{ m/sec}$$

since $MB = \frac{AB}{2}$ according to the condition of the problem.

Position II (Fig. 133*c*). In this position the instantaneous centre of zero velocity is at infinity as the velocities of points A and B are parallel. Therefore

$$\omega_{AB} = \frac{v_A}{\infty} = 0.$$

The connecting rod AB is in translation at the instant considered

$$v_A = v_B = v_M = 2.4\pi \text{ m/sec}.$$

Position III (Fig. 133*d*). Similarly to position *I*, the instantaneous centre of zero velocity is the point B. For this position

$$\omega_{AB} = \frac{v_A}{AB} = \frac{2.4\pi}{2} = 1.2\pi \frac{1}{\text{sec}} ,$$

$$v_M = \omega_{AB} MB = \frac{v_A}{AB} MB = \frac{v_A}{2} = 1.2\pi \text{ m/sec}.$$

Position IV (Fig. 133*e*). For this position we have, similarly to position *II*

$$\omega_{AB} = \frac{v_A}{\infty} = 0,$$

$$v_A = v_B = v_M = 2.4\pi \text{ m/sec}.$$

74. Determination of the Acceleration of Any Point of a Body in Plane Motion

As with velocities, the acceleration of any point of a body in plane motion can be represented as the acceleration in complex motion. In other words, the total acceleration of an

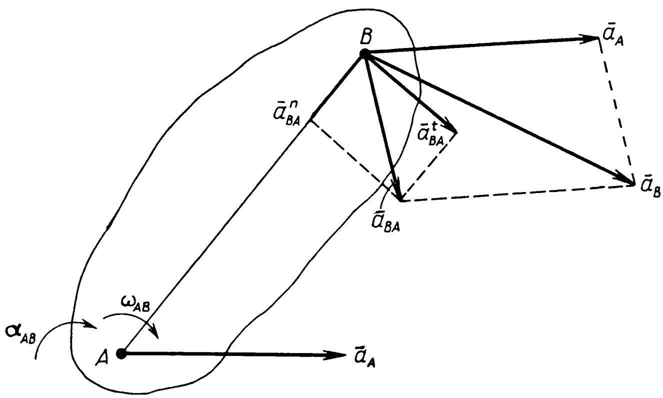


Fig. 134.

arbitrary point B of a body in plane motion is determined as the geometric sum of two accelerations: the base translational acceleration $\bar{a}_A$ of a pole A and the relative rotational acceleration $\bar{a}_{BA}$ with respect to this pole (Fig. 134)

$$\bar{a}_B = \bar{a}_A + \bar{a}_{BA}. \tag{140}$$

In the general case, the relative acceleration in the rotation is in turn composed of the normal and tangential components

$$\bar{a}_{BA} = \bar{a}^n_{BA} + \bar{a}^t_{BA}. \tag{141}$$

Substituting for the relative acceleration in the expression for the total acceleration, we obtain finally

$$\bar{a}_B = \bar{a}_A + \bar{a}^n_{BA} + \bar{a}^t_{BA}. \tag{142}$$

Thus, *the total acceleration of any point of a body in plane motion is determined as the geometric sum of three components*:

(1) base acceleration $\bar{a}_A$ of an arbitrary pole,

(2) normal component of relative acceleration $\bar{a}^n_{BA}$,

(3) tangential component of relative acceleration $\bar{a}^t_{BA}$.

Formula (142) defines the distribution of accelerations in the case of the plane motion of a body.

The normal component of the relative acceleration is directed along the radius of rotation and is equal to

$$a^n_{BA} = \frac{v^2_{BA}}{AB} = \omega^2_{BA} AB,$$

where v_{BA} = relative velocity of point B with respect to pole A,

ω_{BA} = angular velocity of body.

The tangential component of the relative acceleration is perpendicular to the radius of rotation and is equal to

$$a^t_{BA} = \alpha_{BA} BA,$$

where α_{BA} = angular acceleration of body.

The graphical construction of the total acceleration of point B is presented in Fig. 134.

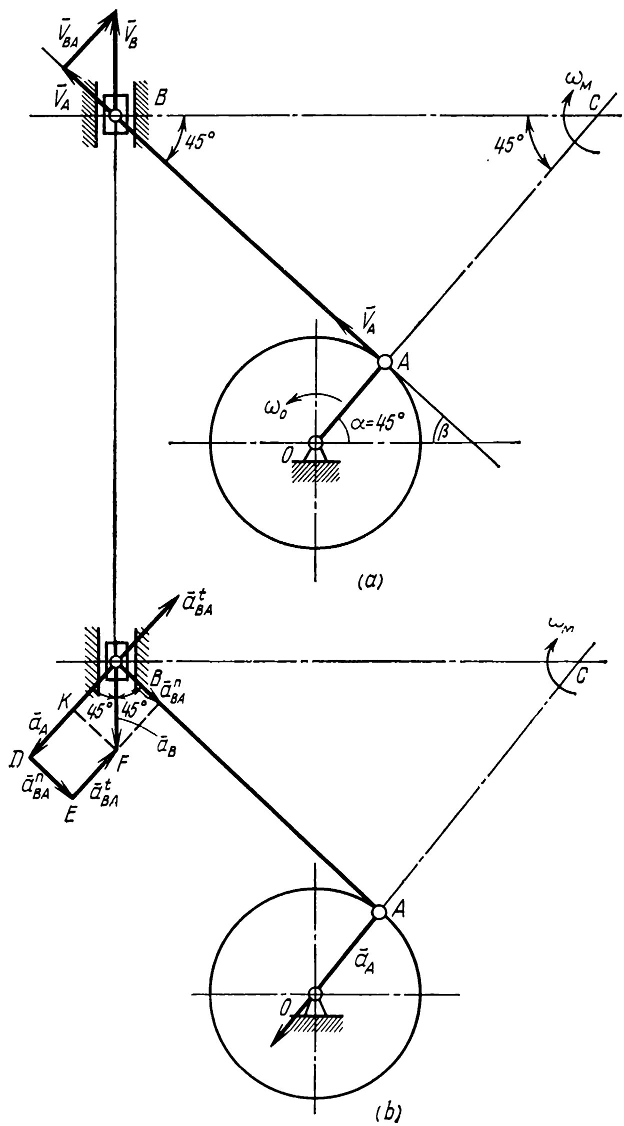
Example 53. A crank OA of length 20 cm rotates uniformly with an angular velocity $\omega = 10 \frac{1}{\text{sec}}$ and drives a connecting rod AB of length 100 cm; the slider B moves along a vertical.

Find the angular velocity and angular acceleration of the connecting rod and the acceleration of the slider B at the time when the crank and the connecting rod are mutually perpendicular and make angles $\alpha = 45°$ and $\beta = 45°$ with a horizontal axis (Fig. 135*a*).

Solution. According to the condition of the problem, $OA = 20$ cm, $\omega_O = 10 \frac{1}{\text{sec}}$, $AB = 100$ cm, $\alpha = \beta = 45°$. The velocity of point A of the crank is

$$v_A = OA\,\omega_O = 20 \times 10 = 200 \text{ cm/sec}.$$

The determination of the instantaneous centre of zero velocity (point C) of the connecting rod AB is shown in Fig. 135*a*. The velocity of point A of the connecting rod AB can be determined by considering the rotation of the rod AB



$\bar{V}_{BA}$
$\bar{V}_B$
$\bar{V}_A$
B
45°
ω_M
C
45°
$\bar{V}_A$
A
ω_0
$\alpha=45°$
β
O
(a)
$\bar{a}^t_{BA}$
ω_M
C
K
B
$\bar{a}^n_{BA}$
45°
45°
$\bar{a}_A$
$\bar{a}_B$
D
F
$\bar{a}^n_{BA}$
$\bar{a}^t_{BA}$
E
A
$\bar{a}_A$
O
(b)

Fig. 135.

about the instantaneous centre

$$v_A = \omega_M AC,$$

but $AC = AB = 100$ cm (see Fig. 135*a*).

Then

$$\omega_M = \frac{v_A}{AC} = \frac{200}{100} = 2 \frac{1}{\text{sec}}.$$

Determine the velocity of point B by using the instantaneous centre of zero velocity

$$BC = \sqrt{AB^2 + AC^2} = \sqrt{100^2 + 100^2} = \sqrt{20{,}000} \cong 141 \text{ cm},$$

then

$$v_B = \omega_M BC = 2 \times 141 = 282 \text{ cm/sec}.$$

The angular velocity of the connecting rod AB is

$$\omega_{AB} = \frac{v_{BA}}{BA} \frac{1}{\text{sec}},$$

where $\bar{v}_{BA}$ is the relative velocity (see Fig. 135*a*)

$$v_{BA} = v_A = 200 \text{ cm/sec}.$$

Substituting the numerical values, we find

$$\omega_{AB} = \frac{200}{100} = 2 \frac{1}{\text{sec}}.$$

In order to find the angular acceleration of the connecting rod AB, consider the accelerations of its points A and B.

Point A rotates uniformly about point O and has only a normal acceleration

$$a_A = OA\,\omega_O^2 = 20 \times 10^2 = 2{,}000 \text{ cm/sec}^2.$$

If point A is chosen as a pole, the acceleration of point B is determined according to the equation

$$\bar{a}_B = \bar{a}_A + \bar{a}_{BA}^n + \bar{a}_{BA}^t.$$

The normal component of the relative acceleration $\bar{a}_{BA}^n$ is directed along the connecting rod AB from point B to point A (Fig. 135*b*) and can easily be calculated

$$a_{BA}^n = AB\,\omega_{BA}^2 = 100 \times 2^2 = 400 \text{ cm/sec}^2.$$

Figure 135*b* illustrates the geometric addition of the vectors $\bar{a}_A$ and $\bar{a}^n_{BA}$. To obtain the total acceleration of point B it is necessary to add to the vectors laid off the tangential acceleration $\bar{a}^t_{BA}$ of point B relative to A. This acceleration is perpendicular to the connecting rod AB, while the absolute acceleration of point B is directed along the vertical, along the guides. Figure 135*b* represents the acceleration polygon $BDEF$ constructed on the basis of the foregoing ($BD = a_A$, $DE = a^n_{BA}$, $EF = a^t_{BA}$, $BF = a_B$).

Using this polygon we calculate the magnitude of the tangential component of the relative acceleration.

From the drawing of Fig. 135*b* we have

$$BK = KF = DE = a^n_{BA} = 400 \text{ cm/sec}^2,$$

but from the same drawing

$$a^t_{BA} = EF = DK = BD - BK = 2{,}000 - 400 = 1{,}600 \text{ cm/sec}^2.$$

On the other hand

$$a^t_{BA} = \alpha\, AB = \alpha \times 100,$$

whence

$$\alpha = \frac{a^t_{BA}}{100} = \frac{1{,}600}{100} = 16 \frac{1}{\text{sec}^2}.$$

The acceleration of the slider B is

$$a_B = \frac{a^n_{BA}}{\cos 45^\circ} = \frac{400 \times 2}{\sqrt{2}} = 565.6 \text{ cm/sec}^2.$$

75. Planetary Gearing

In engineering practice wide use is made of planetary and differential gears some members of which have plane motion. In analysing these gears and determining their gear ratios the laws of plane motion are employed. The most convenient method of analysis is to resolve the motion of the whole mechanism into components.

Gears with one driving member and one or several driven members in which the axes of some wheels move in space are called *planetary gears* (the motion of members of these gears is similar to that of planets).

The axis O_1O_1 (Fig. 136) about which the input and output shafts of the planetary gear rotate is called the *basic axis*, the members A, H, C are called the *basic members*. In a planetary gear, one of the three basic members (other than H) is always stationary, one is the driving and one is the driven member. The basic members A and C, which have the form

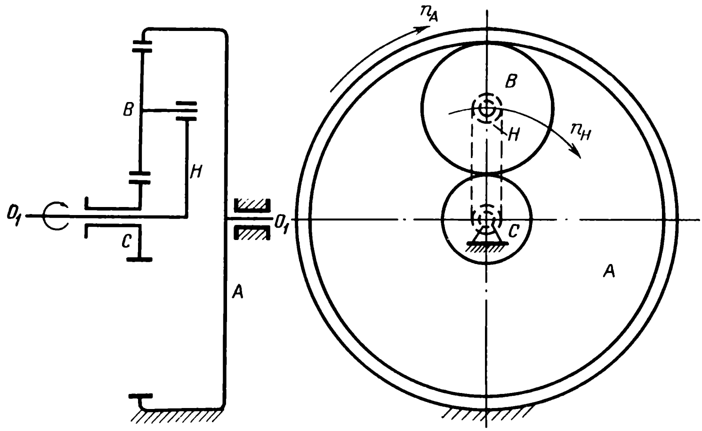


Fig. 136.

of gear wheels, are also referred to as *sun wheels*. The member H, which carries a moving axle, is called a *planet carrier*, and the wheel B, which rotates about the moving axis, is called a *planet wheel*. The motion of the planet wheel is a plane motion and can be resolved into components, viz. a base motion with a pole and a relative motion about this pole. The main purpose of planetary gears is to transmit motion with large gear ratios.

Planetary drives are usually made with spur or helical gears. Determine the gear ratio of a planetary drive by resolving the motion into components and subsequently summing them up.

Consider a planetary mechanism consisting of two sun wheels A and C, a planet wheel B and a planet carrier H.

Let the wheel A be stationary, the planet carrier H the driving, and the wheel C the driven member. The planet wheel B is in plane motion: it moves with the planet carrier (base motion) and relative to the planet carrier in the rotation.

Imagine that the stationary member A is disconnected from the frame and that all the moving members (H, B and C) are locked together and rotated as a unit one complete revolution clockwise. Clearly, all members of the mechanism will make this revolution. The results are entered in the first line of Table 1.

Table 1

Resolution of Motion in Planetary Mechanism

Nature of Rotation	Wheel A	Planet B	Wheel C	Planet Carrier H
Rotation of locked system, one revolution	$+1$	$+1$	$+1$	$+1$
Rotation of wheel A in opposite direction with planet carrier held stationary, one revolution	-1	$+\frac{Z_A}{Z_B}$	$+\frac{Z_A}{Z_C}$	0
Total	0	$1+\frac{Z_A}{Z_B}$	$1+\frac{Z_A}{Z_C}$	1

Since the wheel A, which is stationary in this case, has received one revolution (+1) we should reverse it by one revolution (—1) in order to replace it in the original position. The planet carrier H, which has already made +1 revolution, will then be stationary. We obtain, not a planetary, but an ordinary gear drive, and the determination of the number of revolutions of the wheels B and C presents no difficulty. It is only necessary to take into account the direction of rotation (external gearing changes the direction and sign of rotation whereas internal gearing does not). The results of the computation are summarized in Table 1.

From the third line it is seen that the sun wheel A is stationary, the planet carrier H makes one revolution, the other sun wheel C makes $1 + Z_A/Z_C$ revolutions. It is apparent that the gear ratio between the planet carrier and the wheel C with the wheel A stationary is

$$i_{H-C}=\frac{n_H}{n_C}=\frac{1}{1+\frac{Z_A}{Z_C}}=\frac{Z_C}{Z_C+Z_A}. \tag{143}$$

Thus, if the planet carrier makes n_H rpm, the number of revolutions of the wheel C is

$$n_C = n_H \frac{Z_C + Z_A}{Z_C} \text{ rpm.} \tag{144}$$

76. Differential Gearing

Differentials, or *equalizers*, differ from planetary drives in the absence of stationary wheels. All their members are moving members. This makes it possible to sum on the basic driven member motions transmitted from two independent sources through two basic driving members.

Differential drives may be of the spur gear or bevel gear type. The gear ratio of a differential drive can be determined by the same tabular method but, because of the presence of two driving members, the computation is carried out in two stages (Fig. 136):

(a) find the gear ratio from one driving member (for example, A) assuming the second driving member (for example, H) to be stationary, i^H_{AC};

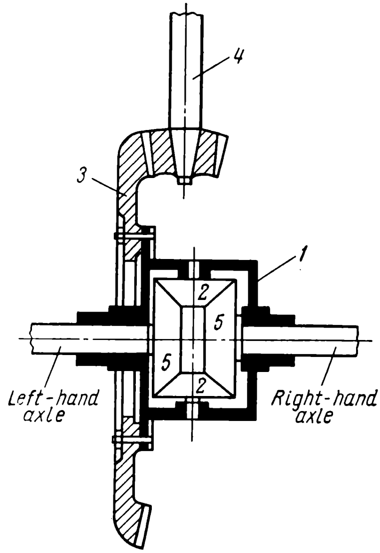


Fig. 137.

(b) find the gear ratio from the second driving member, assuming the first driving member to be stationary, i^A_{HC};

(c) determine the number of revolutions of the driven member C

$$n_C = n_A \frac{1}{i^H_{AC}} + n_H \frac{1}{i^A_{HC}} . \tag{145}$$

Clearly, to find the motion of the driven member one must know the motions of both driving members.

An example of such gearing is the differential (equalizer) of an automobile which makes it possible for the driving wheels to rotate with different numbers of revolutions (for example, when making a turn).

Figure 137 shows a diagrammatic sketch of a bevel gear automobile differential. A carrier *1* is rigidly fastened to a large bevel (driven) gear *3*. The torque is transmitted from the automobile engine through bevel gears *4* and *3* of the

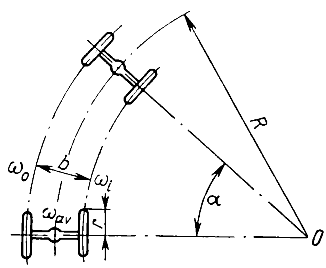


Fig. 138.

final drive to the carrier *1* and then is distributed between axle shafts *5* through planet gears *2*.

When the number of revolutions of one wheel differs from the number of revolutions of the other, gears *2* will rotate about their own axes. Because of this the driving wheels, subjected to a constant torque, can rotate with different angular velocities on curved sections or irregularities of the road. When the automobile is travelling on a straight section of road and the resistance to the motion of the driving wheels (right-hand and left-hand) is the same, the planet gears will not rotate about their axes; they will be "rigidly fastened" to the carrier and axle shafts and so to the wheels. In this position the planet gears will rotate together with the carrier with an angular velocity ω_{av}.

On a straight section of road the distance travelled by the driving wheels per second is

$$S_{av} = r\omega_{av}, \tag{146}$$

where r is the radius of the driving wheel, ω_{av} the angular velocity of the driving wheel on the straight section of road.

When the automobile is travelling on a curved section of road, the driving wheel which rolls along the inner curve with an angular velocity ω_i will rotate slower than the left-hand wheel which rolls along the outer curve with an angular velocity ω_o.

On the curved section of road the right-hand and left-hand driving wheels will travel different distances in one second.

In Fig. 138 it is seen that the right-hand wheel will travel the distance

$$S_i = r\omega_i = \left(R - \frac{b}{2}\right)\alpha \tag{147}$$

and the left-hand wheel will travel the distance

$$S_o = r\omega_o = \left(R + \frac{b}{2}\right)\alpha, \tag{148}$$

where R = radius of curvature of road,
b = width of track.

During the same time interval the mid-point of the drive shaft will travel the distance

$$S_{av} = r\omega_{av} = R\alpha, \tag{149}$$

whence

$$\omega_{av} = \frac{R\alpha}{r}. \tag{150}$$

By adding expressions (147) and (148) member by member, we obtain

$$r\omega_i + r\omega_o = \left(R - \frac{b}{2}\right)\alpha + \left(R + \frac{b}{2}\right)\alpha$$

or

$$\omega_i + \omega_o = 2\frac{R\alpha}{r}. \tag{151}$$

From equation (150) for ω_{av} we find

$$2\omega_{av} = 2\frac{R\alpha}{r}, \tag{152}$$

consequently,

$$\omega_i + \omega_o = 2\omega_{av}. \tag{153}$$

Formula (153) shows that the sum of the angular velocities of the driving wheels is twice the average angular velocity. If, for instance, $\omega_i = 0$, then $\omega_o = 2\omega_{av}$. This is possible only if the left-hand wheel is rolling on hard ground while the right-hand wheel is skidding.

Thus, the angular velocities of the driving wheels may vary from 0 to $2\omega_{av}$.

The kinematic analysis of differentials consists in determining the gear ratios and angular velocities of all rotating members. In differential drives, the relationship between the angular velocities is more complicated than for a simple gear drive.

PART 3. DYNAMICS

CHAPTER XIV

Basic Concepts and Axioms of Dynamics

77. Subject of Dynamics

Dynamics is a part of theoretical mechanics which deals with the motion of a particle or body under the action of applied forces; it establishes the relationship between the applied forces and the motion they produce.

Dynamics studies force systems which are not in a state of equilibrium and in this it differs essentially from statics. Dynamics also differs radically from kinematics where motion is assessed only from the geometrical point of view without reference to the cause of the motion. Dynamics is the most general division of theoretical mechanics, it uses the conclusions of both statics and kinematics and establishes the general laws of motion of particles and bodies as a function of the acting forces.

Dynamics is subdivided into two parts: the first studies the motion of a particle and the second the motion of rigid bodies. All rigid bodies are treated as systems consisting of a large number of separate particles.

The laws of motion of particles and rigid bodies established in dynamics are objective laws of nature. They are supported by numerous observations and experiments. On the basis of these laws it was possible more than once to make predictions which were subsequently confirmed.

The application of the laws of dynamics to the study of natural phenomena and in engineering did not clash with experience until the end of the XIX century, when a number of investigations led to irreconcilable contradictions between the laws of electrodynamics and classical mechanics. These contradictions gave rise to a new mechanics, the relativity theory. The laws of classical mechanics, the laws of dynamics, however, remain valid in engineering practice, in the range of so-called low velocities, i.e., velocities considerably less than the velocity of light.

78. Principle of Inertia

At the basis of dynamics are some axioms suggested by experience. In statics, axioms were presented which are common to statics and dynamics.

Below are given the axioms underlying the laws of dynamics. Some of these do not differ from the axioms of statics.

Consider the first axiom.

Axiom I (*law or principle of inertia*). *A force system applied to a particle is balanced if the particle is in a state of relative rest or moves with constant speed in a straight line under its action.*

This axiom has already been considered in statics. The first axiom as formulated above defines a balanced force system which, as previously noted (Sec. 2), is equivalent to zero and may be discarded.

However, there is one more meaning here which should be discussed in greater detail. In the case of relative rest or uniform rectilinear motion the acceleration of a particle is zero. Therefore, *under the action of a balanced force system or in the absence of force effects a particle is not accelerated and moves with constant speed in a straight line.*

A particle which is not acted upon by other particles or bodies is called an *isolated particle* in mechanics. Consequently, the acceleration of an isolated particle is always zero. A particle cannot change its velocity on its own. This requires an external action of another body.

The first axiom of dynamics expresses the fundamental property of a material body—its inability to impart acceleration to itself.

79. Fundamental Law of Dynamics of Particles

When a particle is acted upon by an unbalanced force system, it moves either in a curved path or with a non-uniform velocity, i.e., it has an acceleration. The relationship between the force acting on the particle and the acceleration caused by this force is established by the second axiom of dynamics.

Axiom II. The acceleration imparted to a particle by a force applied to it is proportional to the magnitude of the force and is in the direction of this force (Fig. 139)

$$\bar{P} = m\bar{a}, \tag{154}$$

where $\bar{P}$ = force (exerted by an external material body on particle) producing acceleration of particle,
$\bar{a}$ = acceleration imparted to particle,
m = factor of proportionality relating force to acceleration resulting from application of force.

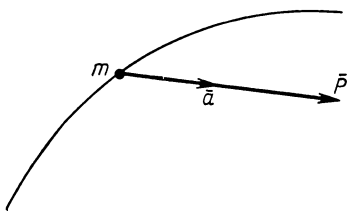


Fig. 139.

Equation (154) is called *the fundamental equation of dynamics* in vector form. It is also valid in scalar form

$$P = ma. \tag{154a}$$

The difference between the first and the second forms is that the vector form takes into account the direction of the force and acceleration vectors and the scalar form only the numerical values of these quantities.

The factor m appearing in the fundamental equation of dynamics has a very important physical meaning, it represents the mass of a particle.

Solving Eq. (154a) for the acceleration, we obtain

$$a = \frac{P}{m}. \tag{155}$$

It is seen from Eq. (155) that the larger the mass, the larger the force required to impart a specified acceleration to the body. Consequently, the mass characterizes the "inertness" of the body, i.e., its reluctance to yield to a force.

Thus, *the mass of a particle is a measure of its "inertness"*.

From Eq. (154a) we find the mass

$$m=\frac{P}{a}. \tag{156}$$

If Eq. (156) is applied to a particle subjected to a force of gravity $\bar{G}$, we obtain

$$m=\frac{G}{g}, \tag{156a}$$

where G = weight of body (force of gravity),
g = acceleration due to gravity, a quantity depending on the latitude (less at the equator and larger at the poles); for middle latitudes it is common to take

$$g=9.81 \text{ m/sec}^2.$$

It is seen from Eq. (156a) that the mass is proportional to the weight of a body. The inert mass of a particle is a scalar quantity which is always positive. From the point of view of theoretical mechanics, the mass of a particle or body is constant and independent of motion. The mass is a more comprehensive characteristic of a body than its weight as the weight is different at different points on the earth while the mass always remains the same. Physically the mass can be thought of as the quantity of matter contained in a body.

80. Systems of Units

The question of dimensions is of paramount importance in dynamics. Clearly, when solving problems one should express all data in a consistent system of units. The system generally employed at present is the international system of units (SI) the fundamental units in which are the units of length, time and mass. Sometimes use is still made of the engineers' system of units (mkgfs) the fundamental units in which are the units of length, time and force.

All other quantities encountered in engineering mechanics are measured in units derived from the fundamental units.

Consider the international system of units, SI.

In this system the unit of length is the metre (m), the unit of mass is the kilogram (kg), the unit of time is the second (sec).

The kilogram equals the mass of one litre of distilled water at 4°C. The unit of force is a derived unit in the SI system.

The unit of force in this system is defined as the force which gives an acceleration of 1 m/sec^2 to a mass of 1 kg. It is called the newton (N).

If we put $m = 1$ kg and $a = 1$ m/sec^2 in the formula $P = ma$, the dimension of force in the SI system is expressed as

$$[P] = [ma] = [1 \text{ kg} \times 1 \text{ m/sec}^2] = [\text{kg-m/sec}^2] = [\text{N}].$$

For convenience, multiples of the newton are also used, viz. kilonewtons (kN) and meganewtons (MN). One kilonewton equals a thousand newtons: 1 kN = 1,000 N; one meganewton a thousand kilonewtons or a million newtons: 1 MN = 1,000 kN = 10^6 N.

Consider now the engineers' system of units (mkgfs). In this system the unit of length is the metre (m), the unit of time is the second (sec) and the unit of force is the kilogram force (kgf).

The kilogram force (kgf) equals the weight of one litre of distilled water at 4°C.

In distinction to the units of mass we always include the letter f in the units of force. Thus, the kilogram force is denoted by kgf, the gram force by gf (the gram mass is denoted by g).

In the engineers' system of units the mass is measured in derived units. The unit of mass is defined as the mass of a body to which a force of one kilogram gives an acceleration of 1 m/sec^2. This unit is known as *the engineers' unit of mass.*

If we put $P = 1$ kgf and $a = 1$ m/sec^2 in the formula $m = \frac{P}{a}$, the dimension of mass is expressed as

$$[m] = \left[\frac{P}{a}\right] = \left[\frac{\text{kgf-sec}^2}{\text{m}}\right].$$

In engineering calculations it is sometimes necessary to determine the mass of a body from its weight or the weight from the mass.

If the weight G is given, the mass of the body is

$$m = \frac{G}{g},$$

where $g = 9.81$ m/sec² is the acceleration of a freely falling body.

When solving problems in dynamics it is essential to check dimensions carefully to avoid errors. Sometimes units of one system have to be converted into units of another system. This conversion is based on the fact known from physics that a mass of 1 kg weighs 1 kgf. The force of gravity acting on a mass of 1 kg is expressed in newtons as

$$G = mg = 1 \text{ kg} \times 9.81 \text{ m/sec}^2 = 9.81 \text{ kg-m/sec}^2 = 9.81 \text{ N}.$$

Thus, one kilogram force is equivalent to 9.81 newtons, i.e.,

$$1 \text{ kgf} = 9.81 \text{ N or } 1 \text{ N} = 0.102 \text{ kgf}.$$

If very high accuracy is not essential, it is conventionally taken that

$$1 \text{ kgf} \cong 10 \text{ N and } 1 \text{ N} \cong 0.1 \text{ kgf}.$$

81. Axiom of Superposition

An important axiom of dynamics is the axiom of superposition.

Axiom III. If several forces act on a particle, the acceleration received by the particle is the same as when acted upon by a single force equal to the geometric sum of these forces (Fig. 140).

A force which imparts to a particle the same acceleration as a given force system $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots, \bar{P}_n$ is called the *resultant force* in dynamics. Since the forces applied to a particle are concurrent, their resultant is always applied to the same particle and is determined (see Sec. 6) by the polygon rule (Fig. 140).

The concepts of the resultant are equivalent in statics and dynamics.

Thus, the significance of the third axiom of dynamics resides in the fact that several forces $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots, \bar{P}_n$ acting on a particle can be replaced by their resultant $\bar{R}$

$$\bar{R} = \bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \ldots + \bar{P}_n.$$

The fundamental law of dynamics (Axiom II) becomes in this case

$$m\bar{a}=\bar{R}.$$

Substituting the expression for the resultant $\bar{R}$ in the last equality, we obtain

$$m\bar{a}=\bar{P}_1+\bar{P}_2+\bar{P}_3+\ldots+\bar{P}_n. \tag{157}$$

Dividing all terms of Eq. (157) by m gives

$$\bar{a}=\frac{\bar{P}_1}{m}+\frac{\bar{P}_2}{m}+\frac{\bar{P}_3}{m}+\ldots+\frac{\bar{P}_n}{m}. \tag{158}$$

The terms on the right-hand side of the last equation represent the accelerations of the particle produced by each of

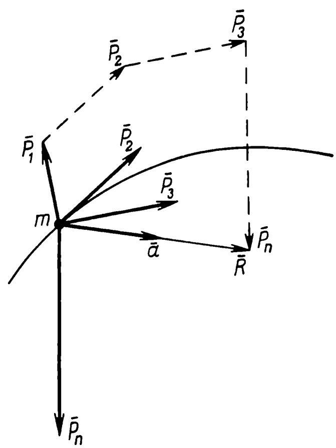


Fig. 140.

the applied forces acting separately

$$\frac{\bar{P}_1}{m}=\bar{a}_1,\quad \frac{\bar{P}_2}{m}=\bar{a}_2,\quad \frac{\bar{P}_3}{m}=\bar{a}_3,\quad \ldots,\quad \frac{\bar{P}_n}{m}=\bar{a}_n.$$

Taking this into account, we obtain finally

$$\bar{a}=\bar{a}_1+\bar{a}_2+\bar{a}_3+\ldots+\bar{a}_n, \tag{159}$$

i.e., *the acceleration received by a particle under the action of several forces can be determined as the geometric sum of the accelerations produced by each of the forces acting separately.*

82. Axiom of Interaction

Consider the last axiom of dynamics.

Axiom IV. To every action corresponds an equal and opposite reaction. In other words, the forces exerted by particles on

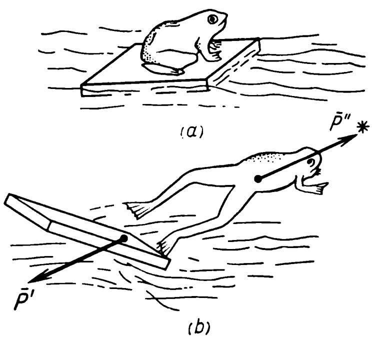


Fig. 141.

each other are always equal in magnitude and directed along a straight line joining these particles in opposite sense.

This axiom is known from statics. Without repeating the elementary explanations given in Sec. 2, we shall nevertheless discuss this question in some detail. The great importance of this axiom resides in the fact that it provides the basis for the explanation of the nature of so-called reaction propulsion, which will be illustrated by a simple example.

Imagine a frog resting on a piece of wood which lies on the still surface of a pond (Fig. 141*a*). Suppose that the frog and the piece of wood have the same weight. Noticing a fly at some distance, the frog jumps to catch it (Fig. 141*b*). At the same instant the piece of wood moves away in a direction opposite to that of the frog's jump. If the water resistance is neglected, the piece of wood will move away from the initial position through the same distance as the frog provided their weights are equal. The motion of both the frog and the piece of wood has been caused by the forces of interaction between them. The force $\overline{P}'$ exerted by the frog on the piece of wood during the jump is the action and the reactive force $\overline{P}''$ is the reaction (Fig. 141*b*).

The forces $\overline{P}'$ and $\overline{P}''$ are equal in magnitude but are applied to different bodies. The cause of the frog's motion is the reactive force, $\overline{P}''$, therefore this type of motion is called *reaction propulsion*.

Thus, we have

$$P' = P''. \tag{160}$$

According to the fundamental law of dynamics we can write

$$P' = m_1 a_1, \quad P'' = m_2 a_2,$$

where m_1 and m_2 are the masses of the piece of wood and the frog, respectively, a_1 and a_2 the accelerations of the piece of wood and the frog.

Substituting the values of the forces P' and P'' in Eq. (160), we obtain

$$m_1 a_1 = m_2 a_2,$$

whence

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}, \tag{161}$$

i.e., *the accelerations due to the interaction of two bodies are inversely proportional to their masses.*

The propulsion of a rocket occurs in much the same way as the motion of the frog. Just as the frog pushes off from the piece of wood, so the rocket pushes off from the gases ejected from its nozzle. When ejected from the rocket's nozzle, the gases throw it in an opposite direction and this happens not only in air but also in vacuum. This jet propulsion has no connection with "pushing off from the air". That is why the principle of jet propulsion is used in space rockets and in launching artificial satellites into space.

83. Two Basic Problems of Dynamics

As noted above (Sec. 77), dynamics establishes the relationship between the applied forces and the motion they cause. Two basic problems are distinguished which are solved on the basis of the axioms of dynamics set forth in Sections 79 and 81.

The first problem of dynamics is: Given the motion of a particle, determine the forces acting on it. This is the so-called direct problem of dynamics. To solve it, one should first determine the acceleration of the particle. The methods of determining the acceleration of a particle depend on the methods of specifying its motion, they are described in Sec. 52. Once the acceleration of the particle is determined, we next use the fundamental law of dynamics (Axiom II) and find the acting force. If several forces act on a particle and only some of them are unknown, we have to use the axiom of superposition (Axiom III).

The second problem of dynamics is: Given the forces, determine the motion of a particle. This is termed the inverse problem of dynamics and its solution is generally much more complicated than that of the direct problem.

Here again, use should be made of the fundamental law of dynamics (Axiom II). From this law we determine the acceleration in terms of the acting force and the given mass of the particle.

Although the acceleration of the particle may be known, it is not yet possible to determine completely its motion or to calculate, if required, the distance travelled by the particle or the time of motion. To solve the inverse problem it is necessary to have some additional data which are called initial conditions. They should define the velocity and position of the particle at a certain instant. Knowing the acceleration, we can then find the velocity, the distance travelled and other kinematic characteristics for any instant. The concept of initial conditions will be illustrated by detailed examples below.

