EIGHT LECTURES ON MATHEMATICAL ANALYSIS

A. YA. KHINCHIN

D. C. Heath and Company

Eight Lectures on Mathematical Analysis

A. Ya. Khinchin

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SURVEY OF

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A project conducted by

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PREFACE TO THE AMERICAN EDITION

THIS BOOK is a translation and adaptation of one by the outstanding Russian mathematician A. Ya. Khinchin, whose original contributions to mathematical analysis, the theory of numbers, probability theory, and statistics have left a mark on modern mathematics. The excerpt given below from the preface to the Russian edition explains the origin and aims of this book in his own words.

We frequently encounter a situation in which an engineer, teacher, or economist who has at some time studied higher mathematics in a "simplified" course begins to feel the need for a broader and, what is more important, a more solid foundation for his mathematical knowledge. This need, whether it arises out of specific research by the specialist in his own scientific field or comes as an inevitable consequence of the general widening of his scientific and cultural horizons, must of course be satisfied. It might be supposed that the specialist might easily satisfy his need; he could merely take any comprehensive text on mathematical analysis and study it systematically, making use of the rudimentary knowledge he has already acquired. However, experience shows that this method, which seems so natural, almost never leads to the desired goal but instead often brings disillusionment and a consequent paralysis of any further effort. For such a student usually has only limited time at his disposal and therefore cannot undertake to work systematically through a full-length textbook. On the other hand (and this is probably the most important factor) he does not yet have a firm grounding in mathematics, and therefore he cannot, without outside help, pick out the essentials. He will be compelled instead to devote his attention to irrelevant details, and in these he will finally get lost, unable to see the forest for the trees.

And yet very little is needed to satisfy fully the needs of this student. A few years ago I had an opportunity to give a special course of lectures devoted to this purpose. The course consisted of only twelve two-hour lectures and was one of the series of courses offered by the University of Moscow to raise the mathematical qualifications of engineers. I must confess that at the beginning my task seemed to me to be almost hopeless; and yet I have reason to believe that my course, in spite of its brevity, did satisfy the needs of the audience. The secret of this success consisted in finding the right key to the pedagogical problem that faced me. I renounced from the very beginning any idea of presenting even a single topic in full detail: instead, I limited myself to a vivid and concrete presentation of the essential points and spoke more of goals and perspectives, of problems and methods, of the connections of the fundamental notions of analysis with each other and with their applications, than of individual theorems and their proofs. I did not hesitate, on numerous occasions, to refer my students to a text for details not of fundamental significance (sometimes even for entire sequences of theorems and proofs). But, in return, I begrudged no time in elucidating concepts, methods, and ideas

that have a leading and essential significance and tried in every way and with the most varied descriptions and images to impress these fundamental notions as vividly and effectively as possible upon the consciousness of my listeners. I have grounds for believing that after this preparation any student who felt the desire or need for a deeper study of a particular topic in analysis was already independently able, first, to find the material he needed, and second, to approach its study economically, knowing how to distinguish the primary and basic from the secondary and nonessential.

The many discussions I have had with individual students and groups have firmly convinced me that the path I chose was correct. In this connection, I would mention that the very large audiences that attended all of these lectures and the small number of drop-outs are the best proof of how widespread among engineers is the need for raising the level of their mathematical knowledge.

This book has the same goal as the course of lectures I have just described and tries to realize it by the same means. The reader should therefore be warned from the very beginning that he will not find here a complete presentation of a university course in analysis, or even of individual topics selected from such a course. I have set myself the task only of giving a general sketch of the basic ideas, concepts, and methods of mathematical analysis. But I have tried to make this sketch as simple and as easy to retain as possible, to make it something that can be read and assimilated by anyone familiar with even the crudest exposition of the subject, and one which, once assimilated, should enable the student to study the details of any part of the subject independently and effectively.

At the same time, I hope that this book may also be of real benefit to many students in the mathematics departments of universities. Neither a text nor a lecturer, limited as they both are by the exigencies of time and the program, can pay enough attention to the discussion of fundamental questions; both are compelled to concentrate on the exposition of all the details of the material they cover. And yet everyone knows how useful it is sometimes to turn one's eyes away from the trees and look at the forest. I would like to believe that this book will help to reveal that broader view to more than one future mathematician who is studying analysis for the first time.

The Survey of Recent East European Mathematical Literature wishes to express its appreciation to Mrs. Irena Zygmund for translating the book and to Messrs. Louis I. Gordon and C. Clark Kissinger for reading the manuscript and for their valuable suggestions.

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1. The Continuum

1. WHY BEGIN WITH THE CONTINUUM?

DEFINITION. *We shall call the variable y a function of the variable x if to each value of the variable x there corresponds a uniquely determined value of y.*

This sentence is like a gateway leading into the domain of higher mathematics. With it we define the most important and basic concept of mathematical analysis, the concept of *functional dependence.*[1] In this concept there appears in embryo the prospect of mastering natural phenomena and technological processes by means of a mathematical apparatus. That is why we must require of this definition absolute clarity; not a single word should leave the shadow of a doubt. The least ambiguity in the definition would threaten the entire edifice constructed upon this fundamental concept and would necessitate a complete reconstruction.

Nevertheless, it turns out on closer inspection that this concise formulation with which we began is in many respects incomplete and open to various interpretations. We shall dwell only on one such doubtful point here, and in clarifying its content we shall be led directly to the topic of our first lecture.

Our definition contains the words *to each value of the variable x.* If there is to be no ambiguity, we must clarify the meaning of the term *value of the variable.* But this is not enough. Our definition speaks of *each value.* It follows that for a function to be defined, it is not enough to know some individual values of the variable x to which there corresponds a y. We must know the whole collection of values to each of which there corresponds a definite value of y. In other words, we must know what is called in analysis the *domain of definition* (or simply the *domain*) of the given function. The set of corresponding values of y is called the *range* of the function.

[1] A detailed discussion of various points of view regarding *functions* is given in Lecture 3.

What can we say about the particular values of a variable? We know that they are numbers, and so the set of these values is a set of numbers. But what is this set and what numbers does it contain? From the very beginning we shall exclude the complex numbers from consideration and assume that all values of x and y are real numbers.

But can all real numbers serve as values of the quantity x, and if not, which can and which cannot? Nothing was said about this in our definition. But this is completely understandable because one cannot give the same answer to this question for all functions (and, as a matter of fact, not even for the same function in different problems). The domain of a function depends on the nature of the function as well as on the special problem in which the function is employed. One and the same function may have to be considered in different problems for different sets of values of the *independent variable* x.

Examples of functions whose domain of definition is not the set of all real numbers are abundant. For instance, the function $y = x!$ makes sense (at least within the scope of elementary mathematics) only for positive integers x; the function $y = \log x$, only for $x > 0$; etc. One can also construct examples in which the natural domain of definition will be a number set of considerably more complicated structure.

If, however, we ask ourselves what are the domains of definition which occur most often in mathematical analysis, we shall have to say that in the overwhelming majority of cases the domain of a function is an *interval* (open or closed), that is, the set of all real numbers contained between two given numbers (with the given numbers either included or excluded). Sometimes this interval is represented by a half-line (for example, the set of all $x > 0$). A half-line obviously represents the set of all real numbers which are greater (or less) than a certain given number (while the condition $>$ or $<$ is sometimes replaced by $\geq$ or $\leq$). Finally, there are cases in which the interval is represented by the whole straight line; that is, all real numbers may serve as values of the variable x. We then say that the domain of definition of the function is represented by the whole real axis, or the *number line.*

In any case, we see that in mathematical analysis the environment in which functions exist and unfold their individual characteristics is the set of all real numbers. This set is called in mathe-

matics the *continuum*[1] (or, more precisely, the linear continuum). And just as the careful gardener examines the soil before planting, we shall examine the environment in which this development of mathematical analysis from the concept of functional dependence is to take place. This is why the continuum is the first topic of study in any serious and well-constructed course in mathematical analysis. Only after the nature of the continuum has been sufficiently clarified can we go on to study the concept of functional dependence. And the structure of the continuum turns out to be not so simple as it might appear at first glance. The world of real numbers will unfold before our eyes as a complicated structure abounding in the most diverse details whose investigation cannot be considered complete even to this day.

2. NEED FOR A THEORY OF REAL NUMBERS

Why is it impossible to study the continuum before constructing a complete theory of real numbers? What, then, is the continuum? What real numbers exist? And when and how can we be sure that we have actually included *all* the real numbers in our theory?

In elementary algebra we have at our command the set of all *rational* numbers (all integers and fractions, positive, negative, and zero). But very soon we begin to notice that these numbers are insufficient. For example, among the rational numbers there is no number $\sqrt{2}$; that is, there is no rational number whose square is equal to the number 2. But why is it necessary to have such a number? It would be needed if only to represent the length of the diagonal of a square whose side equals 1. Consequently, if we were to forego the existence of such a number, we would have to reconcile ourselves to the fact that the lengths of certain segments which arise so naturally and simply in geometry would not be expressible by any number. It is clear that metric geometry could not develop on such foundations. This means that $\sqrt{2}$ must find a place among the real numbers. But as it does not appear among the rational numbers, we call this number *irrational.* However, the $\sqrt{2}$ is by no means satisfied with the mere recognition of its existence; it im-

[1] In some contexts in higher mathematics, the term continuum is reserved for compact connected sets. Thus the student may sometimes see a closed interval, for example [0,1], referred to as a continuum.

mediately demands, first, that we assign to it a definite place among the rational numbers, that is, indicate precisely which rational numbers are less than $\sqrt{2}$ and which are greater, and second, that we learn to perform the fundamental operations with it. For example, $\sqrt{2} > 1$ because the diagonal of the square is greater than its side, and the sum of the side and the diagonal of the square is equal to $1 + \sqrt{2}$. Thus we are compelled to assign a meaning to the number $1 + \sqrt{2}$ also, as it is not rational, and we must also include this number in the set of real numbers. These demands of the new number are well-founded and justified, and if at this moment we do not respond to them, it is only because we are about to introduce into our system many other new numbers. All of these without exception will present to us the same demands, and it will be simpler to satisfy all of them simultaneously than to treat in detail each new number separately.

Following the number $\sqrt{2}$, all (positive and negative) square roots of positive rational numbers enter our system in a natural and unavoidable way, then the cube roots, and finally all numbers of the form

$$r^{\frac{1}{n}}, \tag{1}$$

where r is any positive rational number and n is any integer greater than 1.

As we know, however, the matter does not end here. Needs just as concrete as the case of the diagonal of a square force us in numerous other instances to introduce new numbers in the form of roots of algebraic equations. This occurs whenever a given equation does not have roots among the numbers which we have already introduced, and yet we cannot deny the existence of these roots without depriving ourselves of a numerical description for some concrete physical entity.

Now let us go all the way in this direction. We denote by the term *algebraic number* any (real) root of an equation of the form $P(x) = 0$, where $P(x)$ is any polynomial with integral coefficients, and we introduce into our system all real algebraic numbers. In particular, we thus introduce all numbers of the form (1) above since the number $r^{\frac{1}{n}}$ is defined as a root of the equation $qx^n - p = 0$, where $\frac{p}{q} = r$ is a representation of the rational number r in the

form of a simple fraction. As a still more special case, every rational number $r = \frac{p}{q}$ also belongs to the set of algebraic numbers, since it is the root of the equation $qx - p = 0$.

It is very easy to *order* this system of algebraic numbers, that is, to formulate a rule allowing us to determine which of two arbitrary algebraic numbers is the greater and which is the lesser. It is somewhat more difficult, although still not too complicated, to formulate rules for performing the ordinary algebraic operations on these numbers and to show that the results of these operations are again algebraic numbers, so that, and this is a very important point, algebraic operations with algebraic numbers always lead to algebraic numbers and therefore can never require the introduction of new numbers.

Can we perhaps stop here and consider the construction of the system of real numbers finished? Can we now recognize the set of all algebraic numbers as the continuum? We know well that we cannot do that, and we know why we cannot. Although the numbers we have so far introduced are sufficient for many algebraic theories, it is precisely for analysis that they fail. In its first steps, mathematical analysis adds to the elementary operations of algebra the fundamental and most important operation of *passage to the limit*. A number of cases exist in which concrete considerations force us to recognize the existence of the limit of this or that sequence of numbers. What is more, this limit appears to us as a number which has a definite and real meaning, a number on which, in turn, we should like to perform algebraic and analytic operations.

If for every sequence of algebraic numbers to which we found it desirable to assign a definite limit, such a limit actually existed within the domain of algebraic numbers, then we could freely accept this domain as being indeed the continuum and conclude that mathematical analysis needs no real numbers other than the algebraic numbers. But this is not the situation. If we take a unit circle and begin to inscribe regular polygons in it, starting with a triangle or a square and proceeding by doubling the number of sides indefinitely, the perimeters of all such polygons are algebraic numbers and the limit of this numerical sequence is called the circumference of the circle. To admit that this limit does not exist would forbid geometry to speak of the circumference of a circle. We can easily imagine the loss caused by such a restriction not

only to geometry but also to all other sciences that make use of the concept of a circle.

Yet it is possible to prove that among the algebraic numbers such a limit does not exist. What then is the solution to this impasse? It is clear: we must recognize that for the purposes of mathematical analysis, algebraic numbers alone do not suffice, and that it is necessary to adjoin to them real numbers of a new kind. All such nonalgebraic real numbers are called *transcendental.* We denote the number constructed above (the circumference of the unit circle) by 2π; thus π is a transcendental number. Another important example of a transcendental number is the familiar $e = 2.718\ldots$, which, as you know, is generated by a simple passage to the limit of a sequence of rational numbers. The transcendence of the numbers e and π was established quite late, in fact not until the second half of the last century. However, the necessity of introducing transcendental numbers was established somewhat earlier, in the middle of the last century, by the French mathematician Liouville on the basis of other examples which were simpler but also less important.

Thus the construction of our continuum is not yet finished. How, then, are we to proceed? For example, can we stop here and say, "The continuum is the set of all algebraic numbers to which, as the need arises, we adjoin other numbers (called transcendental) which we obtain (as in the case of e and π) from algebraic numbers by passage to the limit?" We ask this question because it is on just such a basis, though not made explicit, that the majority of *simplified* courses on mathematical analysis are constructed, courses which avoid the exposition of a complete theory of irrational numbers. But the answer to our question is, of course, no. We cannot stop at the point at which we now find ourselves, and for numerous quite simple and convincing reasons.

First of all, the continuum, as the *totality* of real numbers, should be defined once and for all as a given fixed set (on the pattern of the definition cited above for the set of all algebraic numbers) without leaving open any possibility of subsequently adding more numbers to it. Further, the words "as the need arises" in our provisional definition obviously have no precise meaning. If we have a sequence of algebraic numbers which does not have an algebraic limit, and if the question arises whether to assign to it a transcendental limit or to treat it as a sequence without a limit, we

have the right, from a formal point of view, to follow our own judgment. To make the decision, guided not by formal considerations but by concrete and real ones, no matter how important, is to reject the concept of a mathematical definition. We can refuse formal existence to the number π. In this particular case, it would be highly inconvenient, but in other cases such a refusal might lead to no inconvenience at all. It is clear that a criterion which would force us to introduce a transcendental number *whenever it would be awkward to get along without it* is not, nor can it by any reformulation become, a mathematical criterion. Finally, it is not at all certain that the numbers introduced in this manner would be sufficient. For we may have to add the new numbers, multiply them, and make them pass to a limit (for mathematical analysis has no use for numbers which do not permit such operations). How can we be sure that the results of all such operations will be real numbers belonging to our continuum? For if not, then we would again have to add new numbers to our set and thus our continuum would still not contain *all* the real numbers.

We see then that the position we hypothetically took above is untenable. We cannot construct one or two transcendental numbers as examples and say "and so on," and let it go at that. For by this procedure, nothing is really defined at all.

So we see that we cannot lay a solid foundation for mathematical analysis without a general theory of real numbers; a theory not limited to individual constructions of new numbers, but containing the general principle for all such constructions and yielding in one stroke the whole set of real numbers.

3. CONSTRUCTION OF THE IRRATIONAL NUMBERS

There exist in mathematics a number of different theories for the continuum. However, all these theories, it is important to remember, approach the problem in conceptually identical manners. In comparison with this essential identity, those details in which they differ are as the structural details of a building compared with its overall architectural plan.

Once the rational numbers are given, all these theories aim at obtaining from them, at one stroke, the entire set of real numbers by means of a single principle of construction. And in the different theories this principle takes different forms, but the similarity of these

theories is not limited to having a single principle of construction. All principles which lead to the construction of the new (irrational) numbers are, in spite of considerable formal differences, based on the same idea in all the theories. This idea is the fundamental analytical operation of *passage to the limit;* all known methods are reducible to it and may be considered as different forms of it. For example, you know that square roots of natural numbers can be realized as limits of appropriately selected sequences of rational numbers (approximate square roots). This is true in other cases as well.

In view of the foregoing, to obtain a full understanding of the system of real numbers one need not examine all the different theories in detail. It will be sufficient to take as an example any one of them, as everything of fundamental importance which we thus discover applies equally well in all the other theories. We choose to treat the theory of Dedekind, not because it has any essential advantages over others, but solely for the practical reason that this theory is adopted in a majority of the most widely used textbooks. Thus you will have no difficulty in obtaining a textbook where you may follow the details omitted from our exposition.

Before we introduce irrational numbers, we must examine a little more carefully the set of rational numbers (which we denote by R). First, let us note a very elementary property of this set: *between any two rational numbers r_1 and r_2 there always exists a third rational number.* We can see this most easily by noting that the arithmetic mean of r_1 and r_2 $\left(\text{that is, } \frac{r_1 + r_2}{2}\right)$ is a rational number with the desired property. Repeated application of this fact at once establishes that between r_1 and r_2 there are infinitely many rational numbers.

We shall now examine carefully the situation arising from our attempts to find or to define $\sqrt{2}$. (By this symbol we mean the positive square root.) First we look among the rational numbers (any others do not exist for us at the moment) for a number whose square would equal 2. We can easily establish that such a rational number does not exist, but we shall not give here the familiar proof of this fact. Thus, if we choose any rational number r, then we shall have either $r^2 < 2$ or $r^2 > 2$.

Now let us consider only the positive rational numbers. They fall naturally into two *classes:* class A consisting of all the positive

rationals r_1 such that $r_1^2 < 2$, and class B consisting of all the positive rationals r_2 such that $r_2^2 > 2$. Since r_1 and r_2 are positive, it follows from the inequalities $r_1^2 < 2 < r_2^2$ that $r_1 < r_2$, that is, *that every number of class A is smaller than every number of class B.* It is evident that if we now adjoin to class A the number zero and all the negative rational numbers, then we shall have a separation of the *whole set* R into two classes, A and B, where every number of class A will be smaller than every number of class B. By the term *cut* (or more precisely, a *cut in the set* R), we shall mean any division of the set R into two nonempty[1] classes for which the above condition holds (the condition that $A < B$). Thus the separation of the set R into the class B, consisting of all positive rationals whose square is greater than two, and the class A, consisting of all rationals not in B, determines a cut in R.

We can construct cuts in R in widely differing ways, some quite elementary. For example, by including in class A all rational numbers $r_1 \leq 5$ and in the class B all rational numbers $r_2 > 5$, we clearly obtain a cut in R. If we represent numbers in the usual manner by points on a straight line, then, of course, every cut will be represented by a separation of the (rational) points of the line into two sets, the first of which is situated entirely to the left of the other set.

At first glance it may appear that all cuts in R are essentially the same, that two different cuts differ only by the location at which they are made, and that, because of this, one can be converted into another by a simple translation. It is extremely important to realize that *this notion is wrong,* and that in the very structure of the cuts there may occur profound, and for our purposes fundamental, differences.

Notice that the cut in our last example has the property that there exists a number (a rational number, since we do not as yet have any others) such that all numbers less than it belong to class A, and all numbers greater than it belong to class B; in our example the number 5 is obviously such a number. We shall call the number which has this property the *edge* of the given cut. Thus the cut in our last example has an edge.

On the contrary, *in our first example* (concerning $\sqrt{2}$) *there is no edge.* We shall now prove this. Let us assume that there exists a rational number r which is an edge. Then we must have either

[1] A set is called nonempty if it contains at least one element.

$r^2 < 2$ or $r^2 > 2$. To be definite, let us suppose that $r^2 < 2$. Since r is an edge, it follows that for each $r' > r$ we must have $r'^2 > 2$.

If $r < 1$, then by taking $r' = 1$, we arrive immediately at a contradiction. On the other hand, if $r \geq 1$, then $r^2 \geq r$. By taking $2 - r^2 = c > 0$ and $r' = r + \frac{c}{4}$, we have

$$r'^2 = r^2 + \frac{rc}{2} + \frac{c^2}{16} \leq r^2 + \frac{r^2 c}{2} + \frac{c^2}{16} = 2 - \frac{7}{16} c^2 < 2,$$

which again leads to a contradiction, since $r' > r$.

Thus all the cuts in R can be classified into two types: those that have an edge and those that do not. Moreover, we should keep in mind the following:

(*a*) A cut cannot have two edges, for if r and r' were both edges of a cut and $r < r'$, then by virtue of the discussion on page 8, there would exist a number r'' such that $r < r'' < r'$. But since r is an edge and $r'' > r$, it would follow that $r'' \in B$.[1] At the same time, from the fact that r' is an edge and $r'' < r'$, it would follow that $r'' \in A$, which is contradictory.

(*b*) The edge of a cut, if it exists, is either the greatest number of the class A or the least number of the class B; if, however, there is no edge, then there exists neither a greatest number of class A nor a least number of class B.

(*c*) Every rational number r_0 is the edge of two different cuts. In one of them the class A consists of all numbers $r \leq r_0$ (and the class B of all numbers $r' > r_0$), and in the other the class A consists of all numbers $r < r_0$ (while the class B consists of all numbers $r' \geq r_0$).

(*d*) The classification of the totality of all cuts in R into two types, cuts which have an edge and cuts which do not, is, of course, an intrinsic structural property of the set R; this would still exist even if we were not at all interested in introducing nonrational numbers.

At this point, the example involving $\sqrt{2}$ suggests our subsequent course of action. The intuitive picture is clear: we see before us the number line (the axis of real numbers) cut into two parts at a point to which there corresponds no rational number. To renounce the

[1] $r'' \in B$ means that r'' is an element of the set B.

existence of such a point would intuitively imply a gap in the number line and the continuum would lose its continuity (its solid and gap-free character), that very characteristic to which the continuum owes its name. And from the practical point of view, as we have already stated, all applied sciences (and first of all geometry) would be subjected to a very considerable inconvenience if we were to accept the lack of an edge between our two classes. Therefore, stimulated by the demands of our intuitive perception as well as by very serious practical considerations, we introduce into our system of numbers a new number $\sqrt{2}$, which we define to serve as the edge of this cut. Such numbers we call *irrational numbers.*

But the particular cut we have selected does not differ in principle from any other cut in R of the same type, that is, a cut which has no (rational) edge. Therefore, in constructing the general theory, we extend our definition in a natural way to any cut of this type. To every cut in R with no rational edge we assign a new, irrational number, which by definition will be its edge.