In practical applications various modifications of the above problems may be encountered. When solving a problem, one should determine the type to which it belongs and carry out the solution accordingly.

Consider some examples.

Example 54. A particle of weight $G = 100$ N moves along a smooth horizontal surface (Fig. 142) with an acceleration $a = 1.5$ m/sec^2.

Neglecting resistance, determine the force $\bar{P}$ required to produce the motion. The acceleration of a freely falling body is $g = 9.81$ m/sec².

Solution. Given here is the motion of the particle and it is required to determine the force (the first, direct, problem of dynamics). The particle is acted upon by three forces:

(1) the weight $\bar{G}$,

(2) the reaction of the smooth horizontal plane $\bar{N}$,

(3) the moving force $\bar{P}$.

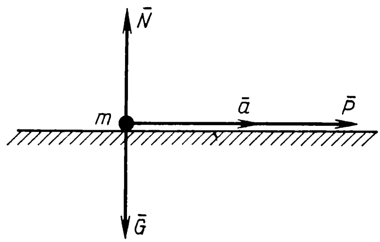


Fig. 142.

The forces $\bar{G}$ and $\bar{N}$ are balanced; consequently, the fundamental equation of dynamics can be written as

$$ma = P.$$

The mass of the body is determined as the quotient of the weight and the acceleration of gravity

$$m = \frac{G}{g} = \frac{100}{9.81} = 10.2 \text{ kg}.$$

Substituting the value of the mass in the fundamental equation, we obtain

$$P = ma = 10.2 \times 1.5 = 15.3 \text{ N}.$$

Example 55. A body M falling from a height $H = 1{,}500$ m by gravity overcomes an air resistance $\bar{R}$. Assuming the resistance to be constant and equal to half the weight, find the acceleration a of the body and its velocity v, 5 sec after the motion starts, if the initial velocity $\bar{v}_0$ is zero, i.e., $v_0 = 0$; $g = 9.81$ m/sec² (Fig. 143). Also determine the time of fall of the body.

Solution. Given here are the forces and it is required to determine the elements of motion: acceleration, velocity and time of motion (the inverse problem of dynamics). From the fundamental equation we have

$$ma = G - R,$$

but

$$m = \frac{G}{g}, \quad R = 0.5G,$$

then

$$\frac{G}{g} a = G - 0.5G = 0.5G,$$

whence

$$a = 0.5g = 0.5 \times 9.81 = 4.9 \text{ m/sec}^2.$$

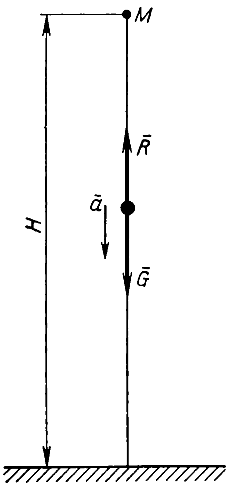


Fig. 143.

The magnitude of the acceleration a does not vary as the acting forces are constant. Consequently, the body is in uniformly accelerated motion, a = constant. For uniformly accelerated motion, the velocity is given by the formula

$$v = v_0 + at,$$

where $v_0 = 0$ according to the condition of the problem. This is the initial condition from which we can find the magnitude of the velocity at any instant; at $t = 5$ sec

$$v = v_0 + at = 0 + 4.9 \times 5 = 24.5 \text{ m/sec}.$$

To determine the time of fall we write the equation for the distance travelled in uniformly accelerated motion

$$H = v_0 t + \frac{at^2}{2}.$$

Since $v_0 = 0$, we obtain

$$t = \sqrt{\frac{2H}{a}} = \sqrt{\frac{2 \times 1{,}500}{4.9}} = 24.8 \text{ sec}.$$

Thus, the body will fall on the earth in 24.8 sec.

As follows from the foregoing discussion and examples, the fundamental law of dynamics is used to solve the direct and inverse problems of dynamics. It is often found more convenient (especially when a particle is in curvilinear motion) to apply this law in an alternate form, in projections on the co-ordinate axes.

Let us write the fundamental law of dynamics as a vector equality

$$m\bar{a} = \bar{P}.$$

Assuming for simplicity that the motion of a particle occurs in the plane of the drawing, we project both members

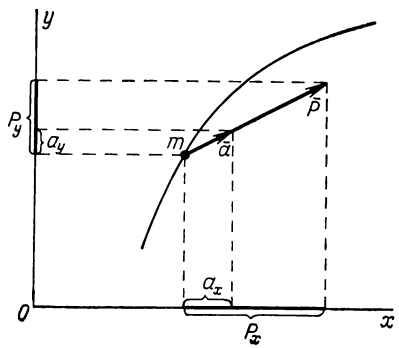


Fig. 144.

of this vector equality on two co-ordinate axes (Fig. 144) and obtain

$$\left.\begin{aligned} ma_x &= P_x, \\ ma_y &= P_y. \end{aligned}\right\} \qquad (162)$$

As is known from kinematics (Sec. 52), the projections of the acceleration of a particle on the co-ordinate axes a_x and a_y are expressed in terms of the second derivatives of its co-ordinates

$$a_x = \frac{d^2x}{dt^2}, \quad a_y = \frac{d^2y}{dt^2}.$$

Substituting the projections of the acceleration in the preceding equations, we have

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= P_x, \\ m\frac{d^2y}{dt^2} &= P_y. \end{aligned}\right\} \qquad (163)$$

Relations (163) are called the *differential equations of motion of a particle*; they express the fundamental law of motion of a particle in co-ordinate form.

If several forces act on a particle simultaneously, its differential equations of motion should be represented, according to Axiom III, as

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= R_x, \\ m\frac{d^2y}{dt^2} &= R_y, \end{aligned}\right\} \tag{164}$$

where R_x and R_y are the projections on the x and y axes of the resultant of all forces applied to the particle.

The differential equations of motion, as well as the fundamental law of dynamics, enable one to solve both the direct and inverse problems of dynamics. The direct problem is solved by differentiating given equations of motion of a particle. The inverse problem is reduced to the integration of the differential equations of motion of a particle. The integration involves arbitrary constants whose values are determined from the given initial conditions discussed above.

The solution of two basic problems of dynamics by using differential equations is illustrated below by examples.

Example 56. The motion of a particle of mass $m = 3$ kg is defined by the equations

$$x = 2\sin 2t, \quad y = 2\cos 2t \quad (x \text{ and } y \text{ in m, } t \text{ in sec}).$$

Determine the magnitude and direction of the force acting on the particle.

Solution. This is a direct problem. Differentiate the given equations of motion twice

$$\frac{d^2x}{dt^2} = -8\sin 2t,$$

$$\frac{d^2y}{dt^2} = -8\cos 2t.$$

Substitute the values of the derivatives and the mass in the differential equations of motion of the particle

$$P_x = \frac{m\,d^2x}{dt^2} = -3\times 8\sin 2t =$$

$$= -24\sin 2t \text{ kg-m/sec}^2 = -24\sin 2t\text{ N},$$

$$P_y = \frac{m\,d^2y}{dt^2} = -3 \times 8 \cos 2t =$$
$$= -24 \cos 2t \text{ kg-m/sec}^2 = -24 \cos 2t \text{ N}.$$

From the known projections of the force we determine its magnitude and direction

$$P = \sqrt{P_x^2 + P_y^2} = \sqrt{(-24 \sin 2t)^2 + (-24 \cos 2t)^2} = 24 \text{ N},$$

$$\cos(\bar{P}, x) = \frac{P_x}{P} = -\frac{24 \sin 2t}{24} = -\sin 2t.$$

Example 57. A particle of mass m is projected with a velocity $\bar{v}_0$ at an angle α to the horizon (Fig. 145).

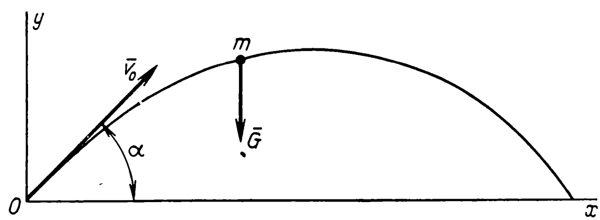


Fig. 145.

Neglecting air resistance, determine the path followed by the particle.

Solution. Here we have an inverse problem. Consider the particle in an arbitrary position and set up the differential equations of motion. The only force acting on the particle is its weight, $G = mg$. Hence the differential equations take the form

$$\frac{m\,d^2x}{dt^2} = P_x = 0,$$

$$\frac{m\,d^2y}{dt^2} = P_y = -G = -mg.$$

Cancelling the constant factor, we obtain

$$\frac{d^2x}{dt^2} = 0,$$

$$\frac{d^2y}{dt^2} = -g.$$

Integrate these equations once

$$\left.\begin{aligned}\frac{dx}{dt}&=C_1,\\ \frac{dy}{dt}&=-gt+C_2.\end{aligned}\right\}\tag{a}$$

Integrate a second time

$$\left.\begin{aligned}x&=C_1t+D_1,\\ y&=-\frac{gt^2}{2}+C_2t+D_2.\end{aligned}\right\}\tag{b}$$

Here C_1, C_2, D_1, D_2 are constants of integration which are determined from the initial conditions of the problem.

Since the particle starts from the origin of co-ordinates, we have: when $t=0$, (1) $x=x_0=0$ and (2) $y=y_0=0$. We also know the initial velocity of the particle v_0 and its direction—the angle α. Consequently, when $t=0$, (3) $v_x = v_0\cos\alpha$ and (4) $v_y=v_0\sin\alpha$.

Substituting $t=0$ first in equations (b) and then in equations (a), we obtain

$$D_1=x_0=0,\quad C_1=v_0\cos\alpha,$$
$$D_2=y_0=0,\quad C_2=v_0\sin\alpha.$$

We have finally

$$x=v_0t\cos\alpha,$$
$$y=-\frac{gt^2}{2}+v_0t\sin\alpha.$$

Eliminate the time t from these equations. From the first equation we find

$$t=\frac{x}{v_0\cos\alpha}.$$

Substituting in the second equation, we determine the equation of the path

$$y=-\frac{gx^2}{2v_0^2\cos^2\alpha}+x\tan\alpha.$$

This is the equation of a parabola.

Consequently, a body projected at an angle to the horizon moves by gravity in a parabolic path.

The equations of motion derived above can be used to determine some parameters of the path. Let it be required, for example, to determine the horizontal range of the body. As is seen from Fig. 145, when the particle is at a maximum distance from the origin it is located on the x axis and the co-ordinate y is zero. We set y equal to zero

$$y = -\frac{gt^2}{2} + v_0 t \sin\alpha = 0.$$

By solving this quadratic equation, we obtain two roots

$$t_1 = 0, \quad t_2 = \frac{2v_0 \sin\alpha}{g}.$$

The first root corresponds to the initial, the second to the final position of the body.

Substituting the value of t_2 in the equation for x, we find

$$x_{max} = \frac{2v_0^2 \sin\alpha\cos\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}$$

which is the horizontal range of the body.

CHAPTER XV

Motion of Particles. Method of Kinetostatics

84. Ideal and Real Constraints

It has already been indicated in the statics section that constraints restrict the freedom of motion of a particle or body. If no constraints are imposed on a particle, it can move freely in space. Problems of theoretical mechanics are usually

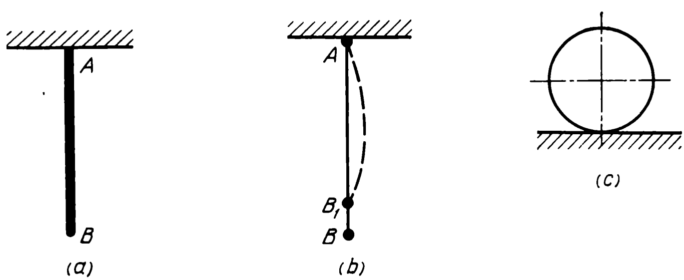


Fig. 146.

concerned with bodies and particles on which certain constraints are imposed. Such particles are called *constrained particles.* When solving problems involving the motion of a constrained particle, one should add constraining forces to those acting on the particle and consider the particle as moving freely under the action of the whole set of forces.

In dynamics, constraints are classified according to several characteristics. If a constraint prevents the movement of a body in two opposite directions, it is referred to as *restrain-*

ing, or *two-sided*, as, for example, the constraint effected by a rigid bar (Fig. 146*a*). Here the point *B* can move neither up nor down along the bar. If a constraint prevents movement in one direction but permits movement in the opposite direction, it is referred to as *non-restraining*, or *one-sided*. An example is a flexible string, a supporting plane, etc. (Fig. 146*b* and *c*).

When determining the reactions of constraints it is usually assumed that they are frictionless, the reactions are then perpendicular to the supporting surface (Fig. 147*a*). Such constraints are called *ideal constraints*. *Real constraints* always

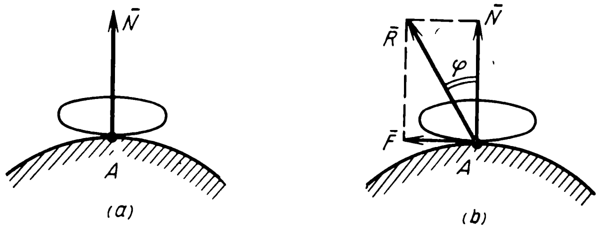


Fig. 147.

involve friction and the reaction $\bar{R}$ deviates from the normal (Fig. 147*b*). The effects of friction are often insignificant, however, and may be neglected; in such cases, too, the constraints are regarded as ideal.

85. D'Alembert's Principle

Consider a particle *A* acted upon by an arbitrary number of forces $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots, \bar{P}_n$ (Fig. 148). Among the forces there may be given active forces as well as constraining forces.

According to Axiom III of dynamics, the particle *A* receives the same acceleration under the action of these forces as when acted upon by a single force equal to the geometric sum of the given forces

$$m\bar{a} = \bar{R} = \bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \ldots + \bar{P}_n,$$

where $\bar{a}$ = acceleration of particle *A*,

m = mass of particle *A*,

$\bar{R}$ = resultant of force system $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots, \bar{P}_n$.

Suppose that, in addition to the forces $\overline{P}_1$, $\overline{P}_2$, $\overline{P}_3$, . . ., $\overline{P}_n$, there is a fictitious force $\overline{W}$ acting on the particle, which is equal and opposite to the resultant $\overline{R}$

$$\overline{W} = -\overline{R} = -m\overline{a} = -[\overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \ldots + \overline{P}_n]. \quad (165)$$

Equation (165) can be represented as

$$(-m\bar{a}) + \bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \ldots + \bar{P}_n = 0$$

or

$$\overline{W} + \bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \ldots + \bar{P}_n = 0. \quad (166)$$

It is evident from Eq. (166) that all forces, including the force $\overline{W}$, must be balanced as the forces $\overline{W}$ and $\overline{R}$ are equal and opposite and have the same line of action.

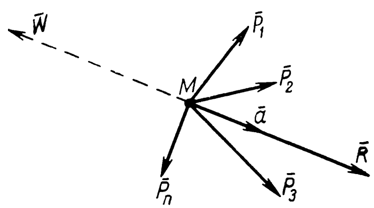


Fig 148.

The inertia force for a moving particle is defined as the product of the mass of the particle and its acceleration with the sign reversed.

The above statement is known as *D'Alembert's principle* and can be formulated as follows: *at any given instant the forces applied to a particle are balanced by inertia forces.*

D'Alembert's principle can be applied not only to a particle but also to a rigid body. Thus, dynamic problems involving a particle, a rigid body or a system of bodies can be reduced to problems of statics.

The method based on D'Alembert's principle can be formulated thus: *if to all forces actually acting on the particles of a moving body are added fictitious inertia forces, the body can be treated as if in equilibrium under the action of all these forces.*

It should be emphasized that inertia forces actually exist but they are applied not to a moving body but to constraints. That is why we speak of their fictitious nature.

86. Inertia Force for a Particle in Rectilinear Motion

Let a particle A move in a straight line with an acceleration $\bar{a}$ (Fig. 149). The weight of the particle is denoted by $\bar{G}$. In rectilinear motion, the direction of acceleration coincides with the path. The inertia force is directed opposite to the acceleration and its magnitude is defined by the formula

$$W = ma = \frac{G}{g} a.$$

If the particle is accelerated (Fig. 149*a*), the directions of acceleration and velocity coincide and the inertia force

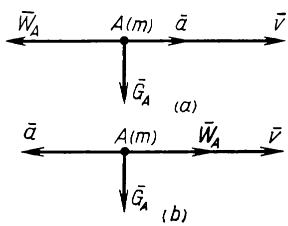


Fig. 149.

acts in a direction opposite to the motion. In decelerated motion (Fig. 149*b*), when acceleration is opposite to velocity, the inertia force acts in the direction of the motion.

87. Inertia Force for a Particle in Curvilinear Motion

Let a particle A (Fig. 150) of mass m move along an arbitrary path with an acceleration

$$\bar{a}_A = \bar{a}_A^n + \bar{a}_A^t.$$

The magnitude of the acceleration $\bar{a}_A$ can be represented analytically as

$$a_A = \sqrt{(a_A^n)^2 + (a_A^t)^2},$$

where $\bar{a}_A^n$ = normal component of acceleration,
$\bar{a}_A^t$ = tangential component of acceleration.

Proceeding to the determination of the inertia force for the particle A, it should be noted that it is also made up of two components: the normal, or centrifugal, component, and the tangential component.

The normal, or *centrifugal, component of the inertia force* is equal to the product of the mass concentrated in the particle A and the normal component of the acceleration of the particle A

$$W^n = ma_A^n. \tag{167}$$

The force $\overline{W}^n$ has a sense opposite to that of the acceleration $\bar{a}_A^n$.

The mass m concentrated in the particle A can be represented as the ratio of the weight G to the acceleration of gravity g.

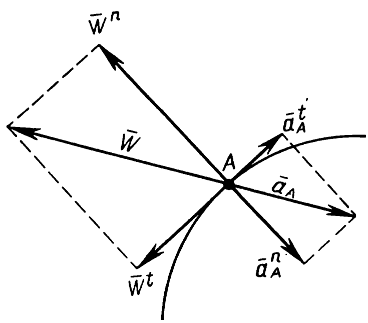


Fig. 150.

Equation (167) can be written as

$$W^n = ma_A^n = \frac{G}{g} a_A^n. \tag{167a}$$

The tangential component of the inertia force is equal to the product of the mass concentrated in the particle A and the tangential component of the acceleration of the particle A

$$W^t = ma_A^t \tag{168}$$

or

$$W^t = \frac{G}{g} a_A^t. \tag{168a}$$

The sense of the force $\overline{W}^t$ is always opposite to that of the acceleration $\bar{a}_A^t$.

Having determined the normal and tangential components of the inertia force, we can find the total inertia force for the particle A.

The total inertia force $\overline{W}$ for a particle is defined as the geometric sum of the normal and tangential components

$$\overline{W} = \overline{W}^n + \overline{W}^t. \tag{169}$$

Since the tangential and normal components are mutually perpendicular, the magnitude of the total inertia force can be represented analytically as

$$W = \sqrt{(W^n)^2 + (W^t)^2} = \sqrt{\left(\frac{G}{g} a_A^n\right)^2 + \left(\frac{G}{g} a_A^t\right)^2}. \tag{170}$$

Consider a special case when a particle rotates about an axis. The angular velocity ω and the angular acceleration α are assumed to be known.

Also known are:

(1) the normal component of the acceleration given by

$$a_A^n = \frac{v_A^2}{\rho} = \omega^2 \rho,$$

where ρ = radius of curvature of path of particle A,
v_A = linear velocity of particle A;

(2) the tangential component of the acceleration

$$a_A^t = \alpha\rho.$$

The magnitude of the total acceleration of the particle A is then

$$a_A = \sqrt{(a_A^n)^2 + (a_A^t)^2} = \sqrt{(\omega^2\rho)^2 + (\alpha\rho)^2}$$

or

$$a_A = \rho\sqrt{\omega^4 + \alpha^2}.$$

The magnitude of the normal, or centrifugal, component of the inertia force for the particle A is defined by the formula

$$W^n = m\omega^2\rho = \frac{G}{g}\omega^2\rho. \tag{171}$$

The magnitude of the tangential component of the inertia force is

$$W^t = m\alpha\rho = \frac{G}{g}\alpha\rho \tag{172}$$

and the magnitude of the total inertia force is determined in terms of the normal and tangential components

$$W=\sqrt{(W^n)^2+(W^t)^2}=\sqrt{\left(\frac{G}{g}\,\omega^2\rho\right)^2+\left(\frac{G}{g}\,\alpha\rho\right)^2}$$

or

$$W=\frac{G}{g}\,\rho\sqrt{\omega^4+\alpha^2}. \qquad (173)$$

It has been shown above that $\rho\sqrt{\omega^4+\alpha^2}=a_A$, hence

$$W=\frac{G}{g}\,a_A=ma_A. \qquad (173a)$$

The total inertia force $\overline{W}$ is always directed opposite to the total acceleration $\bar{a}_A$ (Fig. 150).

88. Inertia Force for a Rigid Body

As previously noted, a rigid body is a system of particles. Therefore, the inertia force for a rigid body is the geometric

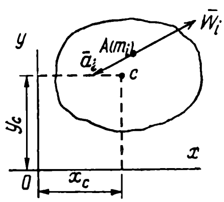


Fig. 151.

sum of the inertia forces for all the particles of the body

$$\overline{W}=\sum_{i=1}^{n}-m_i\bar{a}_i=-\sum_{i=1}^{n}m_i\bar{a}_i, \qquad (174)$$

where m_i is the mass of an arbitrary particle, $\bar{a}_i$ the acceleration of this particle (Fig. 151).

The accelerations of different particles of a body may have different directions depending upon the type of motion. Hence the determination of the total inertia force for a rigid body is reduced to the summation of the inertia vectors for all the particles.

The summation is performed throughout the volume of the body and is rather a complex operation.

In Sec. 118 are considered some special cases of the determination of the inertia forces for members of mechanisms when the law of motion is given.

89. Solution of Problems by the Method of Kinetostatics

D'Alembert's principle is very helpful in solving problems. When considering the fictitious equilibrium of a particle or body under the action of applied forces and inertia forces use can be made of the well-known procedures and methods of

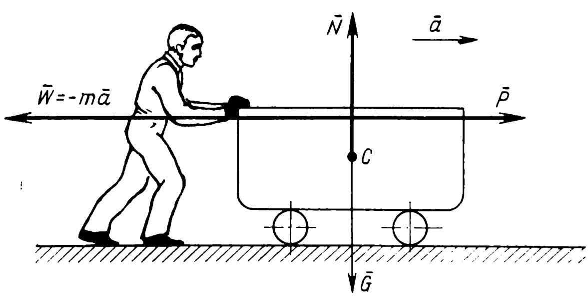


Fig. 152.

statics. This procedure for solving dynamics problems is called *the method of kinetostatics*. When using this method it should be remembered that the inertia forces are actually applied not to a moving body but to the constraints controlling its motion. This will be illustrated by the following example.

A man pushes a waggon with a force $\bar{P}$ imparting an acceleration $\bar{a}$ to it (Fig. 152). If the inertia force $\bar{W} = -m\bar{a}$ is applied to the waggon, it may be considered as being in a state of equilibrium. In actual fact, however, the inertia force is applied not to the waggon but to the hands of the man who moves it. In this case the inertia force represents a reaction corresponding to the action (force $\bar{P}$) exerted by the man on the waggon.

The method of kinetostatics is a formal procedure for solving dynamics problems as in actuality the equilibrium assumed does not exist.

Below is demonstrated the application of the method of kinetostatics for solving problems.

Example 58. A cage is moving down a mine shaft with a constant acceleration $a = 4$ m/sec^2 (Fig. 153), the weight

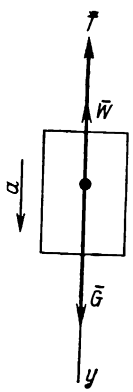


Fig. 153.

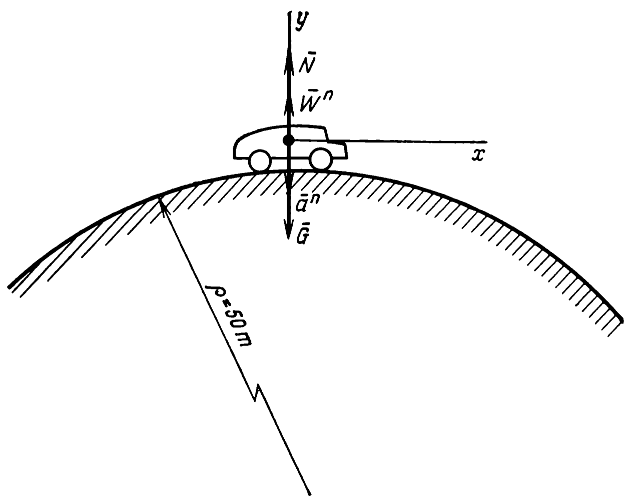


Fig. 154.

of the cage is $G = 2.00$ kN.

Determine the tension $\overline{T}$ in the rope supporting the cage. The acceleration due to gravity is $g = 9.81$ m/sec^2.

Solution. Apply to the cage all given forces and the inertia force and set up an equation of equilibrium.

Projecting all forces on the y axis, we obtain

$$\sum P_{iy} = 0, \quad -T - W + G = 0.$$

The magnitude of the inertia force is given by the formula

$$W = ma = \frac{G}{g} a = \frac{2}{9.81} \times 4 = 0.815 \text{ kN}.$$

From the equation of equilibrium we have

$$T = G - W = 2 - 0.815 = 1.185 \text{ kN}.$$

Example 59. An automobile of weight $G = 10$ kN is travelling on a convex bridge whose radius of curvature at mid-span is $\rho = 50$ m (Fig. 154).

Determine the pressure exerted on the bridge by the automobile as it passes its middle. The speed of the automobile is $v = 36$ km/hr.

Solution. Apply to the centre of gravity of the automobile its weight G, the normal reaction of the bridge $\bar{N}$ which is equal to the pressure exerted by the automobile on the bridge, and the inertia force $\bar{W}$. The automobile is travelling at the constant speed $v = \frac{36{,}000}{3{,}600} = 10$ m/sec, consequently, it has only the normal component of acceleration

$$a^n = \frac{v^2}{\rho} = \frac{10^2}{50} = 2 \text{ m/sec}^2.$$

The magnitude of the normal component of the inertia force is

$$W^n = ma^n = \frac{G}{g} a^n = \frac{10}{9.81} \times 2 = 2.04 \text{ kN}.$$

The tangential component of the inertia force is zero as the automobile has no tangential component of acceleration, $a^t = 0$.

Projecting all forces on the y axis, we obtain

$$\sum P_{iy} = 0, \quad N + W^n - G = 0,$$

whence

$$N = G - W^n = 10 - 2.04 = 7.96 \text{ kN}.$$

The solution of the above problems is no different from the solution of statics problems.

Example 60. A winch lifting a load of mass $m_Q = 2$ tons is mounted on a symmetrical truss (Fig. 155). The mass of the truss is $m_P = 1{,}200$ kg, its span is $l = 8$ m. The winch is mounted at one-fourth the span ($a = 2$ m) from the left support A, the mass of the winch is $m_G = 300$ kg, the diameter of the winch drum is $d = 0.4$ m. The equation of motion of the load is $S = 0.9t^2$ (S in m, t in sec); its acceleration is directed upward.

Determine the reactions at the supports of the truss with due allowance for dynamic loading.

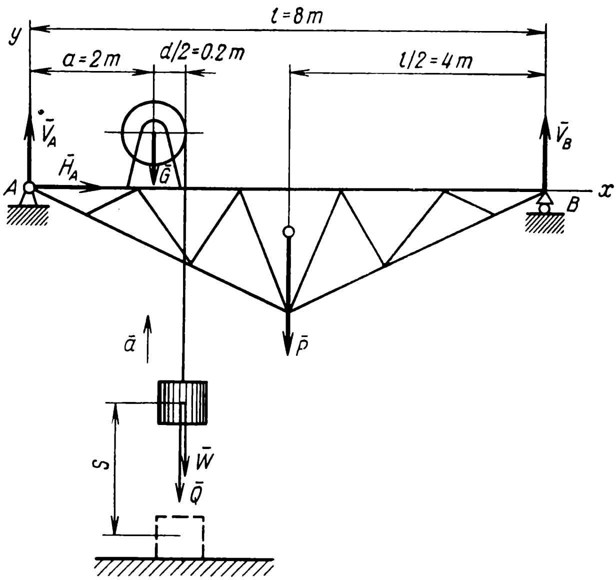


Fig. 155.

Solution. Use the method of kinetostatics. Determine the acceleration of the load which is moving vertically upward

$$a = \frac{dv}{dt} = \frac{d^2S}{dt^2} = 2 \times 0.9 \times 1 = 1.8 \text{ m/sec}^2.$$

The inertia force for the load being lifted is

$$W = m_Q a = 2{,}000 \times 1.8 = 3{,}600 \text{ kg-m/sec}^2 = 3{,}600 \text{ N}.$$

This force is directed downward, i.e., opposite to the acceleration.

Apply all given forces and the inertia force $\overline{W}$ to the truss (Fig. 155).

The given forces are the weights of the load, truss and winch. They can easily be calculated from the known masses.

The weight of the load is

$$Q = m_Q g = 2{,}000 \times 9.81 = 19{,}620 \text{ N}.$$

The weight of the truss

$$P = m_P g = 1{,}200 \times 9.81 = 11{,}772 \text{ N}.$$

The weight of the winch

$$G = m_G g = 300 \times 9.81 = 2{,}943 \text{ N}.$$

Remove the supports at A and B and apply the reactions they cause. The immovable hinge support A involves two reaction components $\bar{V}_A$ and $\bar{H}_A$ while the movable hinge support B involves one reaction component $\bar{V}_B$.

Set up equations of equilibrium

$$\sum P_{ix} = 0, \qquad H_A = 0,$$

$$\sum P_{iy} = 0, \qquad V_A - G - Q - W - P + V_B = 0,$$

$$\sum m_A = 0, \qquad Ga + (Q + W)\left(a + \frac{d}{2}\right) + P\frac{l}{2} - V_B l = 0.$$

From the first equation it follows that the horizontal component of the reaction at the support A is zero. By solving the third and second equations, we find

$$V_B = \frac{Ga + (Q+W)\left(a + \frac{d}{2}\right) + P\frac{l}{2}}{l} =$$

$$= \frac{2{,}943 \times 2 + 23{,}220 \times 2.2 + 11{,}772 \times 4}{8} = \frac{104{,}058}{8} = 13{,}007 \text{ N},$$

$$V_A = G + Q + W + P - V_B = 2{,}943 + 19{,}620 + 3{,}600 + 11{,}772 -$$

$$- 13{,}007 = 24{,}928 \text{ N}.$$

CHAPTER XVI

Work and Power

90. Work of a Constant Force in Rectilinear Motion

In many problems of dynamics the work done by various forces has to be calculated. To begin with, consider the concept of work for a special case when the acting force is constant in magnitude and direction and its point of application moves in a straight path.

We have a particle C subjected to a force $\overline{P}$ which is constant in magnitude and direction (Fig. 156*a*).

Over a certain time interval t, the particle C will have moved through a distance S along the straight path to a position C_1.

The work done by a force $\overline{P}$ when its point of application moves along a straight line is defined as the product of the magnitude of the force $\overline{P}$, the magnitude of the displacement S and the cosine of the angle between the directions of the force and the displacement, i.e.,

$$U = PS \cos(\overline{P}, S), \tag{175}$$

where $\cos(\overline{P}, S) = \cos\alpha$ is the cosine of the angle between the directions of the force and the displacement.

The directions of the rectilinear displacement S and the velocity $\overline{v}$ always coincide, therefore

$$U = PS \cos(\overline{P}, \overline{v}) = PS \cos\alpha. \tag{175a}$$

The angle α may vary from 0 to 180°.

When $\alpha < 90°$ the work is positive, when $\alpha > 90°$ negative and when $\alpha = 90°$ $U = 0$.

When a force makes an acute angle with the direction of the displacement, it is called a *moving force*, its work is always positive. If the angle is obtuse, the force offers some resistance to the motion, it does negative work and is called a *resistance*

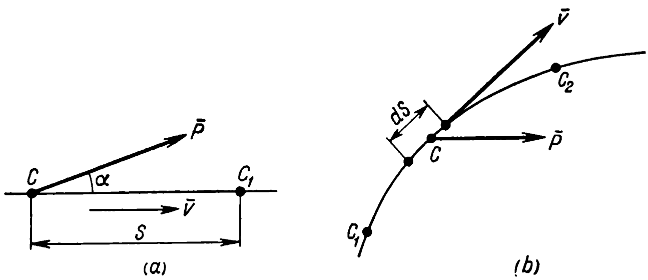


Fig. 156.

force. Examples of resistance forces are cutting forces, frictional forces, air resistance, etc., which are always directed opposite to the motion.

When $\alpha = 0°$, i.e., when the direction of the force coincides with the direction of the velocity, $U = PS$ as $\cos\alpha = \cos 0° = 1$. Clearly, the product $P\cos\alpha$ is the projection of the force $\bar{P}$ on the direction of the displacement of the particle. Consequently, the work of a force $\bar{P}$ can be defined as the product of the distance travelled S and the projection of the force $\bar{P}$ on the direction of the displacement.

The unit of work in the international system of units (SI) is the joule (j) which is defined as the work of a force of one newton (N) on a displacement of one metre (m) coincident in direction with the force

$$1\ \text{j} = 1\ \text{N} \times 1\ \text{m} = 1\ \text{N-m} = 1\ \text{kg-m}^2/\text{sec}^2.$$

A larger unit is the kilojoule (kj)

$$1\ \text{kj} = 1{,}000\ \text{j},$$

The unit commonly used in engineering to measure work is the kilowatt-hour (kW-hr)

$$1 \text{ kW-hr} = 3{,}600 \text{ kj}.$$

The unit of work in the engineers' system of units (mkgfs) is the kilogram-metre (kgf-m), i.e., the work of a force of one kilogram over a distance of one metre.

The relationship between the units of work in the engineers' system and in the international system can easily be established using the relationship between the respective units of force (see Sec. 80)

$$1 \text{ kgf} = 9.81 \text{ N}, \quad 1 \text{ N} = 0.102 \text{ kgf}.$$

By analogy we have

$$1 \text{ kgf-m} = 9.81 \text{ N-m} = 9.81 \text{ j},$$

$$1 \text{ j} = 1 \text{ N-m} = 0.102 \text{ kgf-m}, \quad 1 \text{ kj} = 102 \text{ kgf-m},$$

$$1 \text{ kW-hr} = 3{,}600 \text{ kj} = 367{,}200 \text{ kgf-m}.$$

91. Work of a Variable Force in Curvilinear Motion

Formulas (175) and (175a) derived above are applicable only if the force $\bar{P}$ is constant in magnitude and direction and the displacement on which the force $\bar{P}$ does work is rectilinear.

When the force $\bar{P}$ is variable or the displacement is curvilinear, formulas (175) and (175a) cannot be used. In this case the concept of elementary work is applied which is established as follows: consider a very small portion dS of the distance travelled S (Fig. 156*b*), which may be assumed straight. Any change in the force may be neglected and the force may be assumed constant in that portion.

The elementary work of a variable force $\bar{P}$ is defined as the work done by the force during a small displacement dS such that the change in the force during this displacement may be neglected.