In this way, by means of a single principle we define at once the whole set of irrational numbers. Together with the previously existing set of rational numbers they form the *set of all real numbers,* or the *continuum,* which is now completely defined.

4. THEORY OF THE CONTINUUM

The principle of construction of the irrational numbers that we have introduced by no means exhausts the theory of the continuum; on the contrary, the latter really only begins here. The program of development which must be effected before we can speak of a complete theory of the continuum is still quite extensive.

First of all, we have to *order* our continuum, that is, determine precisely under what conditions we shall consider one given real number as greater or less than another. Further, we have to *define operations* on the real numbers, for so far we haven't the slightest idea, for example, what is meant by the sum of the numbers 1 and $\sqrt{2}$. And then we have to check carefully that these operations have the same properties to which we are accustomed in the domain of rational numbers. For example, the invariance of a sum when the order of the summands is changed (the commutative law of addition) is a theorem which we have to prove again for real numbers. Finally, we have to verify that the continuum really satisfies all

those requirements of practice and of our intuitive perception for the sake of which it was constructed: *continuity*.

It is obvious that this program cannot be carried out in complete detail within the framework of these lectures; in any case it would be extremely tedious. In what follows we shall touch upon only some of its most important features.

To begin with, it is very easy to order our continuum. Suppose that we are given two real numbers α_1 and α_2 and we wish to establish which one is greater than the other. If both numbers are rational, this problem finds its solution in arithmetic, about which we assume full knowledge. If α_1 is irrational and α_2 is rational, then the problem is solved at once: the number α_1 is the edge of a certain cut in R; in accordance with the definition of a cut we shall say that $\alpha_2 < \alpha_1$ or $\alpha_2 > \alpha_1$, depending on whether the rational number α_2 belongs to class A or to class B of this cut. Suppose, finally, that the numbers α_1 and α_2 are both irrational. Since these numbers are not identical, the two cuts corresponding to them are different, and hence the lower classes A_1 and A_2 of these cuts are different. This means that one of these sets, say A_2, contains a rational number r which is not in A_1. From $r \in A_2$ it follows that $r < \alpha_2$, and from[1] $r \notin A_1$ it follows that $r \in B_1$; hence $r > \alpha_1$. Thus there exists a rational number r such that $\alpha_1 < r < \alpha_2$. When this is the case, we shall say that $\alpha_1 < \alpha_2$. On the other hand, if we had found an r' such that $\alpha_2 < r' < \alpha_1$, then we would say $\alpha_2 < \alpha_1$. One of these two conditions must exist as we have just shown, and this completes the ordering of the continuum.

However, we have finished only with the definition of the ordering; its properties are yet to be established. We have to show that $\alpha_1 < \alpha_2$ is incompatible with $\alpha_1 > \alpha_2$ and that from the inequalities $\alpha_1 < \alpha_2$ and $\alpha_2 < \alpha_3$, it follows that $\alpha_1 < \alpha_3$. In short, we have yet to show that the inequalities between real numbers obey the same basic laws as the inequalities between rational numbers. But you should have no trouble in proving these propositions.

Among other things, the foregoing argument shows that *between any two irrational numbers there exists a rational number.* We have seen previously that the same is true if both given numbers are rational. Suppose now that r is rational and α is irrational, and

[1] $\notin$ means "is not an element of."

suppose, for definiteness, that $r < \alpha$. We shall show that in this case also there are rational numbers between r and α. The number α is the edge of a cut (A, B) in R, and from $r < \alpha$ it follows that $r \in A$. But in a cut having an irrational edge the class A does not have a greatest number. Hence there exists a rational number $r' > r$ which belongs to the class A and consequently is less than α. Thus we have $r < r' < \alpha$.

It follows that *between any two real numbers there is an infinite set of rational numbers.* We express this important property of the set R of all rational numbers by saying that R is *everywhere dense* (in the continuum). It is easy to show that the set of irrational numbers is also everywhere dense. We need only consider all numbers of the form $r\sqrt{2}$ where r runs through the set of all rational numbers; all such numbers are irrational and they alone already form an everywhere dense set. Strictly speaking, the expression $r\sqrt{2}$ will acquire a precise meaning only after the operations on real numbers have been defined, and so we shall take up this question at once.

We have no reason to consider this problem in complete detail because the method for constructing definitions of the fundamental operations with real numbers will become clear if we carefully examine only one example. Let α_1 and α_2 be two real numbers; we wish to define their sum $\alpha_1 + \alpha_2$. Regardless of whether the numbers α_1 and α_2 are rational or irrational they are both edges of certain cuts in R, which we denote by (A_1, B_1) and (A_2, B_2) respectively. Further, let a_1, b_1, a_2, b_2 denote arbitrary numbers belonging to the sets A_1, B_1, A_2, B_2 respectively. It is evident that every number of the form $a_1 + a_2$ is less than every number of the form $b_1 + b_2$. We shall now show that there exists one and only one real number α such that for arbitrary a_1, b_1, a_2, b_2 (belonging, of course, to the corresponding sets) we have the inequalities

$$a_1 + a_2 \leq \alpha \leq b_1 + b_2.$$

We shall then naturally define the sum of α_1 and α_2 to be equal to α. The existence and uniqueness of the sum will then follow from the existence and uniqueness of the number α.

Let us consider the cut (A, B) in R defined as follows. If a rational number r is smaller than all numbers of the form $b_1 + b_2$, then we shall put it in class A; otherwise, in class B. It is easy to

see that the division of R so determined is actually a cut. We denote by α the edge of this cut. It is evident that all numbers of the form $a_1 + a_2$ belong to the class A, while all numbers of the form $b_1 + b_2$ belong to the class B. Therefore

$$a_1 + a_2 \leq \alpha \leq b_1 + b_2; \tag{2}$$

that is, the number α satisfies the stipulated conditions. We have yet to establish its uniqueness.

For this purpose we must first make sure that it is possible to choose the numbers a_1 and b_1 so that their difference $b_1 - a_1$ is arbitrarily small. To show this, let a be any rational number of class A_1 and let c be an arbitrarily small positive rational number. In the sequence of rational numbers

$$a, \quad a + c, \quad a + 2c, \quad \ldots, \quad a + nc,$$

the first term belongs to class A_1, and, in general, so do several more terms. But since $a + nc$ increases indefinitely with n, beginning at a certain place all the following terms will belong to class B_1. Therefore, there exists an integer k such that

$$a + kc = a_1 \in A_1, \quad a + (k + 1)c = b_1 \in B_1,$$

while $b_1 - a_1 = c$. Hence, a_1 and b_1 may be chosen so that the difference between them is arbitrarily small. Similarly, a_2 and b_2, and hence the numbers $a_1 + a_2$ and $b_1 + b_2$, can be selected in such a way as to be arbitrarily close to each other.

Let us now suppose that there exist two real numbers α and α' satisfying all inequalities of the form (2) and suppose, for definiteness, that $\alpha < \alpha'$. As we know, there exists a pair of rational numbers r, r' such that

$$\alpha < r < r' < \alpha',$$

and together with the inequalities (2) these relations indicate that any number of the form $a_1 + a_2$ differs from any number of the form $b_1 + b_2$ by a quantity exceeding the constant $r' - r$. However, this is contradictory to what was shown above. In this way the uniqueness of α is demonstrated.

The above definition of addition is convenient in that it permits us to extend at once all the basic laws governing the addition of rational numbers to the addition of arbitrary real numbers. Try

to prove, for example, the commutative law and you will see how easily everything comes out.

As we have already stated, we shall not dwell here on the definitions of the other operations or on proofs of the laws governing them. We shall only mention that it is best to define multiplication analogously to addition, and then to define subtraction and division as the inverse operations.

We shall now turn to the last important problem in this area: to show that the continuum that we have defined actually has *continuity,* the gap-free nature it must have in order to serve as the basis of mathematical analysis, and whose absence in the set R of rationals compelled us to introduce irrational numbers.

In order to answer this question, let us recall what led us to speak of the absence of such continuity in R. It was that among the cuts in R there occurred some which did not have an edge belonging to the set R. Thus, if we can show that for the set of real numbers such a thing cannot happen, that is, that *every cut in the set of real numbers has an edge in the real numbers,* we may consider our task completed and be assured that the continuum which we have constructed satisfies the requirements imposed on it.

To avoid any misunderstanding, let us observe that the cuts mentioned in the italicized statement above are not the same as the Dedekind cuts which were used to define real numbers. Previously, we have always spoken of cuts in the set R of rational numbers, and now for the first time we speak of a *cut in the set of real numbers,* or the continuum. However, the formal definition of a cut is unchanged.

We shall now prove the existence of our desired edge. Let (A, B) be an arbitrary cut in the continuum. Every rational number (as indeed every real number) belongs either to class A or to class B; in this way the cut (A, B) in the continuum induces a cut (A', B') in R. Let α be the real number constituting the edge of the cut (A', B'). We shall show that α is also the edge of the cut (A, B) and the proof of our assertion will be complete.

We shall have to show that every real number $\alpha_1 < \alpha$ belongs to class A and every real number $\alpha_2 > \alpha$ belongs to class B; by symmetry it is enough to prove the first part of this statement. Let r be a rational number situated between α_1 and α. Since $r < \alpha$, it follows that $r \in A' \subset A$,[1] and since $\alpha_1 < r$, it follows that $\alpha_1 \in A$.

[1] The notation $A' \subset A$ means that the set A' is contained in (is a subset of) the set A.

5. FUNDAMENTAL LEMMAS OF THE REAL NUMBER SYSTEM

With the definition of real numbers, we have laid the foundation for mathematical analysis. In building the principles of analysis on this foundation, we shall, of course, have to refer frequently to this basic definition. This will entail some inconvenience, as the construction and investigation of the necessary cuts is usually rather cumbersome.

The way in which mathematics finds a solution to this difficulty is most instructive, since it may be considered typical of all logical situations of this kind which are frequently encountered in the mathematical sciences. In the process of developing mathematical analysis it is noticed that although the direct application of the definition of real numbers appears rather frequently in the reasoning, many of these applications are very similar in form to one another. Actually almost all of these applications follow one of three or four formal patterns (but with a different content each time, of course). Given such a situation, it would obviously be very uneconomical and would make the development and mastery of the given branch of science much more difficult, if the same logical constructions had to be worked out dozens of times anew, changing only their specific subject matter each time.

With complete justification, mathematics long ago acquired the habit, in situations of this kind, of formulating such recurring logical patterns as auxiliary propositions, or *lemmas*. Once such a lemma is proved, it becomes unnecessary to repeat on each occasion the formal construction supporting it; one need merely cite the lemma. In our case, after having proved three or four such auxiliary propositions, we shall in the future almost never have to return to the construction of cuts. We shall be able to replace this construction each time by a reference to one of the principal lemmas, which form, so to speak, a number of small bridges connecting mathematical analysis with its logical foundations. It goes without saying that the choice of these basic lemmas may vary in different expositions. However, in every case we can counsel the reader not to begrudge time and effort spent in mastering a great number of such lemmas. The purpose of each one of them is to lighten work in the future and the effort spent will not be wasted.

We shall now illustrate with a few examples the formulation and proof of such lemmas.

A sequence of real numbers,

$$\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots \tag{3}$$

is called *monotonic* if either $\alpha_n \leq \alpha_{n+1}$ for all n, or $\alpha_n \geq \alpha_{n+1}$ for all n. In the first case we speak of a *monotonically nondecreasing* sequence, and in the second of a *monotonically nonincreasing* sequence. The sequence (3) is called a *bounded* sequence if there exists a number c such that

$$|\alpha_n| < c \qquad (n = 1, 2, \ldots).$$

LEMMA 1. *Every monotonic bounded sequence of real numbers has a limit.*

Proof. Suppose for definiteness that

$$\alpha_n \leq \alpha_{n+1} \quad \text{and} \quad |\alpha_n| < c \qquad (n = 1, 2, \ldots).$$

Let us divide the continuum into two sets A and B, including in B every real number which is greater than *all* the α_n (in particular, the number c belongs to the set B) and including in the set A all remaining numbers (in particular, all the numbers α_n). It is evident that such a division of the continuum is a cut. Letting α be the edge of this cut, we shall show that[1] $\lim_{n\to\infty} \alpha_n = \alpha$, which will prove Lemma 1.

First of all, we note that for any n we have $\alpha_n \leq \alpha$. For if we had $\alpha_n > \alpha$, then from the definition of a cut we would have $\alpha_n \in B$, which contradicts the definition of B. If we now assume, contrary to our assertion, that α is not the limit of the sequence (3), then there must exist a positive constant ε such that for an infinite set of the numbers n, we have the inequality

$$\alpha - \alpha_n > \varepsilon,$$

whence $\alpha_n < \alpha - \varepsilon$. But because of the monotonicity of the sequence, if this inequality is satisfied for an infinite set of values of n, then it must be satisfied for *all* n. By virtue of the definition of the set B, it follows that $\alpha - \varepsilon \in B$, while from $\alpha - \varepsilon < \alpha$, it follows that $\alpha - \varepsilon \in A$ (since α is the edge of the cut (A, B)). This contradiction proves our lemma.

It should be noticed that monotonicity cannot be omitted from the hypothesis of the lemma. The existence of the limit does not fol-

[1] Here the limit is for $n = 1, 2, \ldots$. The theory of limits is discussed in detail in Lecture 2.

low from boundedness alone, as can be seen, for example, in the case of the sequence $\alpha_n = (-1)^n$.

The lemma on monotonic sequences finds a number of applications not only in analysis but also in elementary mathematics. In this latter field we often introduce it as an *axiom,* seldom taking into account that it is simply not true unless the number system under consideration includes *all* the real numbers. This defect can be observed not only in elementary mathematics but also in "simplified" courses in mathematical analysis.

We now observe that the existence of the circumference of a circle (the limit of the perimeters of inscribed regular polygons as the number of sides increases indefinitely) and the existence of the number $e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ are proved most simply by using this lemma.

Besides Lemma 1, stating the basic property of bounded monotonic sequences, there is an analogous proposition of no less significance concerning monotonic functions of a continuously changing variable.

LEMMA 1′. *If the variable x tends toward a from the left and if $f(x)$ is bounded and monotonic in an interval whose right-hand end point is the point a, then $f(x)$ tends to a limit as x approaches a.*

The boundedness of the function $f(x)$ within an interval whose end points are $a - \varepsilon$ and a (where $\varepsilon > 0$) implies the existence of a number c such that $|f(x)| < c$ for $a - \varepsilon < x < a$, and its monotonicity implies that the ratio

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

is either ≥ 0 for any pair of numbers x_1, x_2, $(x_1 \neq x_2)$ belonging to this interval, or it is ≤ 0 for any such pair.

Proof. The proof of Lemma 1′ may be carried out very simply by using Lemma 1. To be definite, suppose that $f(x)$ is nondecreasing; that is, $f(x_1) \leq f(x_2)$ for $a - \varepsilon < x_1 < x_2 < a$. Since for $n > \frac{1}{\varepsilon}$ we have

$$a - \varepsilon < a - \frac{1}{n} < a,$$

the increasing sequence $a - \frac{1}{n}\left(n > \frac{1}{\varepsilon}\right)$ is contained within the interval whose end points are $a - \varepsilon$ and a and has as its limit the number a. The corresponding sequence $f\left(a - \frac{1}{n}\right)$ is, of course, bounded and nondecreasing. Thus, by Lemma 1 there exists a limit:

$$\lim_{n\to\infty} f\left(a - \frac{1}{n}\right) = b.$$

If the number x is sufficiently close to a, then we can find an integer $n = n(x)$ such that

$$a - \frac{1}{n} \leq x \leq a - \frac{1}{n+1}.$$

Since the function f is monotonic,

$$f\left(a - \frac{1}{n}\right) \leq f(x) \leq f\left(a - \frac{1}{n+1}\right). \tag{4}$$

But as $x \to a$ it is evident that $n \to \infty$, and consequently the left and the right sides of the inequalities (4) have b as their common limit. Hence $f(x) \to b$ as $x \to a$, and Lemma 1′ is thus established.

We shall denote by $[a, b]$ the *closed interval* with end points a and b (the set of all numbers x satisfying the inequalities $a \leq x \leq b$), and by (a, b) the open interval with the same end points (all x such that $a < x < b$). We shall call a sequence of intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots \tag{5}$$

a *sequence of nested intervals* if this sequence satisfies the following two conditions:

(*a*) $a_n \leq a_{n+1} < b_{n+1} \leq b_n \quad (n = 1, 2, \ldots)$; each successive interval is entirely contained in the preceding one:

(*b*) $\lim_{n\to\infty} (b_n - a_n) = 0$; the lengths of the intervals tend to zero as their indices increase indefinitely.

LEMMA 2. *If the sequence* (5) *is a sequence of nested intervals, then there exists one and only one real number belonging to all these intervals.*

It would, of course, be possible to carry out the proof of this very useful lemma by constructing an appropriate cut: how-

ever, it will be much simpler to employ Lemma 1. The proof runs as follows:

Proof. By virtue of condition (a), the sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

is monotonic and bounded (this latter fact follows from $a_n < b_1$ for all n), and thus by Lemma 1 it has a limit. Let us set

$$\lim_{n\to\infty} a_n = \alpha.$$

Since for any positive integer k we have the inequality

$$a_n \leq b_k \qquad (n = 1, 2, \ldots),$$

it follows that

$$\alpha \leq b_k \qquad (k = 1, 2, \ldots),$$

so that

$$a_n \leq \alpha \leq b_n \qquad (n = 1, 2, \ldots). \tag{6}$$

Thus, the number α is contained in all the intervals $[a_n, b_n]$. Moreover, there can be only one such number. For if there existed two numbers α and β satisfying the inequalities (6), then (assuming that $\alpha < \beta$) we would have

$$a_n \leq \alpha < \beta \leq b_n \qquad (n = 1, 2, \ldots),$$

and, consequently,

$$b_n - a_n > \beta - \alpha \qquad (n = 1, 2, \ldots).$$

But this contradicts property (b) of a sequence of nested intervals, and thus establishes Lemma 2.

It should be remarked that in the hypothesis of Lemma 2 it is essential to include in every interval its end points. If we omitted this condition and instead of $[a_n, b_n]$ we considered the open intervals (a_n, b_n), then our lemma would be false. For example, the sequence of open intervals $\left(0, \frac{1}{n}\right)$ for $n = 1, 2, \ldots$, has no point common to all the intervals.

The following lemma of more recent origin also serves frequently as a very convenient tool in proving theorems in analysis.

We shall say that the family M (generally infinite) of intervals *covers* a (closed) interval $[a, b]$ if every point of the latter lies in the interior of at least one interval of the family M.

LEMMA 3 (Heine-Borel). *If a family M of intervals covers the closed interval $[a, b]$, then it is possible to select from this family a finite subfamily M' which also covers the interval $[a, b]$.*

Proof. If the interval $\Delta_1 = [a, b]$ cannot be covered by any such finite subfamily of intervals, then dividing it into two equal parts we can assert that at least one of them cannot be covered by a finite subfamily (if both halves could be covered by finite subfamilies, then the whole interval could be so covered). Let us denote this half by Δ_2 (if neither half can be covered by a finite subfamily then let Δ_2 denote the one on the left) and divide it again into two equal parts. We assert again that at least one of them (let us denote it by Δ_3) cannot be covered by a finite subfamily of M. We can repeat this process indefinitely and thus form a sequence of nested intervals, $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n, \ldots$.

By Lemma 2, there exists one and only one point α which belongs to all these intervals. Let Δ be an interval belonging to the family M which contains α as an interior point. Since the length of the interval Δ_n tends to zero as $n \to \infty$, and for each n we have $\alpha \in \Delta_n$, it follows that $\Delta_n \subset \Delta$, for some sufficiently large n.

We thus arrive at a contradiction: the interval Δ_n, which by definition cannot be covered by any finite subfamily of intervals, is covered eventually by a single interval from M. This contradiction proves Lemma 3.

We have now not only constructed a foundation for mathematical analysis, but in having proved three very important lemmas, we have constructed so solid a foundation that the further development of the subject can be carried forward efficiently. The basic concepts and methods, the ideas and logical devices which are used in this development, you will learn in the succeeding lectures.

2. Limits

6. WHAT IS A LIMIT?

The concept of *limit* is one of the most important in mathematical analysis. You are, of course, familiar with many theorems on limits; nevertheless we shall now study this familiar concept thoroughly, both to make it more precise and to broaden it.

Let us first of all consider the meaning of the following sentence: *The variable x* (*in a given phenomenon or process*) *tends to a* (or in other words, *has the limit a*), which we write symbolically in one of two ways:

$$x \to a \quad \text{or} \quad \lim x = a.$$

In attempting to make the definition of a limit satisfy all necessary formal requirements (and without this it is quite impossible to call mathematics a science), we meet at once a peculiar and characteristic difficulty. The fact is that in a precise definition of the concept of limit, we cannot admit such terms as *phenomenon* or *process,* whose mathematical sense is completely unclear. And yet, any formulation of the ordinary definition of a limit is very difficult without the use of these (or equivalent) terms. For we usually say: for any neighborhood[1] U of a, all values of the quantity x from some point on in the given process are contained in U. Or sometimes we say: no matter how small we take the positive number ε, the quantity $x - a$ becomes and remains smaller in absolute value than ε in the course of the given process. We must now find some way to formulate this definition so that it contains no terms whose precise mathematical meaning is questionable. Frankly, we have to say that it is doubtful whether this problem can be solved in any satisfactory manner. Modern mathematicians prefer to renounce its solution entirely and to consider the notation $\lim x = a$ as devoid of any mathematical content. This does not mean, of course, that they are willing to renounce the concept of a limit. How then do they get out of this difficulty?

[1] A neighborhood of the number a is any open interval containing this number.

The fact is that whenever the notion of limit occurs in analysis, we always meet a situation where a function y tends to a limit as the independent variable behaves in some definite manner. For example: *y tends to b as x tends to a, or a_n tends to b as n increases indefinitely;* in symbols,

$$y \underset{x \to a}{\longrightarrow} b \quad \text{or} \quad \lim_{x \to a} y = b$$

$$a_n \underset{n \to \infty}{\longrightarrow} b \quad \text{or} \quad \lim_{n \to \infty} a_n = b.$$

Such a sentence has in fact a very definite meaning. Thus, for example, the expression

$$\lim_{x \to a} y = b$$

means the following: *for every neighborhood V of b there exists a neighborhood U of a such that $y \in V$ whenever $x \in U$, except perhaps when $x = a$.* As you can see, this precise definition of a limit is entirely independent of the notion of a process.[1] It states simply that y is arbitrarily close to b provided x is sufficiently close to a. Of course, for a person not yet used to the typical characteristics of mathematical formalism this might appear strange. How is it possible that the sentence

y approaches the limit b as x approaches the limit a (A)

can have a definite meaning while the phrase *x approaches the limit a* by itself means nothing? Actually there is nothing unacceptable or even unusual in this situation. The sentence (A) must be regarded as something whole and monolithic, and it is not at all necessary that parts of it, taken separately, should make sense; it is only necessary that the sentence taken as a whole have a definite meaning.

It will be seen at once that our definition of a limit includes the two cases most important to analysis: the limit of a sequence, $\lim_{n \to \infty} a_n$, and the limit of a function of a continuously varying quantity, $\lim_{x \to a} y$. In the sentence (A) both a and b can represent either real numbers or $+\infty$ or $-\infty$.

[1] Note that the concept of a limit at a is a property of the function about, but not necessarily at, a. Thus, y at $x = a$ need not equal b, nor need y even be defined at $x = a$. Further, the cases $x \to \infty$, $x \to -\infty$, $y \to \infty$, and $y \to -\infty$ do not require special definitions if by a neighborhood of $+\infty$ (or of $-\infty$) we mean the set of all numbers greater (or smaller) than some arbitrary number.

7. SOME WAYS OF TENDING TOWARD A LIMIT

Let $y = f(x)$ tend to b as x tends to a, where a and b are numbers.[1] Let us assume, for simplicity, that all values of x are greater than a. Symbolically we express this by a convenient notation:

$$x \to a + 0$$

(rather than $x \to a$, $x > a$). Thus, we have

$$\lim_{x \to a+0} f(x) = b. \tag{1}$$

Let us first assume that for all $x > a$ sufficiently close to a we have $f(x) > b$; then in the vicinity of a, $f(x)$ has one of the two geometric representations shown in Figures 1 and 2. In Figure 1

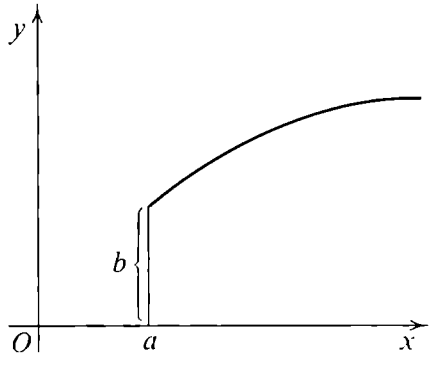

Fig. 1

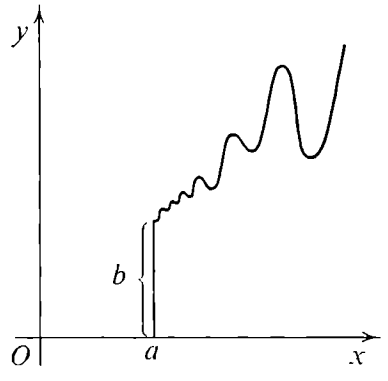

Fig. 2

the closer x is to a the closer $f(x)$ is to b, decreasing monotonically toward b as $x \to a + 0$. In Figure 2 everything is different: as $x \to a + 0$ the function $f(x)$ changes nonmonotonically, sometimes increasing, sometimes decreasing. Of course, these important differences are no obstacle to the analytic representation of both types of variation by the same formula (1), which describes the basic fact common to both: *y is as close as we please to b, provided x is sufficiently close to a.*

The case where $y < b$ for all x sufficiently close to and greater than a is completely analogous to the case above; it is depicted in Figures 3 and 4.

[1] That is, a and b represent finite real numbers in contradistinction to $+\infty$ and $-\infty$.

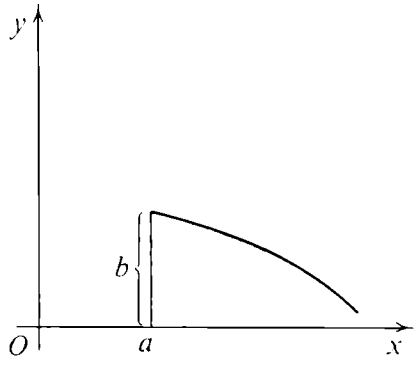

Fig. 3

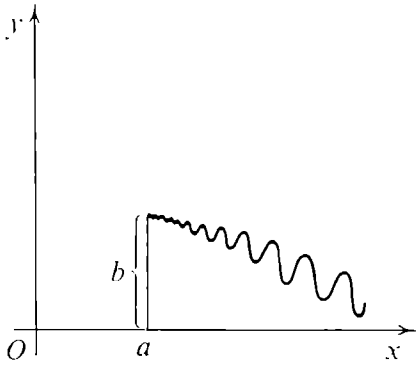

Fig. 4

Finally, it may happen that to the right of $x = a$ and arbitrarily close to it $y = f(x)$ assumes values greater than b, as well as values less than b (Fig. 5). In this case the approach of y toward the limit

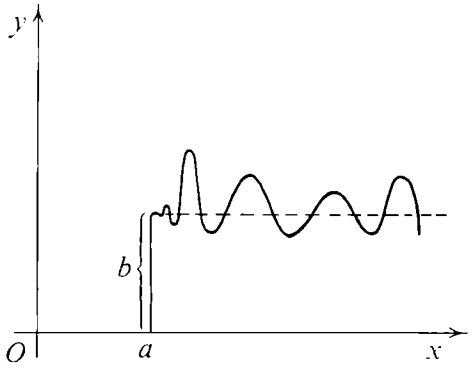

Fig. 5

b as $x \to a + 0$ is, of course, not monotonic. If $f(x)$ is continuous (which we tacitly assumed in all the figures), then it necessarily becomes equal to its limiting value b at an infinite number of points situated to the right of and arbitrarily close to a. In geometric terms, the graph of $y = f(x)$ crosses the line $y = b$ an infinite number of times in the vicinity of $x = a$.

These cases clearly exhaust all the possible ways in which the continuously varying quantity y may tend to b as $x \to a + 0$. We shall not consider separately the cases in which $x \to a - 0$ (i.e., where x approaches a from the left). We obtain the corresponding graphs from Figures 1–5 by reflecting them about the line $x = a$.

If, as we assumed, $y \to b$ as $x \to a$, we have $y \underset{x \to a+0}{\to} b$ as well as $y \underset{x \to a-0}{\to} b$. Moreover, each of the five above-mentioned types of behavior of y to the right of a can occur together with each of the five analogous types to the left of a in the approach of x. Thus, we obtain, in all, 25 different ways in which y may behave as x approaches a.

Let us also observe that sometimes it is convenient to express the cases represented in Figures 1 and 2 in the form

$$y \underset{x \to a+0}{\to} b + 0$$

and those in Figures 3 and 4 in the form

$$y \underset{x \to a+0}{\to} b - 0.$$

Up to now, we have assumed that a and b are numbers. You can enumerate for yourselves without difficulty all the possible types of behavior of $y = f(x)$ where one or both of these letters denote $+\infty$ or $-\infty$. For example, when $a = +\infty$ and b is a number, all possible cases are represented in Figures 1′–5′.

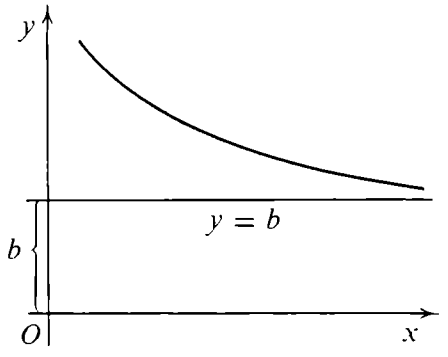

Fig. 1′

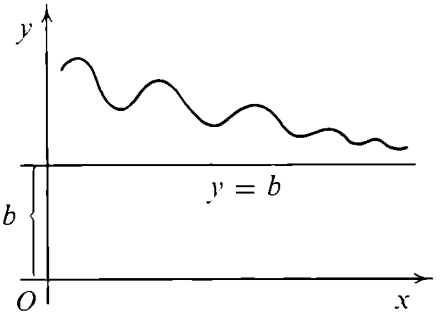

Fig. 2′

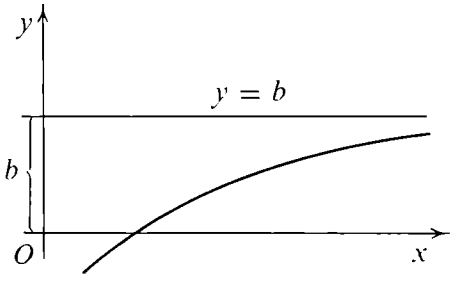

Fig. 3′

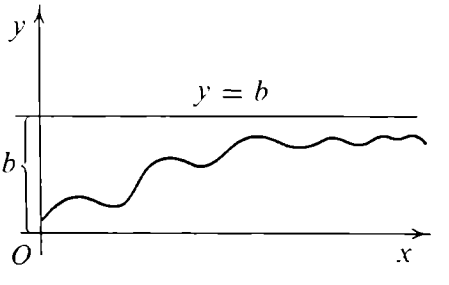

Fig. 4′

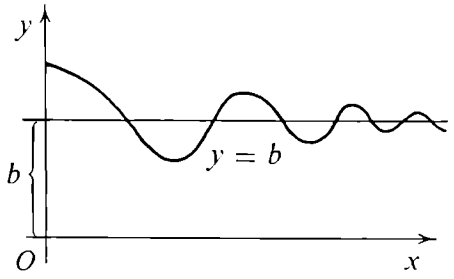

Fig. 5′

The function $f(x)$ is monotonic in Figures 1′ and 3′, but not monotonic in the remaining figures. In Figure 5′, if $f(x)$ is continuous then it becomes equal to its limit b an infinite number of times for arbitrarily large values of x.

When a is a number and $b = +\infty$, there are essentially only two cases as $x \to a + 0$. These are shown in Figures 6–7.

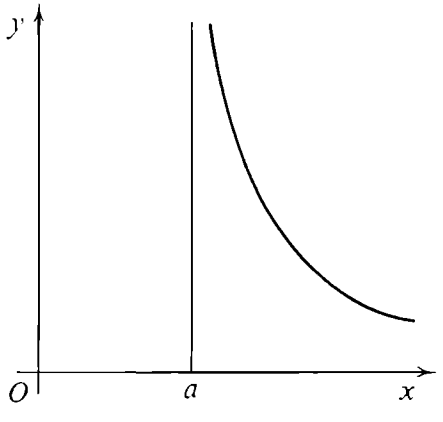

Fig. 6

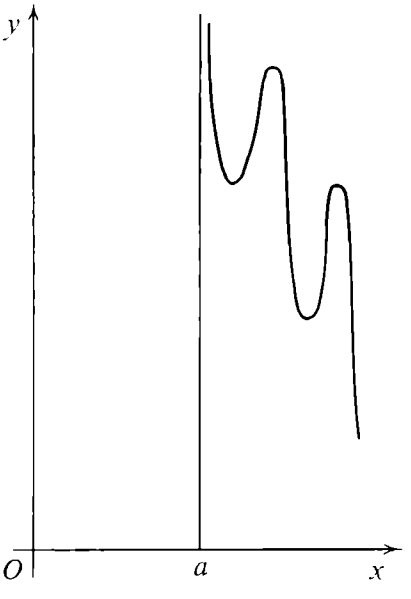

Fig. 7

8. THE LIMIT OF A CONSTANT FUNCTION

As you undoubtedly know, we always consider the limit of a constant (a function which assumes only one value) as existing and equal to its unique value. Such an agreement is sometimes a cause of confusion. The notion of limit arises, after all, in connection with a variable quantity. How then can we speak of the limit of a constant?

A little reflection, however, shows that nothing is amiss. In the first place, such an agreement follows logically from our definition of a limit. For, if $y = b$ for every value of x, then given any neighborhood V of b we have $y \in V$; from which, by the definition of a limit, we have

$$\lim_{x \to a} y = b. \tag{2}$$

Secondly, such an agreement is expedient, in fact, almost unavoidable. Suppose for a moment that we agreed to consider a constant function as not having a limit and suppose for some function $y = f(x)$ the relation (2) is valid; then, as we know, $\lim_{x \to a} (-y) = -b$. But $y + (-y) = 0$ for all values of x; the sum of the functions y and $-y$, each of which has a limit, turns out to be a constant and thus has no limit. Hence, in the formulation of the familiar theorem on the limit of the sum of two variables we would have to add the condition, *providing this sum is not a constant.* Similar artificial and awkward restrictions would also have to be added to almost every theorem on limits. To free ourselves from this, we attribute a limit to every constant function; and since, as we have just seen, we do not thereby violate the general definition of a limit, this convention is adopted in all expositions of mathematical analysis.

9. INFINITELY SMALL AND INFINITELY LARGE QUANTITIES

The quantity $y = f(x)$ is called *infinitely small,* or *infinitesimal,* as $x \to a$, if as $x \to a$ we have $y \to 0$. In many expositions of analysis the notion of the infinitesimal plays a fundamental role in the theory of limits. It is defined prior to the general concept of a limit, and in fact, the limit is often defined in terms of an infinitesimal. In contrast to this, a number of modern scholars, who consider the expression *infinitesimal quantity* as superfluous and liable to cause (indeed as actually causing) confusion and misunderstanding, advise us to dispense entirely with this term.

In this connection we should note that what is being referred to as superfluous is not the *concept* of a quantity tending to zero (this concept plays an essential role in any development of analysis and in all its applications) but only the *term* "infinitesimal quantity." This expression is indeed completely inappropriate and frequently leads to misconceptions. The term seems to be trying to describe the

magnitude of the quantity to which it is applied, and it is misleading to designate as *infinitely small* a quantity which in certain stages of its variation may not be small at all. In reality, of course, the term is intended to describe only the *character of variation* of the quantity and not its magnitude. This apparent misnomer stems from an era when one ascribed a completely different meaning to this concept.

As we have observed, however, quantities tending to zero are so frequently encountered in analysis and its applications that it would be very difficult to do without a special short name for them, while on the other hand, the replacement of a long-established name by some other term would be one of those terminological disasters whose disagreeable effects linger on long afterwards. Therefore, we shall retain the term on the condition that we do not take the notion of an infinitely small quantity as the basis of the theory of limits, but treat it only as a special case of a quantity tending to a limit. The danger of misunderstanding and confusion of concepts will then be greatly minimized.

A quantity $y = f(x)$ is said to be *infinitely large* (as $x \to a$) if

$$\lim_{x \to a} |y| = +\infty.$$

Thus, an infinitely large quantity tends either to $+\infty$ or to $-\infty$, or does not tend to any limit, taking alternately positive and negative values which increase indefinitely in absolute value. An example of such behavior is given by the function $y = n(-1)^n$ as $n \to +\infty$ through integral values. As far as the term *infinitely large* is concerned, we can repeat all that has been said about *infinitely small.*

We have no reason to dwell here on the familiar theorems on the sum, difference, and product of infinitely small quantities. Let us recall only two matters which receive too little attention and therefore frequently lead to misunderstanding:

(1) In the theorems on the *sum* and *product* of quantities involving infinitesimals, the condition that the number of terms (or factors) be finite is really essential; without this restriction both theorems are simply false, as it is easy to show by simple examples.

(2) We cannot formulate any general theorem about the *ratio* of two infinitely small quantities. Such a ratio can vary in an arbitrary manner and, in particular, may not tend to any limit at all.

Before we proceed further we need to make a few remarks concerning the symbols $+\infty$ and $-\infty$, which we have already used quite frequently. You know, of course, that these symbols do not denote a number. The best and clearest answer to the question of the meaning of the symbol $+\infty$ is that this symbol in itself has no meaning. Only the expression *a neighborhood of* $+\infty$ has a meaning, and by this expression we understand the set of all real numbers greater than some (arbitrary) number α. It is convenient to have a short name for such a set, as in analysis we meet with it at almost every step.

Simultaneously, with the definition of a *neighborhood of* $+\infty$, such expressions as $\lim\limits_{x \to a} y = +\infty$ (where a may be either $+\infty$, $-\infty$, or some real number) also acquire a meaning. Indeed, in our definition of the expression $\lim\limits_{x \to a} y = b$ the letters a and b entered only in connection with their neighborhoods. Therefore, each of these letters can reasonably be replaced by the symbol $+\infty$ (or $-\infty$) after we have established what sets are to be neighborhoods of that symbol, and there is no need to ascribe any meaning to the symbol itself.

It is clear from all this how much caution is necessary in using the symbols $+\infty$ and $-\infty$. In particular, it is completely inadmissible to perform any arithmetic operations with them $\left(\frac{1}{\infty} = 0, \text{etc.}\right)$ as is done in some *simplified* courses in analysis. Similarly, all kinds of equalities in which the symbol $+\infty$ or $-\infty$ appears other than explicitly in the role of a limit, as for example $\tan \frac{\pi}{2} = \pm\infty$, have no meaning.[1] On the other hand, we do assign a definite meaning to inequalities of the form $a < +\infty$, $b > -\infty$, and $-\infty < c < +\infty$. They mean, respectively, that a is either a number or the symbol $-\infty$, that b is either a number or the symbol $+\infty$, and that c is a number. Finally, we must decide, when speaking of the existence of the limit of a function $f(x)$, whether this limit must be a number or whether we also accept as a limit one of the symbols $+\infty$ and $-\infty$. It is clear, of course, that in this case we can decide either way and that, consequently, the choice should be made on the basis of expediency. It is usually agreed that, unless explicitly stated otherwise, the sentence $f(x)$ *has a limit* is to mean that this limit is a *number*. We shall hereafter observe this rule.

[1] A correct version would be: $\lim\limits_{x \to \frac{\pi}{2} - 0} \tan x = +\infty$ and $\lim\limits_{x \to \frac{\pi}{2} + 0} \tan x = -\infty$.

10. CAUCHY'S CONDITION FOR THE LIMIT OF A FUNCTION

One of the most important problems in the theory of limits is obviously the following: *Given a function* $y = f(x)$, *determine whether it has a limit as* $x \to a$. Note that in view of the agreement just made, we are concerned with the existence of a *number* b such that $y \to b$ as $x \to a$. Here a can signify either a number or one of the symbols $+\infty$ and $-\infty$, and we are considering only the existence of the limit b; the actual value of this limit is of no concern to us at this moment. The following important criterion for the existence of a limit turns out to be especially useful in theoretical investigations.

CAUCHY'S CONDITION.[1] *A necessary and sufficient condition for a function* $y = f(x)$ *to approach a limit as* $x \to a$ *is the following: For any* $\varepsilon > 0$ *there exists a neighborhood* U *of the number (or of the symbol)* a *such that for any two numbers* x_1 *and* x_2 *in* U *we have*

$$|f(x_1) - f(x_2)| < \varepsilon.$$

Proof. (i) Suppose that y tends to the number b as $x \to a$. By the definition of a limit there exists a neighborhood U of a such that for $x \in U$

$$b - \frac{\varepsilon}{2} < f(x) < b + \frac{\varepsilon}{2}.$$

Therefore, if $x_1 \in U$ and $x_2 \in U$, both $f(x_1)$ and $f(x_2)$ are included between $b - \frac{\varepsilon}{2}$ and $b + \frac{\varepsilon}{2}$ and, consequently, differ from each other by less than ε. This proves the necessity of Cauchy's condition.

(ii) Let us assume now that Cauchy's condition is satisfied. For every positive integer n there exists a neighborhood U_n of a such that for $x_1 \in U_n$ and $x_2 \in U_n$ we have

$$|f(x_1) - f(x_2)| < \frac{1}{n}. \tag{3}$$

Every such neighborhood U_n is either an open interval or a half-line; moreover, we have the right to assume that $U_{n+1} \subset U_n$ $(n = 1, 2, \ldots)$. If the interval (half-line) U_{n+1} were not part of the interval (half-line) U_n, then we could simply replace the interval (half-line) U_{n+1} by the intersection U'_{n+1} of U_n and U_{n+1}. (This

[1] Cauchy's condition is also known as Cauchy's criterion, Cauchy's test, Cauchy's convergence principle, and others.

intersection is, of course, not empty.) It is evident that U'_{n+1} is an interval (half-line). Moreover, $U'_{n+1} \subset U_n$ and for any $x_1 \subset U'_{n+1}$ and $x_2 \subset U'_{n+1}$ we have

$$|f(x_1) - f(x_2)| < \frac{1}{n+1}.$$

It follows that the neighborhood U'_{n+1} can replace the neighborhood U_{n+1} in all respects.

Since $f(x)$ satisfies the condition (3) in U_n, the whole set M_n of its values in this neighborhood is contained within some closed interval Δ_n of length $\frac{2}{n}$. But $U_{n+1} \subset U_n$; consequently, $M_{n+1} \subset M_n$ and, hence, $\Delta_{n+1} \subset \Delta_n$. And since the length $\frac{2}{n}$ of the interval Δ_n tends to zero as $n \to \infty$, the intervals Δ_n form a sequence of nested intervals. By Lemma 2 of Lecture 1 (page 19) we can assert that there exists a unique number b which belongs to all the intervals Δ_n.

Finally, we shall show that $\lim_{x \to a} y = b$. For this purpose we denote by V an arbitrary neighborhood of b. From the definition of this number it follows that $\Delta_n \subset V$ for all sufficiently large n; therefore, if $x \in U_n$, then $f(x) \in M_n \subset \Delta_n \subset V$, and so $\lim_{x \to a} y = b$. This completes the proof that Cauchy's condition is sufficient.

In particular, for a sequence of numbers

$$a_1, a_2, \ldots, a_n, \ldots$$

to have a limit it is necessary and sufficient that for any $\varepsilon > 0$ we have

$$|a_n - a_m| < \varepsilon,$$

for all sufficiently large m and n. Cauchy's condition is seldom used directly to prove the existence of a limit for a specific function. For this purpose we usually apply simpler criteria; however, these criteria are not *characteristic*, that is, necessary and sufficient at the same time. On the other hand, in general theoretical investigations it is just this property that makes Cauchy's condition an almost indispensable tool, as we shall see later on.

11. A REMARK ON THE FUNDAMENTAL THEOREMS ON LIMITS

You are, of course, well acquainted with the fundamental theorems of the theory of limits, such as the theorems on the limit of a sum, difference, and product; there is no need to prove them here or even to state them. However, we shall have to make a remark in connection with this group of theorems. Since this remark will apply in the same degree to all theorems of this group, it will be enough to illustrate it with reference to any one of them.

When we say *the limit of the sum of two quantities equals the sum of their limits* (and this is the way the theorem is stated in most *simplified* courses) we implicitly assume that all three quantities, the two terms and the sum, tend to limits and we are concerned only with the interrelation of these limits. However, we assume more than is necessary: it is enough to assume the existence of a limit for each of the terms, and it will follow that the sum must also have a limit (equal to the sum of the limits of the terms). This requires no additional reasoning, as it follows from any proof of the theorem on the limit of a sum.

On the other hand, from the fact that the sum has a limit it does not necessarily follow that each of the terms has a limit. For example, suppose that $y = f(x)$ does not tend to any limit as $x \to a$; it is evident that the same will be true for $1 - y$, while the sum of these two quantities (being a constant) has the limit 1 as $x \to a$.