Denoting the elementary work by dU, we write

$$dU = P\,dS\cos(\bar{P},\ dS)$$

or

$$dU = P\,dS\cos(\bar{P},\ \bar{v}), \tag{176}$$

where $\bar{v}$ is the velocity of the particle; the direction of the small displacement dS always coincides with the direction of the velocity.

Summing the elementary work over a finite distance, $C_1C_2 = S$, we obtain the total work

$$U = \int_{C_1}^{C_2} P\,dS\cos(\bar{P}, \bar{v}). \tag{177}$$

92. Work of a Resultant Force

Consider a particle M subjected to several forces $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n$ (Fig. 157). The resultant of the applied force system is the force $\bar{R}$. The elementary displacement dS of

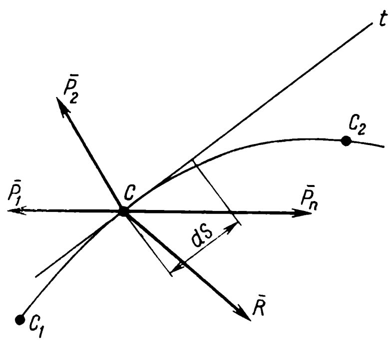


Fig. 157.

the particle is directed along the tangent to the path. Let us project all forces on the tangent. According to the theorem proved in Chapter II, the sum of the projections of the components is equal to the projection of the resultant

$$R\cos(\bar{R}, dS) = P_1\cos(\bar{P}_1, dS) + P_2\cos(\bar{P}_2, dS) + \ldots + P_n\cos(\bar{P}_n, dS).$$

Multiplying all terms in this equation by dS, we obtain

$$R\,dS\cos(\bar{R}, dS) = P_1\,dS\cos(\bar{P}_1, dS) + P_2\,dS\cos(\bar{P}_2, dS) + \ldots + P_n\,dS\cos(\bar{P}_n, dS), \tag{178}$$

but

$$R\,dS\cos(\bar{R},\, dS) = dU,$$
$$P_1\,dS\cos(\bar{P}_1,\, dS) = dU_1,$$
$$P_2\,dS\cos(\bar{P}_2,\, dS) = dU_2,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$P_n\,dS\cos(\bar{P}_n,\, dS) = dU_n,$$

consequently,

$$dU = dU_1 + dU_2 + \ldots + dU_n.$$

By summing the elementary work in the portion C_1C_2, we obtain finally

$$\int_{C_1}^{C_2} dU = \int_{C_1}^{C_2} dU_1 + \int_{C_1}^{C_2} dU_2 + \ldots + \int_{C_1}^{C_2} dU_n$$

or

$$U = U_1 + U_2 + \ldots + U_n. \tag{179}$$

Consequently, *the work of the resultant of a force system applied to a particle is equal to the sum of the work of the component forces during the same displacement.*

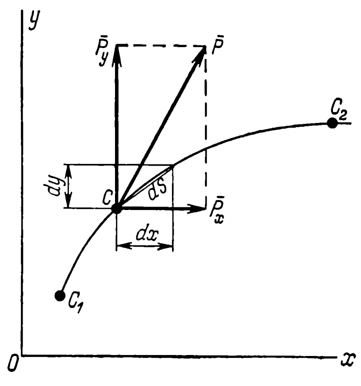


Fig. 158.

On the basis of the above theorem the elementary work of a force can be expressed in terms of its components along the co-ordinate axes. For a force in a plane (Fig. 158)

$$dU = P\,dS\cos(\bar{P},\, dS) = P_x\,dx + P_y\,dy. \tag{180}$$

Accordingly the total work in the portion C_1C_2 is determined as the sum of the elementary work

$$U=\int_{C_1}^{C_2} dU=\int_{C_1}^{C_2} (P_x\,dx+P_y\,dy). \tag{181}$$

93. Work of a Force of Gravity

Let the point of application B of a force of gravity $\bar{G}$ move along a certain path from position B_1 to position B_2 (Fig. 159).

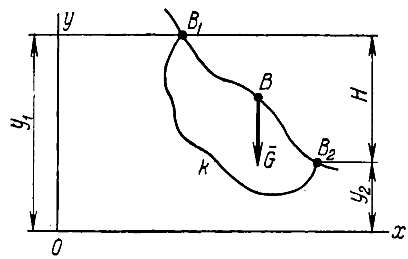


Fig. 159.

Express the work of the gravity force $\bar{G}$ according to formula (181)

$$U=\int_{B_1}^{B_2} (G_x\,dx+G_y\,dy).$$

The projections of the gravity force $\bar{G}$ on the co-ordinate axes are

$$G_x=0,\quad G_y=-G.$$

Substituting these values in the above formula and integrating, we obtain

$$U=\int_{B_1}^{B_2} G_y\,dy=\int_{y_1}^{y_2} -G\,dy=-G\,(y_2-y_1)= \\ =G\,(y_1-y_2)=GH. \tag{182}$$

Thus, we come to the following conclusions:

1. *The work of a force of gravity is equal to the product of the force and the vertical displacement of its point of application.*

2. *The work of a force of gravity depends only on the initial and terminal positions of the point of application of the force and is independent of the actual path followed by that point.*

If, for example, the point B moved along a curve B_1kB_2 (Fig. 159), the work of the force of gravity would be the same.

94. Work of an Elastic Force

An *elastic force* is a force that varies directly as the displacement of the point of application of the force from an equilibrium position. Elastic forces arise, for example, when springs are deformed (Fig. 160). As the deformation of a spring

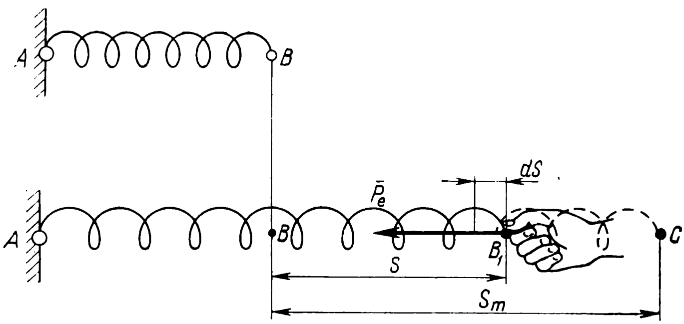


Fig. 160.

increases, so does the elastic force but it is always directed toward the point corresponding to the equilibrium position of the spring. The law of variation of an elastic force is

$$P_e = cS, \tag{183}$$

where S is the displacement of the point of application of the force, c the factor of proportionality, also called the stiffness factor or spring constant.

From formula (183) it follows that

$$c = \frac{P_e}{S}.$$

Thus, *the stiffness factor is the force required to give an elastic body a unit deformation.*

T e stiffness factor is measured in units of force divided by units of length, i.e., in N/cm, kN/mm (in the international system of units) and in kgf/cm, kgf/mm (in the engineers' system of units).

Let us calculate the work of the elastic force for the spring represented in Fig. 160.

Suppose that the spring is first stretched by an amount $BC = S_m$ and then moves back toward its equilibrium position from point C to point B. This means that the elastic force and the displacement of its point of application have the same direction and the work of the elastic force is positive. In this case the elastic force is a moving force. Examining the deformation of the spring from point B to point C it is readily observed that the elastic force has a sense opposite to the displacement of its point of application and the work of the elastic force is negative. Here, the elastic force is a resistance force.

Since the magnitude of an elastic force varies with the displacement, its work during a finite displacement from zero to S_m must be calculated by formula (178)

$$U = \int_0^{S_m} P_e \, dS.$$

Substituting $P_e = cS$ and integrating, we obtain

$$U = \int_0^{S_m} cS \, dS = \left| \frac{cS^2}{2} \right|_0^{S_m} = \frac{cS_m^2}{2}, \tag{184}$$

where S_m is the final elastic deformation produced by the elastic force.

Consequently, *the work of an elastic force equals half the product of the stiffness factor and the square of the deformation produced by the elastic force.*

The work of an elastic force is positive when the deformation disappears under its action. When an elastic system is deformed by an external force the work of the elastic force is always negative as it opposes the deformation.

The relation (183) between the magnitude of an elastic force and the deformation can be represented graphically (Fig. 161*a*).

Clearly, the work of an elastic force is equal to the area of the triangle

$$\text{area } \triangle OAB = \frac{OB \times AB}{2} = \frac{S_m (P_e)_m}{2} = \frac{cS_m^2}{2} = U.$$

In engineering, the relation between the force and displacement is often expressed by a complex function

$$P = f(S).$$

It is then very convenient to calculate work using the graphical representation of that function.

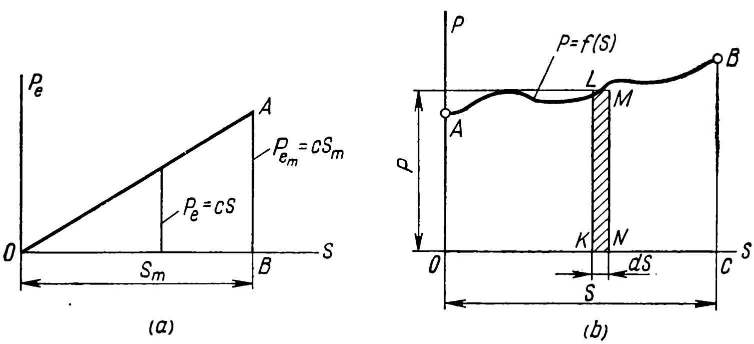


Fig. 161.

Let the relation between a force $\bar{P}$ and the displacement of its point of application be represented graphically by a curve AB (Fig. 161*b*), the direction of the force coinciding with that of the displacement.

The elementary work of the force $\bar{P}$ during the displacement dS is represented by the area of the shaded rectangle

$$dU = \text{area } KLMN = P\,dS.$$

The total work is determined as the sum of the areas of all elementary rectangles which, in the limit, is obviously equal to the area $OABC$ bounded by the axis of abscissas, the curve $P = f(S)$ and two extreme ordinates of this curve.

In engineering, special recording instruments are employed, indicators among others, which draw to a certain scale graphs showing the variation of forces. Through the use of an appropriate scale, work is determined from the area under the graph so drawn whenever direct calculation would involve great difficulties.

95. Concept of Mechanical Efficiency

Forces driving a machine, or so-called moving forces, always do positive work. The machine in turn consumes the accumulated energy in doing some useful work or, as is said, in overcoming the work of useful resistance forces. Clearly, the work of useful resistance forces cannot be equal to the work of moving forces as there are always friction and other parasitic resistance forces in the machine. *The ratio of the work of useful resistance forces $U_{u.\,f}$ (useful work) to the work of moving forces $U_{m.\,f}$ (expended work) is called the mechanical efficiency*

$$\left.\begin{aligned} \eta &= \frac{U_{u.\,f}}{U_{m.\,f}} \\ \text{or} \quad \eta &= \frac{U_{u.\,f}}{U_{m.\,f}}\,100\%. \end{aligned}\right\} \tag{185}$$

The work of useful resistance forces in steady-state motion, i.e., in motion with constant velocity, is equal to the difference between the work of moving forces and that of parasitic resistance forces, i.e.,

$$U_{u.\,f} = U_{m.\,f} - U_{p.\,f},$$

then

$$\eta = \frac{U_{m.\,f} - U_{p.\,f}}{U_{m.\,f}} = 1 - \frac{U_{p.\,f}}{U_{m.\,f}}\,. \tag{186}$$

Since $U_{p.\,f}$ is never zero and $U_{u.\,f} < U_{m.\,f}$, we always have $\frac{U_{p.\,f}}{U_{m.\,f}} > 0$, i.e., $\frac{U_{p.\,f}}{U_{m.\,f}}$ is a proper fraction, consequently, $\eta < 1$, i.e., the efficiency is always a proper fraction.

It is essential that parasitic resistance forces be reduced to a minimum and the efficiency approach unity.

96. Efficiency of a System of Mechanisms Connected in Series

A machine may consist of a number of elements, simple machines or drives connected in series or in parallel. The connection may also be of a mixed type.

A *series connection* is that in which the useful resistance of each element is the moving force for the next element.

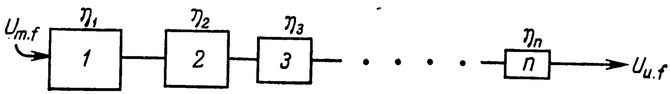


Fig. 162.

The overall efficiency of a complex machine is equal to the product of the efficiencies of the constituent simple machines connected in series

$$\eta = \eta_1 \eta_2 \eta_3 \dots \eta_n.$$

Let us prove this (Fig. 162).

The overall efficiency of the whole machine is

$$\eta = \frac{U_{u.f}}{U_{m.f}}.$$

Denote the work of parasitic resistance forces of the first machine by $U'_{p.f}$; then

$$U'_{m.f} = U_{m.f} - U'_{p.f},$$

where $U'_{m.f}$ is the work of moving forces transmitted from the first to the second machine.

The efficiency of the first machine is

$$\eta_1 = \frac{U_{m.f} - U'_{p.f}}{U_{m.f}} = \frac{U'_{m.f}}{U_{m.f}}.$$

Likewise, for the second machine

$$U''_{m.f} = U'_{m.f} - U''_{p.f}$$

and

$$\eta_2 = \frac{U''_{m.f}}{U'_{m.f}}.$$

For the last machine

$$\eta_n = \frac{U^n_{m.f}}{U^{(n-1)}_{m.f}}.$$

But the work of moving forces of the last machine $U^n_{m.f}$ is equal to the work of useful resistance forces of the whole complex machine $U_{u.f}$, therefore

$$\eta_n = \frac{U^n_{m.f}}{U^{(n-1)}_{m.f}} = \frac{U_{u.f}}{U^{(n-1)}_{m.f}}.$$

Let us find the product of the efficiencies of all the machines forming the complex machine

$$\eta_1\eta_2\eta_3\ldots\eta_n = \frac{U'_{m.f}}{U_{m.f}}\,\frac{U''_{m.f}}{U'_{m.f}}\,\frac{U'''_{m.f}}{U''_{m.f}}\cdots\frac{U_{u.f}}{U^{(n-1)}_{m.f}} = \frac{U_{u.f}}{U_{m.f}},$$

but

$$\frac{U_{u.f}}{U_{m.f}} = \eta,$$

i.e., it is equal to the efficiency of the whole machine; consequently,

$$\eta_1\eta_2\eta_3\ldots\eta_n = \eta. \tag{187}$$

97. Efficiency of a System of Mechanisms Connected in Parallel

In the case of a *parallel connection* each of the constituent machines is independent, i.e., the useful resistance forces and the moving forces of the individual machines forming the complex machine are independent of each other (Fig. 163).

The efficiency of the first machine is

$$\eta_1 = \frac{U'_{u.f}}{U'_{m.f}},$$

for the second machine

$$\eta_2 = \frac{U''_{u.f}}{U''_{m.f}},$$

for the nth machine

$$\eta_n = \frac{U^n_{u.f}}{U^n_{m.f}}.$$

The efficiency of a complex machine is equal to the sum of the work of the useful resistance forces of all its constituent machines divided by the sum of the work of the moving forces

$$\eta = \frac{\sum_{i=1}^{n} (U_{u.f})_i}{\sum_{i=1}^{n} (U_{m.f})_i} = \frac{U'_{u.f} + U''_{u.f} + U'''_{u.f} + \dots + U^n_{u.f}}{U'_{m.f} + U''_{m.f} + U'''_{m.f} + \dots + U^n_{m.f}}. \quad (188)$$

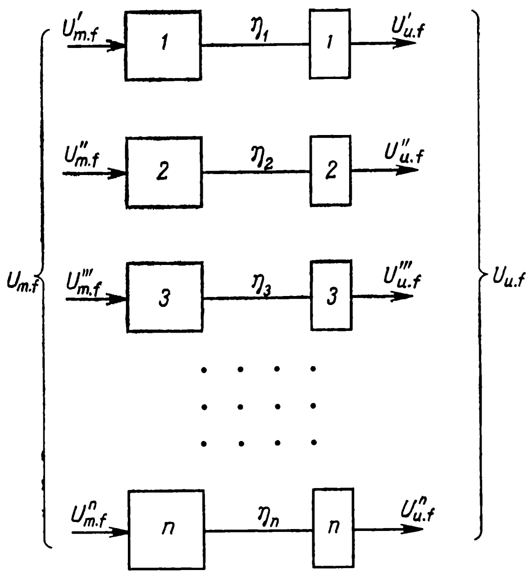


Fig. 163.

Suppose that in a special case the efficiencies of the constituents are equal

$$\eta_1 = \eta_2 = \eta_3 = \dots = \eta_n = \eta.$$

Then for the first machine

$$U'_{u.f} = \eta U'_{m.f},$$

for the second machine

$$U''_{u.\,f} = \eta U''_{m.\,f},$$

. . . .

. . . .

for the nth machine

$$U^{n}_{u.\,f} = \eta U^{n}_{m.\,f}.$$

Substituting these values in the numerator of Eq. (188), we obtain

$$\eta = \frac{\eta U'_{m.\,f} + \eta U''_{m.\,f} + \eta U'''_{m.\,f} + \dots + \eta U^{n}_{m.\,f}}{U'_{m.\,f} + U''_{m.\,f} + U'''_{m.\,f} + \dots + U^{n}_{m.\,f}} = \eta. \qquad (189)$$

Thus, *the efficiency of a parallel-connected complex machine with equal efficiencies of the individual machines forming the complex machine is equal to the efficiency of any one of them.*

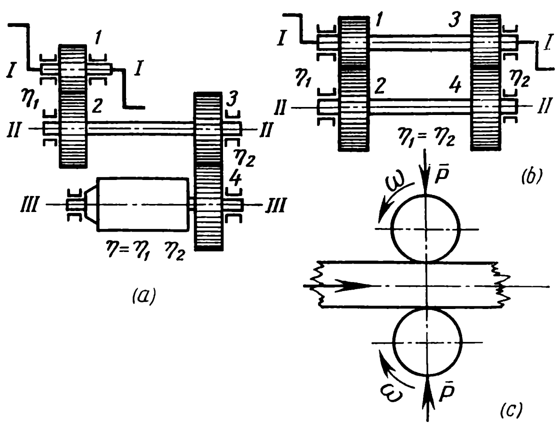


Fig. 164.

To compare the efficiencies of mechanisms connected in series and in parallel, consider the following example.

1. Given are four spur gears *1* to *4* (Fig. 164*a*) connecting three shafts. This gear drive is used in cranes and winches. A handle is fitted at each end of shaft *I-I*. A load to be lifted is supported by a rope wound around a drum which is mounted on shaft *III-III* (load shaft).

The driving moment developed on the handle is transmitted from the first to the second wheel. The moment obtained on shaft *II-II* is transmitted to wheel *4* by gear *3*. The moment transmitted by gear *3* is a driving moment in relation to wheel *4* and the moment of wheel *4* is a moment of useful resistance in relation to gear *3*.

Thus, the mechanisms are connected in series. If the efficiency of each pair of gear wheels is 0.95, the overall efficiency of the whole winch is $\eta = 0.95 \times 0.95 \cong 0.9$.

2. Given are four parallel-connected spur gears *1* to *4* (Fig. 164*b*). This type of gear connection is used, for example, in feeding mechanisms of machine tools.

The two pairs of gears are identical. If the efficiency of each pair is 0.95, the efficiency of the whole gear drive is also 0.95.

Examining both gear drives it may be concluded that the efficiency of the parallel-connected drive is higher than for the series-connected drive but nevertheless it is the first type of drive that is used most widely. This is due to the fact that in the first case a large load of 30 to 40 kN can be lifted by two men without much effort (the force exerted by each of them is 150 to 200 N).

In this case a gain in force is obtained: the moment of useful resistance on the load shaft is 20 to 25 times greater than the driving torque developed by the men. This justifies an energy loss of several per cent by parasitic resistance.

In the second case the purpose of the arrangement is the following. In practice the pressure between feed shafts (Fig. 164*c*) may be very large. If an attempt were made to transmit rotation from shaft *I-I* to shaft *II-II* by a single pair of gear wheels when the forces involved are very large, the wheels would be of too large a size. Therefore, the driving torque of the first shaft is transmitted to the second shaft by two pairs of wheels each designed only for half the torque; the wheel dimensions in each pair are considerably smaller than in a single pair. The dimensions of the whole installation will be more economical. The torque on shaft *II-II* is only 3 to 4 times greater than the driving torque on shaft *I-I*.

Thus, an appreciable gain in force is obtained by using a series connection at the expense of a small reduction in efficiency.

Example 61. An electric winch is used to raise a block of weight $G = 500$ N to a height $H = 10$ m by pulling it up an inclined plane (Fig. 165). The diameter of the load drum of the winch is $d = 0.5$ m, the tractive force produced by the winch is $P = 100$ N. The drum rotates at $n = 30$ rpm. The winch has raised the block in $t = 2$ min.

Determine the efficiency of the inclined plane.

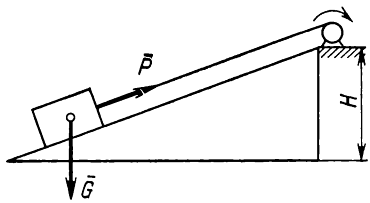


Fig. 165.

Solution. The useful work done in raising the block is determined as the work of the force of gravity

$$U_{u.f} = GH = 500 \times 10 = 5{,}000 \text{ N-m} = 5{,}000 \text{ j}.$$

Calculate now the work of moving forces, i.e., the work of the tractive force $\bar{P}$

$$U_{m.f} = PS.$$

The distance travelled by the block is found from the angular velocity $\omega = \frac{\pi n}{30} = \frac{\pi \times 30}{30} = \pi \frac{1}{\text{sec}}$, the drum diameter $d = 0.5$ m and the time of operation of the winch $t = 2 \text{ min} = 120$ sec (the drum is assumed to be rotating at a constant speed)

$$S = \omega \frac{d}{2} t = \pi \frac{0.5}{2} 120 = 94.25 \text{ m}.$$

Substituting the numerical values of the force $\bar{P}$ and the distance travelled S in the expression for the work of moving forces, we obtain

$$U_{m.f} = PS = 100 \times 94.25 = 9{,}425 \text{ N-m} = 9{,}425 \text{ j}.$$

The efficiency of the inclined plane is

$$\eta = \frac{U_{u.f}}{U_{m.f}} = \frac{5{,}000}{9{,}425} = 0.53 = 53\%.$$

98. Power

Power is defined as the work done by a force per unit time, i.e., *power is the first derivative of work with respect to time*

$$N=\frac{dU}{dt}=\frac{P\,dS\cos\alpha}{dt}. \tag{190}$$

As noted above, $P\cos\alpha$ is the projection of a force $\overline{P}$ on the direction of motion of a particle; denoting $P\cos\alpha$ by P_v, we obtain

$$N=P_v\frac{dS}{dt}.$$

If the force $\overline{P}$ coincides with the direction of motion of a particle, then $\cos\alpha=1$ and

$$N=P\frac{dS}{dt}.$$

It is known from kinematics that $\frac{dS}{dt}$ represents the velocity, i.e., $\frac{dS}{dt}=v$, then

$$N=Pv. \tag{191}$$

Power is measured in units of work divided by units of time.

The units of power in the international system of units (SI) are the watt (W) and the kilowatt (kW). *The watt is power corresponding to work of one joule per second*

$$1\ \text{W}=1\ \text{j/sec}=1\ \text{N-m/sec}=1\ \text{kg-m}^2/\text{sec}^3,$$

$$1\ \text{kW}=1{,}000\ \text{W}.$$

The unit of power in the engineers' system of units is kgf-m/sec; another unit still in use is the horsepower (hp)

$$1\ \text{hp}=75\ \text{kgf-m/sec}.$$

The relationships between the units of power in the international and engineers' systems are similar to the corresponding relationships between the units of force and work

$$1\ \text{kgf-m/sec}=9.81\ \text{j/sec}=9.81\ \text{W},$$

$$1\ \text{W}=0.102\ \text{kgf-m/sec}.$$

Using the above expressions, establish the relationship between the horsepower and the watt or kilowatt

$$1 \text{ hp} = 75 \text{ kgf-m/sec} = 736 \text{ W},$$

$$1 \text{ kW} = 1{,}000 \text{ W} = \frac{1{,}000}{736} = 1.36 \text{ hp}.$$

Example 62. A train of mass 500 tons is moving at constant speed along a horizontal track.

Determine the power developed by the locomotive if the resistance to the motion of the train is 200 N per 1 ton of the mass when the speed of the train is $v = 21.6$ km/hr.

Solution. The resistance to the motion of the train for 500 tons is

$$P = 200 \times 500 = 100{,}000 \text{ N} = 100 \text{ kN}.$$

The power developed by the locomotive is given by the formula

$$N = Pv.$$

In our example $P = 100$ kN,

$$v = 21.6 \text{ km/hr} = \frac{21.6 \times 1{,}000}{3{,}600} = 6 \text{ m/sec}.$$

Substituting the values of P and v in the formula for the power, we obtain

$$N = Pv = 100{,}000 \times 6 = 600{,}000 \text{ N-m/sec} =$$

$$= 600{,}000 \text{ W} = 600 \text{ kW}.$$

Example 63. Determine the capacity of a pumping station discharging 30,000 buckets of water per hour. The water is delivered to a height $H = 30$ m. The efficiency of the installation, i.e., the ratio of the work of useful forces to that of moving forces is $\eta = 0.75$ (one bucket of water weighs 120 N).

Solution. The discharge of the pumping station per second is

$$V = \frac{30{,}000}{3{,}600} = 8.33 \text{ buckets/sec}.$$

The weight of the water raised per second is

$$P = 8.33 \times 120 = 1{,}000 \text{ N}.$$

The power developed by the motor is

$$N = \frac{PH}{\eta} = \frac{1{,}000 \times 30}{0.75} = 40{,}000 \text{ W} = 40 \text{ kW}.$$

99. Work and Efficiency for Bodies Sliding Along an Inclined Plane

Work Done in Moving a Body Along an Inclined Plane. Most of the inclined transporting devices are basically de-

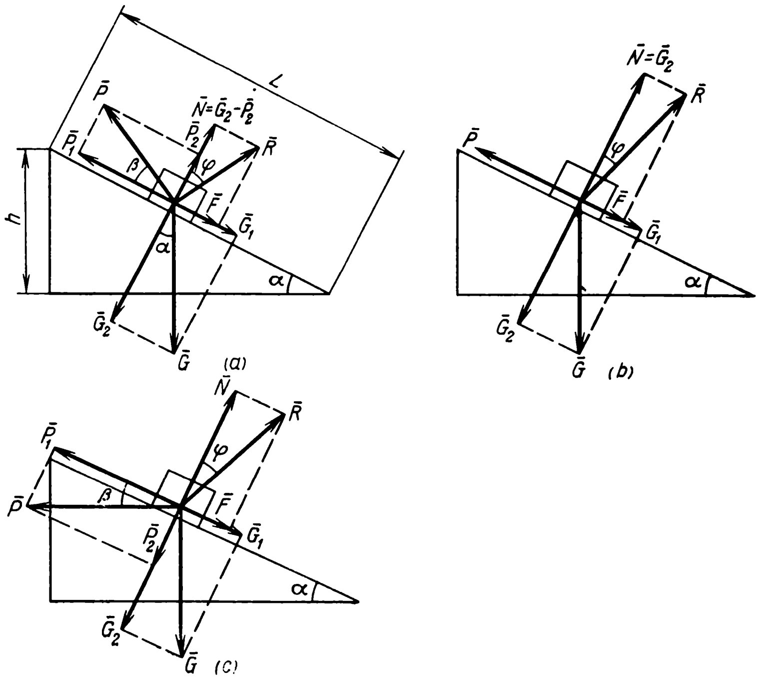


Fig. 166.

signed for transferring loads along an inclined plane (belt and chain conveyors, etc.).

In nearly all branches of industry the inclined plane figures in the calculation of the tractive force, the work of forces and the efficiency of a transporting device.

Considering the weight of a body $\bar{G}$ acting on an inclined plane and the moving force (tractive force) $\bar{P}$ (Fig. 166*a*) and resolving them in two directions—along the normal to the inclined plane and parallel to it, we obtain

$$P_1 = P\cos\beta, \qquad P_2 = P\sin\beta,$$
$$G_1 = G\sin\alpha, \qquad G_2 = G\cos\alpha.$$

Derive the relationship between the weight $\bar{G}$ of the body and the required tractive or moving force $\bar{P}$ (Fig. 166*a*).

If the body moves up the inclined plane with constant velocity, the following equation of equilibrium must be fulfilled

$$P_1 = G_1 + F,$$

where $\bar{F}$ is the friction force

$$F = fN,$$

$\bar{N}$ is the normal pressure

$$N = G_2 - P_2,$$

i.e.,

$$F = f(G_2 - P_2).$$

Consequently,

$$P_1 = G_1 + f(G_2 - P_2).$$

Substituting the values of the forces in this equation, we obtain

$$P\cos\beta = G\sin\alpha + f(G\cos\alpha - P\sin\beta)$$

or

$$P(\cos\beta + f\sin\beta) = G(\sin\alpha + f\cos\alpha),$$

whence

$$P = G\frac{\sin\alpha + f\cos\alpha}{\cos\beta + f\sin\beta}. \qquad (192)$$

It is known that

$$f = \tan\varphi = \frac{\sin\varphi}{\cos\varphi},$$

where φ is the angle of friction.

Hence,

$$P = G\frac{\sin\alpha + \frac{\sin\varphi}{\cos\varphi}\cos\alpha}{\cos\beta + \frac{\sin\varphi}{\cos\varphi}\sin\beta} = G\frac{\sin\alpha\cos\varphi + \sin\varphi\cos\alpha}{\cos\beta\cos\varphi + \sin\varphi\sin\beta}$$

or

$$P = G\frac{\sin(\alpha+\varphi)}{\cos(\beta-\varphi)}. \tag{193}$$

The work done by this moving force in pulling the body up the inclined plane is

$$U_{m.f} = P\cos\beta\, L = G\frac{\sin(\alpha+\varphi)}{\cos(\beta-\varphi)}\cos\beta\, L. \tag{194}$$

When $\beta = 0$, i.e., when the tractive force is parallel to the inclined plane (Fig. 166*b*), we have from formula (193)

$$P = G\frac{\sin(\alpha+\varphi)}{\cos\varphi}. \tag{195}$$

The work done by this force is

$$U_{m.f} = PL = GL\frac{\sin(\alpha+\varphi)}{\cos\varphi}, \tag{196}$$

or

$$U_{m.f} = GL\frac{\sin\alpha\cos\varphi + \sin\varphi\cos\alpha}{\cos\varphi} = GL(\sin\alpha + \tan\varphi\cos\alpha).$$

It is known that $\tan\varphi = f$, then

$$U_{m.f} = GL(\sin\alpha + f\cos\alpha). \tag{197}$$

If the tractive force is horizontal, i.e., $\beta = -\alpha$, we have from formula (194) (Fig. 166*c*)

$$U_{m.f} = GL\cos\alpha\frac{\sin(\alpha+\varphi)}{\cos(\alpha+\varphi)} = GL\cos\alpha\tan(\alpha+\varphi) \tag{198}$$

or

$$U_{m.f} = GL\cos\alpha\frac{\tan\alpha+\tan\varphi}{1-\tan\varphi\tan\alpha} = GL\cos\alpha\frac{\tan\alpha+f}{1-f\tan\alpha}. \tag{199}$$

Efficiency of an Inclined Plane. It is known that the mechanical efficiency is the ratio of the work of useful resistance forces to the work of moving forces, i.e.,

$$\eta = \frac{U_{u.f}}{U_{m.f}}. \tag{200}$$

When a body moves up an inclined plane the work of useful resistance forces is

$$U_{u.f} = Gh,$$

where h is the height to which the body is raised along the inclined plane

$$h = L \sin \alpha.$$

Then

$$U_{u.f} = GL \sin \alpha. \tag{201}$$

Substituting in Eq. (200) the values of the work of moving forces (194) and the work of useful resistance forces, we obtain

$$\eta = \frac{GL \sin \alpha}{\frac{GL \sin(\alpha + \varphi) \cos \beta}{\cos(\beta - \varphi)}} = \frac{\sin \alpha \cos(\beta - \varphi)}{\sin(\alpha + \varphi) \cos \beta}. \tag{202}$$

When $\beta = 0$, i.e., when the moving force is parallel to the inclined plane, we have

$$\eta = \frac{\sin \alpha \cos \varphi}{\sin(\alpha + \varphi)}. \tag{203}$$

When $\beta = -\alpha$, i.e., when the moving force is horizontal, we have

$$\eta = \frac{\sin \alpha \cos(\alpha + \varphi)}{\sin(\alpha + \varphi) \cos \alpha} = \frac{\tan \alpha}{\tan(\alpha + \varphi)}. \tag{204}$$

This equation may be represented as

$$\eta = \frac{\tan \alpha}{\tan(\alpha + \varphi)} = \frac{\tan \alpha (1 - \tan \varphi \tan \alpha)}{\tan \alpha + \tan \varphi} = \frac{\tan \alpha (1 - f \tan \alpha)}{\tan \alpha + f}. \tag{205}$$

Friction of a Wedge. A wedge is a combination of two inclined planes. Wedges are used as a cutting or splitting tool and also as a means for achieving an assigned motion (in presses).

At present wedges are used as a point tool, as an element of detachable joints of parts and machines and, finally, as a means for smooth feeding of a turret in machine tools.

Suppose a wedge is being driven into a body by a force $\overline{P}$ (Fig. 167).

As the wedge advances under the action of moving forces, the reactive forces $\overline{N}$ normal to the jaws of the wedge induce tangential reactions or friction forces $\overline{F}$ directed along the jaws and opposing the motion of the wedge

$$F = fN.$$

The wedge problem is reduced to the determination of the force $\overline{P}$ required to move the wedge and the efficiency of the wedge.

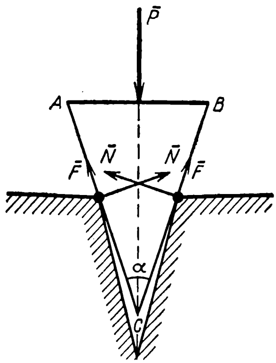


Fig. 167.

Let us find the sum of the projections of all forces on the vertical axis y

$$\sum P_{iy} = 0,$$

$$P - 2N \sin\frac{\alpha}{2} + 2F \cos\frac{\alpha}{2} = 0$$

or

$$P - 2N \sin\frac{\alpha}{2} + 2fN \cos\frac{\alpha}{2} = 0,$$

whence

$$P = 2N\left(\sin\frac{\alpha}{2} + f\cos\frac{\alpha}{2}\right).$$

It is known that

$$f = \tan\varphi = \frac{\sin\varphi}{\cos\varphi},$$

then

$$P = 2N\left(\sin\frac{\alpha}{2} + \frac{\sin\varphi}{\cos\varphi}\cos\frac{\alpha}{2}\right)$$

or

$$P = 2N\frac{\sin\left(\frac{\alpha}{2} + \varphi\right)}{\cos\varphi}. \qquad (206)$$

This formula is similar to that defining the tractive force on an inclined plane. And this is not coincidental as the wedge is a combination of two inclined planes.

The efficiency of the wedge is

$$\eta = \frac{U_{u.f}}{U_{m.f}}$$

or, by analogy with an inclined plane,

$$\eta = \frac{\tan\frac{\alpha}{2}}{\tan\frac{\alpha}{2} + \tan\varphi} = \frac{\tan\frac{\alpha}{2}}{\tan\frac{\alpha}{2} + f}. \qquad (207)$$

Friction in V-Shaped Guides. Wedge-shaped sliders are extensively used as guides in prime movers, metal-working and wood-working machines, etc.

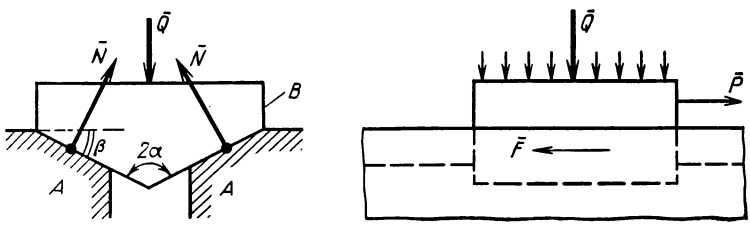


Fig. 168

A slider B moves at constant speed in a wedge-shaped groove A (Fig. 168) in a direction perpendicular to the plane of the drawing.

Let us find the relation between the friction force $\bar{F}$ and the pressure $\bar{Q}$ exerted on the slider. To maintain uniform motion, the moving force must overcome the friction force $\bar{F}$ so that

$$P = F = 2fN, \qquad (a)$$

where $\bar{N}$ is the normal reaction, f the coefficient of friction between the contact surfaces of the slider and groove.

Let the angle at the vertex of the slider be 2α. By projecting the normal reactions $\bar{N}$ on the direction of the force $\bar{Q}$, we obtain

$$Q - 2N \sin\alpha = 0,$$

whence

$$N = \frac{Q}{2 \sin\alpha}. \tag{b}$$

Substituting the above value of $\overline{N}$ in equation (a), we have

$$P = F = Q\left(\frac{f}{\sin\alpha}\right) \tag{208}$$

or

$$F = Qf',$$

where f' is the reduced coefficient of friction for the wedge-shaped slider

$$f' = \frac{f}{\sin\alpha}.$$

Since $\sin\alpha$ is always less than unity, f' is always greater than f.

If $\alpha = 90°$, $\sin\alpha = 1$, i.e., the motion of the slider occurs on a plane and $f' = f$.

The reduced coefficient of friction may also be expressed as (Fig. 168)

$$f' = \frac{f}{\sin\alpha} = \frac{f}{\cos\beta}. \tag{209}$$

Example 64. A body of weight $G = 100$ N is moving along an inclined plane under the influence of a horizontal force $\overline{P}$. The length of the incline is 2 m, the angle of inclination of the plane $\alpha = 10°$. The coefficient of friction between the body and the plane is $f = 0.15$.

Determine the work and efficiency for the body moving along the inclined plane.

Solution. 1. To determine the work of moving forces use Eq. (198)

$$U_{m.f} = GL \cos\alpha \tan(\alpha + \varphi),$$

where φ is the angle of friction;

$$\varphi = \text{arc} \tan f = \text{arc} \tan\ 0.15 = 8°30',$$

then

$$U_{m.f} = 100 \times 2 \cos 10° \tan(10° + 8°30') =$$
$$= 100 \times 2 \times 0.985 \times 0.335 \simeq 66 \text{ N-m} = 66 \text{ j}.$$

2. To determine the efficiency use Eq. (204)

$$\eta = \frac{\tan\alpha}{\tan(\alpha+\varphi)} = \frac{\tan 10^\circ}{\tan(10^\circ + 8^\circ 30')} = \frac{0.176}{0.335} = 0.525.$$

Example 65. A wedge-shaped slider of weight $\bar{G}$ is moving up inclined guides at constant speed under the action of a horizontal force $P = 5$ kN.

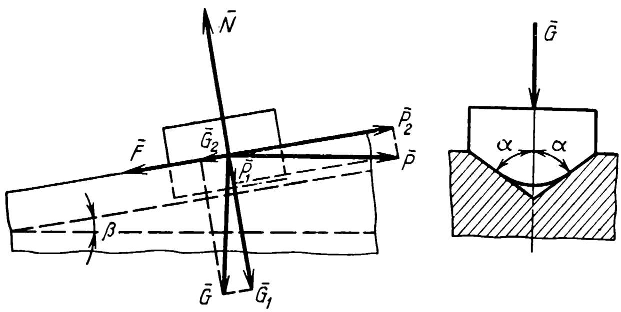


Fig. 169.

Determine the weight of the slider $\bar{G}$ if $\alpha = 60^\circ$, $\beta = 10^\circ$ and the coefficient of friction between the slider and the guides is $f = 0.15$ (Fig. 169).

Solution. As the slider moves up at constant speed

$$P_2 = F + G_2. \tag{a}$$

Here $P_2 = P\cos\beta$, $G_2 = G\sin\beta$; $F = f_1 N$, where $f_1 = \frac{f}{\sin\alpha}$.

The normal reaction $N = G_1 + P_1$, but since $G_1 = G\cos\beta$ and $P_1 = P\sin\beta$, we have $N = G\cos\beta + P\sin\beta$.

Consequently,

$$F = \frac{f}{\sin\alpha}(G\cos\beta + P\sin\beta).$$

Substituting the values of $\bar{P}_2$, $\bar{F}$ and $\bar{G}_2$ in equation (a), we obtain

$$P\cos\beta = \frac{f}{\sin\alpha}(G\cos\beta + P\sin\beta) + G\sin\beta$$

or

$$P\cos\beta = \frac{Gf\cos\beta}{\sin\alpha} + \frac{Pf\sin\beta}{\sin\alpha} + G\sin\beta,$$

then

$$G\left(\sin\beta+\frac{f\cos\beta}{\sin\alpha}\right)=P\left(\cos\beta-\frac{f\sin\beta}{\sin\alpha}\right),$$

whence, dividing through by cos β, we obtain

$$G=P\frac{\sin\alpha\cos\beta-f\sin\beta}{\sin\beta\sin\alpha+f\cos\beta}=P\frac{\sin\alpha-f\tan\beta}{\tan\beta\sin\alpha+f}.$$

Substituting the given data, we obtain

$$G=5\frac{\sin 60^\circ-0.15\tan 10^\circ}{\sin 60^\circ\tan 10^\circ+0.15}=5\frac{0.866-0.15\times 0.176}{0.866\times 0.176+0.15}=13.9\text{ kN}.$$

100. Work and Power in Rotation

Many machines incorporate rotating bodies, such as shafts, pulleys, flywheels. Rotation is caused by a torque applied to a body (Fig. 170). This torque may be produced by a cir-

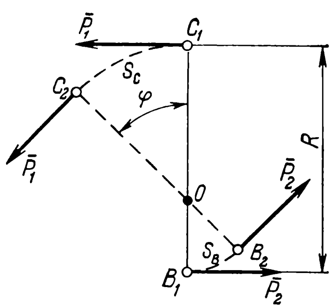


Fig. 170.

cumferential force $\bar{P}$ applied at a distance R (radius) from the axis of rotation. The magnitude of the torque is then defined by the formula

$$M=PR.$$

Knowing the magnitude of the torque and the radius R, we can easily find the circumferential force

$$P=\frac{M}{R}.$$

Let us calculate the work done by a couple of constant moment $M = PR$. A body to which this couple is applied will rotate about a point O (Fig. 170).

If the body rotates through an angle φ, the point C will move from position C_1 to position C_2 having travelled a distance $S_C = OC\,\varphi$, and the point B will travel a distance $S_B = OB\,\varphi$. In the rotation, the forces $\overline{P}_1$ and $\overline{P}_2$ are directed along the tangent to the path, i.e., the angles between the forces $\overline{P}_1$ and $\overline{P}_2$ and the respective displacements S_C and S_B on which they do work are zero.

It follows that the work done by the force $\overline{P}_1$ is $U_1 = P_1 S_C = P_1\,OC\,\varphi$ and the work done by the force $\overline{P}_2$ is $U_2 = P_2 S_B = P_2\,OB\,\varphi$.

The total work done by the couple is equal to the sum of the work done by the forces $\overline{P}_1$ and $\overline{P}_2$

$$U = U_1 + U_2 = P_1\,OC\,\varphi + P_2\,OB\,\varphi.$$

Taking into account that $P_1 = P_2 = P$, $OC + OB = R$ and factoring out $P\varphi$, we obtain

$$U = P\varphi\,(OC + OB) = PR\varphi = M\varphi. \tag{210}$$

The work of a couple is equal to the product of the moment of the couple and the angle of rotation expressed in radians.

Naturally, the moment and the work must be expressed in a consistent system of units.

Let us derive a formula for determining the power of a torque. To do this, we differentiate the expression for its work with respect to time

$$N = \frac{dU}{dt} = \frac{d\,(M\varphi)}{dt}.$$

Putting the constant value of the torque before the differential sign and noting that $\frac{d\varphi}{dt} = \omega$, we obtain finally

$$N = M\,\frac{d\varphi}{dt} = M\omega. \tag{211}$$

Thus, *power in rotation is equal to the product of torque by angular velocity.*

If the number of revolutions per minute is known, the angular velocity of a rotating body can be determined from the formula

$$\omega = \frac{\pi n}{30} .$$

Substitute the value of the angular velocity in the expression for power

$$N = M\omega = M \frac{\pi n}{30} .$$

From the last equation we can find the torque in terms of power and revolutions

$$M = \frac{30}{\pi} \frac{N}{n} = 9.56 \frac{N}{n} . \quad (212)$$

Thus, *the torque is directly proportional to power and inversely proportional to the number of revolutions per minute.*

Hence, for any fixed power of a motor the maximum torque that the motor is capable of resisting can be changed by varying the revolutions. By reducing the number of revolutions per minute we increase the torque, and vice versa. The number of revolutions is changed by means of transmission mechanisms.

Example 66. A shaft transmits a power $N = 75$ kW. Determine the required number of revolutions if the torque on the shaft is not to be less than $M = 180$ N-m.

Solution. From formula (212) we have

$$n = 9.56 \frac{N}{M} = 9.56 \frac{75 \times 1{,}000}{180} = 3{,}983 \text{ rpm}.$$

Consequently, in order to provide the required torque the shaft must make no more than 3,983 revolutions per minute.

101. Work in Rolling Motion

It is well known that rolling friction is substantially lower than sliding friction. It is therefore advantageous to use rollers and wheels to move heavy loads. An appreciable amount of work is then gained as compared to the transference of these bodies by drawing (sliding).

Consider the transference of a load on *rollers* which are employed when the distances involved are small (Fig. 171).

Let a load be placed on a platform resting on rollers.

Denote

$\bar{Q}$ = weight of platform and load,
$\bar{q}$ = weight of roller,
$\bar{P}$ = moving force,
$\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_n$ = forces acting respectively on the first, second and so on rollers; their sum is equal to the total weight of platform and load, i.e., $Q_1 + Q_2 + \ldots + Q_n = Q$,

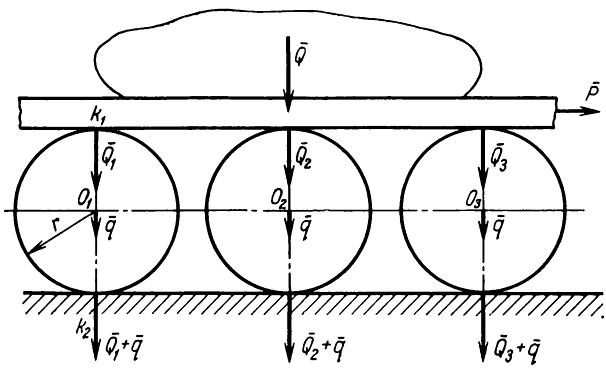


Fig. 171.

k_1 = coefficient of rolling friction between platform and roller,
k_2 = coefficient of rolling friction between roller and supporting surface,
r = roller radius,
n = number of rollers.

The moving force $\bar{P}$ overcomes the resistance due to rolling friction.

We write the expression for the tractive force $\bar{p}$ for one (the first) roller

$$p = \frac{k_1 Q_1}{2r} + \frac{k_2 (Q_1 + q)}{2r} = \frac{(k_1 + k_2) Q_1}{2r} + \frac{q k_2}{2r}, \qquad (213)$$

for all n rollers

$$P = \sum_{i=1}^{n} p = \frac{(k_1 + k_2)(Q_1 + Q_2 + \ldots + Q_n)}{2r} + \frac{q n k_2}{2r}, \qquad (214)$$

but

$$Q_1 + Q_2 + \ldots + Q_n = Q,$$

then

$$P = \frac{(k_1 + k_2) Q}{2r} + \frac{qnk_2}{2r}. \tag{215}$$

If the quality and the physical condition of the materials of the platform and the supporting surface are the same, i.e., $k_1 = k_2 = k$, then

$$P = \frac{2kQ}{2r} + \frac{qnk}{2r} = \frac{k(2Q + qn)}{2r}. \tag{216}$$

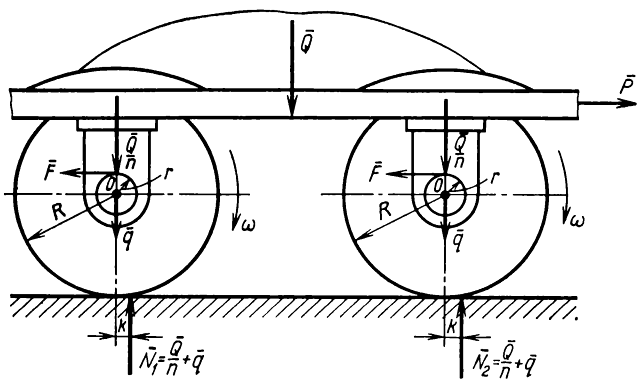


Fig. 172.

Neglecting the rolling resistance due to the weight of the rollers, we have

$$P = \frac{kQ}{r}. \tag{217}$$

We can now evaluate the work expended in rolling the load and the platform through a distance S

$$U = PS = \frac{kQ}{r} S. \tag{218}$$

When the distances involved are great, *wheels* are used to transfer heavy loads.

Consider the transference of a load on wheels.

Denote (Fig. 172)

$\overline{Q}$ = weight of car and load less wheels,
$\overline{q}$ = weight of one wheel,
n = number of wheels,
$\frac{\overline{Q}}{n}$ = weight of car and load per wheel,
R = wheel radius,
r = radius of wheel axle journal,
$\overline{F}$ = friction force between journal and bearing,
f = coefficient of friction between journal and bearing,
k = coefficient of rolling friction between wheel and supporting surface,
$\overline{P}$ = total moving force,
$\frac{\overline{P}}{n}$ = moving force per wheel.

We write the equation of equilibrium for one wheel

$$\sum M_O = 0,$$

$$\frac{P}{n}R - \left(\frac{Q}{n} + q\right)k - Fr = 0.$$

It is known that $F = \frac{Q}{n}f$, then

$$\frac{P}{n}R = \left(\frac{Q}{n} + q\right)k + \frac{Q}{n}fr.$$

To obtain the equation of equilibrium for all wheels it is necessary to multiply both sides of the equality by n, then

$$PR = (Q + qn)k + Qfr$$

or

$$P = \frac{(Q + qn)k + Qfr}{R}. \qquad (219)$$

The rolling resistance due to the weight of the wheels $\left(\frac{qnk}{R}\right)$ is very small compared to the rolling resistance of the car and the load; neglecting this resistance, we obtain

$$P = \frac{Qk + Qfr}{R}$$

or

$$P = Q\frac{k + fr}{R}. \qquad (220)$$

The quantity $\frac{k+fr}{R}$ is called the *traction coefficient* and denoted by f_0, then

$$P = Qf_0. \qquad (221)$$

The work done by the tractive force $\bar{P}$ in rolling the load on wheels through a distance S is

$$U = PS = Qf_0S = Q\frac{k+fr}{R}S. \qquad (222)$$

Example 67. A steam cylinder of weight G = 15 kN is being hoisted at constant velocity on rollers of diameter d = = 100 mm along a wooden floor of slope α = 30° (Fig. 173).

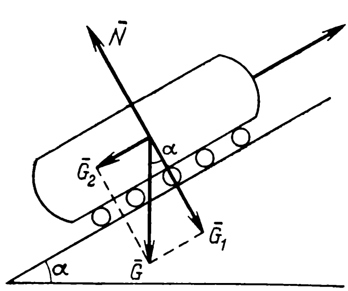


Fig. 173.

Determine the tension in the rope and the work expended in raising the cylinder if the coefficient of rolling friction between the cylinder and the rollers is k_1 = 0.02 cm, and between the roller and the floor k_2 = 0.4 cm. Neglect the weight of the rollers. The height to which the cylinder is raised is h = 3 m.

Solution. As the cylinder moves up at constant velocity

$$P = G_2 + P',$$

where $\bar{P}$ = tension in rope,

$\bar{G}_2$ = tangential component of cylinder weight,

$\bar{P}'$ = force expended in overcoming friction.

From the drawing

$$G_2 = G\sin\alpha = 15 \times 0.5 = 7.5 \text{ kN},$$
$$G_1 = N = G\cos\alpha = 15 \times 0.866 = 12.9 \text{ kN}$$

The friction force is

$$P' = \frac{G_1 (k_1 + k_2)}{2r} = \frac{12.9\,(0.02 + 0.4)}{10} = 0.542 \text{ kN}.$$

The tension in the rope is

$$P = G_2 + P' = 7.5 + 0.542 = 8.042 \text{ kN}.$$

Before calculating the work expended in rolling the cylinder, it is necessary to find the distance travelled

$$S = \frac{h}{\sin\alpha} = \frac{3}{0.5} = 6 \text{ m}.$$

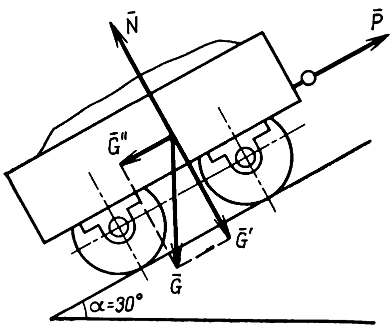


Fig. 174.

The work expended in rolling the cylinder is

$$U = PS = 8.042 \times 6 = 48.252 \text{ kN-m} = 48.252 \text{ kj}.$$

Example 68. A car of weight $G_1 = 10$ kN with a load $G_2 = 30$ kN is being hoisted with a velocity $v = 1$ m/sec along a ramp inclined at $\alpha = 30°$. The diameter of the car wheels is $D = 400$ mm, the diameter of the car axle journals $d = 60$ mm (Fig. 174).

Determine the tension in the rope of the winch and the power developed by the winch if the coefficient of friction for the journals is $f = 0.1$ and the coefficient of rolling friction $k = 0.05$ cm.

Solution. As the car moves up at constant velocity

$$P = P' + G'',$$

where $\overline{P'}$ = force required to overcome kinetic-friction force,
$\overline{G''}$ = tangential component of weight of car and load,
$\overline{P}$ = tractive force (tension in rope).

The weight of the car and the load is

$$G = G_1 + G_2 = 40 \text{ kN},$$

$$G'' = G \sin\alpha = 40 \times 0.5 = 20 \text{ kN},$$

$$G' = G \cos\alpha = 40 \times 0.866 = 35.0 \text{ kN}.$$

The force required to overcome the kinetic-friction force is

$$P' = G' \frac{k + fr}{R},$$

where r = radius of car axle journal,
R = radius of car wheel.

$$P' = 35.0 \frac{0.05 + 0.1 \times 3}{20} = 0.613 \text{ kN}.$$

The tractive force is

$$P = G'' + P' = 20 + 0.613 = 20.613 \text{ kN} \cong 20.6 \text{ kN}.$$

The power developed by the winch is

$$N = Pv = 20.6 \times 1 = 20.6 \text{ kN-m/sec} = 20.6 \text{ kW}.$$

CHAPTER XVII

The Laws of Dynamics

102. Concept of a System of Particles

So far we have considered the motion of separate particles or the motion of bodies reduced to this case. In applied problems of mechanics one often deals with a system of particles rather than a single particle. *A system of material particles, or a material system, is a set of particles and bodies in which the motion of each particle depends on the motion of all the other particles.* An example is any rigid body or an arbitrary combination of rigid bodies.

The number of particles forming any rigid body will be assumed infinite. On this basis we shall extend the laws of dynamics for a particle to the motion of separate bodies or systems. These laws for systems of particles reflect in the most general form the mechanical phenomena occurring in nature.

In the study of systems of particles two categories of forces are distinguished, external and internal.

External forces are exerted on the particles of a given system by other bodies and particles not belonging to the system.

Internal forces are forces of interaction between particles within a system.

A very important property of the internal forces of a system is that the internal forces within each system are mutually balanced, i.e., the resultant force and the resultant couple are zero. As an illustration let us consider a system consisting of two bodies: a support and a shaft fixed in it. The forces external to this system are the forces of interaction with the earth—

the forces of weight, and also the forces of interaction with the motor—the moving forces, etc.

The internal forces are the pressures exerted by the support on the shaft and by the shaft on the support. According to Axiom IV of action and reaction, these forces are equal and opposite; they are balanced within the system being considered.

Another typical example of a system of particles is the solar system. The motions of all planets of the solar system are interrelated. A conclusive evidence of this interrelation is the discovery of the planet Neptune. Based on preliminary calculations and a study of irregularities in the motion of the planet Uranus, the French astronomer Le Verrier established in 1845 that the cause of these deviations could only lie in the existence of an unknown planet outside the orbit of the planet Uranus. Le Verrier calculated the position of this planet and with these data it was indeed discovered in 1846 by means of a powerful telescope and named Neptune. This was a triumph of theoretical mechanics.

The forces of interaction between the planets of the solar system are internal forces, and the forces exerted on them by other planets and stars not belonging to the solar system are external forces. Since the solar system is considerably removed from other celestial bodies, the external forces acting on the planets of the solar system are vanishingly small and generally neglected.

It should be borne in mind that one and the same force may be external relative to one system and internal relative to another. Thus, the force of gravity is an external force relative to all terrestrial bodies but must be classified as an internal force if both the body in question and the earth are included in the system under consideration.

Forces acting on the particles of a system are commonly divided into given forces and reactions of constraints. Given forces, as the name suggests, are known before the motion o a system is analysed. By contrast, reactions of constraints can be determined only in the course of considering the motion of a system.

As pointed out in the statics section, the direction of reactions of constraints can be indicated in some cases.

103. Law of Momentum for a Particle

The momentum of a particle is a vector quantity equal to the product of the mass of the particle and its velocity

$$\overline{q} = m\overline{v}. \tag{223}$$

The momentum vector has the same direction as the velocity. The momentum of a particle can be projected on the co-ordinate axes.

The projection on the x axis is mv_x, on the y axis mv_y, and on the z axis mv_z.

The dimension of momentum in the international system of units (SI) is

$$[mv] = [\text{kg-m/sec}]$$

and in the engineers' system of units (mkgfs)

$$[mv] = [\text{kgf-sec}^2/\text{m-m/sec}] = [\text{kgf-sec}].$$

We introduce another new concept—*the impulse of a force. The impulse of a constant force is a vector of magnitude equal to the product of the force by the time it acts and of the same direction as the force*

$$\overline{S} = \overline{P}\ (t_2 - t_1). \tag{224}$$

The impulse of a force varying in time is defined as the sum of elementary impulses

$$\overline{S} = \int_{t_1}^{t_2} d\overline{S} = \int_{t_1}^{t_2} \overline{P}\,dt, \tag{225}$$

where $d\overline{S} = \overline{P}\,dt$ is the elementary impulse during an infinitesimal interval of time. The impulse of a force can also be projected on the co-ordinate axes

$$\text{on } x \text{ axis } S_x = \int_{t_1}^{t_2} P_x\,dt,$$

$$\text{on } y \text{ axis } S_y = \int_{t_1}^{t_2} P_y\,dt,$$

$$\text{on } z \text{ axis } S_z = \int_{t_1}^{t_2} P_z\,dt.$$

The dimension of the impulse of a force in the international system of units (SI) is

$$[S] = [Pt] = [\text{N-sec}] = [\text{kg-m/sec}^2\text{-sec}] = [\text{kg-m/sec}]$$

and in the engineers' system of units (mkgfs)

$$[S] = [Pt] = [\text{kgf-sec}].$$

The impulse of a force is expressed in the same units as momentum.

Let us derive the law of momentum for the case when a particle A moves in a straight line under the action of a constant force. According to the fundamental equation of dynamics, the acceleration of the particle is then a constant

$$a = \text{constant}.$$

The velocity of the particle A at any instant is given by the formula

$$v_2 = v_1 + at. \qquad \text{(a)}$$

Write the fundamental law of dynamics in scalar form

$$P = ma.$$

Multiplying both sides by a time interval $t = t_2 - t_1$ gives

$$Pt = mat. \qquad \text{(b)}$$

Substituting the value of at from formula (a), $at = v_2 - v_1$, in the right-hand member of equation (b), we find

$$Pt = mv_2 - mv_1.$$

Taking into account that the product Pt is the impulse of the acting force, $Pt = S$, we have finally

$$S = mv_2 - mv_1. \qquad (226)$$

It follows that *the algebraic increment of the momentum of a particle during a time interval* $t = t_2 - t_1$ *is equal to the impulse of the acting force during the same time interval.*

The law of momentum in the most general form can be derived by using the differential equations of motion of a particle

$$m\frac{d^2x}{dt^2} = P_x,$$

$$m\frac{d^2y}{dt^2}=P_y,$$
$$m\frac{d^2z}{dt^2}=P_z.$$

Taking into account that the second derivatives of the co-ordinates are equal to the first derivatives of the corresponding projections of the velocities

$$\frac{d^2x}{dt^2}=\frac{dv_x}{dt},$$
$$\frac{d^2y}{dt^2}=\frac{dv_y}{dt},$$
$$\frac{d^2z}{dt^2}=\frac{dv_z}{dt}$$

and putting the constant m under the differential sign, we obtain

$$\left.\begin{aligned}\frac{d(mv_x)}{dt}&=P_x,\\ \frac{d(mv_y)}{dt}&=P_y,\\ \frac{d(mv_z)}{dt}&=P_z.\end{aligned}\right\}\qquad(227)$$

These equations express the law of momentum for a particle in differential form. *The derivative of the projection of the momentum of a particle is equal to the corresponding projection of the force acting on the particle.*

We integrate Eqs. (227) over a time interval from t_1 to t_2. The velocity of the particle varies from v_1 to v_2.

Upon integration we obtain

$$\left.\begin{aligned}mv_{2x}-mv_{1x}&=\int_{t_1}^{t_2}P_x\,dt=S_x,\\ mv_{2y}-mv_{1y}&=\int_{t_1}^{t_2}P_y\,dt=S_y,\\ mv_{2z}-mv_{1z}&=\int_{t_1}^{t_2}P_z\,dt=S_z.\end{aligned}\right\}\qquad(228)$$

Three scalar equations (228) in the projections on the co-ordinate axes are obviously equivalent to the vector equality

$$m\bar{v}_2 - m\bar{v}_1 = \int_{t_1}^{t_2} \bar{P}\,dt = \bar{S}. \tag{229}$$

Thus, we come to the conclusions:

1. A change in the projection of the momentum of a particle on any axis is equal to the projection on the same axis of the impulse of the acting force during the same time interval.

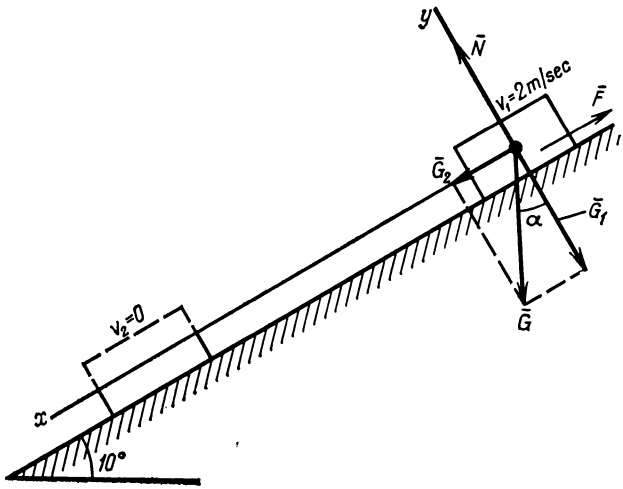


Fig. 175.

2. A change in the momentum vector of a particle is equal to the impulse vector of the acting force during the same time interval.

When solving problems one can use either the scalar (228) or the vector (229) form of the law.

Example 69. A case dropped onto an inclined plane of slope $\alpha = 10^\circ$ had a velocity $v_1 = 2$ m/sec (Fig. 175). After $t = 5$ sec, it came to rest ($v_2 = 0$) because of friction.

Determine the coefficient of friction between the case and the plane.

Solution. Set up the momentum equation in the direction of the x axis

$$mv_2 - mv_1 = S. \tag{a}$$

Express the mass of the case in terms of its weight

$$m = \frac{G}{g}.$$

Calculate the impulse of the acting forces along the x axis

$$S = (G_2 - F)\,t. \tag{b}$$

Resolving the weight of the case $\bar{G}$ along and perpendicular to the inclined plane, we obtain

$$G_1 = G\cos\alpha,$$

$$G_2 = G\sin\alpha.$$

Setting up the equation in the projections of all forces on the y axis, we find

$$\sum P_{iy} = 0, \quad N - G_1 = 0, \text{ whence } N = G_1 = G\cos\alpha.$$

The friction force $\bar{F}$ is proportional to the normal pressure

$$F = fN = Gf\cos\alpha.$$

Substituting the values of the forces $\bar{G}_2$ and $\bar{F}$ in the expression (b) for the impulse, we obtain

$$S = (G\sin\alpha - Gf\cos\alpha)\,t = G(\sin\alpha - f\cos\alpha)\,t.$$

Equate the value of the impulse to the change in momentum in the equation (a)

$$\frac{G}{g}v_2 - \frac{G}{g}v_1 = G(\sin\alpha - f\cos\alpha)\,t.$$

Dividing through by G and noting that $v_2 = 0$, we obtain

$$-\frac{v_1}{g} = t\sin\alpha - ft\cos\alpha.$$

Solve the resulting equation for the unknown coefficient of friction f

$$f = \frac{\sin\alpha}{\cos\alpha} + \frac{v_1}{gt\cos\alpha} = \tan\alpha + \frac{v_1}{gt\cos\alpha} =$$

$$= \tan 10^\circ + \frac{2}{9.81 \times 5 \times \cos 10^\circ} = 0.176 + 0.042 = 0.218.$$

104. Law of Momentum for a System of Particles

The momentum of a system of particles is the geometric sum of the momenta of all particles entering into the system

$$\overline{Q}=\sum_{i=1}^{n}\overline{q}_i=\sum_{i=1}^{n}m_i\overline{v}_i. \tag{230}$$

The momentum vector of a system can be projected on the co-ordinate axes

$$\left.\begin{aligned} Q_x&=\sum_{i=1}^{n}q_{ix}=\sum_{i=1}^{n}m_iv_{ix},\\ Q_y&=\sum_{i=1}^{n}q_{iy}=\sum_{i=1}^{n}m_iv_{iy},\\ Q_z&=\sum_{i=1}^{n}q_{iz}=\sum_{i=1}^{n}m_iv_{iz}.\end{aligned}\right\} \tag{231}$$

Each particle of the system is in motion under the action of external and internal forces whose resultants will be denoted, respectively, by $\overline{P}_i^e$ and $\overline{P}_i^i$.

We write the law of momentum in differential form for an arbitrary particle m_i of the system in the projections on an axis x

$$\frac{d(m_iv_{ix})}{dt}=P_{ix}^e+P_{ix}^i.$$

Sum up the equation for all the particles of the system

$$\sum_{i=1}^{n}\frac{d(m_iv_{ix})}{dt}=\sum_{i=1}^{n}P_{ix}^e+\sum_{i=1}^{n}P_{ix}^i.$$

We put the differentials in the first term of the equation before the summation sign. The sum will then represent the projection of the momentum of the system on the x axis

$$Q_x=\sum_{i=1}^{n}m_iv_{ix}.$$

The second term can be replaced by the projection of the resultant of the external forces on the x axis

$$\sum_{i=1}^{n}P_{ix}^e=R_x^e.$$

The third term is zero as the internal forces are balanced within the system

$$\sum_{i=1}^{n} P_{ix}^{i} = 0.$$

The equation takes the form

$$\frac{dQ_x}{dt} = R_x^e. \tag{232a}$$

The momentum equations for the system in the projections on the other two co-ordinate axes are derived in a similar way

$$\frac{dQ_y}{dt} = R_y^e, \tag{232b}$$

$$\frac{dQ_z}{dt} = R_z^e. \tag{232c}$$

Consequently, *the derivative of the projection of the momentum of a system on any axis is equal to the projection of the resultant of the external forces on the same axis.*

Integrating Eq. (232a) over a time interval from t_1 to t_2, we obtain

$$Q_{2x} - Q_{1x} = \int_{t_1}^{t_2} R_x^e \, dt.$$

The right-hand member of this equation represents the projection of the impulse of the external forces on the x axis

$$\int_{t_1}^{t_2} R_x^e \, dt = S_x^e.$$

We have finally

$$Q_{2x} - Q_{1x} = S_x^e. \tag{233a}$$

Similarly, we obtain

$$Q_{2y} - Q_{1y} = S_y^e, \tag{233b}$$

$$Q_{2z} - Q_{1z} = S_z^e. \tag{233c}$$

Three scalar equations (233a), (233b) and (233c) are obviously equivalent to a single vector equation

$$\overline{Q}_2 - \overline{Q}_1 = \overline{S}^e. \tag{234}$$

Thus, we come to the conclusions:

1. A change in the projection of the momentum of a system on any axis is equal to the projection on the same axis of the impulse of the external forces during the same time interval.

2. A change in the momentum vector of a system is equal to the impulse vector of the external forces during the same time interval.

Consider the case when the resultant of the external forces has zero projection on an axis x, $R_x^e = 0$, then $\frac{dQ_x}{dt} = R_x^e = 0$ and consequently $Q_x =$ constant, i.e., if the projection of the resultant of the external forces on any axis is zero, the projection of the momentum of the system on this axis remains constant.

105. Potential and Kinetic Energy

In the study of physical phenomena it is found that there exist two basic forms of mechanical energy. These are *potential energy*, or the *energy of position*, and *kinetic energy*, or the *energy of motion*. We mostly have to deal with potential energy due to gravity.

The potential energy of a particle or body with respect to the force of gravity is defined as the capacity of this particle or body to do work as it moves down from a certain height to sea level (*zero level*). The potential energy is numerically equal to the work done by the gravity force during a displacement from the zero level to a given position. Denoting the potential energy by V, we obtain

$$V = GH, \tag{235}$$

where $G =$ weight of particle (or body),

$H =$ height of its centre of gravity above sea level.

Potential energy is expressed in the same units as work, i.e., in joules (j) in the international system and in kgf-m in the engineers' system of units.

The kinetic energy is defined as the capacity of a moving body (*or particle*) *to do work*. The kinetic energy of a particle is numerically equal to half the product of its mass by the square of the velocity, i.e., $\frac{mv^2}{2}$.