Thus the theorem on the limit of a sum should read:

THEOREM. *If each of a finite set of quantities tends to a limit as $x \to a$, then the sum of these quantities also tends to a limit and this limit is equal to the sum of the limits of the individual terms.*

Analogous formulations may be made of all the other theorems of this group.

12. PARTIAL LIMITS; THE UPPER AND LOWER LIMITS

We shall now investigate in detail the behavior of a function of x in the neighborhood of $x = a$ when it fails to have a limit as $x \to a$. The so-called *simplified* courses usually leave this question untouched.

We shall call a number b a *partial limit*[1] of $y = f(x)$ as $x \to a$ if for any neighborhood U of a and any neighborhood V of b there exists $x \in U$ such that x is different from a and $f(x) \in V$. This simple definition can also be applied when b is one of the symbols $+\infty$ and $-\infty$. Less formally, a partial limit b as $x \to a$ is a number b such that arbitrarily close to a there are values of x for which y differs from b by as little as we please. (Of course, when a or b is $+\infty$, instead of speaking of numbers *as close to a (or b) as we please,* we speak of numbers *as large as we please;* and, analogously, when a or b is $-\infty$.)

We define analogously the partial limit of a sequence $a_1, a_2, \ldots, a_n, \ldots$ as $n \to \infty$.

If $\lim_{x \to a} y$ exists, then by a direct application of the definition we see that it is also a partial limit of y (and, in fact, the only one). This is true regardless of whether this limit is a number or one of the symbols $+\infty$ and $-\infty$. But if there is no limit, then y has at least two partial limits, and a necessary and sufficient condition for the existence of one and only one partial limit is the existence of $\lim_{x \to a} y$. To establish these assertions we shall prove the following:

PROPOSITION 1. *The function $y = f(x)$ has at least one partial limit as $x \to a$.*

PROPOSITION 2. *If b is the only partial limit of the function $y = f(x)$ as $x \to a$, then $\lim_{x \to a} y = b$.*

In all cases the limits and partial limits can be either numbers or the symbols $+\infty$ and $-\infty$.

Proof of Proposition 1. If one of the symbols $+\infty$ or $-\infty$ is a partial limit of $f(x)$ as $x \to a$, then the theorem is established. Therefore, we may assume that this is not the case and we shall then show that there exists a number b which is a partial limit of $f(x)$ as $x \to a$.

Since neither $+\infty$ nor $-\infty$ is a partial limit of $f(x)$ as $x \to a$, there exists a pair of numbers α and β ($\alpha < \beta$) such that for all x sufficiently close to a,

$$\alpha \leq f(x) \leq \beta.$$

[1] Since the existence of a partial limit b of a sequence implies the existence of a subsequence having b as its limit, such partial limits are often referred to as *subsequential* limits.

Thus, in any neighborhood of a there exists an x such that $f(x)$ belongs to the interval $[\alpha, \beta]$, denoted by Δ_1. Let us agree to denote by (A) this property of Δ_1. If we divide this interval into halves, then at least one of these halves must have property (A). For if some neighborhood U_1 of a contains no x for which $f(x)$ belongs to the right half of Δ_1, and some neighborhood U_2 contains no x such that $f(x)$ belongs to the left half of this interval, then the neighborhood U consisting of the intersection of U_1 and U_2 cannot, of course, contain any x for which $f(x) \in \Delta_1$. In other words the interval Δ_1 does not possess property (A). Therefore, we can choose a half of the interval Δ_1 which has property (A), denote it by Δ_2, and deal with it in the same way we have dealt with Δ_1; that is, we shall divide it into halves and denote by Δ_3 the one which has property (A).

Continuing this process indefinitely, we clearly obtain a sequence of nested intervals $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n, \ldots$. Let b be the unique number common to all these intervals and let U and V denote arbitrary neighborhoods of a and b respectively. By the definition of b there exists an interval $\Delta_n \subset V$; but every interval Δ_n has property (A) and, consequently, there exists an $x \in U$ such that $f(x) \in \Delta_n \subset V$. But this means that b is a partial limit of y as $x \to a$.

Proof of Proposition 2. By virtue of Proposition 1, the quantity y has at least one partial limit b as $x \to a$; if the relation $\lim_{x \to a} y = b$ is not true, then there exists a neighborhood V of the number (or symbol) b having the property that in any neighborhood U of a there exists an x such that $f(x)$ is outside of V. Hence, there must clearly occur one of the following two situations: either one of the symbols $+\infty$ and $-\infty$ (other than b) is a partial limit of y as $x \to a$, or there exists an interval $[\alpha, \beta]$ *situated outside* V such that any neighborhood U of a contains an x for which $f(x) \in [\alpha, \beta]$. In the first case, Proposition 2 is established. In the second case, we can show by precisely the same method as in the proof of Proposition 1 that, as $x \to a$, the quantity y has a partial limit b' contained in $[\alpha, \beta]$ and, consequently, different from b. Thus Proposition 2 is proved in this case also.

We have established, then, that every quantity which does not have a limit as $x \to a$ (where a may be $+\infty$ or $-\infty$) must have at least two partial limits. In general, we can assert nothing more.

Thus, the quantity $y = (-1)^n$ (where n is a positive integer) obviously has exactly two partial limits, $+1$ and -1, as $n \to \infty$. On the other hand, there are cases where there exist an infinite number of partial limits. For example, the function $y = \sin \frac{1}{x}$ as $x \to 0$ has for partial limits all numbers in the interval $[-1, +1]$. For as $x \to 0$, the quantity $\sin \frac{1}{x}$ changes continuously an infinite number of times from -1 to $+1$ and back again; hence, for any number β of the interval $[-1, +1]$, we can find an arbitrarily small number α (that is, α belonging to an arbitrarily small neighborhood of zero) for which $\sin \frac{1}{\alpha} = \beta$.

In spite of the fact that the sets of partial limits for different functions may differ so widely, they nevertheless have certain features in common which are of essential importance in analysis. We shall now consider some of them.

PROPERTY I. *If a number (or symbol) b is not a partial limit of $y = f(x)$ as $x \to a$, then there exists a neighborhood V of b which does not contain any such partial limit.*

Proof. If b is not a partial limit of y as $x \to a$, then there exists a neighborhood V of b and a neighborhood U of a such that y cannot belong to V for $x \in U$. But since the open interval V is a neighborhood of each of its points, it follows that no point of V can be a partial limit of y as $x \to a$, which proves Property I.

Before going on to Property II, let us observe the following: If among the partial limits of $y = f(x)$ as $x \to a$ there occurs the symbol $+\infty$, we naturally regard this partial limit as the greatest; similarly, if the symbol $-\infty$ is a partial limit, we consider it to be the least. Having agreed on this we can now formulate Property II.

PROPERTY II. *Among the partial limits of $y = f(x)$ as $x \to a$ there always exists a greatest and a least.*

You understand, of course, that this statement is far from obvious; there is, for example, neither a greatest nor a least number in the open interval (0, 1) and, in general, not every set of numbers has this property. Thus (as is already evident from Property I), not every set can qualify as a set of partial limits; it must have certain specific traits, and in particular must contain a greatest and a least number (or symbol).

Proof. First of all, we may restrict ourselves to the *greatest* partial limit, as the proof for the least partial limit is completely analogous. Further, we may assume that the symbol $+\infty$ is not a partial limit (otherwise it would be the greatest partial limit and Property II would be established). By virtue of Property I, it follows that there are also no partial limits in a neighborhood of the symbol $+\infty$: in other words, all partial limits are smaller than a certain number. If the symbol $-\infty$ is the only partial limit, then it is also the greatest partial limit and Property II is again established. We may therefore assume that among the partial limits there exists some number b.

Let us now divide the set of all real numbers into two classes A and B according to the following rule: if to the right of x there exists at least one partial limit, then $x \in A$; otherwise $x \in B$. It is easy to see that this division is a cut. Let α be the edge of this cut; we shall show that α is the greatest partial limit. First, α is a partial limit, as otherwise, by virtue of Property I, some neighborhood (α_1, α_2) of α $(\alpha_1 < \alpha < \alpha_2)$ would contain no partial limit. But from the inequality $\alpha_1 < \alpha$, we have $\alpha_1 \in A$, so that there would be a partial limit to the right of α_1; since it is not in the interval $[\alpha_1, \alpha_2]$, it must lie to the right of α_2, which is impossible since the inequality $\alpha_2 > \alpha$ implies that $\alpha_2 \in B$. Second, there can be no partial limits to the right of α. For if $\beta > \alpha$ were a partial limit, then for any γ situated between α and β, $\alpha < \gamma < \beta$, we would have $\gamma \in B$ from $\gamma > \alpha$ and, simultaneously, $\gamma \in A$ from $\beta > \gamma$. Thus α is the greatest partial limit and Property II is established.

The greatest and least partial limits, whose existence is asserted in Property II, are of great importance: we call them the *upper* and *lower limits* of $y = f(x)$ as $x \to a$ and denote them by

$$\overline{\lim_{x \to a}} f(x) \quad \text{and} \quad \underline{\lim_{x \to a}} f(x),$$

or by

$$\limsup_{x \to a} f(x) \quad \text{and} \quad \liminf_{x \to a} f(x).$$

Obviously,

$$\limsup_{x \to a} f(x) \geq \liminf_{x \to a} f(x)$$

in all cases. These two numbers can assume any value, and each of them may turn out to be $+\infty$ or $-\infty$. We say that $f(x)$ is *bounded as* $x \to a$ if its upper and lower limits as $x \to a$ are numbers, and *unbounded* if at least one of them is the symbol $+\infty$ or $-\infty$. Evi-

dently the boundedness of $f(x)$ as $x \to a$ means the existence of numbers α and β ($\alpha < \beta$) and a neighborhood U of a such that for any $x \in U$ we have

$$\alpha < f(x) < \beta.$$

A necessary and sufficient condition for the existence of $\lim\limits_{x \to a} y$ (in the sense of a number *or* a symbol) is the relation

$$\overline{\lim_{x \to a}}\, y = \underline{\lim_{x \to a}}\, y,$$

since coincidence of the upper and lower limits is equivalent to the existence of only one partial limit.

Numbers situated between $\underline{\lim\limits_{x \to a}}\, y$ and $\overline{\lim\limits_{x \to a}}\, y$ may or may not be partial limits as $x \to a$; we have seen above examples of two extreme cases where (1) none of these numbers was a partial limit and where (2) each of them was a partial limit. In the general case some of them will be partial limits, others not.

Sometimes upper and lower limits are defined differently, and it is useful to know these different definitions in order to broaden and make more concrete our understanding of these concepts, as well as to facilitate their application.

DEFINITION. *A number b is called the upper limit of $y = f(x)$ as $x \to a$ if for every interval $[\alpha, \beta]$ containing b in its interior there exists an arbitrarily small neighborhood U of a such that $f(x) < \beta$ for all $x \in U$ and $\alpha < f(x_0) < \beta$ for at least one number x_0 ($x_0 \neq a$) contained in U.*

To prove that the new definition is equivalent to the previous ones, we observe first of all that if a number b is the upper limit according to the new definition, then b is a partial limit. If there were a partial limit $b' > b$, we could find numbers α, β, and α' such that

$$\alpha < b < \beta < \alpha' < b'.$$

Then, since b' is a partial limit, any neighborhood U of a will contain a number x for which $\alpha' < y$, whence $y > \beta$. But this obviously contradicts the fact that b is the upper limit according to the new definition.

Conversely, suppose now that b is the greatest partial limit of $y = f(x)$ as $x \to a$ according to the previous definition; if $\alpha < b < \beta$, then any neighborhood U of a contains a number x for

which $\alpha < y < \beta$. To prove that b is the upper limit of y in the sense of the new definition we therefore have only to show that with a proper choice of U we shall have $y < \beta$ for every $x \in U$.

Since the number b is the *greatest* partial limit, it follows that $+\infty$ cannot be a partial limit. Therefore, there exists a number y_0 and a neighborhood U_0 of a such that $y < y_0$ for all $x \in U_0$. If $y_0 \leq \beta$, there is nothing further to prove. If $y_0 > \beta$, then every number λ in the closed interval $[\beta, y_0]$, not being a partial limit, has a neighborhood V_λ to which there corresponds a neighborhood U_λ of a such that for any $x \in U_\lambda$, $y = f(x)$ is outside V_λ. The family $\{V_\lambda\}$ covers the interval $[\beta, y_0]$. Applying the Heine-Borel lemma (Lemma 3 of Lecture 1), we can find a finite subfamily $V_{\lambda_1}, V_{\lambda_2}, \ldots, V_{\lambda_n}$ covering $[\beta, y]$. Let U denote the intersection of the neighborhoods $U_0, U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_n}$; it is clear that U is a neighborhood of a and that if $x \in U$, the corresponding value of y cannot belong to any of the intervals $V_{\lambda_1}, V_{\lambda_2}, \ldots, V_{\lambda_n}$. In other words, $y < \beta$ for every $x \in U$, which proves our assertion.

It goes without saying that the lower limit can also be redefined in the same way and that the equivalence of the old and new definitions can be proved in a completely analogous manner. Similar definitions can be formulated for the case when b is $+\infty$ or $-\infty$.

If we wish to view this whole abstract scheme in a more concrete way, a picture presents itself which we may attempt to describe without any pretense to formal precision. A variable quantity under some process of change approaches now one number, now another, now a third, and so on. We call the number b a *partial limit* of the given quantity if this quantity will after any point of time, however late, still approach arbitrarily close to the number b. It is quite possible that in the interval between two such approaches it departs extremely far from b: what is important is only that sooner or later it again approaches arbitrarily close to b. If b is the *only* such point of gravitation, then our quantity y not only becomes but finally *stays* arbitrarily close to b and we have $\lim_{x \to a} y = b$.

Generally speaking, y does not tend to a limit, but rather, like an undamped pendulum it continues to oscillate within certain limits, no matter how long we watch the process of its variation. These limits (among which may be $+\infty$ and $-\infty$) are precisely what we have called the upper and lower limits of the given quantity.

13. LIMITS OF FUNCTIONS OF SEVERAL VARIABLES

There yet remain to be considered a few uncomplicated problems in the theory of limits connected with functions of several variables. To simplify our exposition we shall speak only of functions of *two* independent variables; all that is said in this section can also be applied, with appropriate and self-evident changes, to functions of any number of variables.

Every pair of values of the independent variables x and y is represented by a point in the coordinate plane. In what follows, for brevity, we shall mean by *the point* (a, b) a pair of values for these variables, $x = a$ and $y = b$. By a neighborhood of the point (a, b) we shall understand any open set[1] in the coordinate plane which contains this point in its interior; this set may be the interior of a circle, the interior of a rectangle, or may have a more complicated form. The term *a neighborhood of the point* (a, b) can also be given a meaning when a or b (or both) denotes $+\infty$ or $-\infty$. For example, if a is a number and b is $+\infty$, then we shall call every domain of the form $\alpha < x < \beta, y > \gamma$, where $\alpha < a < \beta$ and the number γ is arbitrary (Fig. 8), a neighborhood of the point (a, b). For

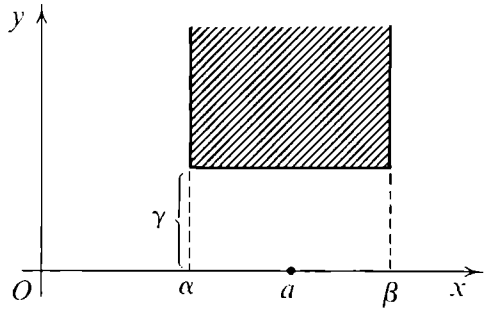

Fig. 8

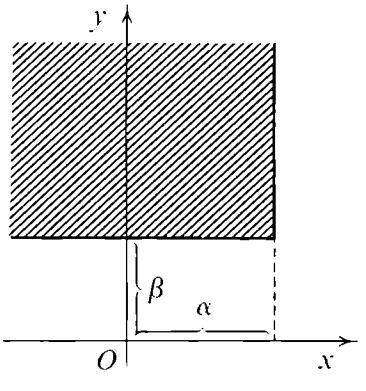

Fig. 9

$a = -\infty$ and $b = +\infty$, a neighborhood of (a, b) is any domain of the form $x < \alpha$, $y > \beta$, where α and β are arbitrary numbers (Fig. 9).

[1] An open set in the plane is a set with the following property: if a point belongs to this set, then there exists a rectangle with its center at this point which is wholly contained in this set.

We say that the number (or symbol) c is the limit of the function $z = f(x, y)$ as $x \to a$, $y \to b$, if for every neighborhood V of c, there exists a neighborhood U of (a, b) such that at every point (x, y) contained in U and different from (a, b) we have $z \in V$. We write this as follows:

$$\lim_{\substack{x \to a \\ y \to b}} z = c \quad \text{or} \quad z \underset{\substack{x \to a \\ y \to b}}{\longrightarrow} c. \tag{4}$$

With this definition the whole theory of limits for functions of one independent variable can easily be extended to the two-dimensional case; in particular, the concepts of upper and lower limit retain their definitions and their properties. Cauchy's condition likewise remains valid.

As for proof of the latter, it is entirely analogous to the one-dimensional case; the only difference is that the lemmas used must now be applied in their two-dimensional forms. We need, for example, the lemma on nested two-dimensional intervals and the two-dimensional form of the Heine-Borel lemma. We can prove these in the same way as for the corresponding one-dimensional case, and there is no need to dwell on them here.

It is necessary to differentiate strictly between the *double limit* (4) and the *iterated limits*

$$\lim_{y \to b} \left(\lim_{x \to a} z\right) \quad \text{and} \quad \lim_{x \to a} \left(\lim_{y \to b} z\right). \tag{5}$$

Here, instead of one two-dimensional passage to the limit, we have two successive one-dimensional passages to a limit. The geometric picture connected with neighborhoods of the point (a, b) in the coordinate plane now disappears completely. Thus to obtain $\lim_{y \to b} \left(\lim_{x \to a} z\right)$ we first have to assign to the variable y any constant value and, having transformed in this way the quantity z into a function of one variable x, look for its limit as $x \to a$. This limit can exist for some values of y and not exist for others. If it exists for all values of y situated in a neighborhood V of b (with the possible exception of $y = b$), then in this neighborhood $\lim_{x \to a} f(x, y)$ is a function of one variable y, and, in turn, we can inquire about its limit as $y \to b$. If this limit exists, we denote it by $\lim_{y \to b} \left(\lim_{x \to a} z\right)$.

It may happen that both iterated limits (5) exist while the double limit (4) does not exist. Consider, for example, the behavior of the function

$$z = \frac{2xy}{x^2 + y^2}$$

in a neighborhood of the origin. (The fact that the function is not defined at the origin makes no difference, as in the definition of the double limit as well as of the iterated limits the quantity $f(a, b)$ plays no part; if desired, we may assign $f(0, 0)$ an arbitrary value.) For x constant and unequal to zero we have $\lim_{y \to 0} z = 0$, and for y constant and unequal to zero we have $\lim_{x \to 0} z = 0$; hence

$$\lim_{x \to 0} (\lim_{y \to 0} z) = \lim_{y \to 0} (\lim_{x \to 0} z) = 0.$$

On the other hand, an arbitrary neighborhood of the point (0, 0) contains points for which $x = 0$ and $y \neq 0$, as well as points for which $x = y \neq 0$. Since at points of the first kind $z = 0$ and at points of the second kind $z = 1$, both the numbers 0 and 1 are partial limits of the function z as $x \to 0, y \to 0$; and, consequently, $\lim_{\substack{x \to 0 \\ y \to 0}} z$ does not exist.

We can also have the opposite case where the double limit (4) exists while neither of the iterated limits exists. Let us consider, for example, the function

$$z = \begin{cases} (x^2 + y^2) \sin \dfrac{1}{xy} & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0, \end{cases}$$

in the vicinity of the origin. Since $|z| < \varepsilon^2$ for all points in the disc $x^2 + y^2 < \varepsilon^2$, it follows that $\lim_{\substack{x \to 0 \\ y \to 0}} z = 0$. On the other hand, for a constant $x \neq 0$ and for $y \to 0$ (as also for a constant $y \neq 0$ and $x \to 0$) $\sin \dfrac{1}{xy}$, and hence the function z, do not tend to a limit. Thus the limits $\lim_{x \to 0} z\,(y \neq 0)$ and $\lim_{y \to 0} z\,(x \neq 0)$ do not exist, and we cannot even begin to speak of the existence of the iterated limits (5).

However, it is possible to show that if all three limits (4) and (5) exist, they must be equal. For, suppose that

$$\lim_{x \to a} (\lim_{y \to b} z) = c. \tag{6}$$

Let V be any neighborhood of c, and U any neighborhood of (a, b). Then by virtue of (6), there exists a neighborhood A of a such that for all $x \in A$ $(x \neq a)$, we have

$$\lim_{y \to b} z \in V;$$

this neighborhood A is an interval (or half-line) on the x-axis. Obviously, we can select in this interval a number x_0 situated so close to a that the point (x_0, b) is contained in U (Fig. 10). Fixing this

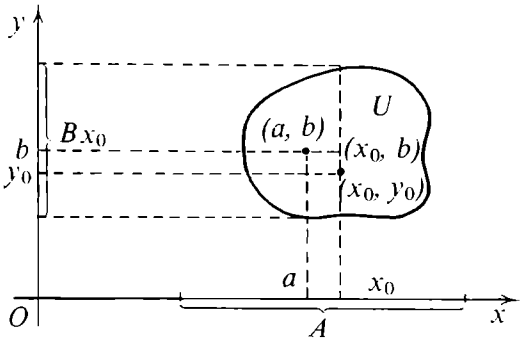

Fig. 10

value x_0 and taking into account that $x_0 \in A$ we have:

$$c_{x_0} = \lim_{\substack{y \to b \\ x = x_0}} z \in V.$$

But by the definition of a limit, this means that for an arbitrary neighborhood V' of c_{x_0} it is possible to find a neighborhood B_{x_0} of b such that $z \in V'$ for $x = x_0$ and any $y \in B_{x_0}$. But nothing prevents us from selecting the neighborhood V' so small that $V' \subset V$ (since $c_{x_0} \in V$); and, on the other hand, since the point $(x_0, b) \in U$, we can choose a number $y_0 \in B_{x_0}$ so close to b that the point (x_0, y_0) will also be in U (Fig. 10).

Since $y_0 \in B_{x_0}$, we have $z \in V' \subset V$ at the point (x_0, y_0). Let us recall now that U is *any* neighborhood of the point (a, b) and that V is *any* neighborhood of c. Thus, the fact that we can always find a point $(x_0, y_0) \in U$ at which $z \in V$ means that c is a partial limit of z as $x \to a$, $y \to b$. If, as we assumed, the limit $\lim\limits_{\substack{x \to a \\ y \to b}} z$ exists, then it is the only partial limit and, therefore, it has to be equal to c.