Kinetic energy is also expressed in the same units as work

$$\left[\frac{mv^2}{2}\right] = [m]\,[v^2] = \left[\text{kg}\,\frac{\text{m}^2}{\text{sec}^2}\right] = \left[\frac{\text{kg-m}}{\text{sec}^2}\,\text{m}\right] = [\text{N-m}] = [\text{j}].$$

As pointed out in Sec. 102, every rigid body or mechanical system is composed of a large number of separate particles. Hence the kinetic energy of a rigid body or any mechanical system can be represented as the sum of the kinetic energies of all the particles forming this body or system. Denoting the kinetic energy of a body or system by E, we obtain

$$E = \sum_{i=1}^{n} \frac{m_i v_i^2}{2}, \tag{236}$$

where m_i = mass of arbitrary particle,
v_i = velocity of particle,
n = number of particles in the system under consideration, commonly $n \to \infty$.

Potential and kinetic energies are scalar quantities.

106. Kinetic Energy of a Body in Various Types of Motion

Every system, and a rigid body too, consists of separate particles. Hence the kinetic energy of a system or rigid body can be represented as the sum of the kinetic energies of all the particles forming the body

$$E = \sum_{i=1}^{n} \frac{m_i v_i^2}{2}.$$

Let us see how the kinetic energy of a rigid body is determined in some cases of motion.

Kinetic Energy of a Rigid Body in Translation. The translation of a rigid body is characterized by the fact that the velocities of all its points are equal and have the same direction (Fig. 176*a*), i.e.,

$$\bar{v}_i = \bar{v}_A = \bar{v}_B = \ldots = \bar{v}_C,$$

where $\bar{v}_C$ is the velocity of the centre of gravity or any other point of the body.

The kinetic energy of the body is

$$E=\sum_{i=1}^{n}\frac{m_i v_i^2}{2}=\frac{v_C^2}{2}\sum_{i=1}^{n}m_i=\frac{Mv_C^2}{2}, \qquad (237)$$

where M is the entire mass of the rigid body

$$M=\sum_{i=1}^{n}m_i.$$

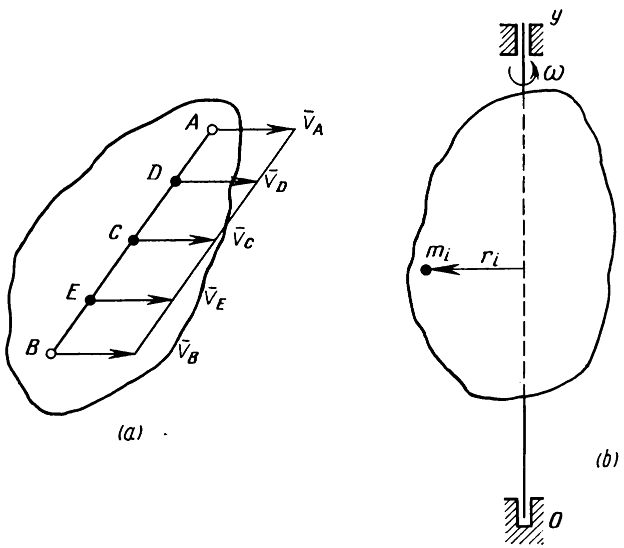


Fig. 176.

Consequently, *the kinetic energy of a rigid body in translation is equal to the product of half the square of the velocity of any point of the body and the entire mass of the body.*

Kinetic Energy of a Rigid Body Rotating About a Fixed Axis. If a body rotates about an axis y with an angular velocity ω, the velocity of any point i of the body is proportional to the distance r_i of this point from the axis of rotation (Fig. 176*b*)

$$v_i=\omega r_i,$$

where r_i = distance of the point considered from axis of rotation (a variable quantity),
ω = angular velocity (the same for all points of the body).

Substituting the value of v_i in the kinetic energy formula and putting the constant quantity ω before the summation sign, we obtain

$$E=\sum_{i=1}^{n}\frac{m_i v_i^2}{2}=\sum_{i=1}^{n}\frac{m_i(\omega r_i)^2}{2}=\frac{\omega^2}{2}\sum_{i=1}^{n}m_i r_i^2.$$

The quantity $\sum_{i=1}^{n} m_i r_i^2$ *which represents the sum of the products of the mass of each particle and the square of its distance from the axis of rotation y is called the moment of inertia of the body with respect to this axis and is denoted by* I_y. This quantity plays an important part in the dynamics of rigid bodies.

Consequently, *the kinetic energy of a rigid body rotating about an axis is equal to the product of half the square of the angular velocity of the body and the moment of inertia of the body with respect to the axis of rotation*

$$E=I_y\frac{\omega^2}{2}. \tag{238}$$

Kinetic Energy of a Rigid Body in Plane Motion. As demonstrated in the kinematics section, any plane motion may be resolved into two motions: a translation with an arbitrary pole and a rotation about the pole. Accordingly, *the kinetic energy of a body in plane motion is made up of the translational kinetic energy and the rotational kinetic energy*

$$E=\frac{Mv^2}{2}+I\frac{\omega^2}{2}, \tag{239}$$

where v = velocity of pole in translation,
ω = angular velocity of body which is independent of choice of pole.

107. Moments of Inertia of Homogeneous Bodies of Simple Shape

In the preceding section we met with a new concept, the moment of inertia of a rigid body. This quantity is a very important characteristic and is frequently encountered in the solution of problems.

The moment of inertia of a rigid body is equal to the sum of the products of the mass m_i of each particle of the body and the square of its distance from the axis of rotation y

$$I_y = \sum_{i=1}^{n} m_i r_i^2, \tag{240}$$

where n is the number of elementary particles into which the body is broken up.

When a rigid body is bounded by curved surfaces, its moment of inertia can be computed by breaking it up into an infinite number of elementary particles and performing integration

$$I_y = \int_M r_i^2 \, dm. \tag{241}$$

The index M indicates that the integration is extended over the entire mass of the body.

Determine the dimension of the moment of inertia

$$[I_y] = [m_i] [r_i^2].$$

Substituting the dimensions from the international system of units (SI), we obtain

$$[I_y] = [\text{kg-m}^2].$$

Similarly, in the engineers' system of units (mkgfs) we have $[m] = \text{kgf-sec}^2/\text{m}$ and $[r] = \text{m}$

$$[I_y] = [m_i] [r_i^2] = [\text{kgf-sec}^2\text{-m}^2/\text{m}] = [\text{kgf-sec}^2\text{-m}].$$

We present (without derivation) formulas for computing the moments of inertia of simple bodies with respect to the axis of rotation.

For a homogeneous rod, with respect to an axis perpendicular to the axis of the rod and passing through its end (Fig. 177*a*)

$$I = \frac{ml^2}{3}, \tag{242}$$

where m = mass of rod,
l = length of rod.

For a homogeneous rod, with respect to an axis through its centre of gravity

$$I_C = \frac{ml^2}{12}. \tag{243}$$

For a homogeneous cylinder (Fig. 177*b*)

$$I = \frac{mR^2}{2}, \tag{244}$$

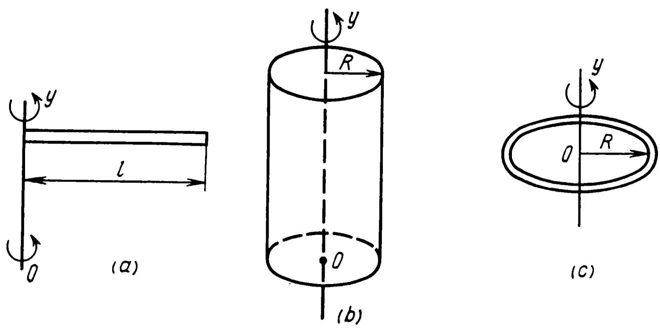


Fig. 177.

where m = mass of cylinder,
R = radius of cylinder.

For a circumference or a thin ring, if its thickness is negligible (Fig. 177*c*)

$$I = mR^2. \tag{245}$$

For a homogeneous sphere, with respect to any centroidal axis

$$I = \frac{2mR^2}{5}. \tag{246}$$

In computing the moments of inertia for bodies of irregular shape, which is a very laborious and at times insoluble problem, it is common to use the concept of the radius of gyration k. The radius of gyration is defined as a quantity equal to the square root of the ratio of the moment of inertia of the body to its mass

$$k = \sqrt{\frac{I}{m}}, \tag{247}$$

whence

$$I = mk^2. \tag{248}$$

There are various experimental methods available for determining the moments of inertia and the radii of gyration of bodies of irregular shape.

108. Law of Kinetic Energy for a Particle

Let us derive the law of kinetic energy for a particle of mass m acted upon by a constant force $\overline{P}$. In this case the particle has a constant acceleration, $a = \frac{P}{m} =$ constant, and

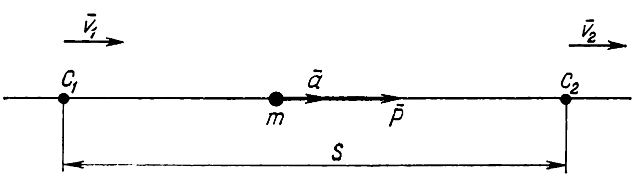


Fig. 178.

its motion is uniformly accelerated. The direction of motion coincides with that of the force $\overline{P}$ (Fig. 178), i.e., the particle moves in a straight line.

Let the particle move from position C_1 to position C_2 under the action of the force $\bar{P}$ (Fig. 178).

The velocity v_2 of the particle at the end of the motion is given by the formula

$$v_2 = v_1 + at,$$

where $v_1 =$ initial velocity of particle in position C_1,
$t =$ time of motion.

The distance travelled by the particle can be expressed as

$$S = v_1 t + \frac{at^2}{2}.$$

Since the direction of the force coincides with that of motion, the work done by the force $\overline{P}$ is

$$U = PS = ma\,S = mav_1 t + ma\,\frac{at^2}{2}.$$

We add to and subtract from the right-hand member of this equation the expression for the initial kinetic energy $\frac{mv_1^2}{2}$, then

$$PS = mav_1t + ma\frac{at^2}{2} + \frac{mv_1^2}{2} - \frac{mv_1^2}{2}$$

or

$$PS = \frac{2mav_1t + m(at)^2 + mv_1^2}{2} - \frac{mv_1^2}{2} =$$

$$= \frac{m}{2}[2av_1t + (at)^2 + v_1^2] - \frac{mv_1^2}{2}.$$

The bracketed expression represents the square of the sum $(v_1 + at)^2$, which in turn equals v_2^2, then

$$PS = \frac{mv_2^2}{2} - \frac{mv_1^2}{2}. \tag{249}$$

This equation shows that *the change in kinetic energy of a particle during a displacement is equal to the work done during the same displacement by the force acting on the particle.*

Below is given an alternate derivation of the law of kinetic energy for the general case of an arbitrary force and a curvilinear motion.

Write the fundamental law of dynamics

$$m\bar{a} = \bar{P},$$

where $\bar{P}$ is the resultant of a force system applied to a particle.

Project this vector equation on the direction of the velocity of the particle, i.e., on the direction of the tangent to the path. As is known from kinematics, the projection of the total acceleration on the tangent to the path is equal to the tangential component of the acceleration $\bar{a}_t$ (Fig. 179a)

$$ma_t = P\cos(\bar{P}, \bar{v}). \tag{a}$$

The tangential component of the acceleration is determined as the first derivative of the velocity

$$a_t = \frac{dv}{dt}.$$

Substitute the value of the tangential component of the acceleration in equation (a) and multiply both sides by the velocity

$$\frac{mv\,dv}{dt} = Pv\cos(\overline{P}, \overline{v}).$$

Separate the variables

$$mv\,dv = Pv\,dt\cos(\overline{P}, \overline{v}).$$

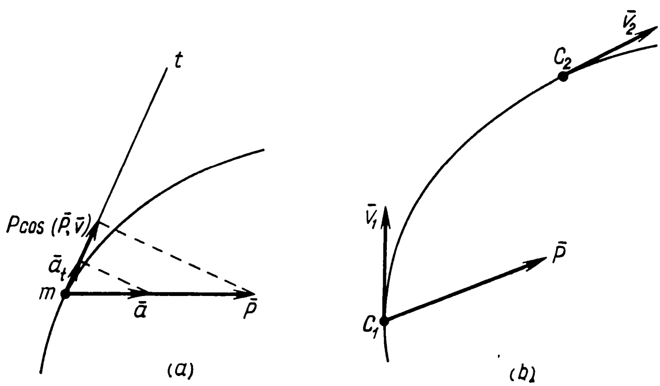


Fig. 179.

Noting that $mv\,dv = d\left(\frac{mv^2}{2}\right)$ and $v\,dt = dS$, where dS is an elementary displacement of the particle, we have

$$d\left(\frac{mv^2}{2}\right) = P\,dS\cos(\overline{P}, \overline{v}). \tag{250}$$

The expression on the right-hand side of Eq. (250) represents the elementary work

$$P\,dS\cos(\overline{P}, \overline{v}) = dU.$$

Let the particle move from position C_1 to position C_2 under the action of the force $\overline{P}$ (Fig. 179*b*), its velocity changing from v_1 to v_2.

Integrating Eq. (250) between the above limits, we obtain finally

$$\frac{mv_2^2}{2}-\frac{mv_1^2}{2}=\int_{C_1}^{C_2} P\,dS\cos(\overline{P},\overline{v})=U. \qquad (251)$$

Consequently, *the change in kinetic energy of a particle during a certain time interval is equal to the work done on the corresponding displacement by the resultant of all the forces applied to the particle.*

In solving problems of dynamics it is well to remember that the law of kinetic energy can be used to advantage in all cases where acting forces or velocities depend on the position of a particle.

109. Law of Kinetic Energy for a System of Particles

Consider a system of n particles. Separate an arbitrary particle of mass m_i and apply to it all acting forces. Denote by $\overline{P}_i$ the resultant of all given forces applied to the particle and by $\overline{N}_i$ the resultant of the reactions of constraints. Write the law of kinetic energy for the particle in differential form

$$d\left(\frac{m_i v_i^2}{2}\right)=dU_{P_i}+dU_{N_i}.$$

Add up the kinetic energy equations for all the particles of the system

$$\sum_{i=1}^{n} d\left(\frac{m_i v_i^2}{2}\right)=d\sum_{i=1}^{n}\frac{m_i v_i^2}{2}=\sum_{i=1}^{n} dU_{P_i}+\sum_{i=1}^{n} dU_{N_i}.$$

Integrating this equation over a time interval from 0 to t, we obtain

$$E_2-E_1=\sum_{i=1}^{n} U_{P_i}+\sum_{i=1}^{n} U_{N_i},$$

where $\sum U_{P_i}$ = work of given forces during the displacement considered,

$\sum U_{N_i}$ = work of reactions of constraints.

When the constraints of a system are frictionless, the total work done by the reactions of the constraints during any displacement of the system is zero

$$\sum_{i=1}^{n} U_{N_i} = 0.$$

The law of kinetic energy for the system of particles is then written as

$$E_2 - E_1 = \sum_{i=1}^{n} U_{P_i}, \tag{252}$$

i.e., *the change in kinetic energy of a system of particles during a certain time interval is equal to the sum of the work done on the corresponding displacement by the given forces applied to the system.*

If friction in the constraints imposed on a system cannot be neglected, the kinetic energy equations must take into account the work done by friction forces.

It should be noted that, when deriving the law of kinetic energy for a system of particles, we classified the forces acting on each particle not as *internal* and *external* forces but as *given forces* and *reactions of constraints. Here the work done by internal forces is not zero in general.*

This may be verified by considering a system of particles connected by elastic springs. The internal forces of such a system are elastic forces due to deformation of the springs. The work done by these internal forces is not zero, however.

Example 70. Determine the kinetic energy of a homogeneous cylinder rolling without friction on a straight track. The mass of the cylinder is $M = 200$ kg, the velocity of its centre $v_O = 2$ m/sec (Fig. 180).

Solution. As is known from kinematics, a cylinder rolling on a straight track is in plane motion. Consequently, its kinetic energy is given by the formula

$$E = \frac{Mv_O^2}{2} + \frac{I\omega^2}{2}.$$

The moment of inertia of a homogeneous cylinder is

$$I = \frac{MR^2}{2}.$$

To determine the angular velocity of the cylinder, we consider the motion of a point O relative to the instantaneous centre of zero velocity: $v_O = \omega R$, whence $\omega = \frac{v_O}{R}$.

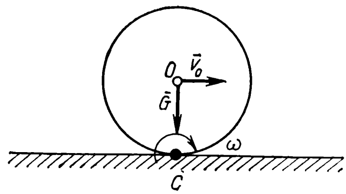


Fig. 180.

Substituting the values of M, I and ω in the kinetic energy formula, we obtain

$$E = \frac{Mv_O^2}{2} + \frac{MR^2 \frac{v_O^2}{R^2}}{2 \times 2} = \frac{Mv_O^2}{2} + \frac{Mv_O^2}{4} = \frac{3}{4} Mv_O^2 = \frac{3}{4} 200 \times 2^2 =$$
$$= 600 \text{ kg-m}^2/\text{sec}^2 = 600 \text{ j}.$$

Example 71. An automobile is travelling at a speed $v_1 = 72$ km/hr $= 20$ m/sec. The maximum braking force of the

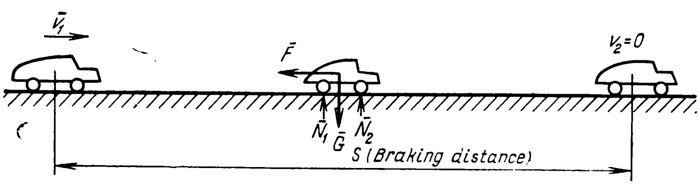


Fig. 181.

automobile amounts to half its weight, i.e., the reduced coefficient of resistance to the motion of the automobile is $f = 0.5$.

Determine the distance in which the automobile can be brought to a stop ($v_2 = 0$) by applying the brakes (Fig. 181); assume that the automobile moves in translation.

Solution. Set up the kinetic energy equation

$$E_2 - E_1 = \sum U_{P_i}.$$

The kinetic energy in translation is

$$E=\frac{Mv^2}{2}=\frac{Gv^2}{2g}.$$

Calculate the work done by the given forces (Fig. 181)

$$\sum U_{P_i}=U_G+U_{N_1}+U_{N_2}+U_F=-FS=-fGS,$$

where $F = fG$ is the resistance to the motion of the automobile.

The work of the weight $\overline{G}$ and of the reactions $\overline{N}_1$ and $\overline{N}_2$ is zero as these forces are perpendicular to the displacement.

Substituting the value of $\sum U_{P_i}$ and E_1 in the kinetic energy equation and noting that $v_2 = 0$ and hence $E_2 = 0$, we obtain

$$-\frac{Gv_1^2}{2g}=-fGS.$$

Multiply the equation by -1 (to reverse the signs) and cancel out G

$$S=\frac{v_1^2}{2gf}=\frac{20^2}{2\times 9.81\times 0.5}=40.8\text{ m}.$$

Example 72. A rope supporting two loads $P_2 = 100$ kN and $P_3 = 30$ kN passes over a pulley of weight $P_1 = 20$ kN and radius $r = 0.2$ m. The load P_2 moves vertically down and the load P_3 moves up a smooth inclined plane of slope $\alpha = 45°$ (Fig. 182). The pulley has a radius of gyration $k = 0.1$ m.

Neglecting the mass of the rope and the frictional resistance of the supports, determine (1) the distance S the load P_2 must move downward to attain a velocity $v = 3$ m/sec if the initial velocity is zero; (2) the acceleration of the loads.

Solution. (1) The velocity of the loads is numerically equal to the velocity on the periphery of the pulley. Consequently, the angular velocity of the pulley can be determined from the formula

$$\omega=\frac{v}{r}.$$

Write the law of kinetic energy for the system under consideration

$$E_2 - E_1 = \sum_{i=1}^{n} U_{P_i}, \tag{a}$$

$E_1 = 0$ as the initial velocity is zero,

$$E_2 = \frac{I_1\omega^2}{2} + \frac{m_2 v^2}{2} + \frac{m_3 v^2}{2}, \tag{b}$$

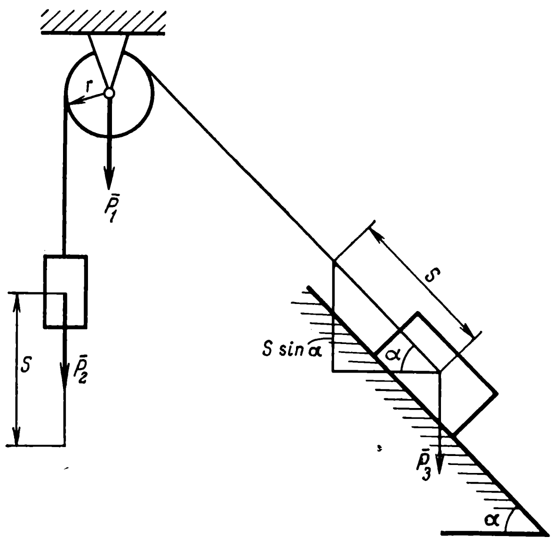


Fig. 182.

where the moment of inertia of the pulley

$$I_1 = \frac{P_1}{g} k^2.$$

Substitute the values of I_1, ω, $m_2 = \frac{P_2}{g}$ and $m_3 = \frac{P_3}{g}$ in equation (b)

$$E_2 = \frac{I_1 v^2}{2r^2} + \frac{P_2 v^2}{2g} + \frac{P_3 v^2}{2g} = \frac{v^2}{2g}\left(\frac{P_1 k^2}{r^2} + P_2 + P_3\right). \tag{c}$$

Calculate the work of the given forces. The work is done by only two forces, $\overline{P}_2$ and $\overline{P}_3$

$$\sum_{i=1}^{n} U_{P_i} = P_2S - P_3S \sin\alpha = S(P_2 - P_3 \sin\alpha).$$

Substituting the calculated values in equation (a), we obtain

$$\frac{v^2}{2g}\left(\frac{P_1k^2}{r^2} + P_2 + P_3\right) = S(P_2 - P_3 \sin\alpha), \qquad \text{(d)}$$

whence

$$S = \frac{v^2\left(\frac{P_1k^2}{r^2} + P_2 + P_3\right)}{2g(P_2 - P_3 \sin\alpha)} = \frac{3^2\left(\frac{20\times 0.1^2}{0.2^2} + 100 + 30\right)}{2\times 9.81\,(100 - 30\times 0.707)} = 0.787 \text{ m}.$$

(2) In order to calculate the acceleration of the loads we differentiate equation (d) with respect to time. Noting that the expressions in parentheses are time independent, we have

$$\frac{2v\,dv}{2g\,dt}\left(\frac{P_1k^2}{r^2} + P_2 + P_3\right) = \frac{dS}{dt}(P_2 + P_3 \sin\alpha).$$

But $\frac{dS}{dt} = v$ and $\frac{dv}{dt} = a$, i.e., the acceleration of the loads P_2 and P_3.

We find ultimately

$$a = \frac{dv}{dt} = \frac{(P_2 + P_3 \sin\alpha)\,g}{\frac{P_1k^2}{r^2} + P_2 + P_3} = \frac{772}{135} = 5.73 \text{ m/sec}^2.$$

110. Fundamental Equation of Dynamics for a Rigid Body in Rotation

Let a body rotate about a fixed axis Oy (Fig. 183) with an angular acceleration α.

The body is subjected to forces $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_n$. Derive the relation between the forces applied to the body and the angular acceleration α imparted to it. Use D'Alembert's principle.

Consider an elementary particle of the body, say, a particle i. Apply to it the normal and tangential components of the inertia force. Similarly, applying the inertia forces to all the particles of the body, we obtain a balanced force system according to D'Alembert's principle. Write equations of equilibrium for this system. Set up the sum of the moments of all forces about the axis of rotation Oy. The algebraic sum of the moments of the external forces $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n$ about the axis of rotation Oy will be denoted by $\sum M_y$ and called a *torque*.

The centrifugal inertia forces intersect the axis of rotation and give no moment about it. The tangential components of the inertia forces will enter into the equation.

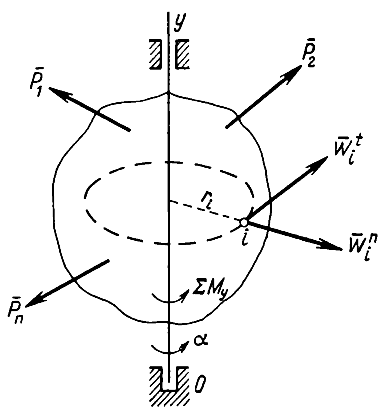


Fig. 183.

The moment equation takes the form

$$\sum W_i^t r_i = \sum M_y. \tag{a}$$

Write out the left-hand member of equation (a)

$$W_i^t = m_i r_i \alpha.$$

Substituting in equation (a), we obtain

$$\sum m_i r_i^2 \alpha = \sum M_y.$$

Put the angular acceleration before the summation sign as a quantity common to all particles of the body

$$\alpha \sum m_i r_i^2 = \sum M_y.$$

The coefficient of α is the moment of inertia of the body about the axis Oy already familiar to us

$$\sum m_i r_i^2 = I_y.$$

We obtain finally

$$I_y \alpha = \sum M_y. \tag{253}$$

This is the fundamental equation of dynamics for a rigid body in rotation. It states that the product of the moment of inertia of a body by the angular acceleration in its rotation about any axis is equal to the sum of the moments of all forces about this axis.

This equation is similar to the fundamental equation of dynamics for a particle

$$ma = P.$$

The characteristic of inertness of a rotating body is its moment of inertia and the cause of rotation about an axis is the moment of external forces.

From Eq. (253) it follows that

$$I_y = \frac{\sum M_y}{\alpha}, \tag{254}$$

$$\alpha = \frac{\sum M_y}{I_y}. \tag{255}$$

The last formula helps in understanding the physical meaning of the moment of inertia of a body. This quantity characterizes the inertness of a body in rotation. The larger the moment of inertia of a body, the larger the torque required to impart to the body a certain angular acceleration α.

Consequently, the moment of inertia can be regarded as a *measure of inertness of a rigid body in the rotation about a fixed axis* just as the mass is a measure of inertness of a particle or body in translation.

Let us see how the fundamental equation of rotation of a rigid body can be used for the solution of problems.

Example 73. A flywheel, which is assumed to be a thin ring of mass $m = 300$ kg and radius $R = 0.6$ m, is rotating according to the law $\varphi = 2t^2$ (φ in radians, t in seconds).

Neglecting the mass of the shaft and friction in the bearings, determine the torque M_O which must be applied to the flywheel to maintain the prescribed law of rotation (Fig. 184).

Solution. Use the fundamental equation of dynamics for a body in rotation

$$I\alpha = \sum M.$$

Indicate all external forces acting on the flywheel and shaft. These are the weight of the flywheel $\overline{G}$, the reactions at the bearings $\overline{N}_1$ and $\overline{N}_2$ and the torque M_O. The weight $\overline{G}$ and the reactions at the bearings intersect the axis of rotation and therefore give no moment about it. Consequently, $\sum M = M_O$.

Calculate the moment of inertia of the flywheel

$$I = mR^2 = 300 \times 0.6^2 = 108 \text{ kg-m}^2.$$

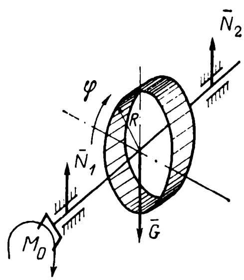


Fig. 184.

Determine the angular acceleration of the flywheel from the given equation of rotation

$$\omega = \frac{d\varphi}{dt} = \frac{d\,(2t^2)}{dt} = 4t\,\frac{1}{\text{sec}},$$

$$\alpha = \frac{d\omega}{dt} = \frac{d\,(4t)}{dt} = 4\,\frac{1}{\text{sec}^2}.$$

Substituting the numerical values in the fundamental law of rotation of a body, we obtain

$$M_O = I\alpha = 108 \times 4 = 432 \text{ kg-m}^2/\text{sec}^2 = 432 \text{ N-m}.$$

Example 74. A pulley representing a homogeneous cylinder of mass $m_2 = 400$ kg and radius $R = 1$ m is mounted on a solid shaft of mass $m_1 = 50$ kg and radius $r = 0.2$ m.

The shaft and pulley were rotating with an angular velocity $\omega_0 = 4\,\frac{1}{\text{sec}}$. Then the shaft was stopped by means of a brake shoe pressed against the pulley rim (Fig. 185). The coefficient of friction between the shoe and the pulley rim is $f = 0.2$.

Determine the magnitude of the pressing force $\overline{Q}$ if the shaft was brought to rest in $t = 2$ sec. Neglect friction in the bearings and assume the rotation of the shaft during the slow-down period to be uniformly decelerated.

Solution. Use the fundamental equation of rotation of a body

$$I\alpha = \sum M.$$

Figure 185 shows the external forces acting on the shaft and pulley. Except for the friction force $\overline{F}$, all forces intersect

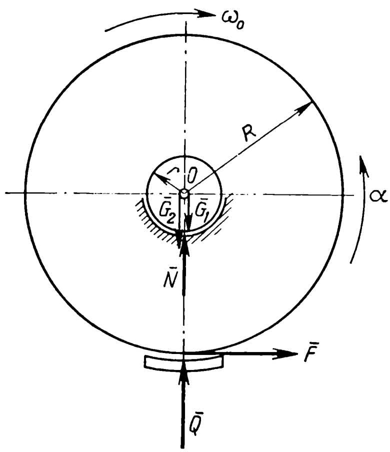


Fig. 185.

the axis of rotation and give no moment about it.

Therefore,

$$\sum M = FR.$$

The friction force $\overline{F}$ can be expressed in terms of the normal pressure $F = fQ$, then

$$\sum M = fQR.$$

The fundamental equation takes the form $fQR = I\alpha$, whence

$$Q = \frac{I\alpha}{fR}.$$

Calculate the moment of inertia of the system under consideration

$$I = I_{sh} + I_p = \frac{m_1 r^2}{2} + \frac{m_2 R^2}{2} = \frac{50 \times 0.2^2}{2} + \frac{400 \times 1^2}{2} = 201 \text{ kg-m}^2.$$

To determine the angular acceleration, we use the equation of uniformly variable rotation

$$\omega = \omega_0 + \alpha t,$$

whence

$$\alpha = \frac{\omega - \omega_0}{t}.$$

In our case the final angular velocity is zero and consequently

$$\alpha = -\frac{\omega_0}{t} = -\frac{4}{2} = -2 \frac{1}{\sec^2}.$$

The angular acceleration is substituted in the fundamental equation of rotation with a positive sign if it has the same sense as the torque (as in our case). The torque and the angular acceleration are both counterclockwise (Fig. 185). Substituting the numerical values in the expression for the pressing force $\overline{Q}$, we obtain

$$Q = \frac{I\alpha}{fR} = \frac{201 \times 2}{0.2 \times 1} = 2{,}010 \text{ N}.$$

Using the engineers' system of units, $\bar{Q}$ may be expressed in kilograms force as

$$Q = 0.102 \times 2{,}010 = 205 \text{ kgf}.$$

Example 75. A pulley of radius $r = 0.5$ m representing a homogeneous cylinder is driven by a belt (Fig. 186*a*); the tension in the two parts of the belt is, respectively, $T_1 = 200$ kN and $T_2 = 100$ kN. The weight of the pulley is $P = 600$ kN.

Neglecting friction in the journals, determine the total acceleration of point A on the rim of the pulley in $t = 2$ sec after the motion starts.

Solution. Determine the sum of the moments of the external forces $\overline{T}_1$ and $\overline{T}_2$ about the axis of rotation

$$\sum M_O = (T_1 - T_2)\, r = (200 - 100)\, 0.5 = 50 \text{ kN-m}.$$

The moment of inertia of the pulley is

$$I = \frac{P}{g}\frac{r^2}{2} = \frac{600 \times 0.5^2}{9.81 \times 2} = 7.64 \text{ kg-m}^2.$$

Calculate the angular acceleration of the pulley

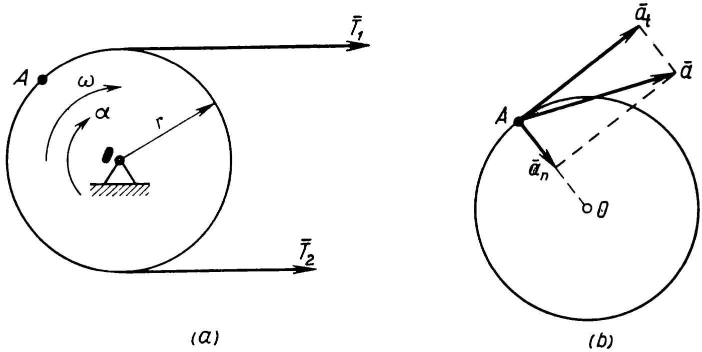


Fig. 186.

$$\alpha = \frac{\sum M_O}{I} = \frac{50}{7.64} = 6.55 \frac{1}{\text{sec}^2}.$$

The moment of the external forces is constant and so is the angular acceleration. Consequently, the pulley is in uniformly accelerated rotation. The angular velocity 2 sec after the motion starts is given by the formula

$$\omega = \omega_0 + \alpha t = 6.55 \times 2 = 13.1 \frac{1}{\text{sec}}.$$

Calculate now the total acceleration of point A (Fig. 186*b*)

$$a = r\sqrt{\omega^4 + \alpha^2} = 0.5\sqrt{13.1^4 + 6.55^2} = 85.6 \text{ m/sec}^2.$$

CHAPTER XVIII

Application of the Laws of Kinematics and Dynamics to the Analysis of Mechanisms*

111. Principles and Definitions

Classification of Machines. This chapter provides a theoretical basis for designing machines and mechanisms used in various branches of industry.

A machine is a combination of bodies connected together so that to a given motion of one of the bodies corresponds a definite motion of each of the remaining bodies. A machine is intended for a continual and indefinitely long transformation of some type of energy into mechanical work or for reverse transformation of work into another form of energy.

Consequently, a machine is characterized by:

(1) the transformation of energy into mechanical work or the transformation of mechanical work into another form of energy;

(2) definite motion of all its parts when any one part executes a specified motion.

All the variety of machines can be divided according to the nature of operation into four classes: prime movers, transforming machines, transporting machines, machine tools or working machines.

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* This chapter includes material which is beyond the scope of courses for non-mechanical vocational schools. However, these topics are on the curriculum for certain technological trades trained in the fundamentals of technical mechanics.

The importance of the material is in the emphasis on practical application, and that is why the chapter is included in the textbook.

Every machine consists of the following principal components: (a) receiver, which is directly affected by the external forces driving the machine; (b) tools, or actuators, doing the work the machine is designed to produce; (c) transmission mechanisms, or drives, which transmit work and a particular type of motion from the receiver to the tool.

Besides these principal parts, a machine has parts for control and regulation of motion, as well as a frame and foundation which serve to support the moving parts of the machine.

As an example, a diagram of an engine is presented in Fig. 187. Piston *1* is the receiver upon which steam pres-

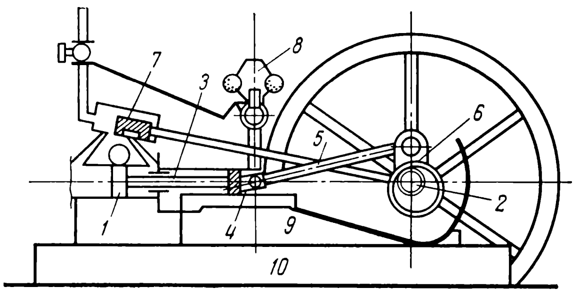


Fig. 187.

sure is exerted, and shaft *2* is the tool, or actuator. The transmission mechanism consists of piston rod *3*, slider *4*, connecting rod *5* and crank *6*. Slide mechanism 7 and governor *8* control and regulate motion. Frame *9* and foundation *10* support all elements of the machine.