3. Functions

14. WHAT IS A FUNCTION?

The definition of *functional dependence*, which we discussed to a certain extent at the very beginning of Lecture 1, originated and finally won acceptance amid strong opposition. Even today the echoes of this conflict have not yet completely died down, although it seems that the scientific value of this definition is no longer disputed.[1]

The basic aim of this struggle was to overcome the predominance of the analytical apparatus which, since the eighteenth century, has weighed heavily upon the idea of functional dependence. This predominance, which transformed the analytic expression from a convenient instrument into a despotic ruler over the idea of a function, has been more or less completely eliminated within mathematics itself. But in the applied sciences and in the schools (even in higher technical education) it still prevails to a considerable degree. Almost every engineer, while accepting the formal scientific definition of functional dependence (in which not a word is said about an analytic expression), still visualizes a function primarily as a formula, as an analytic expression, and, in general, cannot think about functional dependence except in these terms. Thus he differs in an essential way from the mathematician, who is accustomed to taking the definitions of his concepts seriously and, therefore, at the word *function* thinks always of a correspondence between two sets without associating it with any analytical apparatus. In view of this difference, we shall have to dwell at some length on the idea of functional dependence and other related notions.

It will perhaps be best, instead of entering into a detailed and systematic analysis (for which we have not enough time anyway), simply to touch on a few of the more typical sharp conflicts which arise in this connection between the notion of functional dependence entertained by a mathematician and that entertained by a man

[1] In the United States of America a number of writers prefer a set-theoretic definition of *function*. Various points of view are discussed further in this lecture.

educated in the older traditions. University professors experience conflicts of this type every year with their freshmen who bring from their secondary schools educational habits and traditions of past centuries.

Let us consider the analytic expression of the function represented graphically in Figure 11. One glance at this simple graph

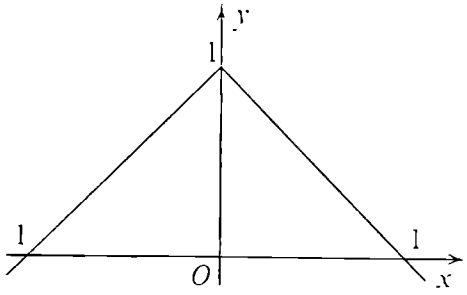

Fig. 11

will convince us that the idea of a real quantity y which varies in accordance with the law indicated by the graph does not contain in itself anything unacceptable, but an engineer or physicist will feel even more confident of this statement than a mathematician. When set to the task of representing the given function analytically, a mathematician immediately writes down the only solution acceptable from his point of view:

$$y = \begin{cases} 1 - x & \text{if } x \geq 0, \\ 1 + x & \text{if } x \leq 0, \end{cases} \tag{1}$$

and right here the conflict begins. An engineer (and let us, for brevity, call by this name a representative of the obsolete traditions—our engineer friends in the audience please forgive us) immediately objects, declaring to the mathematician that he has "written not one but two functions." To this the mathematician replies:

(*a*) It is impossible to *write a function;* one can only write an analytic expression.

(*b*) He, the mathematician, actually wrote two analytic expressions, indicating at the same time precisely for what values of the independent variable x one should apply the one or the other of these expressions in order to compute the corresponding values of the function y.

(*c*) In accordance with the definition adopted for functional dependence, the two analytic expressions in (1) determine exactly *one* function, since they assign to each value of x only one value of y.

(*d*) The expression (1) exactly represents the dependence given graphically in Figure 11, and, therefore, completely solves the stated problem.

The mathematician might also add (but does not do so for pedagogical reasons) that in case of insistent demand he could also represent the function shown in Figure 11 by a single formula

$$y = 1 - |x| \tag{2}$$

valid for all x. He simply considers the expression (1) to be more convenient than the expression (2) and, at the same time, equally legitimate.

Recalling the definition of functional dependence, the mathematician may go even further and insist, for example, that the expression

$$y = \begin{cases} x & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 1 + x & \text{if } x > 0, \end{cases}$$

legitimately represents one single function (Fig. 12). We do not

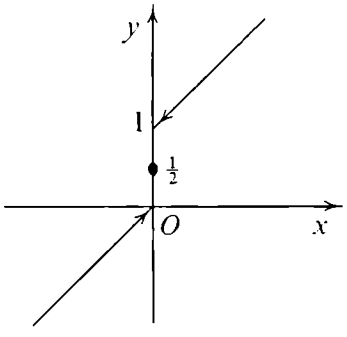

Fig. 12

know what objections the engineer might raise to this, but on the basis of personal experience we are inclined to make the following prognosis: even if he does not express any objections, he will still remain unsatisfied. Habits acquired over the years cannot be eradicated by a single short discussion.

On another occasion the mathematician might define the so-called *Dirichlet function:*

$$y = \begin{cases} 1 & \text{when } x \text{ is a rational number,} \\ 0 & \text{when } x \text{ is an irrational number.} \end{cases}$$

The perplexed engineer will ask, "But what sort of function is this? It cannot be written by means of a formula, nor represented by a graph." To this the mathematician will reply: "In the definition of functional dependence nothing is said either of an analytical expression or of the geometric representation of the function, and, therefore, the question of whether this is a genuine function in no way depends on whether the function can be given analytically or represented geometrically. The definition of the Dirichlet function just formulated attributes to every value of the quantity x a unique value of the quantity y and, therefore, is irreproachable. Moreover, although the geometric representation of the Dirichlet function is difficult, its expression by means of a formula is very simple; it is enough to denote it by $f(x)$."

The engineer, who was listening quietly at first, feels sincere indignation at this last remark, which results in the following dialogue:

E. But is this a formula?

M. What do you call a formula?

E. Well, some analytic expression such as $y = x^2 - x^3$ or $y = \sin x$; but what sort of expression is this $y = f(x)$?

M. Very well. You mean that the notation $y = \sin x$ for the familiar function called sine is, in your view, an analytic expression, while the notation $y = f(x)$ for the Dirichlet function is not an analytic expression? What, then, is the essential difference between the symbols "sin" and "$f(\ \ )$"?

E. But every literate person knows what the formula $y = \sin x$ means, while the notation $y = f(x)$ for what you call the Dirichlet function has just been devised by you and is unknown to anyone else.

M. Now it seems that we understand each other. The distinction which you indicate (and which I do not deny) is, as you yourself realize, not fundamental but historical. For there must have been a moment when someone for the first time proposed to denote by $\sin x$ a function for which no generally accepted notation had existed until then. That moment was for the function $y = \sin x$ the same as this moment is for the Dirichlet function $y = f(x)$. Would you say that the function $y = \sin x$ was not analytically expressible until the suggested notation took root and was generally accepted? And should the notation $y = f(x)$, which I have

just proposed for the Dirichlet function, become accepted by the whole scientific world in a few years, would you then say that it became a formula, an analytic expression, and that the Dirichlet function itself became analytically expressible? It is clear, of course, that an interpretation of analytic expressibility that is based on scientific fashion has not the slightest mathematical value. But if we reject this interpretation, you cannot but admit that the symbol "$f(\ )$" is fundamentally as legitimate as the symbol "sin"; and therefore the Dirichlet function is analytically expressible to the same degree and in the same sense as the sine and cosine functions. Generally speaking, a discussion of the analytic expressibility of a function must in all cases be recognized as pointless, since we can designate the function by any symbol and rightfully consider this designation as its analytic expression. Finally, I can tell you that we know how to express the Dirichlet function by means of symbols familiar to you. But we almost never use this expression because it is complicated and does not offer us the possibility of learning anything essential about the properties of this function, whereas the definition which we gave above shows these properties very clearly. In general, we do not like making fetishes out of analytic expressions; we gladly use them in those cases where they are helpful in studying a given functional dependence, and reject them without regret if the investigation is simpler without them, which is true, for example, in the case we have just considered.

At this point, we must terminate the discussion, since we fail to see what further objections could be raised by our engineer. On the other hand, our hypothetical mathematician has so clearly characterized the attitude of contemporary mathematics to the relationship between functions and analytic expressions that the conclusions will be evident without further comment.[1]

[1] Although the author goes to great length to rule out *analytic expressibility* as a meaningful qualifying condition for a function, the reader should be warned that in higher mathematics the term *analytic* alone, when applied to a function, has a definite and precise meaning.

15. THE DOMAIN OF A FUNCTION

We must now concern ourselves with one more question closely connected with the foregoing. For a function $y = f(x)$ to be considered as given, it is by no means necessary to define it for all values of the variable x. It often happens that only a portion of these values is of any interest, so that to define the function $f(x)$ outside this portion appears pointless. The reasons for such a restriction of the set of values of x can be most varied, purely logical as well as practical; we have already discussed this, in part, in Lecture 1. If, for example, we define $f(x)$ as the perimeter of a regular polygon of x sides inscribed in a circle of radius 1, it is self-evident that the function is determined by this condition for all integral $x \geq 3$, and *only* for these values; we say that $y = f(x)$ is in this case a function of x defined on the set of all integers $x \geq 3$. Formally, of course, nothing prevents us from completing the function $f(x)$ by assigning to it arbitrary values for the remaining values of x; but if there is no need for this we do not do it, leaving the function $f(x)$ completely undefined outside this set. Another example: if in a given physical investigation x designates the temperature of an object expressed in degrees centigrade, it would be absurd to define a function $f(x)$ for values less than -273.

It would, therefore, serve no purpose to require that every function $y = f(x)$ be defined for *all* values of x. To insist on this would force us in practice to extend most functions in an artificial and completely useless manner beyond their domains of original definition. This forces us to introduce a necessary clarification into the definition of functional dependence.

Let us agree that $y = f(x)$ *is a function on the set* M *if to each value* $x \in M$ *there corresponds a unique value of* y. This definition, which is somewhat more precise than the one we gave at the beginning of Lecture 1, immediately resolves all doubts and allows us in studying any individual function to restrict ourselves to that set of values of x which is dictated by the aim of the given investigation (this set being the natural *domain* of the function).

The choice of the domain for a function can be dictated, as we already know, by purely mathematical as well as by other considerations. It is indispensable to caution here against one regrettable confusion which results from the persistence of the tradition of

identifying a function with an analytic expression. Frequently, people want to make the domain of definition of a function depend on the set in which this or that analytic expression has a meaning. They say, for example, that *the domain of the function* $\sqrt{1-x^2}$ *is the interval* $[-1, +1]$ or that *the domain of the function* $\log x$ *is the half-line* $x > 0$, when, in fact, they are talking not about the domain of definition of a function but about the set in which a given analytic expression makes sense. For example, it can easily happen (and a number of these examples are known in mathematics) that we have to deal with a function $y = f(x)$ defined on the interval [0, 2] and of real interest to us in this whole interval, and which for $0 \leq x \leq 1$ can be expressed by the formula $y = \sqrt{1-x^2}$. It does not follow from this, however, that the interval (1, 2] lies outside the domain of our function and is of no interest to us. On the contrary, we shall look for another analytic expression of this function (based, of course, on its definition) for the interval (1, 2]; and if we fail to find one, we shall investigate the function in this interval by other, nonanalytic, methods.

We choose the domain of a function on the basis of considerations either purely mathematical or dictated by its applications, but in every case these considerations are based on the essence of the matter and should never be tied to the purely formal characteristics of any particular analytical apparatus.

16. CONTINUITY OF A FUNCTION

In beginning the investigation of functional dependence, we must first of all introduce, with the help of an appropriate system of classification, a certain amount of order into the diversity of our subject matter. The first such classifying and organizing principle is usually (and justly) the separation of all functions into *continuous* and *discontinuous* functions. Actually, mathematical analysis deals almost exclusively with continuous functions, taking into consideration only in relatively rare instances a few of the simpler types of discontinuous functions. Continuous functions have a number of special properties which discontinuous functions in general do not have. As a consequence of these properties, the study and application of continuous functions is considerably simplified, and thus the investigation of these properties is extremely important for analysis.

We say that *a function* $y = f(x)$ *is continuous at* $x = a$ *(or continuous at the point* a*) if*

$$\lim_{x \to a} f(x) = f(a),$$

or, equivalently, by the definition of a limit, *if for an arbitrary neighborhood* V *of* $f(a)$ *there exists a neighborhood* U *of* a *such that for any* $x \in U$ *we have* $f(x) \in V$. Thus, for the continuity of a function $f(x)$ at the point a it is necessary first that the limit

$$\lim_{x \to a} f(x)$$

exist, and second that this limit coincide with the value of the function at $x = a$. It is clear that the second condition does not follow from the first, as the example of the function

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \tag{3}$$

shows.

With regard to this definition, we should note first of all that continuity understood in this way is a *local property* of the function, that is, a property which may hold at one point and not at another. For example, the function (3) is discontinuous at $x = 0$ and continuous at every other value of x. This distinction between local and global (nonlocal) properties of a function is very important and should always be kept in mind.

Further, we call a function *continuous in a closed interval* $[a, b]$ if it is continuous, in the sense stated above, at every point of this interval. However, at the end point a we require continuity only from the right, that is,

$$\lim_{x \to a+0} f(x) = f(a),$$

and at the end point b continuity only from the left, defined analogously. In the case of an *open* interval (a, b) nothing, of course, is required of the function at the points a and b.

It may be noted, incidentally, that mathematicians have long used the very convenient notation

$$f(c + 0) = \lim_{x \to c+0} f(x)$$

and

$$f(c - 0) = \lim_{x \to c-0} f(x),$$

by means of which it is possible to write the definition of continuity of a function $f(x)$ at the point c in the form of a very simple relation:

$$f(c - 0) = f(c) = f(c + 0).$$

This notation cannot lead to any misunderstanding if we simply remember that $f(c + 0)$ and $f(c - 0)$ are not values of the function at any point, but limits of these values as x varies in a certain definite manner.

17. BOUNDED FUNCTIONS

We now have to get acquainted with another property of functions which, in contrast to continuity, is not a *local* property but a *global* one; that is, it is defined for a set of values of the independent variable without having previously been defined for individual values (at individual points).

A function $y = f(x)$ is said to be *bounded on the set* M if all the values it assumes on this set are contained in some finite interval. We can replace this definition by another requirement which is entirely equivalent: the existence of a positive number c such that $|f(x)| < c$ for all $x \in M$. In addition, we say that $f(x)$ is bounded above (or below) on the set M if there exists a number c such that

$$f(x) < c \quad (f(x) > c) \qquad \text{for all } x \in M.$$

For a function to be bounded it is of course necessary that it be bounded above and below.

The property of boundedness does not, like the property of continuity on a set, merely mean that certain requirements are satisfied at each individual point. If we only wanted to find for each individual point x in the domain of the function a number c such that $|f(x)| < c$, then this could always be trivially accomplished by taking $c = |f(x)| + 1$, and thus every function is bounded at each individual point in its domain. This does not mean, however, that the function is bounded. For this to be so, it is necessary to find a number c which will at once serve as a bound at all points in the domain. To see how a function may be defined at every point of an interval without being bounded on this interval, let us

recall that $\tan x$ increases indefinitely as $x \to \frac{\pi}{2} - 0$ and consequently the function

$$y = \begin{cases} \tan x & \text{if } 0 \leq x < \frac{\pi}{2}, \\ 0 & \text{if } x = \frac{\pi}{2}, \end{cases}$$

is not bounded on the interval $\left[0, \frac{\pi}{2}\right]$

As in the case of many global properties, it is possible to find for the boundedness of a function *in a given interval* a local property whose existence at every point of the interval is equivalent to the existence of the global property. We shall call a function $y = f(x)$ *bounded at the point* x if this function is bounded in a neighborhood U of x. (Note that the local property here is defined in terms of the previously defined global property; in the case of continuity the situation was exactly the reverse.) We can now assert that *a necessary and sufficient condition for the boundedness of a function* $y = f(x)$ *on the closed interval* $[a, b]$ *is that it be bounded at each point of that interval.* The necessity of this condition follows immediately from the definition. To show its sufficiency, suppose that every point x of $[a, b]$ is contained in a neighborhood U_x in which $y = f(x)$ is bounded. Applying Heine-Borel's lemma, we find that the interval $[a, b]$ can be covered by a finite number of intervals $\Delta_1, \Delta_2, \ldots, \Delta_n$ in each of which y is bounded. If $|y| < c_i$ in the interval Δ_i $(i = 1, 2, \ldots, n)$ and if c is the largest of the numbers $c_1, c_2, \ldots, c_n$, then $|y| < c$ for all $x \in [a, b]$, which proves our assertion.

Let us agree to call a set of numbers N *bounded* if all the numbers belonging to it are contained in some interval. It is obvious that the boundedness of $y = f(x)$ on the set M is equivalent to the boundedness of the set N of all values assumed by this function on M. The meaning of the expressions *the set* N *is bounded below* (*or on the left*) and *the set* N *is bounded above* (*or on the right*) is self-evident.

Let us agree to call the number β the *least upper bound* (l.u.b.) of N if:

(1) the set N does not contain numbers greater than β, and

(2) in every neighborhood of β there is a number belonging to N.

Similarly, we call a number α the *greatest lower bound* (g.l.b.) of the set N if:

(1) there are no numbers in N less than α and

(2) in every neighborhood of α there is a number belonging to N.

It is evident that a set which has a least upper (greatest lower) bound is bounded above (below). In analysis, the converse theorem plays a significant role.

THEOREM 1. *Every nonempty set of numbers bounded above (below) has precisely one least upper (greatest lower) bound. In particular, any nonempty bounded set has both a least upper bound and a greatest lower bound.*

Proof. Denoting our bounded set by N, let us divide the set of all real numbers into two classes A and B according to the following principle: $x \in A$ if to the right of x there exists at least one point of the set N, and $x \in B$ otherwise. One can readily see that this partition is a cut. Let α be its edge. We show that α is the l.u.b. of N.

First, we shall show that there are no points of N to the right of α. For, if $\beta \in N$ and $\beta > \alpha$, then setting $\gamma = \frac{1}{2}(\alpha + \beta)$ we have $\alpha < \gamma < \beta$. From $\gamma > \alpha$ it follows that $\gamma \in B$, while from $\beta > \gamma$ and $\beta \in N$ it follows that $\gamma \in A$; we thus arrive at a contradiction.

Further, let $U = (\alpha_1, \alpha_2)$ be any neighborhood of α, $\alpha_1 < \alpha < \alpha_2$. It is obvious that $\alpha_1 \in A$ and $\alpha_2 \in B$. By virtue of the first of these relations there exist points of N situated to the right of α_1, while, as was just shown, there are no points of N to the right of α. Hence all points situated to the right of α, belong to the interval (α_1, α_2). Thus, the number has both of the properties of a l.u.b. and is established as such.

This bound is unique. Indeed, if the set N had two least upper bounds β and β', $\beta < \beta'$, we would arrive at a contradiction immediately upon noting that, on the one hand, N cannot contain any numbers larger than β (as β is an upper bound) and, on the other hand, it would have to contain such numbers because an arbitrarily small neighborhood of β' would have to contain them. The proof of the theorem for the case of the greatest lower bound is, of course, completely analogous.

We can visualize these bounds of a set as the end points of the smallest interval containing all the numbers of this set. Evidently, we can also define the l.u.b. of a set N as the least number c such

that $x \leq c$ for all $x \in N$. An analogous definition can be given for the g.l.b. It is important to remember that each of these bounds of a set may or may not belong to this set. For example, a closed interval contains its bounds, but an open one does not; the set of numbers $\frac{1}{n}$, where n is any positive integer, contains its l.u.b. (the number 1) but does not contain its g.l.b. (the number 0).

If $y = f(x)$ is bounded on a set M then, as we have already mentioned, the set of values assumed by this function on M is bounded, and thus, by the theorem proved above, has least upper and greatest lower bounds. These are called the *bounds of* $f(x)$ *on* M. Hence, *every function bounded on a given set has exactly one l.u.b. and one g.l.b. on this set.* Each of these bounds may turn out to be a value of the function; then it is the greatest or the least value assumed by this function on the given set. It may happen, however, that one bound or the other is not among the values of the function on the given set. In such a case the function does not assume a greatest or a least value on this set. For example,

$$y = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1, \end{cases} \tag{4}$$

clearly has the bounds 0 and 1 on the interval [0, 1]. The g.l.b. is a value (the least) of the function, but the l.u.b. (the number 1) does not belong to the values of the function and the function does not assume a greatest value on the interval [0, 1].

18. BASIC PROPERTIES OF CONTINUOUS FUNCTIONS

We shall now establish four very important properties of continuous functions.

LEMMA. *If* $y = f(x)$ *is continuous at the point* a *and if* $f(a) < b$, *then there exists a neighborhood* U *of* a *such that* $f(x) < b$ *for all* $x \in U$.

Proof. This is an almost trivial conclusion drawn from the very definition of continuity of a function at a point. Indeed, if α and β are selected so that $\alpha < f(a) < \beta < b$ then, by the definition just mentioned, we shall have $\alpha < f(x) < \beta < b$ for all x in some neighborhood of a. And it is obvious that the lemma remains valid if we replace the sign $<$ by the sign $>$.

THEOREM 2. *A function $y = f(x)$ continuous in a closed interval $[a, b]$ is bounded in this interval.*

Proof. Let us observe that $f(x)$, being continuous in $[a, b]$, is continuous at each point of this interval. Let λ be a point of $[a, b]$; from the definition of continuity it follows that there exists a neighborhood U of λ such that for any $x \in U$ we have

$$f(\lambda) - 1 < f(x) < f(\lambda) + 1.$$

But this means that $f(x)$ is bounded in U (all of its values are included in a certain interval). Thus, the function is bounded at the point λ, since we have defined boundedness at a point as boundedness in a neighborhood of this point. And since $f(x)$ is bounded at each point of the interval $[a, b]$, it must also be bounded on this whole interval (as we have already learned on p. 53).

THEOREM 3. *A function $y = f(x)$ continuous in a closed interval $[a, b]$ assumes on this interval a greatest and a least value.*

Proof. By Theorem 2, y is bounded in $[a, b]$ and therefore has a g.l.b. α and a l.u.b. β; we have only to show that these bounds are values assumed by y. It is sufficient to show this for the l.u.b. β.

If β were not a value of $f(x)$ for some $x \in [a, b]$, we would have $f(x) < \beta$ for all x in this interval. For each x let β_x be any number between $f(x)$ and β, so that $f(x) < \beta_x < \beta$. By virtue of our lemma we can find a neighborhood U_x of each x such that $f(x') < \beta_x$ for all $x' \in U_x$. The family of neighborhoods $\{U_x\}$ constructed in this manner covers the interval $[a, b]$, and by the Heine-Borel lemma it contains a finite subfamily $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ which also covers $[a, b]$. But for every $x' \in U_{x_k}$ we have $f(x') < \beta_{x_k}$; denoting by β' the largest of the numbers $\beta_{x_1}, \beta_{x_2}, \ldots, \beta_{x_n}$, we see that for all $x \in [a, b]$ we have

$$f(x) < \beta' < \beta.$$

Consequently, β could not be the least upper bound of $f(x)$ on the interval $[a, b]$, and this contradiction proves our theorem.

We saw above (function (4)) an example of a bounded function which does not assume a greatest value on a given interval. We now know that this is possible only for discontinuous functions, and indeed the function (4) is discontinuous at $x = 1$.

It is also important to emphasize that Theorem 3 is valid only for closed intervals. For example, even such simple functions as $y = x$ and $y = x^2$ need not assume a greatest and a least value on open intervals.

THEOREM 4. *If a function $f(x)$ is continuous on a closed interval $[a, b]$ and if*

$$f(a) < \mu < f(b)$$

or

$$f(a) > \mu > f(b),$$

then there exists a number c in the interval $[a, b]$ such that $f(c) = \mu$.

In short, a continuous function assumes all intermediate values between any two of its values. It is clear that discontinuous functions, in general, do not have this property. For example, the Dirichlet function assumes in every interval the values 0 and 1, but does not assume any intermediate value. For the function (4) we have $f\left(\frac{1}{2}\right) = \frac{1}{2}$ and $f(1) = 0$, but at no point in the interval $\left[\frac{1}{2}, 1\right]$ does the function assume the intermediate value $\frac{1}{4}$.

Proof of Theorem 4. Let us suppose first that $\mu = 0$, $f(a) < 0$, and $f(b) > 0$; we have to prove that at some interior point of the interval $[a, b]$ the function $f(x)$ assumes the value zero. Let us assume the contrary and divide the interval $[a, b]$ into two halves. It is clear that in one of the two halves the function assumes at the end points values of opposite sign; we divide this half again into two equal parts and we select again that half at whose end points the function assumes values of opposite sign, and so on. In this way we obtain a sequence of nested intervals; let α be their common point. By our assumption, $f(\alpha) \neq 0$. Suppose that $f(\alpha) < 0$. Then by the lemma on p. 55 we have $f(x) < 0$ for all x in some neighborhood U of α. This, however, is impossible as U contains an infinite number of our nested intervals at the end points of which $f(x)$ assumes values of opposite sign. In the same manner, we can prove that the inequality $f(\alpha) > 0$ is impossible. The contradiction thus obtained proves our theorem. For if $\mu \neq 0$, it is sufficient to apply the result just obtained to the function $f(x) - \mu$, and our theorem is established.

Before we formulate the next property of continuous functions, we have to introduce a new concept of great importance to the study of this class of functions. We remember, of course, that continuity is a local property of a function. This situation is not changed at all by the fact that we speak of *a function continuous in a given interval,* as continuity in an interval means nothing more than continuity at each point of this interval and, therefore, does not at all change the local character of this concept. It is possible, however, to express the idea of continuity of a function in a given *interval* as a global property, that is, a property which is not stated in terms of the behavior of the function in the neighborhood of each point, but rather one which describes the behavior of the function on the interval as a whole.

DEFINITION. *We shall call a function* $y = f(x)$ uniformly continuous *on the interval* $[a, b]$ *if this function has the following property: for any positive number* ε*, no matter how small, there exists another positive number* δ *such that for any two numbers* x_1 *and* x_2 *in* $[a, b]$ *differing from each other by less than* δ*, we have*

$$|f(x_1) - f(x_2)| < \varepsilon.$$

As you see, this definition does not refer to any individual point but attempts to characterize the behavior of a function on the whole interval $[a, b]$, stating that at any two points sufficiently close to each other the values assumed by the function differ by as little as we please. It is evident why we call this kind of continuity uniform. Its specific character is that it requires some uniformity in the behavior of the function in every part of the given interval; the points x_1 and x_2 can be taken anywhere in the interval $[a, b]$ as long as the distance between them does not exceed δ.

It is quite clear that a function uniformly continuous on $[a, b]$ is necessarily continuous at each point of this interval (and thus is continuous on the whole interval). For by virtue of uniform continuity, the inequality

$$|x - \alpha| < \delta$$

implies the inequality

$$|f(x) - f(\alpha)| < \varepsilon;$$

that is, for any neighborhood V, $(f(\alpha) - \varepsilon, f(\alpha) + \varepsilon)$, of $f(\alpha)$ there exists a neighborhood U, $(\alpha - \delta, \alpha + \delta)$, of α such that if $x \in U$,

then $f(x) \in V$. And this, of course, means that the function $f(x)$ is continuous at $x = \alpha$.

It is extremely important that the converse theorem is also valid: that is, from the continuity of a function at each point of a closed interval $[a, b]$, there follows the uniform continuity of the function on this interval. Thus, the requirement of uniform continuity does not narrow the class of continuous functions (so long as we are referring to a closed interval).

THEOREM 5. *A function $y = f(x)$ continuous at every point of a closed interval $[a, b]$ is uniformly continuous on this interval.*

For open intervals this theorem is false. For example, the function $y = \sin\frac{1}{x}$, clearly continuous at every point of the open interval $(0, 1)$, cannot be uniformly continuous in this interval, since no matter how small we choose δ there exist (in the vicinity of zero) two points whose mutual distance is less than δ, while the difference of the values assumed by the function y at these points is greater than one.

Proof of Theorem 5. Let us assume that $y = f(x)$ is continuous at each point of the closed interval $[a, b]$. Then, for every point x of this interval, there exists a $\delta_x > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ when x_1 and x_2 are contained in the interval $[x - \delta_x, x + \delta_x]$. Let us denote by Δ_x the interval $\left[x - \frac{1}{2}\delta_x, x + \frac{1}{2}\delta_x\right]$. It is obvious that the family of intervals $\{\Delta_x\}$ covers $[a, b]$. Therefore, by the Heine-Borel lemma, there exists a finite subfamily M of the intervals $\{\Delta_x\}$ also covering the interval $[a, b]$. Let δ be the length of the smallest interval in the set M and let x_1 and x_2 be any two points of $[a, b]$ whose distance from each other is less than $\frac{1}{2}\delta$. We must show that $|f(x_1) - f(x_2)| < \varepsilon$. Since the point x_1 belongs to some interval $\left[x - \frac{1}{2}\delta_x, x + \frac{1}{2}\delta_x\right]$ of the family M, the point x_2, whose distance from x_1 is less than $\frac{1}{2}\delta \leq \frac{1}{2}\delta_x$, belongs to the interval $[x - \delta_x, x + \delta_x]$ which, of course, also contains x_1. Whence, by the definition of δ_x we have $|f(x_1) - f(x_2)| < \varepsilon$.

19. CONTINUITY OF THE ELEMENTARY FUNCTIONS

You know, of course, that the sum and product of any finite number of continuous functions is also a continuous function, and that the quotient of two continuous functions is continuous in every interval in which the denominator does not assume the value zero. All these theorems are valid in the local as well as in the global sense, that is, regardless of whether we have in mind continuity at a point, continuity on an interval, or uniform continuity. The proofs of all these theorems can be found in any textbook; they are not of fundamental interest and there is no need to dwell on them here.

Much more interesting is the question of the continuity of the so-called elementary functions, that rather small class of functions which are used in elementary mathematics, but retain their significance in higher mathematics. To this class belong, first of all, all those functions whose values can be obtained from the independent variable by the application of the six basic algebraic operations, and further, a small number of transcendental functions: the trigonometric functions (direct and inverse), the exponential and logarithmic functions, and all possible combinations of the above mentioned functions (for example, $x^2(1 - x \sin x)$ or $\cos 2^{x^2}$).

All elementary functions are continuous except for possible breaks in continuity at individual isolated points, called *points of discontinuity.* (For example, $\frac{1}{x}$ at $x = 0$ or $\tan x$ at $x = \frac{\pi}{2}$. Note, however, that the functions $\frac{1}{x}$ and $\tan x$ are continuous in their domains of definition. The points of discontinuity of which we are speaking are the points at which these functions are not defined. Regarding such points of discontinuity, see p. 65.) However, the proof of this assertion is not easy for all of the elementary functions. In most analysis courses this situation is not investigated fully. It is of great importance, however, and we shall have to make several remarks on the subject.

First of all, it is very important to establish the continuity of all the polynomial functions. This is done as follows: Since for every polynomial $P(x)$ and for every number a the difference $P(x) - P(a)$ is divisible by $x - a$ (Bézout's theorem), we have $P(x) - P(a) = (x - a)Q(x)$, where $Q(x)$ is a polynomial. Since $Q(x)$ is clearly bounded at the point a, we have $P(x) \to P(a)$ as $x \to a$; that is, the

polynomial $P(x)$ is continuous at a. Furthermore, from the continuity of the polynomials there follows directly the continuity of every rational function $\frac{P_1(x)}{P_2(x)}$ in every interval where the polynomial $P_2(x)$ is never zero.

The problem is solved equally simply for the trigonometric functions. The formula

$$\sin x - \sin a = 2 \cos \frac{x + a}{2} \sin \frac{x - a}{2}$$

and the consequent inequalities

$$|\sin x - \sin a| \leq 2\left|\sin \frac{x - a}{2}\right| \leq 2\left|\frac{x - a}{2}\right| = |x - a|$$

show that $\sin x \to \sin a$ as $x \to a$. Thus $\sin x$ is a continuous function, and quite analogously we establish the continuity of the cosine function. Hence the function

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous at all values of x where $\cos x \neq 0$, that is, where $x \neq (2k + 1)\frac{\pi}{2}$, k an integer.

The proof of the continuity of the exponential function a^x is more complicated. Let us assume, for definiteness, that $a > 1$ so that the function $y = a^x$ increases with x.[1] We shall show first that $a^x \to a^0 = 1$ as $x \to 0$. Again for definiteness, let us assume that $x \to +0$, that is, we shall consider a^x only for positive values of x. The case of $x \to -0$ is completely analogous. Thus for $x > 0$ we have $a^x > 1$, and in order to prove that the function $a^x \to 1$ as $x \to +0$ we have only to show that, no matter how small the positive number ε may be, we have $a^x < 1 + \varepsilon$ for sufficiently small x. For every positive integer n we have the inequality

$$(1 + \varepsilon)^n = 1 + n\varepsilon + \cdots > n\varepsilon,$$

whence $(1 + \varepsilon)^n \to +\infty$ as $n \to \infty$. This means that for sufficiently large n we have the inequalities

$$(1 + \varepsilon)^n > a \quad \text{and} \quad a^{\frac{1}{n}} < 1 + \varepsilon.$$

[1] The case of $a < 1$ may be reduced to that of $a > 1$ by writing $a^x = \frac{1}{b^x}$ with $b > 1$.

And since a^x is an increasing function, for all sufficiently small x we have the inequalities

$$1 < a^x < 1 + \varepsilon,$$

or, a^x tends to 1 as $x \to 0$.

The rest of the proof follows easily. The relation

$$a^x - a^\alpha = a^\alpha(a^{x-\alpha} - 1)$$

shows that as $x \to \alpha$ the function a^x tends to a^α. Hence this function is continuous at all values of x.

The question of the continuity of the logarithmic and the inverse trigonometric functions can now be settled on the basis of a general theorem on the continuity of inverse functions.

THEOREM 6. *If a function $y = f(x)$ is continuous and increasing*[1] *on a closed interval $[a, b]$ then the inverse function $x = \varphi(y)$ is also continuous on the interval $[\alpha, \beta]$, where $\alpha = f(a)$ and $\beta = f(b)$.*

Proof. To show this, let γ be any point of $[\alpha, \beta]$. We have to prove that

$$\lim_{y \to \gamma} \varphi(y) = \varphi(\gamma)$$

We first let $y \to \gamma - 0$; since the inverse function $\varphi(y)$, like the function $f(x)$, is increasing, $\lim_{y \to \gamma - 0} \varphi(y)$ is sure to exist. Let us denote this limit by c, and we shall show that $c = \varphi(\gamma)$.

Since $f(x)$ is continuous and $\varphi(y) \to c$ as $y \to \gamma - 0$, it follows that as $y \to \gamma - 0$ we have $y = f[\varphi(y)] \to f(c)$, whence $f(c) = \gamma$. But it follows directly from this that

$$c = \varphi[f(c)] = \varphi(\gamma),$$

that is,

$$\lim_{y \to \gamma - 0} \varphi(y) = \varphi(\gamma).$$

And since in the same way we can also show that

$$\lim_{y \to \gamma + 0} \varphi(y) = \varphi(\gamma),$$

we have finally

$$\lim_{y \to \gamma} \varphi(y) = \varphi(\gamma), \qquad \text{q.e.d.}$$

[1] The conclusion is, of course, equally valid when $f(x)$ is continuous and decreasing.

In order to be able to assert without special proof that composite elementary functions such as $\log \sin x$ and 2^{3x^2-4x} are continuous, we need the following *theorem on the continuity of composite functions* (*functions of functions*).

THEOREM 7. *If* $y = f(x)$ *is continuous on* $[a, b]$ *and* $z = \varphi(y)$ *is continuous on* $[\alpha, \beta]$, *where* α *and* β *are, respectively, the least and the greatest values of* $f(x)$ *on the closed interval* $[a, b]$, *then the function* $z = \varphi[f(x)]$ *is continuous on* $[a, b]$.

Proof. Let c be any point in $[a, b]$, let $\gamma = f(c)$ and $\zeta = \varphi(\gamma) = \varphi[f(c)]$, and let W be any neighborhood of the point ζ. By the continuity of $\varphi(y)$, there exists a neighborhood V of γ such that $\varphi(y) \in W$ whenever $y \in V$. Finally, owing to the continuity of $f(x)$, there exists a neighborhood U of c such that if $x \in U$, then $f(x) \in V$, and consequently, $\varphi[f(x)] \in W$. Since W is an arbitrary neighborhood of $\zeta = \varphi[f(c)]$, it follows that

$$\lim_{x \to c} \varphi[f(x)] = \varphi[f(c)],$$

and the continuity of $\varphi[f(x)]$ at any point c in $[a, b]$ is established.

20. OSCILLATION OF A FUNCTION AT A POINT

The most convenient approach to the study of discontinuous functions is by introducing the concepts of *oscillation in a given interval* and *oscillation at a given point.*

Suppose that a completely arbitrary function $f(x)$ is defined on a closed interval $[a, b]$ with the exception, perhaps, of a finite number of points. If this function is not bounded in the given interval, we shall say that its oscillation in this interval is equal to $+\infty$; if, however, the function is bounded, then by the *oscillation* of the function in $[a, b]$ we shall mean the difference $M - m$ between the l.u.b. M and the g.l.b. m of the function on this interval. In either case, we shall denote the oscillation of $f(x)$ in $[a, b]$ by the symbol $\omega_f(a, b)$.

If the interval $[a', b']$ is contained in the interval $[a, b]$ $(a \leq a' < b' \leq b)$, then the bounds M' and m' of $f(x)$ on $[a', b']$ will evidently satisfy the inequalities $m \leq m' \leq M' \leq M$ and hence, $\omega_f(a', b') \leq \omega_f(a, b)$. Consequently, if we select in the interval $[a, b]$ any point c (the function may not even be defined at

this point), take a neighborhood $[\alpha, \beta]$ of this point, and cause the end points α and β of this neighborhood to approach c, then $\omega_f(\alpha, \beta)$, as a function of the interval, is nonincreasing.[1] And, since the oscillation $\omega_f(\alpha, \beta)$, being a nonnegative quantity, is bounded below, it follows by an argument similar to the proof of Lemma 1 in Lecture 1 (p. 17) that these oscillations tend to a limit. We shall call this limit *the oscillation of* $f(x)$ *at* c, and denote it by $\omega_f(c)$. Thus

$$\omega_f(c) = \lim_{\substack{\alpha \to c-0 \\ \beta \to c+0}} \omega_f(\alpha, \beta).$$

The introduction of this concept allows us, first of all, to look in a new way at the continuity of a function at a given point, as the following theorem shows.

THEOREM 8. *A necessary and sufficient condition for a function* $f(x)$ *defined in a neighborhood of the point* c *to be continuous at that point is that* $\omega_f(c) = 0$.

Proof. (i) Let $\omega_f(c) = 0$. This means that in a sufficiently small neighborhood $U = (a, b)$ of c, the oscillation of $f(x)$ becomes arbitrarily small. And since for every $x \in U$ we have

$$|f(x) - f(c)| \leq M - m = \omega_f(a, b),$$

where M and m are the bounds of $f(x)$ in U, it follows that the value of $|f(x) - f(c)|$ will be as small as we please if x belongs to a sufficiently small neighborhood U of c. This, however, means that $f(x)$ is continuous at c.

(ii) If $\omega_f(c) = \omega > 0$, then in every neighborhood U of c we have $M - m \geq \omega$. Therefore, there exist α and β in U such that $f(\alpha) < m + \frac{\omega}{4}$ and $f(\beta) > M - \frac{\omega}{4}$, and consequently, $f(\beta) - f(\alpha) > \frac{\omega}{2}$. But

$$f(\beta) - f(\alpha) = [f(\beta) - f(c)] + [f(c) - f(\alpha)],$$

and from $f(\beta) - f(\alpha) > \frac{\omega}{2}$ it follows that at least one of the two terms on the right side must be greater than $\frac{\omega}{4}$. Thus there ex-

[1] Here we understand by the term *neighborhood* of c, any set which contains an open interval containing c.

ists a point x in U such that $|f(x) - f(c)| > \frac{\omega}{4}$ and since the neighborhood U is arbitrary, $f(x)$ cannot be continuous at c.

Points at which a given function $f(x)$ is continuous are called *points of continuity* of $f(x)$, and points at which it is discontinuous, *points of discontinuity*. Our theorem then asserts that $\omega_f(c) = 0$ at points of continuity and $\omega_f(c) > 0$ at points of discontinuity.

21. POINTS OF DISCONTINUITY

We are now in a position to attempt the introduction of some order into the multitude of exceedingly diverse types of discontinuity. Since the condition $\omega_f(c) = 0$ distinguishes points of continuity from points of discontinuity, we naturally expect that the quantity $\omega_f(c)$ will serve as a reasonable and convenient *measure* of the discontinuity of the function at this point. Our expectation becomes even stronger as we recall the definition of $\omega_f(c)$. Clearly, $\omega_f(\alpha, \beta)$ measures the spread of the values assumed by $f(x)$ in the interval $[\alpha, \beta]$. We may therefore regard the limit of this quantity, as the interval contracts to the point c, as measuring the spread of the values assumed by $f(x)$ at points arbitrarily close to c, that is, as measuring that characteristic of the function which is responsible for the discontinuity at c. Let us agree to call the quantity $\omega_f(c)$ the *measure of discontinuity* of $f(x)$ at c. This makes it possible to compare different points of discontinuity according to the *degree of discontinuity* at these points. In particular, a function has its greatest discontinuity at those points where the oscillation is not bounded ($\omega_f(c) = +\infty$).

Associated with the concept of measure of discontinuity (or of oscillation at a point) is an important proposition which is a direct generalization of Theorem 5 on uniform continuity (p. 59).

THEOREM 9. *If the measure of discontinuity (the oscillation) of $f(x)$ at every point of the interval $[a, b]$ does not exceed $\lambda \geq 0$, then, no matter how small the positive number ε, there exists a $\delta > 0$ such that the oscillation of $f(x)$ in any subinterval of length less than δ does not exceed $\lambda + \varepsilon$.*

Such a function might be called *continuous within a tolerance of* λ: in particular, setting $\lambda = 0$ we obtain Theorem 5. You can prove

the present theorem for yourself along the same lines which were used in the proof of Theorem 5, only here the number $\lambda + \varepsilon$ will play the part of ε in the earlier proof.

Thus we see that not only continuity, but also the local property consisting of boundedness of the measure of discontinuity at a given point, has its equivalent global property. The theorem which we have just stated has a number of applications, particularly in the integral calculus, and we shall meet with it again in one of the following lectures.

Also very important is another principle for the classification of points of discontinuity. The essence of this principle is the comparison of discontinuities, not according to magnitude, but according to the *form* of the discontinuity. As we know, a criterion of continuity of $f(x)$ at a is given by the equalities $f(a + 0) = f(a) = f(a - 0)$. It may happen that both one-sided limits $f(a + 0)$ and $f(a - 0)$ exist, but at least one of them differs from $f(a)$. In such a case, we shall say that a is a *point of discontinuity of the first kind* of $f(x)$. Thus, a point of discontinuity of the first kind is characterized by the fact that limits of $f(x)$ exist as we approach this point from either side, but either these limits are unequal or, though equal, differ from the value of the function at this point. An example of this first type is portrayed graphically in Figure 13 (here the value of the function at the point of discontinuity can be arbitrary); an example of the second type is shown in Figure 14.

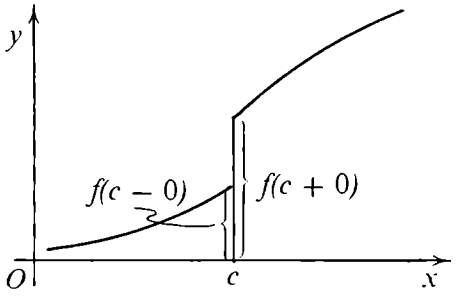

Fig. 13

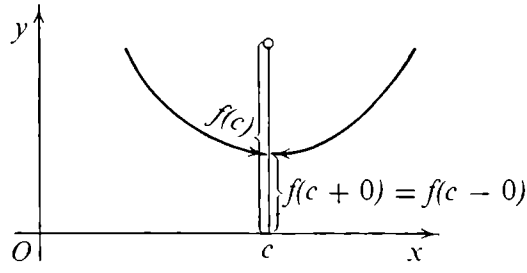

Fig. 14

The simplest type of point of discontinuity is that of the first kind; the relative ease of its investigation is, of course, the result of the existence of the limits $f(a + 0)$ and $f(a - 0)$. All other points of discontinuity are called points of discontinuity of the *second kind.* That is to say, at a point of discontinuity of the second kind the function fails to tend to a limit as the point is approached from at least one direction.

An example of a discontinuity of the second kind is furnished by the behavior of the very instructive function $y = \sin \frac{1}{x}$ as $x \to 0$ (Fig. 15), which we have already mentioned more than once. And

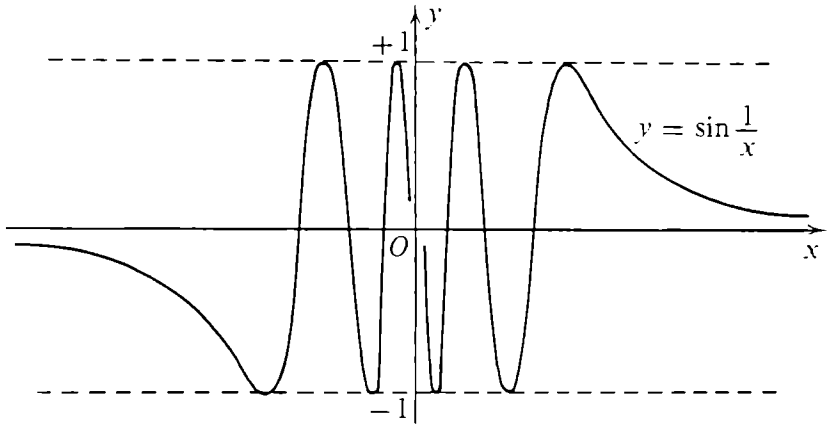

Fig. 15

the Dirichlet function, which we mentioned at the beginning of this lecture, has a discontinuity of the second kind at every point.