Kinematic Pairs *A kinematic pair consists of two bodies in contact which are joined together so that the relative motion between these two bodies is consistent*; for example, piston and cylinder, shaft and bearing, etc.

The bodies constituting a kinematic pair are called *links* or *members*. Parts of a machine may be fastened together permanently; a combination of such parts represents a single member.

Classification of Kinematic Pairs According to the Nature of Contact Between Elements. Kinematic pairs are divided into two main classes: lower and higher pairs. In lower kinematic pairs contact between members takes place over surfaces, in higher pairs along lines or at points.

Lower kinematic pairs.

1. Sliding pairs: (a) cylinder *I* and piston with rod *II* (Fig. 188*a*); (b) shaft *I* with guide key and hub *II* (Fig. 188*b*); (c) slider *I* and straight guides *II* (Fig. 188*c*); (d) slide valve gear—slide valve *I* and slide valve face *II* (Fig. 188*d*).
2. Turning pairs: (a) pin joint (Fig. 189*a*); (b) fork joint (Fig. 189*b*); (c) shaft and bearing (Fig. 189*c*); (d) ball joint of a lever (Fig. 189*d*).
3. Screw pairs: screw and nut (Fig. 189*e*).

Higher kinematic pairs.

1. Shaped rollers—contact along a line perpendicular to the plane of the drawing (Fig. 190*a*).
2. Wheel and rail—contact along a line (Fig. 190*b*).
3. Friction rollers—contact along a line (Fig. 190*c*).
4. Gear wheels—contact along a line (Fig. 190*d*).
5. Cam with a pointed follower—contact at points (Fig. 190*e*).
6. Cam with a roller follower—contact along a line (Fig. 190*f*).

Lower pairs are more wear-resistant since the pressure of one member on the other is distributed over the contact surface, whereas in higher pairs contact between members takes place at points or along lines and the force is transmitted through a small number of points of contact.

Kinematic Chains. *A kinematic chain is a series connection of members constituting kinematic pairs.*

Kinematic chains each member of which is part of two kinematic pairs are referred to as *closed*.

A closed chain may be exemplified by a slider-crank mechanism (Fig. 191). The diagram of the closed chain is: bearing *1*—crank *2*—connecting rod *3*—slider *4*—guides *1*—bearing *1*. The bearing and the guides are rigidly connected to the frame, therefore they are denoted by the same figure *1*.

Kinematic chains involving members which belong to a single kinematic pair are called *unclosed* kinematic chains. An unclosed chain may be exemplified by the diagram of the slider-crank mechanism just discussed, except that some connection, say that between the last and the first element, should be removed.

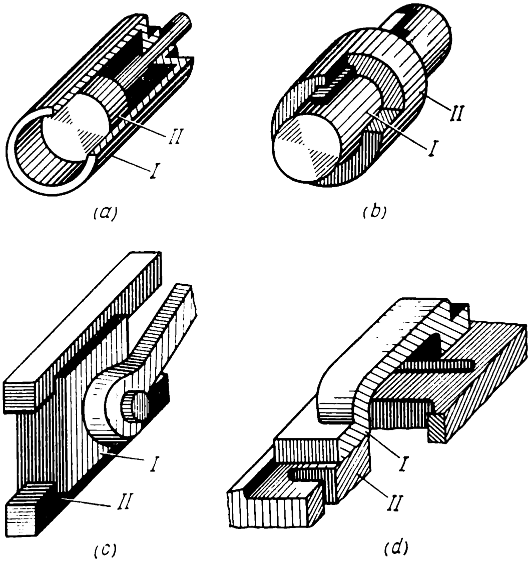


Fig. 188.

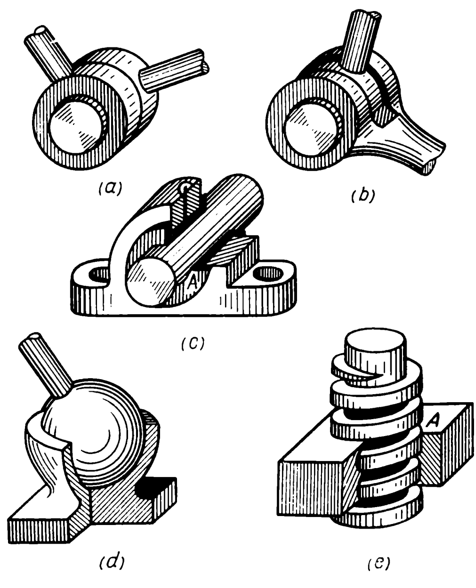


Fig. 189.

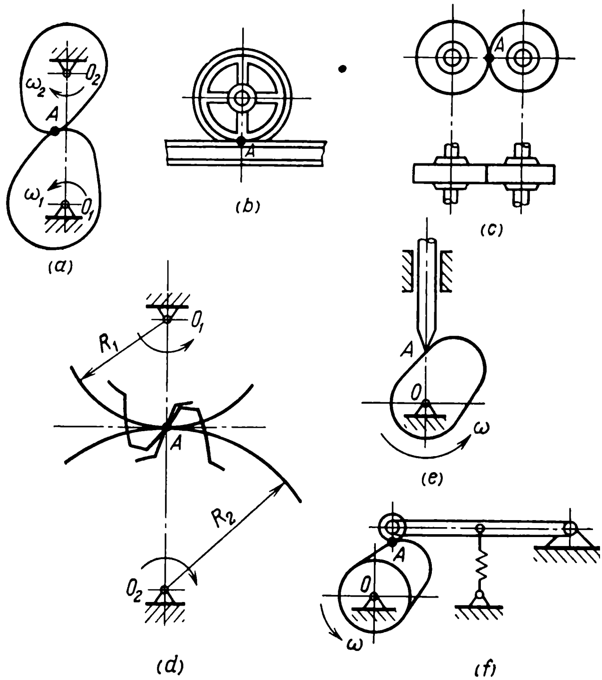


Fig. 190.

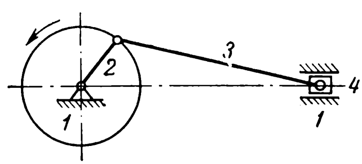


Fig. 191.

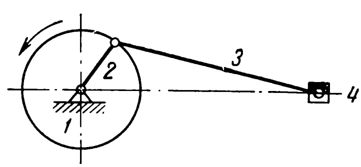


Fig. 192.

In Fig. 192 the slider guides are eliminated and the closing connection with the bearing is destroyed. Thus, the slider no longer follows a definite path.

Kinematic chains may be *simple* or *compound.* In a simple kinematic chain, each member is part of no more than two kinematic pairs, as in the slider crank (Fig. 191).

A compound kinematic chain is a chain in which there is at least one member belonging to more than two kine-

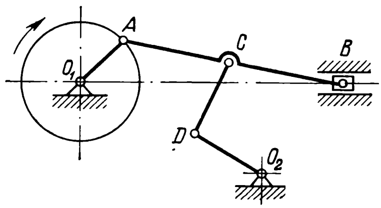


Fig. 193.

matic pairs. In Fig. 193 link *AB* (connecting rod) is connected to three links: the crank, the slider and link *CD*. Consequently, this chain is compound.

A kinematic chain in which, when one or several members execute a specified motion all other members have well-defined motions, is called a mechanism.

Hence the necessity of distinguishing stationary, driving and driven members in a mechanism.

A stationary member is also called a *frame.*

The presence of a frame and other constraints imposed by kinematic pairs makes it possible to obtain specific motions of the driven members in a mechanism when the driving member follows a prescribed law of motion.

A *driving member* is a member which transmits a given motion.

A *driven member* is a member which takes up the motion.

112. Fundamentals of Kinematics of Mechanisms

Basic Problems in Kinematics of Mechanisms. Kinematics of mechanisms deals with the motion of individual members of machines without reference to the factors causing the motion, i.e., the forces acting in machines; the motion of a machine is considered from a purely geometric point of view, the only factor taken into account being the time.

Every motion of a body is characterized by its displacement in space, the velocity and acceleration of its points. Hence the basic problems in the kinematic analysis of mechanisms:

(1) the construction of kinematic diagrams and successive positions of a mechanism;

(2) the construction of paths of points of a mechanism and the tracing of paths in time;

(3) the determination of linear velocities and accelerations of any point of a mechanism and of angular velocities and accelerations of its members.

The following methods are available for the kinematic analysis of mechanisms: experimental, graphical, semigraphical and analytic methods.

The semigraphical method provides means of solving all basic problems in the kinematic analysis of mechanisms. For most practical problems the accuracy of the graphical method is quite sufficient.

In analysing complex mechanisms the graphical procedures greatly simplify the computations and save much time.

Most of the mechanisms encountered in modern machines are two-dimensional, therefore in the following discussion we shall only consider the methods of kinematic analysis of two-dimensional mechanisms.

Construction of Kinematic Diagrams. The basic problem in constructing kinematic diagrams is to represent on a sheet of paper a diagram which gives an idea of the kinematic relationships between individual members of mechanisms. There is no need to depict the complicated shapes of all the parts involved. It is always possible to represent a mechanism by a combination of simple lines since it is composed of rigid unchangeable members.

The construction of kinematic diagrams is of great importance, particularly in the study and analysis of complex machines (automatic and semiautomatic machines, engines, metal- and wood-working machines, etc.).

In representing kinematic diagrams in orthogonal projections use is made of conventional designations according to the USSR State Standard GOST 3462-61.

Choice of Scales. The accuracy of the semigraphical method of kinematic analysis of mechanisms depends primarily on the care taken in graphical constructions and also on the scales adopted.

The choice of scales is governed, on the one hand, by the dimensions of the members of the mechanism under consideration and, on the other, by the necessity to use only standard scales.

According to GOST 3451-59, the following scales are adopted: for reduction 1 : 2, 1 : 5, 1 : 10 (1 : 2.5, 1 : 4 are permitted but not recommended), for enlargement 2 : 1, 5 : 1, 10 : 1.

It is permissible to increase or decrease any of these scales by a factor multiple of 10.

In contrast to scales in machine drawings, in technical mechanics scales have dimensions since in analysing mechanisms it is necessary to represent in a drawing not only the lengths of members but also linear and angular velocities and accelerations, forces, moments of forces, etc.

A *scale* is defined as the ratio of an actual quantity to the length of the segment which represents it in the drawing.

For example,
space scale

$$k_S = \frac{l}{\bar{l}} \text{ m/mm},$$

velocity scale

$$k_v = \frac{v}{\bar{v}} \text{ m/sec per mm},$$

acceleration scale

$$k_a = \frac{a}{\bar{a}} \text{ m/sec}^2 \text{ per mm},$$

force scale

$$k_F = \frac{F}{\bar{F}} \text{ N/mm},$$

where l = true length of mechanism member,

$\bar{l}$ = length of segment representing true length of member in drawing,

v = true velocity of a point of mechanism member,

$\bar{v}$ = length of segment representing true velocity of the point in drawing,
a = true acceleration of a point of mechanism member,
$\bar{a}$ = length of segment representing true acceleration of the point in drawing, etc.

In choosing scales k_v and k_a, it is recommended that the scaled velocity and acceleration vectors for the driving member be not less than 50 mm.

Consider some examples.

1. A crank is 250 mm long. Representing the length of the crank in the drawing by a 50-mm segment, we obtain the scale

$$k_S = \frac{0.25}{50} = 0.005 \text{ m/mm.}$$

2. The velocity of a crank pin obtained by calculation is

$$v = 0.65 \text{ m/sec.}$$

Representing this velocity in the drawing by a segment of length 65 mm, we obtain the standard scale

$$k_v = \frac{0.65 \text{ m/sec}}{65 \text{ mm}} = 0.01 \text{ m/sec per mm.}$$

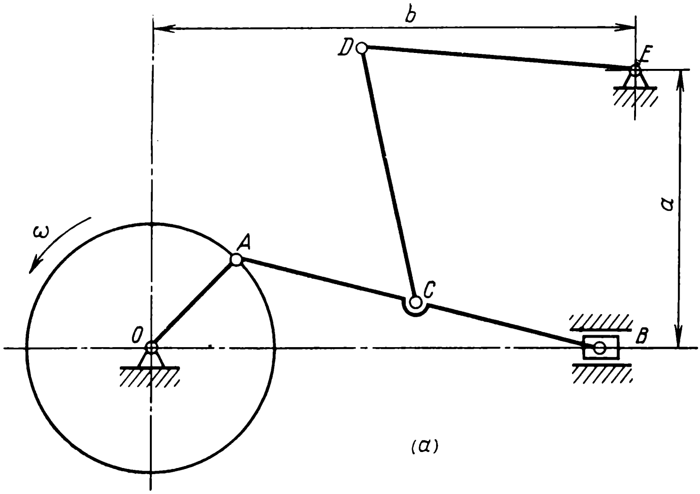
Construction and Tracing of Paths of Particles of a Two-Dimensional Mechanism. The problem is to find from a given position of a mechanism its successive positions; this involves tracing the paths of particles of the mechanism which are not defined by the diagram of the mechanism.

The paths of moving particles, as well as their velocities and accelerations, are an important kinematic characteristic of a mechanism.

In some cases the paths of motion in a mechanism are all-important and their shape determines the principal function of the mechanism; examples are the ellipsograph and straight guiding mechanisms.

Consider the basic method of constructing paths, *the method of markings.*

The motion of each member of a mechanism must be considered successively, beginning with the driving, or main, member and proceeding to the other members of the kinematic chain.


b
D
E
a
ω
A
C
O
B
(a)

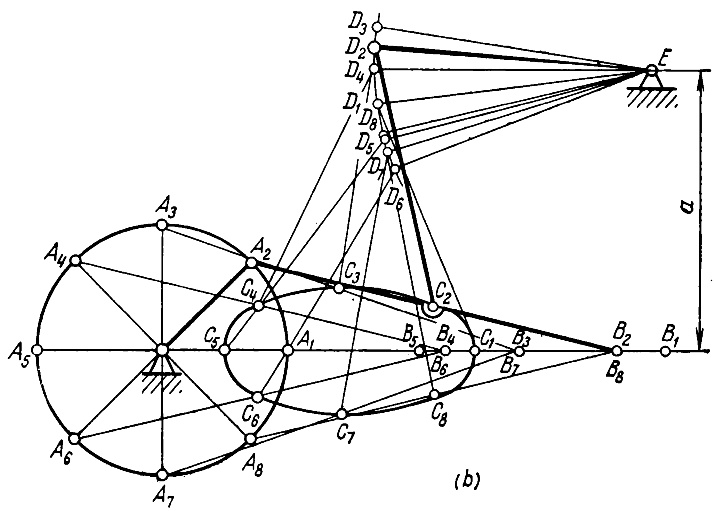

D3
D2
D4
D1
D8
D5
D7
D6
E
a
A3
A4
A2
C3
C4
C2
C5
A1
B5
B4
C1
B3
B2
B1
A5
B6
B7
B8
C6
C8
C7
A6
A8
A7
(b)

Fig. 194.

In mechanisms and machines, members are treated as material rigid bodies, i.e., the distance between the individual particles of a member remains unchanged during the motion.

Let a two-dimensional linkage mechanism be specified by its dimensions and the co-ordinates of fixed particles (Fig. 194*a*). The path of particle A on the main, driving, link OA is known. The centre of the pin A describes a circle of radius OA during one revolution of the crank. If the crank rotates with a constant angular velocity and makes n revolutions per minute, the time for one revolution is

$$t=\frac{60}{n}\ \text{sec.}$$

The path of another particle, B, on the mechanism is also known. The path of particle B is a straight line as it is part of the slider which moves in parallel guides. In order to find by the method of arc markings the paths of all the particles of the mechanism and to trace them, it is necessary to divide the path of particle A into several equal parts, eight in the case under consideration. Thus, the passage from one position to the next is performed during the time interval

$$\frac{t}{8}=\frac{60}{n\times 8}.$$

Let us mark in Fig. 194*b* the positions of particle A corresponding to the following successive angles

0°	45°	90°	135°	180°	225°	270°	315°
A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8

Determine the position of particle B on link AB, moving in a straight line. From the positions of particle A—A_1, A_2, . . ., A_8, at the instants 1, 2, . . ., 8 we find the corresponding positions of particle B—B_1, B_2, . . ., B_8. The length of the connecting rod AB remains unchanged during the motion, therefore we mark the path of particle B by circular arcs of radius AB from each position of particle A as a centre; the markings obtained represent the corresponding positions of particle B.

Next determine the path of particle C. From the foregoing we have the positions of the connecting rod, A_1B_1, A_2B_2, . . ., A_8B_8.

Point C is known to divide the length of the connecting rod into two parts. Swinging circular arcs of radius AC or BC by means of a compass from the corresponding positions of particle A (A_1, A_2, A_3, etc.) or particle B (B_1, B_2,

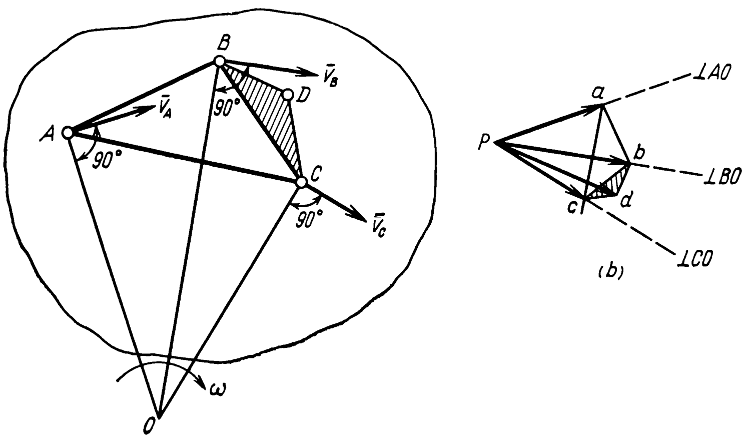


Fig. 195.

B_3, etc.), we mark points C_1, C_2, C_3, . . ., C_8 on the positions of the connecting rod. Joining the obtained points C_1, C_2, C_3, . . ., C_8 by a smooth curve, we find the path of particle C on link AB.

Then we determine the path of particle D. Particle D is common to links CD and ED and, being part of link ED, has a path in the form of a circular arc of radius ED.

Swinging circular arcs of radius CD from the corresponding positions of particle C (C_1, C_2, C_3, etc.), we find the corresponding positions of particle D at the intersection of these arcs and the arc of radius ED.

Velocity Diagram and Its Properties. A two-dimensional figure ABC moves in its plane (Fig. 195*a*).

For a given angular velocity ω the velocity of point A is proportional to the radius OA, i.e.,

$$v_A = \omega\, OA.$$

The velocity of point B is

$$v_B = \omega\, OB.$$

The velocity of point C is

$$v_C = \omega\, OC.$$

The directions of these velocities are perpendicular to the corresponding radii, i.e.,

$$\bar{v}_A \perp OA, \quad \bar{v}_B \perp OB, \quad \bar{v}_C \perp OC.$$

For a multimember mechanism and its successive positions, the geometric addition of velocity vectors illustrated in Sec. 72 (Fig. 126*b*) is rather cumbersome as the vector velocity polygons for various points of the mechanism will be superimposed, the vectors will intersect and it will be difficult to interpret such vector polygons and determine velocities therefrom.

Hence the necessity of constructing a velocity diagram in the plane of the drawing for each successive position of the mechanism.

The velocity diagram is constructed as follows:

1. Choose an arbitrary point p in the plane of the drawing as the centre (pole) of the velocity diagram (Fig. 195*b*).

2. Through point p, draw to a chosen scale straight lines parallel and equal to the segments representing the velocity vectors of points A, B and C.

The obtained figure *pabc*, called a velocity diagram, is similar to figure $OABC$ as the segments representing the velocity vectors of its points are proportional and perpendicular to the respective lines of figure $OABC$.

The following properties of the velocity diagram result from this construction:

1. To points a, b, c, . . ., p in the velocity diagram correspond like points A, B, C, . . ., O in the diagram of the figure and vice versa.

2. To polygon *pabc* in the velocity diagram corresponds a like and similar polygon $OABC$ in the diagram of the figure, these similar polygons being displaced by 90 degrees with respect to each other.

This property of the velocity diagram is called the *image property*.

3. Segments $\overline{ab}$, $\overline{ac}$, $\overline{bc}$, etc. joining the vertices of the velocity diagram are perpendicular to like segments AB, AC, BC, etc. in the diagram of the figure.

4. Segments $\overline{pa}$, $\overline{pb}$, $\overline{pc}$, etc. in the velocity diagram represent in magnitude and direction the velocities of the corresponding points A, B, C, . . . of the moving plane figure.

5. Segments $\overline{ab}$, $\overline{ac}$, $\overline{bc}$ which do not pass through the pole represent in Fig. 195*b* the relative velocities of the corresponding points of the mechanism.

We proceed to determine the velocity of point D. Point D is rigidly connected to points B and C. This rigid connection is indicated by the hatched rigid triangle BCD.

On the face of it, the problem of determining the velocity of point D is complicated by the fact that the path for this point is not known and therefore it is impossible to locate immediately the direction line of the velocity of point D. However, this impression is false.

Consider the motion of point D as the sum of a translation with pole B and a rotation about pole B. According to the theorem on the addition of velocities we find

$$\bar{v}_D = v_B + \bar{v}_{DB}. \tag{256}$$

The magnitude of the velocity $\bar{v}_{DB}$ is not known, only its direction is known, being perpendicular to member BD. The direction line is drawn through the tip of the velocity vector $\bar{v}_B$, i.e., through point b in the velocity diagram.

On the other hand, point D moves with point C.

According to the theorem on the addition of velocities we may write

$$\bar{v}_D = \bar{v}_C + \bar{v}_{DC}. \tag{257}$$

In this case, too, the magnitude of $\bar{v}_{DC}$ is not known; its direction is perpendicular to member DC.

The direction line is drawn through the tip of the velocity vector $\bar{v}_C$, i.e., through point c in the velocity diagram.

The tip d of the desired velocity vector $\bar{v}_D$ of point D will be at the intersection of the directions of $\bar{v}_{DB}$ and $\bar{v}_{DC}$.

As a result, a triangle bcd is constructed in the velocity diagram which is similar to triangle BCD and displaced by

90 degrees with respect to the figure in the diagram of the mechanism.

The velocity of point D can be found without using formulas (256) and (257) by constructing a triangle bdc similar to triangle BDC on segment bc in the velocity diagram. Its vertex d joined to pole p would give the velocity vector of point D.

113. Examples of Constructing Velocity Diagrams

Construction of Velocity Diagram for a Two-Dimensional Four-Bar Linkage with Turning Kinematic Pairs. A four-bar linkage is specified by the dimensions of its links and the law of motion of the driving, main, link O_1A (Fig. 196*a*).

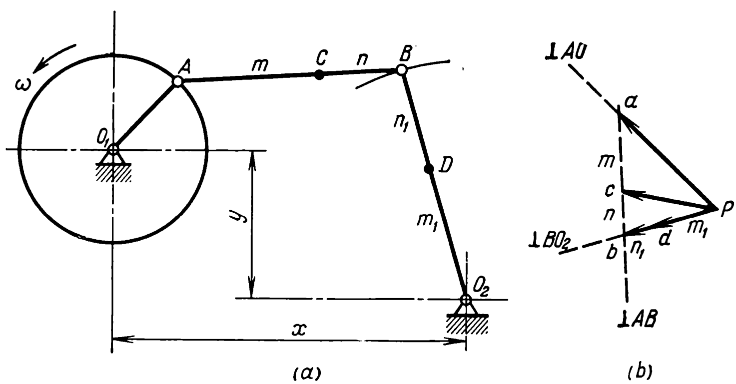


Fig. 196.

For a given angular velocity ω and a given length of crank O_1A, particle A on crank O_1A has a velocity

$$v_A = \omega\, O_1A.$$

The velocity $\bar{v}_A$ is perpendicular to crank O_1A. Through the centre, or pole, p of the velocity diagram we draw a line $\overline{pa}$ perpendicular to crank O_1A (Fig. 196*b*); it represents to a chosen scale the velocity of particle A.

We proceed to determine the velocity of particle B having a complex motion. Particles A and B are in a common link and particle A is in base motion. Consequently, the absolute velocity of particle B is equal to the geometric sum of the base velocity (of particle A) and the relative velocity of particle B with respect to particle A, i.e.,

$$\bar{v}_B = \bar{v}_b + \bar{v}_r$$

but

$$\bar{v}_b = \bar{v}_A,$$

$$\bar{v}_r = \bar{v}_{BA},$$

therefore

$$\bar{v}_B = \bar{v}_A + \bar{v}_{BA}.$$

To construct the velocity of particle B, we draw in the velocity diagram a line perpendicular to the direction of the connecting rod AB through point a, the tip of the base velocity vector, i.e., of the velocity of particle A. This line indicates the direction of the relative velocity $\bar{v}_{BA}$ of particle B with respect to particle A.

On the other hand, particle B as part of arm BO_2 oscillating about a fixed pin O_2 will have its absolute velocity in a direction perpendicular to link BO_2. Therefore, we draw a line perpendicular to BO_2 through pole p until it intersects the direction of the relative velocity $\bar{v}_{BA}$; the intersection gives point b.

Conclusions. 1. Segment $\overline{pb}$ represents to an appropriate scale the absolute velocity $\bar{v}_B$ of particle B.

2. Segment $\overline{ab}$ represents to the same scale the relative velocity $\bar{v}_{BA}$ of particle B with respect to particle A.

If it is necessary to determine the velocity of any particle C on the connecting rod (Fig. 196a) which divides it into parts m and n, use the similarity method. Following this method, point c is found by dividing segment $\overline{ab}$ representing the relative velocity $\bar{v}_{BA}$ into parts proportional to the segments m and n, i.e.,

$$\frac{AC}{CB} = \frac{m}{n} = \frac{ac}{cb}.$$

Joining the point c thus found to the centre of the velocity diagram, we obtain in Fig. 196b the length of segment $\overline{pc}$

representing to an appropriate scale the absolute velocity of particle C.

If we have to determine the velocity of any particle D which divides the arm BO_2 into parts m_1 and n_1, we must also use the similarity method.

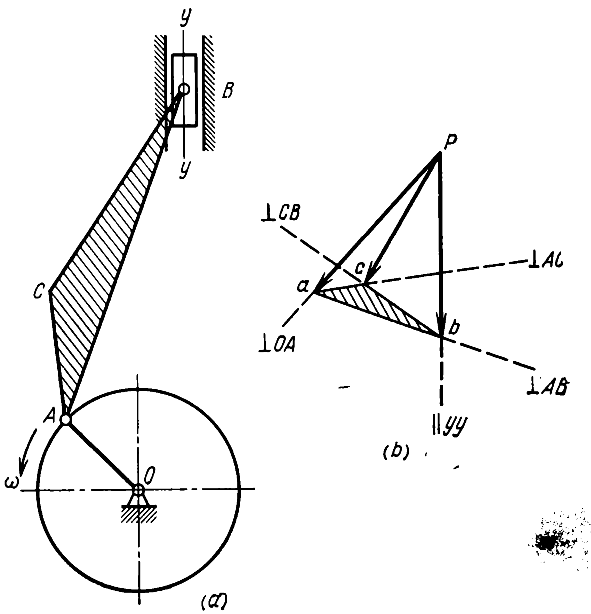


Fig. 197.

Dividing the segment representing the velocity of particle B into parts proportional to the segments m_1 and n_1, we find point d.

Line $\overline{pd}$ will represent to scale the absolute velocity vector of particle D.

Construction of Velocity Diagram for a Two-Dimensional Four-Bar Linkage with a Sliding Kinematic Pair. A slider-crank mechanism is specified by the dimensions of its links and the law of motion of the main, driving, link OA (Fig. 197a).

For a given angular velocity ω and a given length of crank OA, particle A on crank OA has the velocity

$$v_A = \omega OA.$$

This velocity is directed along the perpendicular to the direction of crank OA.

We draw through pole p of the velocity diagram a line $\overline{pa}$, perpendicular to crank OA, which represents to a chosen scale the velocity of particle A.

Particle B (slider) is connected to particle A (crank pin) which is in base motion. Consequently, the absolute velocity of particle B is equal to the geometric sum of the base velocity (of particle A) and the relative velocity of particle B with respect to particle A, i.e.,

$$\bar{v}_B = \bar{v}_b + \bar{v}_r$$

but

$$\bar{v}_b = \bar{v}_A,$$

$$\bar{v}_r = \bar{v}_{BA},$$

hence

$$\bar{v}_B = \bar{v}_A + \bar{v}_{BA}.$$

To construct the velocity of particle B, we draw in the velocity diagram a line perpendicular to the direction of the connecting rod AB through point a, the tip of the base velocity vector, i.e., of the velocity of particle A. This line indicates the direction of the relative velocity $\bar{v}_{BA}$ of particle B with respect to particle A.

On the other hand, the slider, particle B of the mechanism, has a direction of the velocity parallel to the yy axis in its absolute motion, i.e., in the motion relative to the stationary guides. Hence the direction of the absolute velocity vector is determined by a line passing through the pole of the velocity diagram parallel to the yy axis.

The magnitude of the absolute velocity vector of particle B is determined by a segment of that line measured from the pole to the point of intersection b with the direction of the relative velocity of particle B with respect to particle A.

Conclusions. 1. Segment $\overline{pb}$ represents in Fig. 197b to an appropriate scale the absolute velocity $\bar{v}_B$ of particle B.

2. Segment $\overline{ab}$ represents in Fig. 197*b* to the same scale the relative velocity $\overline{v}_{BA}$.

To determine the velocity of particle C it is necessary to construct on segment $\overline{ab}$ of the velocity diagram a triangle abc similar to triangle ABC and located in a like manner ($ac \perp AC$, $bc \perp BC$).

The desired absolute velocity $\overline{v}_C$ is given by a segment $\overline{pc}$ directed away from the pole.

Let us determine the angular velocity ω_2 of the connecting rod AB.

In the velocity diagram we have segment $\overline{ab}$. The relative velocity $v_{BA} = \overline{ab} \times k_v$, where k_v is the velocity scale.

On the other hand,

$$v_{BA} = \omega_2 AB,$$

then

$$\omega_2 = \frac{\overline{ab} \times k_v}{AB}.$$

114. Distribution of Accelerations in a Body in Plane Motion

It is known that the absolute acceleration of any given point of a body in plane motion is equal to the geometric sum of two accelerations: the base acceleration defined by the acceleration of an arbitrarily chosen pole A and the relative acceleration in the rotation about the pole, i.e.,

$$\overline{a}_a = \overline{a}_b + \overline{a}_r.$$

The relative acceleration is in turn made up of two components: normal, or centripetal, and tangential.

The normal component of relative acceleration of a point is equal to the ratio of the square of the relative velocity to the radius of its relative rotation, or to the product of the square of the angular velocity and the radius of its relative rotation, i.e.,

$$a_r^n = \frac{(v_r)^2}{R_r} = \omega_r^2 R_r.$$

The normal component of acceleration is directed along the radius of relative rotation toward the centre.

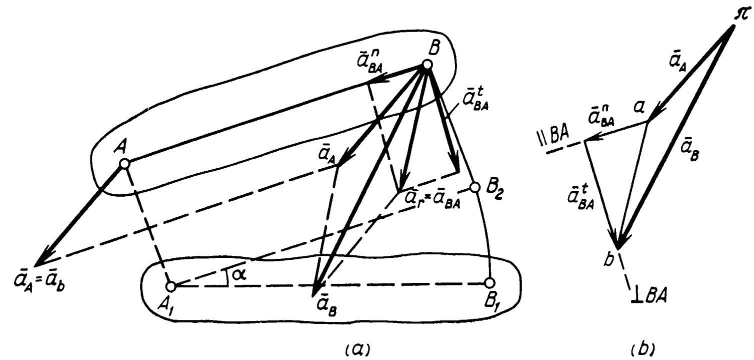


Fig. 198.

The tangential component of relative acceleration is equal to the product of the angular acceleration and the radius of rotation, i.e.,

$$a_r^t = \alpha R_r,$$

but

$$\alpha = \frac{d\omega_r}{dt},$$

consequently,

$$a_r^t = \frac{d\omega_r}{dt} R_r.$$

Conclusion. *The total acceleration of any point of a body in plane motion is geometrically composed of three accelerations:*

(*1*) *base acceleration* $\bar{a}_b$,

(*2*) *normal component of relative acceleration*

$$a_r^n = \frac{(v_r)^2}{R_r} = \omega_r^2 R_r,$$

(*3*) *tangential component of relative acceleration*

$$a_r^t = \frac{d\omega_r}{dt} R_r = \alpha R_r.$$

Let the base motion of a plane figure be specified by the velocity and acceleration of a point A (Fig. 198*a*). This means that in the base motion all points have the same accelerations. We propose to determine the acceleration of point B which is in complex motion.

The acceleration of point B, as noted above, is composed of the base and relative accelerations, i.e.,

$$\bar{a}_B = \bar{a}_b + \bar{a}_r = \bar{a}_b + \bar{a}_r^n + \bar{a}_r^t. \quad (258)$$

This expression is known as the *acceleration distribution formula*.

Consider the components of the vector sum of formula (258):

(1) the base acceleration is the same for all points of the member, i.e.,

$$\bar{a}_b = \bar{a}_A;$$

(2) the normal, or centripetal, component of the acceleration of point B relative to point A is directed along the radius of relative rotation and is equal to

$$a_r^n = a_{BA}^n = \frac{v_{BA}^2}{BA},$$

where v_{BA} is the relative velocity of point B with respect to point A (the magnitude of this velocity is determined from the velocity diagram);

(3) the tangential component of the acceleration of point B relative to point A is directed along the perpendicular to the radius of relative rotation and is equal to

$$a_r^t = a_{BA}^t = \alpha BA,$$

where α is the angular acceleration in the rotation of point B about point A.

Substituting the values obtained in Eq. (258), we have

$$\bar{a}_B = \bar{a}_A + \bar{a}_{BA}^n + \bar{a}_{BA}^t. \quad (258a)$$

We proceed to graphical construction (Fig. 198*a*).

The base acceleration $\bar{a}_b$ is drawn through point B, the vector representing this acceleration is equal in magnitude and parallel to the acceleration of point A.

The normal component $\bar{a}_{BA}^n$ of relative acceleration is drawn through point B (in the direction of the radius of relative rotation BA) from B to A. The magnitude of this acceleration is

$$a_{BA}^{n}=\frac{v_{BA}^{2}}{BA}.$$

The tangential component $\bar{a}_{BA}^{t}$ of relative acceleration is drawn through point B in a direction perpendicular to the radius of relative rotation BA. The magnitude of this acceleration is

$$a_{BA}^{t}=\alpha BA.$$

Adding $\bar{a}_{BA}^{n}$ and $\bar{a}_{BA}^{t}$ geometrically, i.e., drawing the diagonal of the parallelogram constructed on the vectors $\bar{a}_{BA}^{n}$ and $\bar{a}_{BA}^{t}$, we obtain

$$\bar{a}_{BA}=\bar{a}_{r}=\bar{a}_{BA}^{n}+\bar{a}_{BA}^{t}.$$

Adding the base acceleration vector $\bar{a}_{b}$ and the relative acceleration vector $\bar{a}_{r}$ geometrically, we obtain the total acceleration of point B

$$\bar{a}_{B}=\bar{a}_{b}+\bar{a}_{r}.$$

Acceleration Diagram and Its Properties. The geometric addition of acceleration vectors illustrated above (Fig. 198*a*) is inconvenient for multimember mechanisms as the vector acceleration polygons for various points of the mechanism will be superimposed, the vectors will intersect and it will be difficult to interpret such vector polygons and determine accelerations therefrom.

Hence the necessity of constructing acceleration diagrams in the plane of the drawing of the mechanism for its successive positions.

Let us determine the acceleration of point B (formula 258a) by constructing the diagram.

1. Choose an arbitrary point π in the plane of the drawing as the centre (pole) of the acceleration diagram (Fig. 198*b*).

2. Through point π, draw a segment $\overline{\pi a}$ which represents to a chosen scale k_a the base acceleration vector $\bar{a}_b$ equal to the given acceleration of point A.

3. Through the tip (point a) of the base acceleration vector a_b, draw a line parallel to member BA and lay off on it to the same scale as for the base acceleration, the magnitude of the

normal component $\bar{a}^n_{BA}$ of relative acceleration in the direction from B to A is

$$a^n_{BA} = \frac{v^2_{BA}}{BA} \frac{1}{k_a},$$

where the relative velocity (segment $\overline{ba}$) determined from the velocity diagram is

$$v_{BA} = (\overline{ba})\, k_v.$$

Hence

$$a^n_{BA} = \frac{(\overline{ba})^2 k_v}{BAk_a}.$$

4. Through the tip of the normal component $\bar{a}^n_{BA}$ of relative acceleration draw a line perpendicular to member BA and lay off on this perpendicular, to scale, the tangential component $\bar{a}^t_{BA}$ of relative acceleration.

Joining the tail of the first vector (point π) to the tip of the last vector (point b) in the acceleration diagram, we obtain a segment $\overline{\pi b}$ (Fig. 198*b*) which represents to the chosen scale the total acceleration of point B as the geometric sum of the base acceleration and the normal and tangential components of relative acceleration.

Properties of Acceleration Diagram. 1. To points a, b, ... in the acceleration diagram correspond like points A, B, ... in the diagram of the figure.

2. Segments $\overline{\pi a}$, $\overline{\pi b}$, ... in the acceleration diagram represent in magnitude and direction the acceleration vectors of the corresponding points A, B, ... of the moving figure.

3. Segment $\overline{ab}$ joining the tips of the absolute acceleration vectors represents the relative acceleration of the point B with respect to the chosen pole A.

115. Examples of Constructing Acceleration Diagrams

Construction of Acceleration Diagram for a Two-Dimensional Four-Bar Linkage with Turning Pairs. A four-bar linkage is given (Fig. 199*a*).

Suppose that the velocity diagram has been constructed (Fig. 199*b*). We proceed to the construction of the acceleration diagram.

Particle A on link O_1A moves in a circle, therefore the acceleration of particle A is

$$\bar{a}_A = \bar{a}_A^n + \bar{a}_A^t.$$

Link O_1A is the crank. In kinematic analysis it is common to assume that the angular velocity of the crank is constant,

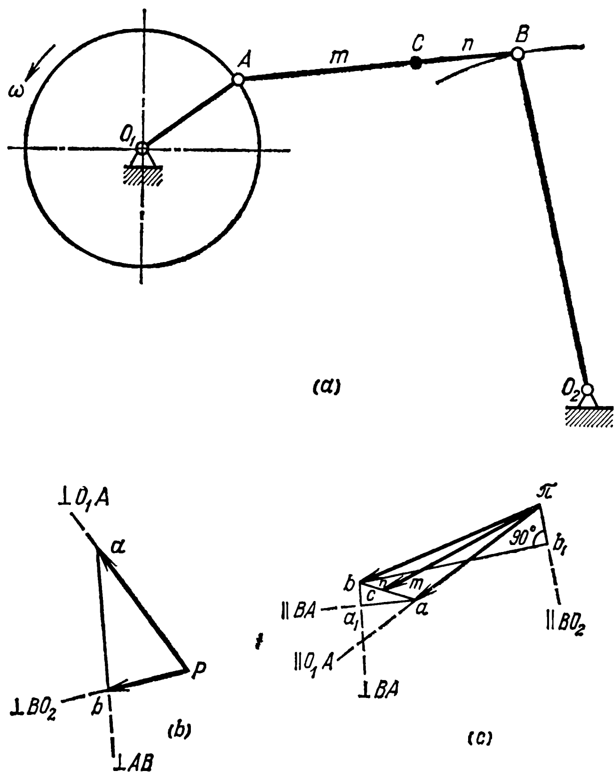


Fig. 199.

i.e., $\omega = \text{constant}$, then the tangential component of acceleration of particle A is zero

$$a_A^t = \alpha\, O_1A = \frac{d\omega}{dt} O_1A = 0.$$

In this case the total acceleration of particle A is equal to the normal component of acceleration

$$a_A = a_A^n = \omega^2 O_1A.$$

The total acceleration of particle A is directed along link O_1A toward the centre of rotation O_1. Choose the centre π of the acceleration diagram in the plane of the drawing (Fig. 199c); through this centre, draw to a chosen scale k_a a segment $\overline{\pi a}$ parallel to link O_1A in the direction from A to O_1 which represents the acceleration vector of particle A.

We now proceed to the determination of the acceleration of particle B which is part of the connecting rod AB and has a complex motion. Particles A and B are in a common link and particle A is in base motion. Consequently, the absolute acceleration of particle B as part of the connecting rod is equal to the geometric sum of the base acceleration (of particle A) and the relative acceleration of particle B with respect to particle A, i.e.,

$$\bar{a}_B = \bar{a}_b + \bar{a}_r$$

but

$$\bar{a}_b = \bar{a}_A,$$

$$\bar{a}_r = \bar{a}_{BA} = \bar{a}_{BA}^n + \bar{a}_{BA}^t,$$

consequently,

$$\bar{a}_B = \bar{a}_A + \bar{a}_{BA}^n + \bar{a}_{BA}^t,$$

where $\bar{a}_A$ = base acceleration; this acceleration has been determined above and is represented by the vector $\overline{\pi a}$ in the acceleration diagram,

$\bar{a}_{BA}^n$ = normal component of acceleration of particle B relative to particle A

$$a_{BA}^n = \frac{v_{BA}^2}{BA}.$$

From the velocity diagram we know the relative velocity $\bar{v}_{BA}$; the length of link AB is given, therefore the magnitude of $\bar{a}_{BA}^n$ can be calculated. This acceleration is directed along link AB from B to A, toward the centre of relative rotation.

To construct $\bar{a}^n_{BA}$, we draw a line parallel to link AB through the tip of the base acceleration vector, i.e., through point a, and lay off a segment $\overline{aa_1}$ which represents to scale the vector $\bar{a}^n_{BA}$

$$\overline{aa_1} = \frac{v^2_{BA}}{BA\, k_a};$$

$\bar{a}^t_{BA}$ = tangential component of acceleration of particle B relative to particle A.

The magnitude of the tangential component of acceleration is not known yet, but the direction of $\bar{a}^t_{BA}$ is known, being always perpendicular to the radius of relative rotation, i.e., to AB.

To construct the direction of $\bar{a}^t_{BA}$, we draw a line perpendicular to link AB through the tip of the normal component of relative acceleration, i.e., through point a_1.

Since the magnitude of the tangential component of acceleration $\bar{a}^t_{BA}$ is not known, we consider the motion of particle B, as part of arm BO_2, about point O_2.

In this motion the acceleration of particle B relative to O_2 is

$$\bar{a}_B = \bar{a}^n_{BO_2} + \bar{a}^t_{BO_2}.$$

The normal component of the acceleration of particle B relative to point O_2 is directed along link O_2B from point B toward the centre of rotation O_2. The magnitude of this acceleration is

$$a^n_{BO_2} = \frac{(v_{BO_2})^2}{BO_2},$$

where

$$v_{BO_2} = (\overline{bp})\, k_v.$$

In the acceleration diagram, $\bar{a}^n_{BO_2}$ is directed along link BO_2 from B to O_2, toward the centre of rotation.

To construct this acceleration, we draw a line parallel to link BO_2 through the centre of the acceleration diagram π and lay off to scale a segment $\overline{\pi b_1}$ equal to $\bar{a}^n_{BO_2}$

$$\bar{a}^n_{BO_2} = \frac{(v_{BO_2})^2}{BO_2\, k_a}.$$

The magnitude of the tangential component $\bar{a}^t_{BO_2}$ of the acceleration of particle B relative to point O_2 is not known. This acceleration is directed along the perpendicular to the radius of rotation, i.e., to BO_2.

To construct the direction of $\bar{a}^t_{BO_2}$, we draw a line perpendicular to link BO_2 through the tip of the normal component of acceleration, i.e., through point b_1. The intersection of the directions of $\bar{a}^t_{BA}$ and $\bar{a}^t_{BO_2}$ gives point b. Joining this point to the centre π of the acceleration diagram, we obtain a segment $\overline{\pi b}$ which represents to the chosen scale the absolute acceleration vector of particle B.

Line $\overline{ab}$ joining points a and b in the acceleration diagram represents the geometric sum of the normal component $\bar{a}^n_{BA}$ and the tangential component $\bar{a}^t_{BA}$ of the acceleration of particle B relative to particle A, consequently,

$$\bar{a}_{BA} = \bar{a}^n_{BA} + \bar{a}^t_{BA}.$$

If it is necessary to determine the acceleration of any particle C on the connecting rod which divides it into parts m and n, the similarity method should be used. For this purpose the vector $\overline{ab}$ representing the relative acceleration $\bar{a}_{BA}$ should be divided into parts proportional to the segments m and n. Joining the point c thus found to the centre of the acceleration diagram, we obtain a vector $\overline{\pi c}$ which represents to an appropriate scale the absolute acceleration vector of particle C.

Construction of Acceleration Diagram for a Two-Dimensional Four-Bar Linkage with a Sliding Pair. A slider-crank mechanism is given (Fig. 200*a*). The velocity diagram has been constructed (Fig. 200*b*).

Particle A on link OA moves in a circle with a constant angular velocity, therefore it has only a normal component of acceleration (tangential component of acceleration is zero when ω = constant), i.e.,

$$a_A = a^n_A = \omega^2 OA.$$

This acceleration is directed along link OA toward the centre of rotation O.

From the pole π of the acceleration diagram (Fig. 200c), we draw a segment $\overline{\pi a}$ parallel to link OA, which represents to a chosen scale k_a the acceleration vector of particle A.

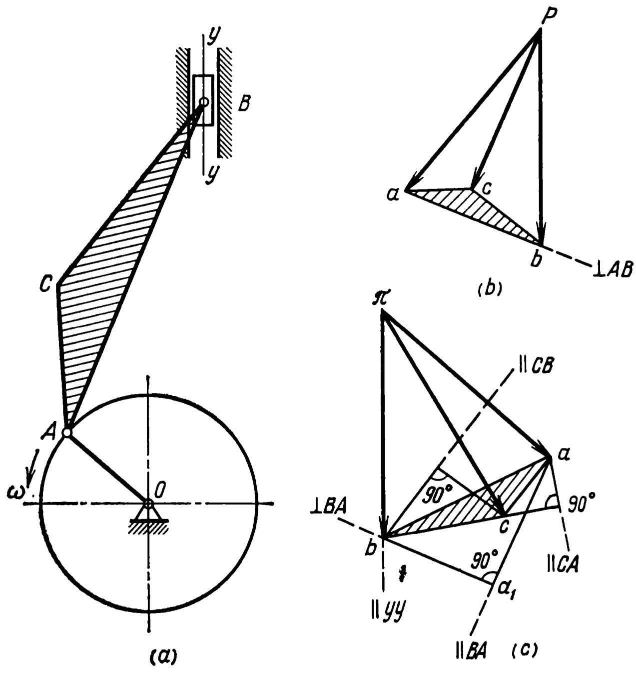


Fig. 200.

Particle B, if it is considered as part of link BA, has a complex motion. Particles A and B are in a common link, consequently,

$$\bar{a}_B = \bar{a}_A + \bar{a}^n_{BA} + \bar{a}^t_{BA},$$

where $\bar{a}_A$ = acceleration of particle A; this acceleration has been determined above and is represented by the vector $\overline{\pi a}$ in the acceleration diagram;

$\bar{a}^n_{BA}$ = normal component of acceleration of particle B relative to particle A

$$a^n_{BA} = \frac{v^2_{BA}}{BA}.$$

This acceleration, whose magnitude is easily determined, has a direction along link BA from B to A, toward the centre of relative rotation.

To construct $\bar{a}^n_{BA}$, we draw a line parallel to link AB through point a, the tip of the acceleration vector of particle A, and lay off a segment $\overline{aa_1}$ which represents to the scale k_a the vector $\bar{a}^n_{BA}$ in the acceleration diagram;

$\bar{a}^t_{BA}$ = tangential component of acceleration of particle B relative to particle A.

This acceleration has a direction perpendicular to link BA, but the magnitude of the acceleration is unknown. To construct the direction of $\bar{a}^t_{BA}$, we draw a line perpendicular to link BA through the tip of the normal component of the relative acceleration vector, i.e., through point a_1.

The acceleration of particle B of the mechanism in its absolute motion, i.e., in the motion relative to the stationary guides, has a direction parallel to the yy axis. Hence the direction of the absolute acceleration is determined by a line passing through the pole π of the acceleration diagram parallel to the yy axis.

The magnitude of the absolute acceleration vector of particle B is determined by a segment $\overline{\pi b}$ of that line measured from the pole to the intersection with the direction of the tangential component of the acceleration of particle B relative to particle A.

Line $\overline{ab}$ in the acceleration diagram represents the relative acceleration vector of particle B with respect to particle A

$$\bar{a}_{BA} = \bar{a}^n_{BA} + \bar{a}^t_{BA}.$$

The acceleration of particle C is determined by two vector equations

$$\bar{a}_C = \bar{a}_A + \bar{a}^n_{CA} + \bar{a}^t_{CA},$$

$$\bar{a}_C = \bar{a}_B + \bar{a}^n_{CB} + \bar{a}^t_{CB}$$

or by the similarity method.

Since particle C is rigidly connected to particles A and B, we can write

$$\triangle abc \backsim \triangle ABC$$

or

$$\frac{AB}{BC}=\frac{ab}{bc} \text{ and } \frac{AB}{AC}=\frac{ab}{ac}.$$

The vertex c of triangle abc is the tip of the acceleration vector $\bar{a}_C$ of particle C. Joining point c to pole π, we find the magnitude of the acceleration vector of particle C.

116. Cam Gears

Cam Gears and Their Significance in the Automation of Manufacturing Processes. Cam gears are used in cases where the displacement, velocity and acceleration of the driven

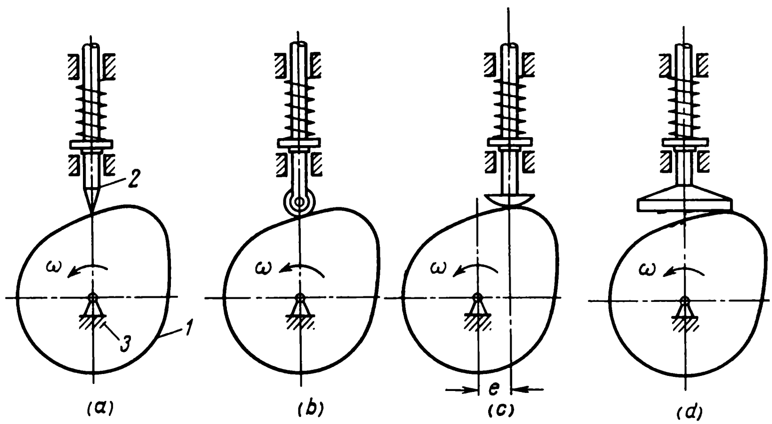


Fig. 201.

member must vary according to a prescribed law and, in particular, in cases where the driven member must move intermittently while the driving member moves continuously.

A cam gear most commonly consists of three members (Fig. 201*a*, *c*, *d*): cam *1*, follower *2* and frame *3*. Figure 201*b* shows a four-member cam gear.

Cam gears are divided into two main groups: two-dimensional and three-dimensional cam gears.

In *two-dimensional* cam gears the cam and the follower move in the same plane or in parallel planes (Fig. 201).

In *three-dimensional* cam gears the cam and the follower move in non-parallel planes. Figure 202 shows a sketch of a three-dimensional cylinder cam gear with a shaped groove on the lateral surface, and Figure 203 shows a sketch of a three-dimensional end cam gear.

By using cam gears the driven member can easily be made to follow a preassigned law of motion.

A cam profile is a sort of operation program for driven mechanisms; this accounts for an extremely wide use of cam

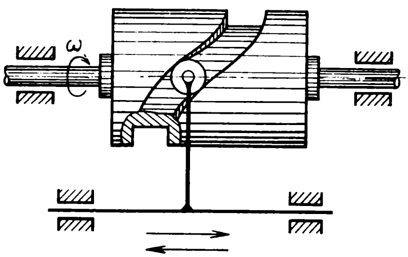


Fig. 202.

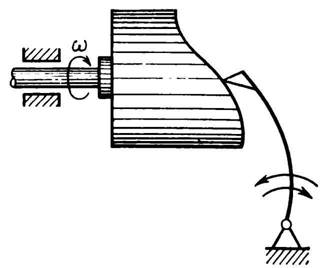


Fig. 203.

gears in various machines and especially in industrial automatic machines. It may be said that cam gears along with gear drives are the most common transmission mechanisms in modern machines. There are no other mechanisms which would provide a practical means for so wide a choice of the law of motion of the driven member.

Cam gears play an important part in piston engines, hydraulic and steam turbines, internal-combustion engines, Diesel engines, textile and printing machines, food automatons, slot-machines, various types of metal-cutting and woodworking machines, instruments and computers.... The list could be prolonged indefinitely.

To increase their durability, cams are made of high-quality steel with a working surface of high hardness.

To reduce friction and wear, the follower is provided with a roller which rotates and rolls without slipping along the working surface of the cam.

The disadvantages of cam gears are fast wear of members and the need to ensure constant contact (engagement) between members.

During operation of a cam gear large forces may develop, primarily inertia forces, tending to separate the working surface of the follower from the cam. To offset these forces either geometric (kinematic) or force engagement of a kinematic chain is used.

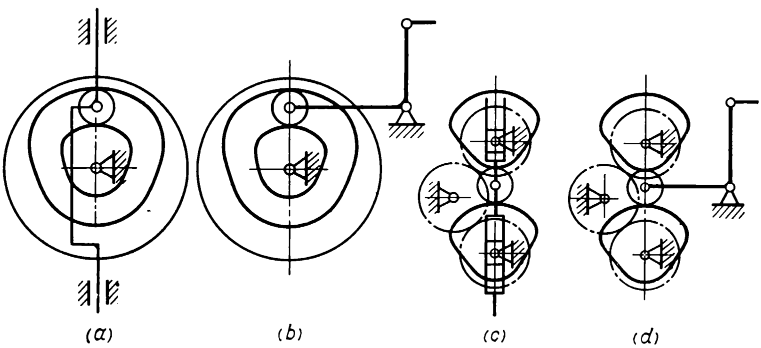


Fig. 204.

Examples of cam gears with geometric (kinematic) engagement are given in Fig. 204*a* which shows a grooved cam with a reciprocating follower and in Fig. 204*b*—a grooved cam with an oscillating follower.

The follower engages a groove cut into the flat surfaces of the cam. The groove provides two working cam profiles which cause the roller to move in both directions.

Figure 204*c* shows the engagement of a single-roller reciprocating follower with conjugate cams on parallel shafts, Figure 204*d*—the engagement of a single-roller oscillating follower with conjugate cams on parallel shafts.

The engagement of a cam-follower kinematic pair may be of the *force* type. In this case the follower is pressed against the cam in all positions with a force which is always larger than the loads tending to separate the follower from the cam. In most cases the engaging force is produced by a spring (Fig. 205*a*) which can develop large forces in spite of its small size (Fig. 205*b*). Pneumatic and especially hydraulic devices are also used, though very rarely, to provide the engaging force.

Among the other disadvantages of cam gears is the difficulty of generating the cam profile of which great accuracy is required, especially for high-speed transmission mechanisms.

It should be noted that cam gears cannot be used in cases where large displacements must be produced.

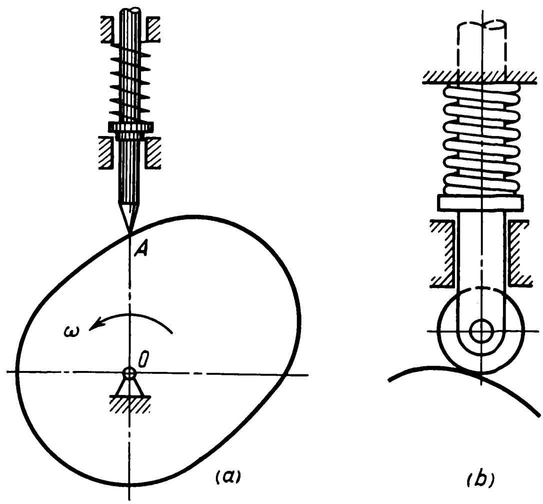


Fig. 205.

Types of Cam Gears. Cam gears differ widely both in the types of motion of the driving and driven members and in their mechanical construction.

Several classifications of cam gears are possible according to different characteristics.

I. Cam gears are divided according to the *motion of the cam* into three types:

sliding cams (Fig. 206*a*),
rotating cams (Fig. 206*b*),
oscillating cams (Fig. 206*c*).

II. Cam gears are divided according to the *motion of the follower* into three types:

Cam gears the driven members (followers) of which have a straight reciprocating motion. For reciprocating followers, it is common to distinguish axial (central) mechanisms in which the direction of motion of the follower passes through

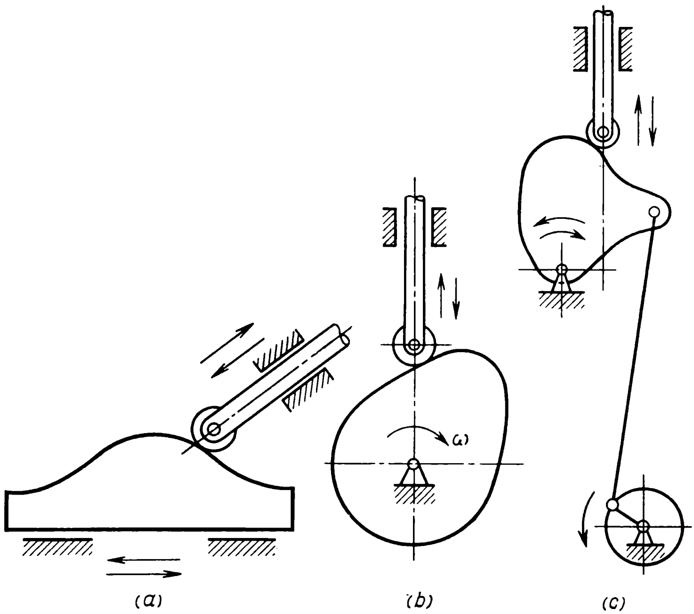


Fig. 206.

the centre of rotation of the cam (Fig. 206*b*), and eccentric mechanisms in which the direction of motion of the follower is offset by an amount e (Fig. 207*a*).

Cam gears the driven members (followers) of which have an oscillatory motion. These mechanisms are usually called rocker cam gears (Fig. 207*b*).

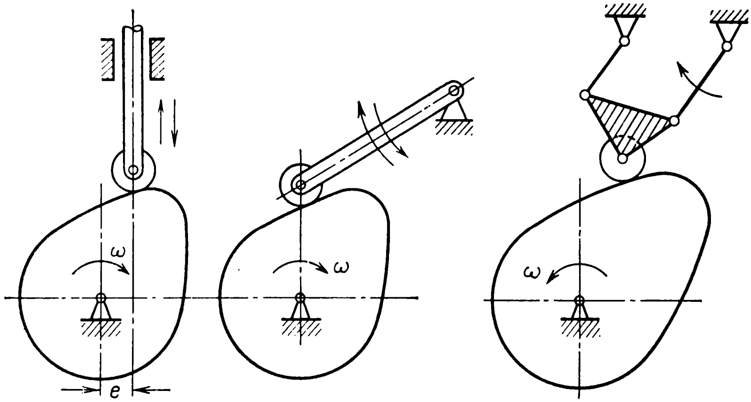
Cam gears the driven members (followers) of which have a complex motion (Fig. 207*c*).

The working surface of the follower, which is in contact with the curve of the cam profile, is called the *shoe*.

III. Cam gears may be divided according to the *profile of the working surface of the follower* into four groups:

cams with a pointed follower (Fig. 201*a*),
cams with a roller follower (Fig. 201*b*),
cams with a flat-faced follower (Fig. 201*d*),
cams with a spherical follower (Fig. 201*c*).

Cam gears with a pointed follower are very rarely used as the wear resistance of such mechanisms is low. Therefore,



(*a*) (*b*) (*c*)

Fig. 207.

a pointed follower is encountered only in light-duty slow-speed mechanisms. In practice such followers are not pointed but rounded off to a small radius.

Cam gears with roller followers are most extensively used since, because of the rolling motion of the follower on the surface of the cam, sliding friction is replaced by rolling friction. The wear resistance of such cam gears is higher than for any others. The efficiency of the mechanism is thus increased.

When the roller follower operates against the periphery of the cam (Fig. 208) the centre of the roller is always at a distance equal to the radius of the roller on the direction of a normal *nn* passing through the point of contact of the curve of the cam profile with the roller.

When rolling along the curve of the cam profile the centre of the roller describes a curve called the *equidistant* (theoretical) *curve of the cam profile*, or *the theoretical cam profile.*

In the kinematic analysis of cam gears, the actual cam profile is replaced by its equidistant curve, then the roller follower can also be replaced by a pointed one whose point coincides with the centre of the roller. Such a replacement does not affect the kinematic structure of the cam gear.

IV. As regards their mechanical construction cam gears differ greatly *according to the type of cam.*

Disk cams are most commonly used. Mechanisms with disk cams are two-dimensional as a rule. Figure 201 shows sketches of disk cams differing in their design. The main types of disk cams are the radial peripheral cam (Fig. 201) and the grooved cam (Fig. 209*a*).

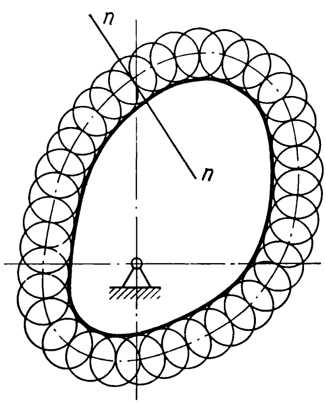


Fig. 208.

Figure 201*b* shows a central mechanism with a roller follower having a straight reciprocating motion.

Figure 201*c* shows an offset mechanism with a spherical, or mushroom, follower having a straight reciprocating motion. A follower with cylindrical, spherical or mushroom surface is no different from a roller follower from the point of view of kinematics.

Figure 210 shows a mechanism with a disk cam and a follower having an oscillatory motion.

Figure 211 shows a mechanism with a disk cam of circular profile. Such cams are called *eccentrics.* The follower in this case may be in the form of a frame or a collar (Figs. 211 and 212).

Figure 209*b* shows a mechanism with a grooved disk cam and an offset follower.

In cases where the follower is to move intermittently the portions of the cam profile corresponding to different types of motion through which the follower passes must be bounded by circular arcs of appropriate radii drawn from the centre of rotation of the cam.

Figure 213 shows a cam used in internal-combustion engines to transmit motion to a valve.

The cam profile consists of the following portions.

1. Portion *AB* corresponds to a rise of the follower. During this period the valve moves and opens the intake or exhaust port.

2. Portion *BC* corresponds to a dwell of the follower and valve in the highest position.

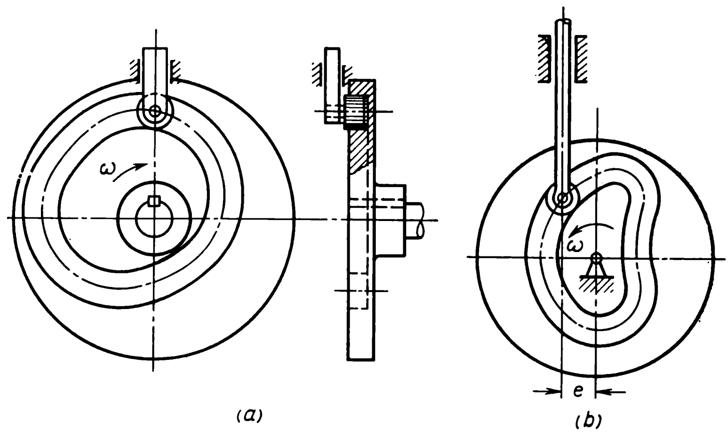


Fig. 209.

3. Portion *CD* corresponds to a fall of the follower. During this period the valve returns to the original position and closes the intake or exhaust port.

4. Portion *DA* corresponds to a dwell of the follower and valve in the lowest position.

117. Introduction to the Dynamics of Machinery

All forces acting on a machine may be divided into two classes: given forces and reactions of constraints. Given forces are in turn divided into five groups: moving forces, forces of useful resistance, forces of parasitic resistance, gravity forces, inertia forces.

Moving forces act on the receiver and drive the machine. The work done by these forces is positive as the moving forces either coincide with the direction of motion or make an acute angle with this direction (gas pressure exerted on

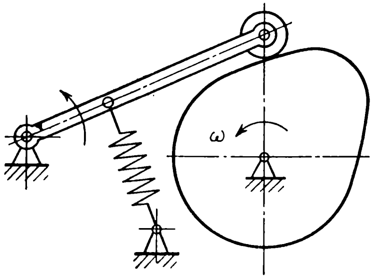


Fig. 210.

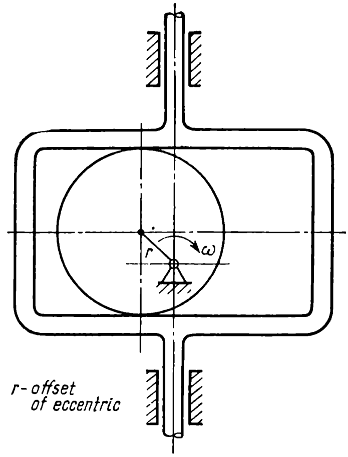


Fig. 211.

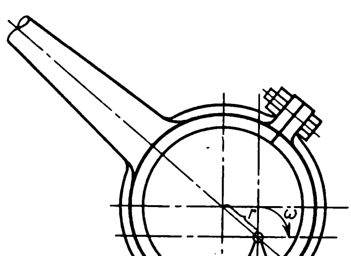


Fig. 212.

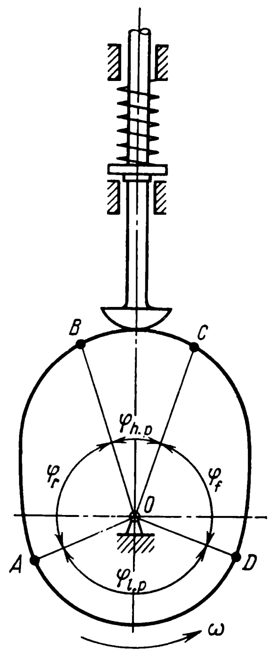


Fig. 213.

a piston of an internal-combustion engine, a force applied on a winch handle).

Forces of useful resistance are those the work of which the machine is designed to overcome. The work done by these forces is negative as they are directed in the sense opposite to the motion or make an obtuse angle with its direction (the resistance of wood or metal to cutting in wood-working or metal-cutting machines, the resistance of soil to implements).

Among the *forces of parasitic resistance* are frictional forces, environmental resistance, etc. The work done by these forces is negative as they are directed in the sense opposite to the motion or make an obtuse angle with its direction.

Gravity forces are taken into account when the centre of gravity of moving parts of a machine changes its position relative to a horizontal plane. The work done by these forces is positive or negative according to whether the centre of gravity of moving parts of the machine is raised or lowered (such as the weight of a lift, the weight of the hook of a crane).

Inertia forces arise in any curvilinear or non-uniform rectilinear motion of machine parts. Taking into consideration these forces is essential in the dynamic design of high-speed machines.

Equation of Motion of a Machine. According to the well-known law, the change in kinetic energy of a material system during a certain time interval is equal to the sum of the work done by the applied forces on the corresponding displacement of the system

$$\sum \frac{m_i v_{i_2}^2}{2} - \sum \frac{m_i v_{i_1}^2}{2} = \sum U, \qquad (259)$$

where m_i = mass of particle,
v_{i_1} = initial velocity,
v_{i_2} = final velocity,
ΣU = sum of work done by forces applied to system.

Denote the work of moving forces by $U_{m.f}$, the work of useful resistance forces by $U_{u.f}$, the work of parasitic resistance forces by $U_{p.f}$ and the work of gravity forces by GH.

We substitute the sum of the work done by the forces acting in a machine in the right-hand member of the expression for the change in kinetic energy

$$\sum\frac{m_i v_{i_2}^2}{2}-\sum\frac{m_i v_{i_1}^2}{2}=\sum U=U_{m\cdot f}-U_{u\cdot f}-U_{p\cdot f}\pm GH. \quad (260)$$

This equation is called the *equation of motion of a machine*.

Let us evaluate the work done by gravity forces.

Consider three cases:

1. A continuous downward motion of the centre of gravity of individual parts or elements of the machine during operation. In this case GH can be incorporated in the work done by moving forces (weights in clocks, the weight of the water in hydraulic wheels).

2. A continuous upward motion of the centre of gravity of individual parts of the machine during operation. In this case GH can be incorporated in the work done by resistance forces (the weight of a chain or rope, the weight of a hook from which a load is suspended).

3. A periodic up-and-down motion of the centre of gravity of individual parts of the machine. In this case, taking the complete period of operation of the machine as a unit time, the quantity GH can be disregarded as it will appear in the equation of motion of the machine with a plus sign for the first half-period and with a minus sign for the second half-period (the weights of a piston, piston rod or connecting rod in vertical engines, the weight of the saw frame in gang-saw mills).

Consequently, the last term $\pm GH$ in the equation of motion of a machine can be omitted, the work of the weight being incorporated in the work of moving forces in the first case, in the work of resistance forces in the second case, and set equal to zero in the third case.

As a result, the equation of motion of a machine takes the following form

$$\sum\frac{m_i v_{i_2}^2}{2}-\sum\frac{m_i v_{i_1}^2}{2}=U_{m\cdot f}-U_{u\cdot f}-U_{p\cdot f}. \quad (261)$$

There are three basic periods in the operation of a machine: starting, steady running and slowing-down.

We shall analyse each of these periods.

Starting Period. When the machine is started the initial velocity $v_{i_1} = 0$. The kinetic energy at the beginning of the motion is then

$$\sum \frac{m_i v_{i_1}^2}{2} = 0$$

and the equation of motion of the machine becomes

$$\sum \frac{m_i v_{i_2}^2}{2} = U_{m \cdot f} - U_{u \cdot f} - U_{p \cdot f}$$

or

$$U_{m \cdot f} = U_{u \cdot f} + U_{p \cdot f} + \sum \frac{m_i v_{i_2}^2}{2} . \qquad (262)$$

Conclusion. *During the starting period the work of moving forces is greater than the sum of the work done by forces of useful and parasitic resistance*

$$U_{m.f} > U_{u \cdot f} + U_{p \cdot f}$$

since some work is expended in imparting acceleration to moving parts of the machine until their speed is brought to a preassigned value.