22. MONOTONIC FUNCTIONS

A function $y = f(x)$ is called *nondecreasing* in a closed interval $[a, b]$, if $f(x_1) \leq f(x_2)$ whenever $a \leq x_1 \leq x_2 \leq b$; if under the same assumptions we always have $f(x_1) \geq f(x_2)$, the function $f(x)$ is said to be *nonincreasing* in $[a, b]$. The nondecreasing and nonincreasing functions together form the class of *monotonic functions*. Monotonic functions have a number of special properties which make them in many instances a very convenient tool of investigation.

First of all, if $f(x)$ is monotonic in an interval $[a, b]$, it is bounded in this interval. (As usual, the interval is taken to be closed. For open intervals the assertion is not true: the function $y = \frac{1}{x}$ is monotonic but not bounded in the open interval $(0, 1)$.) For if $a \leq x \leq b$, $f(x)$ lies between $f(a)$ and $f(b)$, since the greatest and the least values of a monotonic function in the interval $[a, b]$ are assumed at the end points. It follows that $f(a)$ and $f(b)$ are the g.l.b. and the l.u.b. of $f(x)$ in this interval.

A monotonic function can have points of discontinuity, as a glance at Figure 13 shows. But the discontinuities of such a function are limited both as to their character and their number. First, because of their boundedness, monotonic functions cannot have discontinuities of infinite measure. Further, *the number of points within the interval* $[a, b]$ *at which the measure of discontinuity of a monotonic function* $f(x)$ *exceeds any positive number* τ *is not greater than* $\frac{f(b) - f(a)}{\tau}$.

To see this, let us assume arbitrarily that $f(a) < f(b)$ and let us suppose that the function $f(x)$ has within the interval $[a, b]$ n points at which the measure of discontinuity exceeds τ; we denote these n points, from left to right, by $c_1, c_2, \ldots, c_n$ so that for an arbitrarily small $\varepsilon > 0$

$$f(c_k + \varepsilon) - f(c_k - \varepsilon) > \tau \quad (1 \leq k \leq n).$$

If (as is always possible) we choose ε so small that $a < c_1 - \varepsilon$, $c_n + \varepsilon < b$, and

$$c_k + \varepsilon < c_{k+1} - \varepsilon \quad (1 \leq k \leq n - 1),$$

i.e., so that no two of the neighborhoods $(c_k - \varepsilon, c_k + \varepsilon)$ overlap and all of them are wholly contained in $[a, b]$, then we have

$$n\tau < \sum_{k=1}^{n} [f(c_k + \varepsilon) - f(c_k - \varepsilon)] \leq f(b) - f(a),$$

whence

$$n < \frac{f(b) - f(a)}{\tau}, \qquad \text{q.e.d.}$$

A monotonic function can thus have only a finite number of points at which the measure of its discontinuity exceeds a given number. In the case of nonmonotonic functions the situation is different; for example, the measure of discontinuity of the Dirichlet function equals one at every point.

Finally, if a function $f(x)$ is monotonic in a closed interval $[a, b]$, then by Lemma 1′ of Lecture 1 (p. 18) the limits $f(c + 0)$ and $f(c - 0)$ exist at every point c of this interval (except, of course, the end points which have only one-sided limits). It follows from this that a monotonic function can have only points of discontinuity of the first kind.

23. FUNCTIONS OF BOUNDED VARIATION

Suppose $f(x)$ is defined on a closed interval $[a, b]$. Let us subdivide this interval in any manner into n subintervals and denote the points of partition from left to right by $x_1, x_2, \ldots, x_{n-1}$. For convenience, let us set $a = x_0$ and $b = x_n$, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The sum

$$S = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

depends on our partition of the interval $[a, b]$, and this sum will, in general, have different values for different partitions. Performing every possible partition (changing the number n arbitrarily), we shall obtain an infinite set of sums S. The l.u.b. M of this set (which may, of course, equal $+\infty$) is called the *total variation of* $f(x)$ *in the interval* $[a, b]$, and we shall denote it by $V_f(a, b)$.

If $V_f(a, b)$ is a number ($V_f(a, b) < +\infty$), we say that $f(x)$ *is a function of bounded variation on* $[a, b]$. If $V_f(a, b) = +\infty$, we say that the variation of $f(x)$ in $[a, b]$ is infinite (or unbounded).

The class of functions of bounded variation plays a very important role in analysis and its applications. In particular, it is evident that every function monotonic in an interval $[a, b]$ is a function of bounded variation on this interval, since for any monotonic function $f(x)$ the sum S for any partition is equal to $|f(b) - f(a)|$. Hence $V_f(a, b) = |f(b) - f(a)|$.

Of fundamental importance in the investigation of the general properties of functions of bounded variation is a theorem, with the help of which the study of even the most general type of such functions can be completely reduced to the study of monotonic functions.

THEOREM 10. *Every function of bounded variation is the difference of two nondecreasing functions (or equivalently, the sum of two monotonic functions one of which is nondecreasing and the other nonincreasing).*

On the basis of this theorem, all fundamental properties of monotonic functions concerned with the number and character of possible points of discontinuity can at once be extended to functions of bounded variation.

In proving this fundamental theorem let us agree, for brevity, to write $V(x)$ instead of $V_f(a, x)$. Furthermore, we let

$$P(x) = \frac{1}{2}[V(x) + f(x)] \quad \text{and} \quad N(x)\frac{1}{2}[V(x) - f(x)],$$

from which

$$f(x) = P(x) - N(x).$$

Our theorem will then be proved if we show that the functions $N(x)$ and $P(x)$ are nondecreasing in the interval $[a, b]$.

Proof. Let $a \leq x_1 < x_2 \leq b$. Since

$$2[P(x_2) - P(x_1)] = V(x_2) - V(x_1) + [f(x_2) - f(x_1)] \quad \text{and}$$
$$2[N(x_2) - N(x_1)] = V(x_2) - V(x_1) - [f(x_2) - f(x_1)],$$

it is sufficient for our purpose to show that

$$V(x_2) - V(x_1) \geq |f(x_2) - f(x_1)|. \tag{5}$$

Let ε be any positive number. By the definition of $V(x_1)$ as the l.u.b. of the sums S, there exists a partition of the interval $[a, x_1]$ such that the corresponding sum $S = S(a, x_1)$ exceeds $V(x_1) - \varepsilon$. But the sum $S(a, x_1) + |f(x_2) - f(x_1)| = S(a, x_2)$ is obviously one of the sums S for the interval $[a, x_2]$ and therefore does not exceed the l.u.b. $V(x_2)$ of such sums. We thus obtain

$$V(x_1) - \varepsilon + |f(x_2) - f(x_1)| < S(a, x_1) + |f(x_2) - f(x_1)| = S(a, x_2) \leq V(x_2),$$

from which

$$V(x_2) - V(x_1) > |f(x_2) - f(x_1)| - \varepsilon.$$

And, since ε can be taken arbitrarily small, the inequality (5) follows and our theorem is established.

4. Series

24. CONVERGENCE AND THE SUM OF A SERIES

We shall devote this lecture to infinite series, one of the most important tools of mathematical analysis. As you know, in the great problems of mathematical analysis, infinite series appear as a purely technical tool of investigation, very useful and convenient, but rather modest in theoretical significance. Nevertheless, this tool fully deserves a special lecture even in our brief course. Not only do its many applications pervade the whole structure of analysis, as well as almost all applied sciences, but in the relatively simple subject matter of the theory of series, trains of thought and logical patterns typical of the whole of analysis appear especially clearly. It is well known that for a student who has actively and thoroughly mastered the theory of series, the further study of the basic content of analysis will present no unusual difficulty.

An infinite numerical series is an expression of the form

$$u_1 + u_2 + \cdots + u_n + \cdots \tag{1}$$

or

$$\sum_{n=1}^{\infty} u_n,$$

where we shall assume all the u_n to be real numbers. The numbers u_n are called the *terms* of the series. And the finite sums

$$s_n = \sum_{k=1}^{n} u_k$$

are called the *partial sums* of the series (1).

The main question concerning any given series is that of its *convergence.* If the limit

$$\lim_{n\to\infty} s_n = s$$

exists, the series (1) is called *convergent* and the number s is called its *sum.* Otherwise, the series is called *divergent* and has no sum.

When $s_n \to +\infty$ or $s_n \to -\infty$ as $n \to \infty$, we are formally entitled to include the series in either of these categories. But we usually consider such a series as divergent, so that by the sum of a series we always mean a number. However, in giving the name *divergent* to those series with infinite sums as well as to those which have no sum at all, we must remember that these two types of series basically have nothing in common. They are considered together only because both are unlike (although in a different sense) series with finite sums.

In its basic content, analysis deals only with convergent series. Students inevitably form their initial idea of the sum of a series by analogy with ordinary finite sums. And very frequently the lecturer himself encourages, or even directly fosters, such a notion. If approached with adequate caution, this view may actually contribute to our comprehension of the subject. We must not, however, carry this analogy too far, and, above all, we must not attribute to it any power of proof.

The analogy between infinite series and finite sums can be carried to considerable length only for so-called *absolutely convergent series* (which we shall discuss later). For *conditionally convergent* series, our tendency to regard their sums as analogous to finite sums is immediately confronted with contradicting evidence; in particular, it is difficult to regard as *the sum of* all *the terms of a series* a value which can be changed by changing the order of the terms. Actually, the process of forming the sum of a series is not at all similar to the process of finite addition and does not consist (as some would imagine) in adding one term of the series after another *until all of them are exhausted.* The very term *infinite* implies the impossibility of exhausting the terms of a series by successive finite operations. But in spite of this, we can continue to speak of the *sum* of a series if we give up this hopeless process of infinite addition and substitute for it the completely different operation of passage to the limit.

We begin, as in the case of finite sums, with successive addition, forming the sums s_1, s_2, etc. But we do not intend to continue this process without limit. While forming these partial sums, we examine carefully their nature and structure, and, in particular, their dependence on the number n. In other words, the subject of

our study is *the quantity s_n as a function of the variable n.* In many cases, we obtain the whole picture of this functional dependence at once by obtaining a convenient analytical expression for the function s_n (for example, the geometric progression familiar from elementary algebra). More precisely: we try to find out whether the quantity s_n approaches a limit as $n \to \infty$, and if so, what limit. Thus the determination of the sum of an infinite series is composed of two successive phases: (*a*) formation of the partial sums s_n and investigation of their dependence on n, and (*b*) passage to the limit as $n \to \infty$. You can see that this has little resemblance to *addition until the terms are completely exhausted,* and we must keep this difference in mind to avoid the always present danger of being led into error by unfounded analogies.

We see that the problem of the sum of a given series (1) reduces entirely to the problem of the limit of the sequence

$$s_1, s_2, \ldots, s_n, \ldots \tag{2}$$

associated with this series. Conversely, given any sequence (2) we can always reduce the problem of the limit of this sequence to the problem of the sum of the series (1) by setting

$$\begin{aligned} u_1 &= s_1, \\ u_2 &= s_2 - s_1, \\ u_3 &= s_3 - s_2, \\ &\ldots, \\ u_n &= s_n - s_{n-1}. \end{aligned}$$

Viewed in this manner, the basic problem of the theory of series does not differ at all from the basic problem of the theory of sequential limits. In fact, it is only this particular approach to the study of sequences, considering them as the successive partial sums of series, which gives rise to a distinct theory with special features, problems, and methods. Yet this approach is interesting enough from the theoretical point of view and important enough with respect to applications to justify a special theory, the theory of infinite series.

25. CAUCHY'S CONDITION FOR CONVERGENCE

The reader is already familiar with the following proposition:

THEOREM 1. *If a series* (1) *is convergent, then* $u_n \to 0$ *as* $n \to \infty$.

For, since

$$u_n = s_n - s_{n-1} \qquad \text{when } n > 1$$

and the sums s_n and s_{n-1} have the same limit as $n \to \infty$, it follows that $u_n \to 0$. This property of convergent series is important because in many cases it allows us to establish easily the divergence of a series (by showing that u_n is not infinitely small as $n \to \infty$). On the other hand, from the fact that $u_n \to 0$ as $n \to \infty$, it does not follow that the series (1) is convergent. For example, the *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent ($s_n \to +\infty$ as $n \to \infty$) even though its nth term is infinitely small as $n \to \infty$. This one-sidedness of the property in question (necessity without sufficiency) of course limits its scope of application considerably.

The problem of the convergence of a series is, as we have seen, only a special form of the problem of the existence of the limit of a numerical sequence. Therefore, we obtain a necessary and sufficient condition for the convergence of a series by translating into the language of the theory of series the already familiar Cauchy's condition for the existence of the limit of a sequence (Lecture 2, p. 32).

We know that in order for the sequence of partial sums s_n to approach a limit as $n \to \infty$, it is necessary and sufficient that for any $\varepsilon > 0$ the inequality

$$|s_{n+k} - s_n| < \varepsilon$$

be satisfied for all sufficiently large n and any k. Since

$$s_{n+k} - s_n = \sum_{i=n+1}^{n+k} u_i,$$

we have the following theorem.

CAUCHY'S CONDITION. *For the convergence of the series* (1) *it is necessary and sufficient that for any* $\varepsilon > 0$ *the inequality*

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+k}| < \varepsilon \tag{3}$$

be satisfied for all sufficiently large n *and any* k.

In descriptive terms, the requirement of Cauchy's condition is that the absolute value of any *segment* of the series, however long, becomes as small as we please, provided that the *segment* is located far enough to the right. It is evident, as in the theory of limits, that Cauchy's condition can only establish the existence of the sum of a series and does not tell us anything about its value.

Cauchy's condition, although a very powerful tool in general theoretical investigations, is applied relatively seldom to prove the convergence of a specific series (as we have already remarked in an analogous connection in Lecture 2). The reason for this is that it is usually not easy to establish whether condition (3) is satisfied by a given series. Therefore, for use in the theory of series, we construct a large number of other criteria for convergence. These tests, unlike Cauchy's condition, are not characteristic (both necessary and sufficient), but are applied to specific series with incomparably greater ease.

The majority of these criteria pertain to series with positive terms. For expositional reasons, it is more convenient to investigate this simplest, and at the same time most important, class of series at the very beginning. We do this in order to be able later to reduce, as far as possible, the study of series to the study of only this simple type.

26. SERIES WITH POSITIVE TERMS

If all the terms of a series (1) are positive, then s_n is clearly a nondecreasing function of the variable n. By Lemma 1 of Lecture 1, there exist two possibilities: either the series converges or $\lim_{n\to\infty} s_n = +\infty$. In other words, the series converges or diverges, depending on whether the quantity s_n is bounded or unbounded as $n \to \infty$. All the criteria for convergence of series in this class are, therefore, established by showing that under the appropriate conditions the sequence s_n is bounded.

Underlying the majority of these criteria is the following extremely powerful comparison test:

THEOREM 2. *If all the terms of the series*

$$u_1 + u_2 + \cdots + u_n + \cdots \tag{A}$$

and

$$v_1 + v_2 + \cdots + v_n + \cdots \tag{B}$$

are nonnegative and if for $n > n_0$ *we have the inequality*

$$u_n \leq v_n, \tag{C}$$

then the convergence of the series (B) *implies the convergence of the series* (A).

Proof. The proof of this basic fact is very simple. Let us denote by s_n and s_n', respectively, the partial sums of the series (A) and (B). If (B) is convergent, the quantity s_n' is bounded. And since by virtue of condition (C), we have

$$s_n - s_{n_0} \leq s_n' - s_{n_0}',$$

it follows that the quantity $s_n \leq s_n' + (s_{n_0} - s_{n_0}')$ is also bounded and, consequently, the series (A) converges.

Taking the series (B) to be any series whose convergence has already been established, it is often possible to prove that some specific condition on a series (A) implies the relation (C) between its terms and those of the series (B). Such conditions may then serve as criteria for convergence. For example, if for (B) we choose an ordinary geometric progression, we obtain, at once, the simplest tests of convergence, those of d'Alembert, Cauchy, and others.

These, as well as all other tests, in one way or another, require the terms of the series to *decrease rapidly enough as n increases.* For example, the familiar test of d'Alembert requires that the ratio $\frac{U_{n+1}}{U_n}$ be bounded by a number smaller than one for sufficiently large n. This guarantees in an obvious way the sufficiently rapid decrease of u_n as n increases.

We need not discuss here the various tests of this kind; you will find many of them, together with complete proofs, in every textbook. Instead of occupying ourselves with this primarily formal

theory, we shall try to examine more carefully our notions concerning the *rapidity of convergence* of series with positive terms. This problem, though almost never discussed in courses for prospective technicians, actually has a direct practical significance, in addition to its theoretical interest. For if a series converges slowly, that is, if in order to obtain a sum s_n sufficiently close to the limiting value s it is necessary to add a very large number n of terms, such a series cannot be used as a tool for the approximate evaluation of the number s. Therefore, from a practical point of view it is not very useful. But let us observe that occasionally we come across the opposite situation: a divergent series may, under certain conditions, prove to be a very convenient tool for the practical computation of certain quantities. This fact prompted the celebrated scientist Henri Poincaré, who investigated this phenomenon, to express the paradoxical idea that convergent series sometimes turn out to be *practically divergent* and, conversely, divergent series to be *practically convergent.*

For every convergent series, the *remainder* $r_n = s - s_n = u_{n+1} + u_{n+2} + \cdots$ is infinitesimal as n tends to infinity. The rapidity of convergence of the series depends entirely on the order of magnitude of this infinitely small quantity or, more precisely, on how rapidly it tends to zero as n tends to infinity. It is usually considered that a series converges with good rapidity if, as $n \to \infty$, the remainder r_n decreases as rapidly as the nth term of a geometric series. For this to occur, it is sufficient that the terms u_n of the given series should not exceed the corresponding terms of some geometric series for sufficiently large n. Indeed, from the inequality

$$u_n < aq^n \qquad (n \geq n_0),$$

where a and q are constants ($a > 0$, $0 < q < 1$), it follows that for $n \geq n_0$ we also have the inequality

$$r_n < a(q^{n+1} + q^{n+2} + \cdots) = \frac{aq}{1-q} q^n.$$

Much less rapid convergence characterizes the series where the remainder r_n decreases as some negative power of the number n. This convergence is already so slow that the use of such a series for practical purposes often presents difficulties. It should be noted, in this connection, that if u_n is less than an^{-k} (where $a > 0$ and $k > 1$ are constants, and k need not be an integer) for n sufficiently

large, then $r_n < bn^{-k+1}$, where b is some positive constant. For, by applying Lagrange's mean value theorem to the function $f(x) = x^{k-1}$, we obtain for $x > 0$:

$$(x+1)^{k-1} - x^{k-1} = (k-1)(x+\theta)^{k-2} > (k-1)x^{k-2} \quad (0 < \theta < 1).$$

Substituting $x = n - 1$ and dividing both sides by $n^{k-1}(n-1)^{k-1}$, we obtain[1]

$$(n-1)^{-k+1} - n^{-k+1} = \frac{n^{k-1} - (n-1)^{k-1}}{n^{k-1}(n-1)^{k-1}} > \frac{(k-1)(n-1)^{k-2}}{n^{k-1}(n-1)^{k-1}} > \frac{k-1}{n^k}$$

and consequently

$$\frac{1}{n^k} < \frac{1}{k-1}[(n-1)^{-k+1} - n^{-k+1}].$$

If $u_n < an^{-k}$, this gives us

$$u_n < \frac{a}{k-1}[(n-1)^{-k+1} - n^{-k+1}].$$

Therefore,

$$r_n = \sum_{i=1}^{\infty} u_{n+i} < \frac{a}{k-1} \sum_{i=1}^{\infty} [(n+i-1)^{-k+1} - (n+i)^{-k+1}] = \frac{a}{k-1} n^{-k+1},$$

which proves our assertion.

Returning to the problem of tests for convergence, we first make the following observation: in every convergent series we have $u_n \to 0$ (as $n \to \infty$), but even for series with positive terms this convergence of the *general term* u_n to zero need not be monotonic. A simple example is obtained from the geometric series by interchanging the terms u_1 and u_2, u_3 and u_4, and so on, as in the series

$$\frac{1}{2^2} + \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n}} + \frac{1}{2^{2n-1}} + \cdots.$$

However, in the great majority of explicit numerical series with positive terms which occur in analysis, we have $u_{n+1} \leq u_n$

[1] For $1 < k < 2$, the last inequality of the preceding expression must be changed to $(k-1)(x+\theta)^{k-2} > (k-1)(x+1)^{k-2}$. We can, however, still obtain the inequality

$$(n-1)^{-k+1} - n^{-k+1} > \frac{(k-1)}{n^k}.$$

$(n = 1, 2, \ldots)$, that is, u_n decreases monotonically as n increases. For this reason, a group of special tests for convergence of series with positive and monotonically decreasing terms deserves considerable attention; especially so, since this group contains many criteria which are characteristic (necessary and sufficient) and at the same time very convenient to apply. We shall consider closely two tests of this kind.

THEOREM 3. (Cauchy's integral test). *Let $f(x)$ be positive, continuous, and nonincreasing on the half-line $0 \leq x < +\infty$. Then for the convergence of the series*

$$f(1) + f(2) + \cdots + f(n) + \cdots \tag{A}$$

it is necessary and sufficient that the integral

$$\int_0^a f(u)\, du \qquad (a > 0) \tag{B}$$

have a finite limit as $a \to \infty$. (This is obviously equivalent to the condition that the integral *remain bounded as $a \to +\infty$.*)

For convenience in presenting the proof, we have formulated the theorem by taking as our starting point the given function, not the given series. It is clear, however, that for any given series (A) with positive and monotonically decreasing terms one can easily construct any number of positive, continuous, and nonincreasing functions $f(x)$ such that $f(n) = u_n$ (where $n \geq 1$). Hence our theorem is indeed a characteristic test for convergence in series of this type.

Proof. Since $f(x)$ is monotonic (nonincreasing), we have

$$\int_{n-1}^{n} f(u)\, du \geq f(n) \geq \int_n^{n+1} f(u)\, du,$$

from which it follows that

$$\int_0^n f(u)\, du \geq \sum_{k=1}^{n} f(k) \geq \int_1^{n+1} f(u)\, du.$$

These inequalities show immediately that as $n \to \infty$ the quantities

$$\sum_{k=1}^{n} f(k) \text{ and } \int_0^n f(u)\, du$$

are either simultaneously bounded or simultaneously unbounded. This proves the theorem, since for both quantities boundedness is equivalent to the existence of a finite limit.