Stationary machines (metal-working and wood-working machines) are started idle, i.e., with no useful load applied

$$U_{u \cdot f} = 0,$$

then

$$U_{m \cdot f} = U_{p \cdot f} + \sum \frac{m_i v_{i_2}^2}{2} . \qquad (263)$$

In this case the machine attains the desired speed in a shorter time.

Steady Running Period. The steady running of a machine which has no reciprocating masses is characterized by a constant speed over a given time interval

$$v_{i_1} = v_{i_2}.$$

Consequently,

$$\sum \frac{m_i v_{i_2}^2}{2} - \sum \frac{m_i v_{i_1}^2}{2} = 0.$$

Thus, the equation of motion becomes

$$U_{m \cdot f} - U_{u \cdot f} - U_{p \cdot f} = 0,$$

whence

$$U_{m\cdot f} = U_{u\cdot f} + U_{p\cdot f}. \tag{264}$$

Conclusion. *During the steady running period the work of moving forces is equal to the sum of the work done by forces of useful and parasitic resistance.*

Machines with reciprocating parts (grain sorters, internal-combustion engines) have a variable speed over a given time interval but, returning to the original position at definite time intervals, the machine acquires the former speed.

It follows that the equation derived for steady running $U_{m.f} = U_{u.f} + U_{p.f}$ holds for machines with reciprocating parts. In this case the motion is referred to as *periodically steady.*

Slowing-Down Period. The final velocity as the machine is stopped must be zero, i.e., $v_{i_2} = 0$, therefore

$$\sum \frac{m_i v_{i_2}^2}{2} = 0.$$

The equation of motion of the machine is then

$$-\sum \frac{m_i v_{i_1}^2}{2} = U_{m\cdot f} - U_{u\cdot f} - U_{p\cdot f}$$

or

$$U_{m\cdot f} = U_{u\cdot f} + U_{p\cdot f} - \sum \frac{m_i v_{i_1}^2}{2}.$$

Consequently,

$$U_{m\cdot f} < U_{u\cdot f} + U_{p\cdot f}.$$

In practice, to stop the machine, the moving forces and forces of useful resistance are released, i.e.,

$$U_{m\cdot f} = 0$$

and

$$U_{u\cdot f} = 0,$$

then

$$\sum \frac{m_i v_{i_1}^2}{2} = U_{p\cdot f}. \tag{265}$$

The whole reserve of kinetic energy is expended in overcoming the work of parasitic resistance forces.

By increasing the parasitic resistance artificially (by applying the brakes) a machine can be stopped more quickly.

118. Fundamentals of the Dynamic Analysis of Mechanisms

The dynamics of mechanisms and machinery deals with the determination of forces acting on individual members of a mechanism for a given law of motion.

The knowledge of forces is necessary for (a) the design of machine elements for strength, (b) the determination of energy expended during operation of mechanisms, (c) the design for wear and friction, (d) the choice of rational dimensions of members, shapes of elements, etc.

The determination of forces without regard to the effects of motion is called static analysis.

Static analysis is made only for slow-speed machines in which the dynamic effects of the masses of moving members are insignificant. For high-speed machines, it is necessary to take into account the forces due to the motion, i.e., inertia forces.

Analysis which takes account of static and dynamic loads is called dynamic analysis.

The basic method of dynamic analysis is the *method of kinetostatics.*

The method of kinetostatics is based on D'Alembert's principle which is presented in Sec. 85.

The dynamic analysis of mechanisms cannot be restricted to establishing relations between moving and resistance forces. It should include the determination of forces acting on the members and foundation.

These forces occur as reactions of constraints produced by external forces and those induced by dynamic effects (inertia forces) due to the motion of members.

Using the method of kinetostatics, we apply inertia forces to moving members of a mechanism and consider the whole mechanism as being in equilibrium under the action of the given forces, reactions of constraints and inertia forces.

Kinetostatic analysis is of great practical importance since the forces determined serve as the basis for the design of individual members.

A kinetostatic analysis of mechanisms can be carried out when the laws of motion of a mechanism (velocity and acceleration of any point, angular velocity and angular acceleration of all members of the mechanism) are known.

In making kinetostatic analysis, the inertia forces are first determined.

The determination of the magnitude and direction of the inertia force for a particle in rectilinear or curvilinear motion was discussed in Sections 86 and 87.

In Sec. 88 a general method was developed for determining the inertia force $\overline{W}$ for a rigid body, which consists in sum-

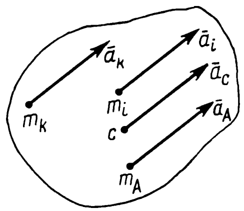


Fig. 214.

ming geometrically the inertia forces $\overline{W}_i = -m_i\bar{a}_i$ for all of its particles

$$\overline{W} = \sum_{i=1}^{n} \overline{W}_i = \sum_{i=1}^{n} -m_i\bar{a}_i = -\sum_{i=1}^{n} m_i\bar{a}_i.$$

Consider the determination of inertia forces in various cases of motion of a rigid body.

I. Translation of a Rigid Body.

In a translation the accelerations of all particles are equal in magnitude and parallel (Fig. 214)

$$\bar{a}_i = \bar{a}_A = \ldots = \bar{a}_C.$$

Substituting the acceleration of the centre of gravity $\bar{a}_C$ in the expression for the inertia force and putting it before the summation sign as a constant quantity, we obtain

$$\overline{W} = -\sum_{i=1}^{n} m_i\bar{a}_i = -\sum_{i=1}^{n} m_i\bar{a}_C = -\bar{a}_C \sum_{i=1}^{n} m_i = -\bar{a}_C M, \quad (266)$$

where $M = \sum_{i=1}^{n} m_i$ is the entire mass of the body.

Consequently, *the inertia force for a rigid body in translation is equal to the entire mass of the body times the acceleration of* *the centre of gravity and is directed in the sense opposite to the acceleration of the centre of gravity.* Obviously, the inertia force for a rigid body in translation is applied at its centre of gravity.

II. Rotation of a Rigid Body About a Fixed Axis.

Consider first the simplest case when the axis of rotation is an axis of symmetry of a rigid body.

Let a rigid body rotate about an axis O which is its axis of symmetry (Fig. 215a) with an angular velocity ω and an angular acceleration α.

Isolate in the body an arbitrary particle m_i at a distance r_i from the axis of rotation. The components of the acceleration of this particle are given by the formulas

$$a_i^n = \omega^2 r_i,$$

$$a_i^t = \alpha r_i.$$

The corresponding inertia forces for the particle are calculated by the formulas of Sec. 87

$$W_i^n = m_i \omega^2 r_i,$$

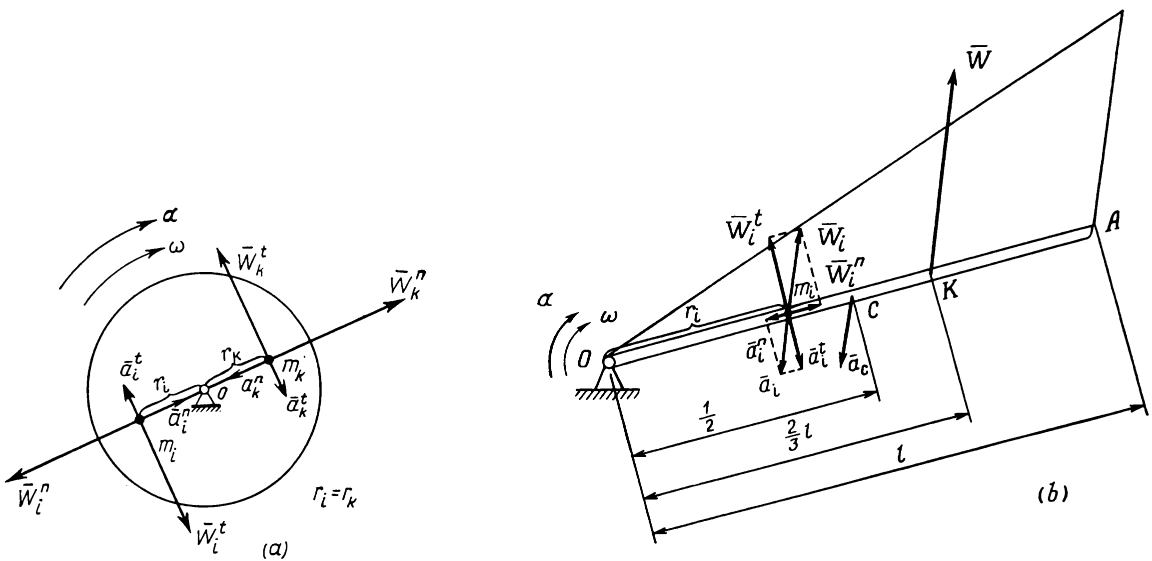
$$W_i^t = m_i \alpha r_i.$$

These forces are opposite to the corresponding accelerations.

As noted, the axis of rotation is an axis of symmetry of the body. Therefore, we can find in the body a particle m_k symmetrical with respect to particle m_i about the axis of rotation, i.e., $r_i = r_k$ (Fig. 215a). The inertia force for the particle m_k is numerically equal to the inertia force for the particle m_i. The normal components of the inertia forces are directed along the same straight line in opposite sense; being equal in magnitude, they cancel. The tangential components of the inertia forces, however, produce a moment about the axis of rotation O. The tangential component of the inertia force for the particle m_i produces about this axis an elementary moment

$$dM_O = m_i \alpha r_i^2.$$

Summing the elementary moments of the tangential components of the inertia forces for all the particles of the



α
ω
$\bar{W}_k^t$
$\bar{W}_k^n$
$\bar{a}_i^t$
r_i
r_k
m_k
a_k^n
$\bar{a}_k^t$
O
$\bar{a}_i^n$
m_i
$\bar{W}_i^n$
$\bar{W}_i^t$
$r_i = r_k$
(a)
$\bar{W}$
A
$\bar{W}_i^t$
$\bar{W}_i$
$\bar{W}_i^n$
m_i
K
C
α
ω
r_i
O
$\bar{a}_i^n$
$\bar{a}_i^t$
$\bar{a}_c$
$\bar{a}_i$
$\frac{1}{2}$
$\frac{2}{3}l$
l
(b)

Fig. 215.

body, we obtain

$$M_O = \sum_{i=1}^{n} dM_O = \sum_{i=1}^{n} m_i \alpha r_i^2.$$

We put the angular acceleration which is the same for all particles of the body before the summation sign

$$M_O = \alpha \sum_{i=1}^{n} m_i r_i^2.$$

The quantity $\sum_{i=1}^{n} m_i r_i^2$, which represents the sum of the products of the mass of each particle and the square of its distance from the axis of rotation, is known as the *moment of inertia* of the body and is denoted by I. The physical significance of this quantity is explained and its values for some bodies are given in Sec. 107.

Thus we obtain

$$M_O = I\alpha, \tag{267}$$

i.e., *the inertia forces of all the particles of a rigid body in its rotation about an axis of symmetry reduce to a couple of moment M_O (inertia moment) equal to the moment of inertia of the body times the angular acceleration.**

Examples of bodies, widely used in engineering, which rotate about their axes of symmetry are shafts, pulleys, gear wheels, flywheels, etc.

Consider another frequently encountered case of a homogeneous rod OA rotating about an axis O through one of its ends (Fig. 215*b*). Isolate in the rod an arbitrary particle m_i at a distance r_i from the axis of rotation O. The inertia force for this particle is defined by the formula (see Sec. 87)

$$W_i = m_i r_i \sqrt{\omega^4 + \alpha^2}.$$

If the rod is divided into a number of small equal portions whose masses are $m_1 = m_2 = \ldots = m_i = \ldots = m_n$,

---

* The inertia moment of a body should not be confused with its moment of inertia. The first quantity is a force factor and is measured in N-m (or kgf-m). The second quantity characterizes the geometric properties of a body and is measured in kg-m² (or kgf-m-sec²).

then it follows from the expression for the inertia force W_i for the particle that its value increases with the distance from the axis of rotation directly as the radius r_i. In other words, the inertia forces are distributed along the length of the rod according to a triangle law (Fig. 215*b*).

The resultant of all the inertia forces is equal to

$$W=\sum_{i=1}^{n} W_i=\sum_{i=1}^{n} m_i r_i \sqrt{\omega^4+\alpha^2}.$$

By putting the constant quantity $\sqrt{\omega^4+\alpha^2}$ before the summation sign, we obtain

$$W=\sqrt{\omega^4+\alpha^2}\sum_{i=1}^{n} m_i r_i.$$

The sum of the products of the mass of each particle of a body and its distance from the axis of rotation is known as the *static moment of the mass of the body*. It is similar to the static moment of an area and can be determined as the product of the entire mass of the body and the distance of its centre of gravity from the axis of rotation (see Sec. 44), i.e.,

$$\sum_{i=1}^{n} m_i r_i = M\,OC = M\frac{l}{2},$$

where $M=\sum_{i=1}^{n} m_i$ is the entire mass of the body, $OC=\frac{l}{2}$ is the distance from the axis of rotation to the centre of gravity of the homogeneous rod.

Substituting $\sum_{i=1}^{n} m_i r_i = M\frac{l}{2}$ in the expression for the inertia force, we obtain

$$W=M\frac{l}{2}\sqrt{\omega^4+\alpha^2},$$

but $\frac{l}{2}\sqrt{\omega^4+\alpha^2}=a_C$ is the acceleration of the centre of gravity of the rod, i.e.,

$$W=Ma_C.$$

Consequently, *the inertia force for a homogeneous rod rotating about an axis through one of its ends is equal to the entire* *mass of the rod times the acceleration of its centre of gravity.* The inertia force is applied at the geometric centre of a triangle which characterizes the distribution of inertia forces (Fig. 215*b*). For a homogeneous rod, this point K is at a distance of two-thirds of its length from the axis of rotation and is called the *centre of oscillation*. The point of application

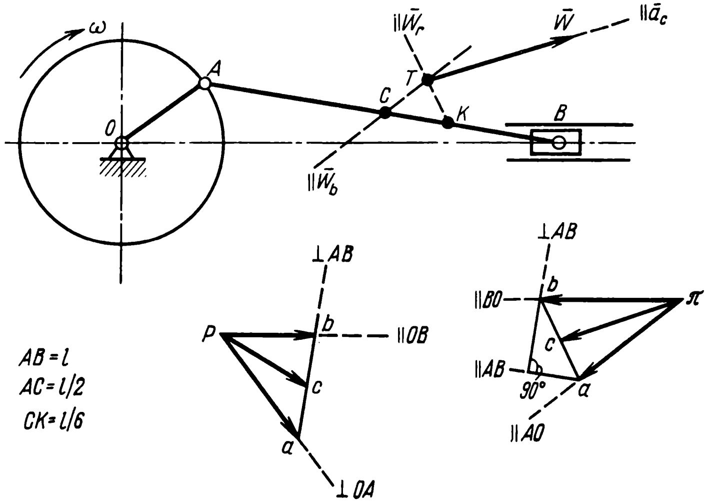


Fig. 216.

of the inertia force does not coincide here with the centre of gravity of the body.

III. Plane Motion of a Rigid Body.

As was shown in Sec. 72, any plane motion may be resolved into a translation with an arbitrary point and a rotation about this point.

The determination of the inertia force for a member in plane motion will be illustrated by taking the connecting rod AB (Fig. 216) of a slider-crank mechanism. The dimensions and masses of all links of the mechanism and the angular velocity ω of the driver OA are assumed to be known.

The connecting rod is assumed to be homogeneous. Find the accelerations of particles A, B and C. For this purpose we construct to appropriate scales the velocity diagram and the acceleration diagram for the mechanism.

The plane motion of the connecting rod AB is resolved into a base motion and a relative motion. The base motion is represented as a translation defined by the motion of the crank pin A.

The base motion of the connecting rod with an acceleration $\bar{a}_A$ involves an inertia force

$$\overline{W}_b = \overline{W}_{tr} = -M\overline{a}_A,$$

where M is the mass of the connecting rod.

This force is applied at the centre of gravity of the connecting rod and has a sense opposite to the acceleration $\bar{a}_A$.

The relative motion is represented as a rotation of the connecting rod about point A.

The relative motion of the connecting rod involves an inertia force

$$\overline{W}_r = \overline{W}_{rot} = -M\overline{a}_{CA}.$$

This force is applied at the centre of oscillation K and has a sense opposite to the acceleration $\bar{a}_{CA}$.

The position of point K for a homogeneous rod (see above) is defined by the formula

$$AK = \frac{2}{3}AB.$$

To find the point of application of the resultant inertia force for the connecting rod it is necessary to extend the directions of $\overline{W}_b$ and $\overline{W}_r$ until they intersect at a point T.

The total inertia force for the connecting rod is applied at point T, has the magnitude

$$\overline{W} = -M\overline{a}_C$$

and has a sense opposite to the acceleration of the centre of gravity of the connecting rod.

After the magnitude and direction of the inertia force are determined it is possible to find the pressure in kinematic pairs. To do this it is sufficient to set up equations of equilibrium for individual links or their combinations, using the well-known methods of statics.

If the connecting rod were a non-homogeneous rod, which is often the case in practice, the determination of the inertia force and its point of application would remain essentially the same, except that the centre of gravity C and the centre of oscillation K would change their positions. The determination of the position of the centre of gravity of non-homogeneous bodies is discussed in Sections 43 and 44. The distance from the centre of gravity of a non-homogeneous rod to its centre of oscillation is given by the formula

$$CK = \frac{I_C}{M\,AC},$$

where I_C is the moment of inertia of the connecting rod with respect to an axis through its centre of gravity, M is the mass of the connecting rod, AC the distance from the point chosen as the pole for the base motion to the centre of gravity.

119. Fundamentals of Regulation

Irregularity of Running. Flywheel. High performance of machines or mechanisms is achieved at particular speeds most efficient in a given situation. On the contrary, changes in speed of machines or mechanisms caused by inequality between the work done by moving forces and by resistance forces upset the operation.

Flywheels and governors offset the irregularity of motion of machines.

Flywheels are used to keep periodic variations (fluctuations) in speed to an acceptable minimum. Governors serve to reduce speed changes under variable loads.

A *flywheel* is a rotating mass of large moment of inertia in the form of a wheel with a heavy rim mounted on a shaft. When the work done by moving forces becomes excessive, the flywheel, which possesses a large moment of inertia, stores up the surplus. When the work done by resistance forces becomes greater than that of moving forces, the flywheel gives up its stored-up kinetic energy. Thus, the flywheel acts as an accumulator of surplus energy of a machine.

The design of a flywheel, i.e., the determination of its moment of inertia, must provide conditions in which variations in angular velocity from an average value will be within permissible limits.

Determination of Moving Forces. The basic condition for good running is the equality of the work done by all resistance forces during a complete period of motion and the work done by all moving forces during the same period.

The law of variation of moving forces during one period of operation of a machine is usually prescribed graphically

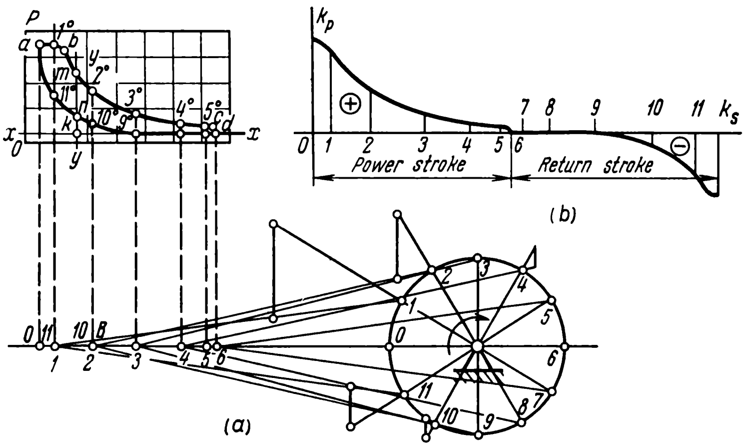


Fig. 217.

in the form of an indicator diagram. Consider, as an example, the indicator diagram for a two-stroke engine (Fig. 217*a*).

The working cycle corresponding to a complete period of motion of the machine is accomplished during one revolution of the engine shaft, i.e., during two piston strokes.

One portion of the indicator diagram, curve *abcd*, shows the variation of gas pressure on the piston during the power stroke, i.e., when the piston moves from left to right; the other portion, curve *dna*, characterizes the pressure variation during the idle stroke, i.e., when the piston moves from right to left.

The ordinates of the indicator diagram represent the unit gas pressure, i.e., the pressure p per unit area of the piston.

The total force exerted on the piston is $P = p\,\frac{\pi D^2}{4}$, where

D is the diameter of the piston.

An arbitrarily taken line *yy* parallel to the axis of ordinates intersects the indicator diagram at two points, *m* and *n*. The segment *mk* measured from the line of gauge pressure represents the gas pressure on the piston during the power stroke, and the segment *nk* represents the gas pressure in reverse motion during the compression stroke.

To simplify the use of the indicator diagram, especially in analysing multicylinder engines, it is drawn so that the position co-ordinates of the piston in forward and backward motions are plotted successively as a function of the crank rotation angle.

A diagram in which segments of the axis of abscissas representing forward and backward strokes of the piston are laid off in succession is called a *developed indicator diagram*. This diagram (Fig. 217*b*) is constructed as follows.

In a system of rectangular co-ordinates, lay off a segment on the axis of abscissas which will represent to a certain scale two piston strokes. Further, starting from the origin of co-ordinates, plot the distances travelled by the piston corresponding to consecutive angles of rotation of the crank.

It is advisable to place the origin on the thrust line *xx* so that the ordinates of the diagram measured from the axis of abscissas represent only the gauge pressure on the piston.

When transferring the ordinates of the indicator diagram to the developed diagram it should be borne in mind that during the compression stroke (curve *da*) the gas pressure is directed opposite to the motion of the piston, therefore part of the mechanical work done by the engine is expended in accomplishing the process of compression and, consequently, cannot be transmitted to the main shaft of the machine.

In order to take into account this loss of mechanical work, the ordinates of the compression curve are assigned a negative sign and laid off below the axis of abscissas (Fig. 217*b*).

Tangential Pressure Diagram. The pressures represented by the ordinates of the developed indicator diagram are pressures exerted on piston *B* (Fig. 218). We resolve the total gas force $\bar{P}$ on the piston into a component $\bar{P}_1$ in the direction of the connecting rod and a component $\bar{N}$ along the normal to the direction of motion of the piston.

As the piston moves the force $\bar{N}$ produces a friction force $F = fN$. This force is incorporated in general resistance forces.

The point of application of the force $\bar{P}_1$ acting along the connecting rod is transmitted to the crank pin, i.e., to point A. Resolve the force $\bar{P}_1$ into a component $\bar{N}_1$ in the direction of the crank and a component $\bar{T}$ along the tangent. The component $\bar{N}_1$ produces friction forces in the journals of the ma-

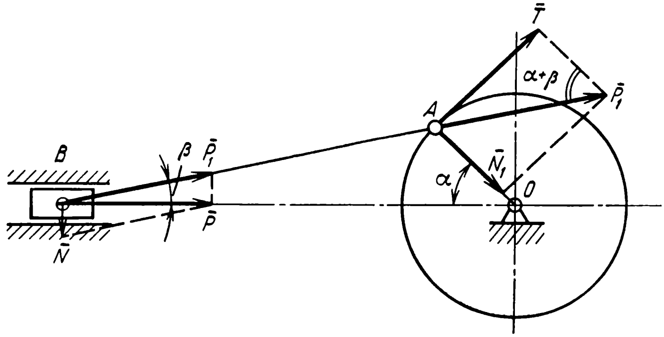


Fig. 218.

chine shaft, which we also incorporate in general resistance forces and do not consider separately.

The component $\bar{T}$ is a tangential force and the work done by this force enters into the final calculation of the weight of a flywheel.

From the resolution of forces we find

$$P_1 = \frac{P}{\cos \beta},$$

$$T = P_1 \sin(\alpha + \beta).$$

Substituting for P_1, we obtain

$$T = P \frac{\sin(\alpha + \beta)}{\cos \beta}. \qquad (268)$$

Examining this formula, we find

if $\alpha = 0$ and $\beta = 0$, $T = 0$,

if $\alpha = 180°$ and $\beta = 0$, $T = 0$, i.e., in the extreme (dead) positions of the crank the tangential force is zero. Consequently, in order to overcome the dead points and set the machine in motion, it is necessary to apply an additional force to turn the shaft of the machine. It is only by turning the crank through a certain angle α that the further motion of the machine under the action of the tangential force can be ensured.

So far we have found the value of the tangential force for the crank angle α. In order to obtain the tangential pressure

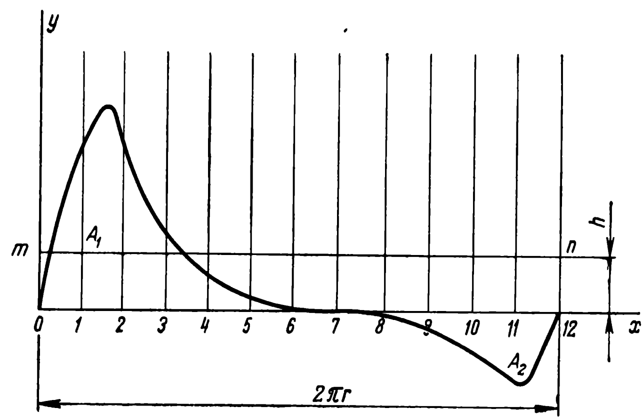


Fig. 219

diagram for a full revolution of the crank (Fig. 219) proceed as follows:

(1) divide the circumference described by the crank pin into several equal parts and determine the value of $\overline{T}$ for each position of the crank from the formula

$$T = P\frac{\sin(\alpha+\beta)}{\cos\beta};$$

(2) unwrap the crank circumference with the divisions marked off on it into a straight line;

(3) draw an ordinate network through the points of division;

(4) lay off the appropriate values of tangential forces on the ordinates;

(5) join the tops of the ordinates by a curve called the *tangential pressure curve.*

We transform the tangential pressure diagram so obtained into an equal rectangle of base $2\pi r$ and height h. The magnitude of the height h is obtained by dividing the entire area under the diagram by the base $2\pi r$. This height gives the value of the required constant resistance.

The resistance line mn separates positive areas from negative areas in the tangential pressure diagram.

Clearly, in steady running the sum of positive areas should equal the sum of negative areas

$$\sum A_p = \sum A_n.$$

Each of the areas, whether positive or negative, characterizes either the surplus of the work done by moving forces over that of resistance forces or the surplus of the work done by resistance forces over that of moving forces.

A flywheel must have a mass sufficient to store up a maximum surplus work done by moving forces and to give up the stored-up kinetic energy in the next time interval when resistance forces exceed moving forces.

We apply to the design of a flywheel the kinetic energy equation

$$\sum \frac{mv_2^2}{2} - \sum \frac{mv_1^2}{2} = U_{m.f} - U_{r.f}, \tag{a}$$

where $U_{r.f}$ is the work done by all resistance forces, $U_{m.f}$ the work done by the moving forces applied to the crank pin. The velocities are $v_2 = \omega_2 r$ and $v_1 = \omega_1 r$. Hence $v_2^2 = \omega_2^2 r^2$ and $v_1^2 = \omega_1^2 r^2$.

Substituting these data in formula (a), we obtain

$$\sum \frac{\omega_2^2}{2} mr^2 - \sum \frac{\omega_1^2}{2} mr^2 = U_{m.f} - U_{r.f}. \tag{b}$$

We factor out $\sum mr^2$

$$\sum mr^2 \left[\frac{\omega_2^2 - \omega_1^2}{2}\right] = U_{m.f} - U_{r.f}. \tag{c}$$

It is known that the sum of the products of an element of mass by the radius squared is the moment of inertia of a rotating mass about the axis of rotation, i.e., $\sum mr^2 = I$. Consequently,

$$I \frac{\omega_2^2 - \omega_1^2}{2} = U_{m.f} - U_{r.f}. \tag{d}$$

The calculation of the mass of a flywheel must be carried out for angular velocities of the wheel ranging from the maximum to the minimum allowable value, i.e., from ω_{max} to ω_{min}, and for the maximum difference between the work done by moving forces and that of resistance forces.

As is known, this difference is represented in the tangential pressure diagram by the maximum positive area, A_{max}.

Taking this into account, the fundamental kinetic energy formula becomes

$$I \frac{\omega_{max}^2 - \omega_{min}^2}{2} = A_{max}$$

or

$$I \frac{(\omega_{max} + \omega_{min})(\omega_{max} - \omega_{min})}{2} = A_{max}. \tag{e}$$

Denote $\frac{\omega_{max} + \omega_{min}}{2}$ by ω_{av} (average angular velocity) and $\frac{\omega_{max} - \omega_{min}}{\omega_{av}}$ by δ (coefficient of fluctuation). Multiplying and dividing the left-hand member of equation (e) by ω_{av}, we obtain

$$I \frac{(\omega_{max} + \omega_{min})(\omega_{max} - \omega_{min})\,\omega_{av}}{2\omega_{av}} = A_{max}$$

or

$$I \omega_{av}^2 \delta = A_{max},$$

whence

$$I = \frac{A_{max}}{\omega_{av}^2 \delta}.$$

Substituting for ω_{av} we have

$$I = \frac{A_{max}\, 900}{\pi^2 n^2 \delta}.$$

The main difficulty in using this formula is that the determination of A_{max} requires very accurate graphical construction. Therefore, for simplicity, we multiply and divide the right-hand member of the equality by A (A is the work done during one complete revolution of the machine). Then

$$I = \frac{A_{max} \times 900\,A}{\pi^2 n^2 \delta\, A};$$

$\frac{A_{max}}{A}$ is the ratio of the maximum surplus area to the entire area under the tangential pressure diagram, hence the expression for I is independent of scales for the diagram. Denote this ratio by μ, i.e.,

$$\frac{A_{max}}{A} = \mu.$$

The work A done during one complete revolution of the machine can be found from the following considerations: if A is the work per revolution, then An is the work per minute (n is the number of revolutions of the crank per minute), $\frac{An}{60}$ is the work per second. The power developed by the machine is then

$$N = \frac{An}{60},$$

whence

$$A = \frac{N \times 60}{n}.$$

Substituting the quantities $\frac{A_{max}}{A} = \mu$, $A = \frac{N \times 60}{n}$ in the equation for the moment of inertia of a flywheel, we obtain

$$I = \frac{A_{max} \times 900\,A}{\pi^2 n^2 \delta\, A} = \frac{900 \times 60 \mu N}{\pi^2 n^3 \delta}. \quad \text{(f)}$$

Let M denote the mass of a flywheel, G its weight, R its radius, g the acceleration of gravity (remembering that the moment of inertia is the product of the mass of a body times the square of its radius of gyration). We have

$$I = MR^2 = \frac{G}{g} R^2 = \frac{900 \times 60 \mu N}{\pi^2 n^3 \delta},$$

whence the weight of the flywheel is

$$G = \frac{900 \times 60 \mu N\, g}{\pi^2 n^3 \delta R^2}.$$

Since $g \cong \pi^2$, we obtain finally

$$G \cong 54{,}000 \cdot \frac{\mu N}{n^3 \delta R^2}, \tag{269}$$

where μ = coefficient determined from tangential pressure diagram,
N = power developed by machine,
n = number of revolutions of machine per minute,
R = radius of flywheel,
δ = coefficient of fluctuation depending on difference of maximum and minimum speeds of machine; in practice coefficient of fluctuation is assigned according to principal function of machine; the rougher the work, the larger δ may be assumed; values of δ are given in Table 2.

Table 2

Coefficient of Fluctuation

Item	δ
Alternating-current generators	0.0003-0.003
Direct-current generators	0.005-0.001
Paper-making machines	0.1-0.025
Gang-saw mills	0.03-0.06
Crushers	0.05-0.2
Presses	0.1-0.15
Elevators	0.13-0.15
Piston engines	0.0066-0.01

The radius of the flywheel is approximately determined as follows. The maximum peripheral speed allowed in practice for cast iron wheels on the basis of their tensile strength under centrifugal loading is

$$v \leqslant 30 \text{ m/sec.}$$

Since

$$v = \omega R$$

or

$$v = \frac{\pi n}{30} R,$$

it follows that

$$\frac{\pi n}{30} R \leqslant 30 \text{ m/sec},$$

whence

$$R \leqslant \frac{900}{\pi n} \text{ m}.$$

The weight G_1 of the flywheel rim is determined from the following empirical formula

$$G_1 = 0.9G.$$

Knowing the weight of the rim, we can determine the area of its cross section

$$S = \frac{G_1}{2\pi R\gamma} \quad (\text{since } S2\pi R\gamma = G_1),$$

where S = cross-sectional area of wheel rim,
γ = specific weight of flywheel material (7.25 for cast iron).

General Principle of Governors. As has already been noted, the basic condition for normal running of machines and mechanisms is the constancy of the speed that ensures the most efficient operating conditions.

If, for example, an engine is designed to serve a twenty-spindle machine and 10 spindles are cut out, the remaining 10 spindles will not take up all the energy and the engine shaft will begin to turn faster.

Governors make it possible to maintain the same speed irrespective of the external load.

The general principle of governors (Fig. 220) is the following.

The release of some of the external loads produces a surplus driving force in the machine (engine) and this results in increased revolutions. As the speed of the machine shaft and, hence, of the governor attached to the shaft increases, so does the centrifugal inertia force for its balls. Under the action of the centrifugal inertia force, the balls are separated and raise the coupling through a lever system. The coupling in turn raises a lever which turns the throttle valve in a steam engine, internal-combustion engine or some other engine and reduces the steam or air-fuel mixture supply.

The reduction in the amount of steam or air-fuel mixture supplied to the machine per unit time slows down the machine. As the revolutions of the machine shaft decrease, so do the revolutions of the governor axle and, hence, the revolutions

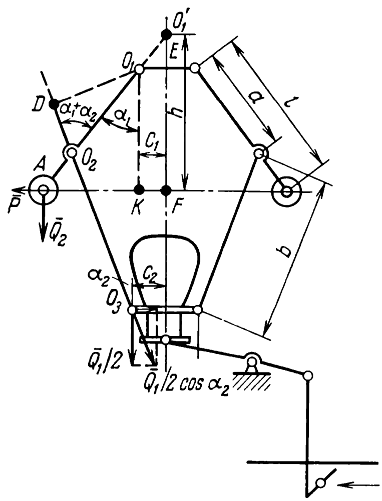


Fig. 220.

of the balls; the centrifugal force decreases, the balls and coupling come down, the throttle opens, the speed of the machine shaft increases and so on.

Without going through the derivations involved, we write the final expression for determining the speed of the machine (engine) shaft as a function of the coupling lift, i.e., the governor configuration

$$n = 30\sqrt{\frac{1+\frac{Q_1 a}{Q_2 l}}{h}}, \tag{270}$$

where Q_1 = weight of coupling with load,
Q_2 = weight of ball,
a, l = lengths of governor members,
h = characteristic of governor configuration.

SUBJECT INDEX

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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