The application of this test to individual series is in many cases quite simple. Suppose, for example, that $u_n = n^{-\alpha}$ $(\alpha > 0)$. Setting

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ x^{-\alpha} & \text{if } 1 \leq x < +\infty, \end{cases}$$

we obviously obtain a function satisfying the conditions of our theorem. If $\alpha \neq 1$, we have

$$\int_0^a f(x)\,dx = 1 + \int_1^a x^{-\alpha}\,dx = 1 + \frac{a^{1-\alpha} - 1}{1 - \alpha}.$$

Thus, by virtue of the integral test, it follows that our series converges when $\alpha > 1$ and diverges when $\alpha < 1$. If $\alpha = 1$, we have

$$\int_0^a f(x)\,dx = 1 + \int_1^a \frac{dx}{x} = 1 + \ln a \to +\infty$$

as $a \to \infty$, which proves that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

THEOREM 4. *If $u_n \geq u_{n+1} > 0$ $(n \geq 1)$, then for the convergence of the series* (1), *it is necessary and sufficient that the series*

$$u_1 + 2u_2 + 4u_4 + 8u_8 + \cdots + 2^n u_{2^n} + \cdots \tag{4}$$

be convergent.

Proof. Since the terms u_n of the series decrease monotonically, it follows that

$$2^{n-1}u_{2^{n-1}} \geq \sum_{k=2^{n-1}+1}^{2^n} u_k \geq 2^{n-1}u_{2^n}$$

(note that the number of terms in the middle part of the inequality is 2^{n-1}), from which we get

$$\sum_{k=0}^{n-1} 2^k u_{2^k} \geq \sum_{k=1}^{2^n} u_k \geq \frac{1}{2} \sum_{k=1}^{n} 2^k u_{2^k}.$$

These inequalities show immediately that the partial sums of the series (1) and (4) are either simultaneously bounded or simultaneously unbounded, thus establishing the theorem.

For our previous example where $u_n = n^{-\alpha}(\alpha > 0)$, the series (4) has the general term $2^{n(1-\alpha)}$. It now follows that the series $\sum_{n=1}^{\infty} n^{-\alpha}$ converges for $\alpha > 1$ and diverges for $\alpha \leq 1$.

27. ABSOLUTE AND CONDITIONAL CONVERGENCE

We now turn to series whose terms have arbitrary sign. As you well know, it is customary to separate convergent series of this general type into two classes with essentially different properties: absolutely convergent series and conditionally convergent series.

We say that the series (1) is *absolutely convergent* if the series

$$|u_1| + |u_2| + \cdots + |u_n| + \cdots \tag{5}$$

converges. If, however, the series (1) converges, but the series (5) diverges. we say that (1) *converges conditionally.*

We should note, first of all, that the absolute convergence of a given series is defined as the convergence of some other series. Therefore, the following theorem (which of course is well known to you) is by no means trivial:

THEOREM 5. *Every absolutely convergent series converges.*

The meaning of the theorem is that the convergence of the series (5) implies the convergence of the series (1), and the simplest way of proving it is by means of Cauchy's condition.

Proof. For any $n > 0$ and $k > 0$ we have

$$\left| \sum_{i=1}^{k} u_{n+i} \right| \leq \sum_{i=1}^{k} \left| u_{n+i} \right|,$$

and by Cauchy's condition the convergence of the series (5) implies that the right-hand side is arbitrarily small for all sufficiently large n and all k. The same is also true for the left-hand side of the inequality, and the convergence of (1) follows from another application of the Cauchy condition.

Among the properties of absolutely convergent series is an extensive resemblance to finite sums. They can be multiplied term by term like finite sums (the distributive law), their terms can be arranged in an arbitrary order (the commutative law), and their terms can be grouped without affecting the convergence of the series or changing its sum (the associative law). In particular, all these properties are obviously possessed by series with positive terms, whose convergence is always absolute.

We shall not dwell here on the proofs of these properties. They are purely formal, and those given in most textbooks will present no difficulty.

Let us look more carefully, however, at conditionally convergent series. Consider, for example, the well-known series of Leibniz

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots;$$

everything turns out well, the partial sums tend to a limit and the remainder tends to zero (and rather rapidly at that), just as for any convergent series. But how unstable this convergence is! It is enough to change suitably the order of the terms and it will have a different sum, and can even become divergent. And all this because the corresponding series (5), which is the harmonic series in this case, is divergent. You can see that a summation whose result depends upon the order of the terms cannot lead to expressions at all analogous to finite sums. The very term *conditional convergence* probably stems from a desire to indicate that the convergence here is not altogether without reservation.

As to changing the sum and even destroying the convergence of the series by rearranging its terms, any conditionally convergent series offers us unlimited possibilities. This is shown clearly by the following simple but remarkable theorem.

THEOREM 6. *If a series* (1) *is conditionally convergent and* α *is an arbitrary number or one of the symbols* $\pm\infty$, *then, by a suitable rearrangement of the terms of the series, we can always construct a new series which will converge to* α.

For the proof, we establish first a lemma which is also of independent interest.

LEMMA. *If the series* (1) *is conditionally convergent, then both the series* (1′) *formed by its positive terms and the series* (1″) *formed by its negative terms are divergent.*

Proof. For let $s_n = s_n' + s_n''$, where s_n' is the sum of the positive terms and s_n'' the sum of the negative terms which occur in the partial sums s_n. If both series (1′) and (1″) were convergent, then both of the partial sums s_n' and s_n'' would tend to a limit. Hence, the difference

$$s_n' - s_n'' = |u_1| + |u_2| + \cdots + |u_n|$$

would also tend to a limit, contradicting the fact that (1) is not absolutely convergent. If one of the series, say (1′), were divergent

and the other convergent, then as $n \to \infty$ we would have $s_n' \to \infty$ while $s_n'' \to c$, where c is a number. But this would imply that $s_n = s_n' + s_n'' \to \infty$ and that (1) is divergent. Hence both series (1′) and 1″) must necessarily be divergent.

We now turn to the proof of the theorem. There are two parts:

Proof of part one of Theorem 6. α is a number. Suppose, for definiteness, that $\alpha > 0$. We shall arrange the terms of the series in the following order: First, we take positive terms in their natural order, that is, the order in which they appear in the series; the sum of such terms will then be increasing. As soon as this sum becomes larger than α, we shall call it the first pivotal sum and begin to add to it the successive negative terms of the series, again in their natural order. The sum of the terms will now be decreasing, and, as soon as this sum becomes less than α, we call it the second pivotal sum and begin to increase it by adding successive positive terms again. Repeating this process indefinitely, we clearly rearrange the terms of the series (1) into a new order. We shall only have to show that the newly formed series has a sum equal to α. Before doing this, however, we must clarify certain details of the construction just described. How do we know that by taking sufficiently many positive terms of the series we can obtain a sum greater than α? Here we use our lemma. The series formed from the positive terms of the series diverges and, therefore, by taking sufficiently many positive terms we can obtain an arbitrarily large sum, and in particular greater than α. Using this lemma repeatedly in the subsequent steps of our construction, we can be certain each time that the adding of positive or negative terms will eventually adjust the sum in the desired manner.

Having thus convinced ourselves of the feasibility of this construction, we must now show that the partial sums of the new series, which oscillate about α, actually approach this number as a limit. It is evident from our construction that if we take two successive pivotal sums of the new series, then all of the intermediate partial sums will be included between them. Hence, it is enough to show that the sequence of pivotal sums has the limit α. But it is clear that each pivotal sum differs from α by not more than the absolute value of its last term, a quantity which certainly tends to zero in view of the convergence of our initial series. Hence our theorem is established in this case.

Proof of part two of Theorem 6. $\alpha = +\infty$ (if $\alpha = -\infty$ the proof is entirely analogous). Since $u_n \to 0$ as $n \to \infty$, the set of numbers $|u_n|$ (where $n = 1, 2, 3, \ldots$) is bounded. Let β be its l.u.b. We take the sequence of positive terms of the series and subdivide it into blocks so that the sum of the terms of each block is greater than 2β (this is possible in view of our lemma). We now rearrange the terms of the series (1) in the following way: We take the first of the blocks just constructed (in its natural order), then add the first negative term, then the second block, then the second negative term, and so on. Since the sum of the terms of each block is greater than 2β and each negative term is in absolute value not greater than β, it follows that by taking n positive blocks and n negative terms we obtain a sum exceeding $2\beta n - \beta n = \beta n$. From this we see that the partial sums of the newly constructed series increase indefinitely as $n \to \infty$.

We have deliberately dwelt on these constructions. They are both interesting and instructive, and compel us to look closely into the structure of the series, to dissect it logically, so to speak. It would be an interesting and useful exercise for you to show that by a suitable change of the order of the terms of a conditionally convergent series we can obtain a series which has no sum, finite or infinite.

28. INFINITE PRODUCTS

Besides addition, yet another arithmetic operation, that of multiplication, can be applied to an infinite sequence of numbers. As we looked previously for the limit of the sum of an infinitely increasing number of terms, so now we pose the problem of the limit of the product of an infinitely increasing number of factors. And we shall, in analogy with the theory of infinite series, build up the theory of *infinite products.* Although this theory, basically developed long ago, does not yet have the extensive range of applications which the theory of series does, its range of applications is nevertheless already so extensive (and grows more so with every decade) that today this theory belongs in the arsenal of every well-trained mathematician.

An infinite product is an expression of the form

$$z_1 z_2 z_3 \cdots z_n \cdots = \prod_{n=1}^{\infty} z_n. \tag{6}$$

We call the quantities $\pi_n = z_1 z_2 \ldots z_n$ $(n = 1, 2, 3, \ldots)$ partial products, and $\lim_{n\to\infty} \pi_n$, if it exists, is called the *value* of the product (6).

In what follows, we shall constantly assume that all the factors z_n are *positive*. Such a limitation, always imposed in the case of real-valued factors, is not brought about by a desire to simplify the investigation, but by more cogent considerations. In the first place, we may change the sign of the factors to make them all positive without affecting anything but the sign of the limit. But more important, just as the nth term of a convergent series tends to zero, so (as it is natural to expect and as we shall soon demonstrate) the nth factor of a convergent infinite product tends to one. Hence, all the terms from some finite point onward must not only be positive, but arbitrarily close to unity.

When all the factors are positive, the study of infinite products can be reduced to the study of infinite series by the simple process of taking logarithms. Setting $v_n = \ln z_n$ and $z_n = e^{v_n}$, we see that the convergence of the product (6) is equivalent to the convergence of the series

$$v_1 + v_2 + \cdots + v_n + \cdots \tag{7}$$

and that the sum of the series (in the case of convergence) is equal to the logarithm of π, the limit of the product (6). There is only one exception to this rule: if $\pi = 0$, the series (7) is obviously divergent (the partial sums tend to $-\infty$). But infinite products with value zero exhibit in many other respects peculiarities which make them resemble divergent, rather than convergent, series. Thus, if $\pi = 0$ it is not necessary that $z_n \to 1$ as $n \to \infty$ (for example, $z_n = \frac{1}{2}$, $n = 1, 2, \ldots$). For these reasons, the product (6) in the case of $\pi = 0$ is customarily called divergent, and the convergence of the product is defined as the existence of a *positive* limit $\pi = \lim_{n\to\infty} \pi_n$. With this definition, the convergence of the product (6) is precisely equivalent to the convergence of the series (7).

The possibility of completely reducing the theory of infinite products to the study of series might lead us to think that there is no need to construct a special theory of infinite products. But as in many similar instances, such a conclusion is false. As often happens in mathematics, a new way of formulating old problems leads not only to new methods of solving these problems, but also serves a heuristic purpose in suggesting entirely new problems.

In the theory of infinite products it is customary to write the factors z_n in the form $1 + u_n$, where $-1 < u_n < +\infty$. Accordingly, $\pi_n = \prod_{k=1}^{n} (1 + u_k)$ and the product (6) is written in the form

$$(1 + u_1)(1 + u_2)\cdots(1 + u_n)\cdots = \prod_{n=1}^{\infty} (1 + u_n). \tag{8}$$

If the product converges, then as $n \to \infty$ we have $\pi_n \to \pi$, where $0 < \pi < +\infty$, so that as $n \to \infty$

$$z_n = \frac{\pi_n}{\pi_{n-1}} \to 1 \quad \text{and} \quad u_n = \frac{\pi_n}{\pi_{n-1}} - 1 \to 0.$$

A comparison of the product (8) with the series (7) indicates that to any series with positive terms there correspond products for which $z_n \geq 1$ or, what is the same, $u_n \geq 0$ $(n = 1, 2, \ldots)$; while to series with negative terms, there correspond products for which $z_n \leq 1$ and $u_n \leq 0$. This leads us to begin our study with those infinite products for which all the u_n are of the same sign. The fundamental and most frequently used theorem concerning such products is the following.

THEOREM 7. *If all u_n have the same sign, then for the convergence of the product* (8) *it is necessary and sufficient that the series*

$$u_1 + u_2 + \cdots + u_n + \cdots \tag{9}$$

converge.

Proof. First of all, we may assume that $u_n \to 0$ as $n \to \infty$, since otherwise both the product (8) and the series (9) would diverge, and the theorem would clearly hold. Hence, we may distinguish two cases.

(i) Suppose $u_n \geq 0$, $n = 1, 2, \ldots$. Since as $x \to 0$ the function e^x differs from $1 + x$ by an infinitely small quantity of the second order with respect to x, it follows that for sufficiently large n we have the inequalities

$$e^{\frac{1}{2}u_n} \leq 1 + u_n \leq e^{u_n}. \tag{10}$$

Denoting by s_n the partial sums of the series (9) and supposing, for simplicity, that the inequalities (10) are satisfied for all n (this does not restrict the generality of the argument since in problems of convergence we can always, without affecting the result, drop any

finite number of initial terms), we obtain by multiplying these inequalities term by term ($n = 1, 2, \ldots, m$)

$$e^{\frac{1}{2}s_m} \leq \pi_m \leq e^{s_m}.$$

If the series (9) converges, we have $s_m \leq s < +\infty$, whence $\pi_m \leq e^s < +\infty$. Thus the quantity π_m remains bounded and, consequently, the product (8) converges. Conversely, if the product (8) converges, then, as $m \to \infty$ we have $\pi_m \leq \pi < \infty$, which means that

$$\frac{1}{2} s_m \leq \ln \pi < +\infty.$$

Hence, the sum s_n remains bounded and the series (10) converges.

(ii) Let $u_n \leq 0$, $n = 1, 2, \ldots$. As before (but here $s_m \leq 0$), we obtain

$$e^{2s_m} \leq \pi_m \leq e^{s_m},$$

from which, as in the preceding case, we can see that when $s_m \geq s > -\infty$ (that is, when the series (9) converges) we have $\pi_m \geq e^{2s} > 0$ and the product (8) converges. Conversely, when $\pi_m \geq \pi > 0$ we have $s_m \geq \ln \pi_m \geq \ln \pi > -\infty$, i.e., from the convergence of the product (8) there follows the convergence of the series (9).

A consideration of the series (7) associated with the product (6) or (8) leads us directly to Cauchy's condition for infinite products:

THEOREM 8. *For the convergence of the product* (8) *it is necessary and sufficient that for an arbitrarily small* $\varepsilon > 0$ *we have*

$$1 - \varepsilon < \prod_{i=n+1}^{n+k} (1 + u_i) < 1 + \varepsilon \tag{11}$$

for n sufficiently large and any $k > 0$.

For those products where u_n is of constant sign, this criterion can be replaced (by virtue of the preceding theorem) by another which is more convenient in the majority of cases, since it is easier to estimate a sum than a product. We may, without changing any of the verbal part of the statement, simply replace the inequality (11) by the inequality

$$\left| \sum_{i=n+1}^{n+k} u_i \right| < \varepsilon.$$

We must remember, however, that this form of Cauchy's condition is valid only in those cases where all the u_n have the same sign.

Unfortunately, within the framework of these lectures we must limit our discussion of the interesting subject of infinite products to what we have already said.

29. SERIES OF FUNCTIONS

So far, the terms of the series we have considered have been fixed numbers. But, as you know, the primary need in analysis is for *series of functions,* that is, series whose terms are functions of a single variable or of several independent variables. All that we have considered so far will serve as preparatory material for the theory of such series.

Let

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots \tag{12}$$

be a series whose terms are functions of the independent variable x defined on the closed interval $[a, b]$. Giving this variable a numerical value $x = \alpha$, we transform the series (12) into an ordinary numerical series $\sum_{n=1}^{\infty} u_n(\alpha)$. From the point of view of the theory of numerical series, we can now say that the formula (12) expresses not a single series, but an entire continuum of different series.

It is clear, of course, that for series of functions the problem of convergence is quite different from that for numerical series. It is meaningless to ask whether the series (12) converges or not, since, in general, this series will converge for some values of x and diverge for others. A sensible formulation of the problem is: *for what values of* x *in* $[a, b]$ *does the series* (12) *converge and for what values does it diverge?* Let us agree to mean by the *domain of convergence* of the given series the set of values x for which the series converges; and by *domain of divergence,* the set of values x for which it diverges. We can see that the problem of convergence for a series of functions consists, first of all, in finding its domain of convergence. We shall not spend any time here with examples, since we shall encounter enough of them later on.

In all analytic applications of the theory of series of functions, the notion of *uniform convergence* is of fundamental importance. The best way to approach the definition of this concept is as fol-

lows: If the series (12) converges at each point of a set M, then the remainder

$$r_n(x) = s(x) - s_n(x)$$

(which, like the sum $s(x)$ and the partial sums $s_n(x)$, is also a function of x) tends to zero as $n \to \infty$ for each $x \in M$. But for our purposes, we must formulate a more detailed description of this fact. Thus, for any $x \in M$ and for any $\varepsilon > 0$ there is an n_0 (depending, of course, on both ε and x) such that for all $n \geq n_0$ we have $|r_n(x)| < \varepsilon$. Now let ε remain fixed and let x vary in M. To each x there corresponds an n_0, i.e., a *place* in the given series, beyond which $|r_n(x)| < \varepsilon$. But does there exist in this series a *place* beyond which the inequality $|r_n(x)| < \varepsilon$ is satisfied for *all* x in M? This, of course, depends on whether the set of numbers n_0 corresponding to the various values of x is bounded or not. If among these numbers there is a greatest one, it can obviously serve as the *place* beyond which we shall have $|r_n(x)| < \varepsilon$ for *all* x in M. If, however, among the numbers n_0 we can find arbitrarily large values, then, however far we proceed in our series, we shall always find values of x in M for which the *place* corresponding to ε has not yet been reached. In this case it is impossible to find an n_0 applicable to all $x \in M$.

Let us look at an example of this latter type. Suppose that

$$s_n(x) = x^n(1 - x^n) \qquad (0 \leq x \leq 1, n \geq 1),$$

that is, let

$$u_n(x) = x^n(1 - x^n) - x^{n-1}(1 - x^{n-1}).$$

Obviously, for each $x \in [0, 1]$ we have $s_n(x) \to 0$ as $n \to \infty$, so that

$$r_n(x) = -s_n(x) = -x^n(1 - x^n),$$

from which

$$r_n(2^{-\frac{1}{n}}) = -\frac{1}{4} \qquad (n = 1, 2, \ldots).$$

Since, for any $n > 0$ the number $2^{-\frac{1}{n}}$ belongs to the interval $[0, 1]$, it follows that however large n may be, the inequality $|r_n(x)| < \frac{1}{4}$ cannot be satisfied for all $x \in [0, 1]$. This means that we actually have here the second of the cases described above.

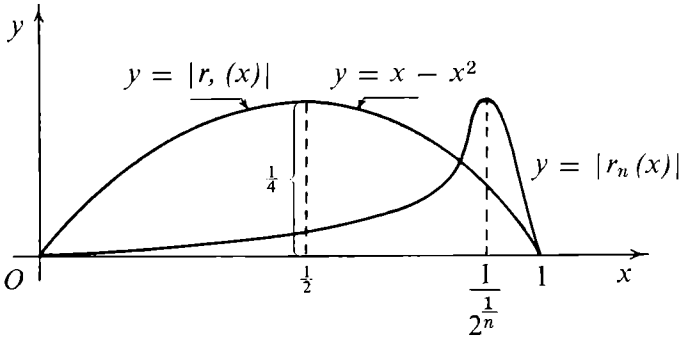

Fig. 16

To illustrate this graphically, we give in Figure 16 the curve $y = |r_1(x)|$ and (schematically) the curve $y = |r_n(x)|$ for a large n. The curve $y = |r_n(x)|$ has a maximum, equal to $\frac{1}{4}$, at $x = 2^{-\frac{1}{n}}$. Everywhere to the left of that point, except for its immediate neighborhood, the quantity $|r_n(x)|$ is extremely small. As n increases the point $2^{-\frac{1}{n}}$ approaches the value 1. From the graph we can see what actually occurs. The fact, paradoxical at first glance, that for each x we have $r_n(x) \to 0$ as $n \to \infty$, while for every n there exists an x such that $|r_n(x)| = \frac{1}{4}$ can be explained by noting that the point at which $|r_n(x)|$ attains this maximum value does not remain unchanged, but rather moves to the right, approaching the value 1 as $n \to \infty$. This *delay* in the approach of $r_n(x)$ to zero moves, therefore, closer and closer to the point 1 as n increases. We naturally expect that at the point 1 itself, the *arrival* of this *delay* will cause the divergence of the series at this point. But in reality everything turns out well since $r_n(1) = 0$ for all n.

Let us now return to the two cases which we previously agreed to distinguish. In the first of them we say that the series (12) converges *uniformly* in the set M, and in the second that it is *nonuniformly* convergent. That is, *the series* (12) *is uniformly convergent in a set* M *if for an arbitrarily small positive* ε, *there exists an* n_0 (*depending only on* ε) *such that for all* $n \geq n_0$ *and* for all $x \in M$ *we have*

$$|r_n(x)| < \varepsilon.$$

We have discussed above in detail an example presenting a typical picture of *nonuniform* convergence.

Let us observe that we can also define *uniform convergence of a sequence of functions*

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots$$

to a limit function $f(x)$ in the same way if $r_n(x)$ is understood to mean the difference $f(x) - f_n(x)$. Thus, we can say that the uniform convergence of a series is equivalent to the uniform convergence of the sequence of its partial sums.

Just as *absolute* convergence is important for arithmetical operations with series, so the hypothesis of *uniform* convergence is not only convenient, but sometimes even indispensable for a great many conclusions of an analytic nature. We shall now look at a few examples demonstrating this.

First, let us consider the following problem: under what conditions can we deduce the continuity of the sum $s(x)$ of the series (12) in the interval $[a, b]$ from the continuity of its terms? That the sum $s(x)$ may be discontinuous even when its terms are continuous is shown by the following example:

$$u_n(x) = x^{n-1} - x^n \qquad (n \geq 1,\ 0 \leq x < 1),\ [u_1(0) = 1]$$
$$s_n(x) = 1 - x^n \qquad (n \geq 1),$$
$$s(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

We shall now prove the following:

THEOREM 9. *If the series of functions* (12) *converges uniformly, then the continuity of its terms implies the continuity of its sum.*

Proof. Let the series (12) converge uniformly in $[a, b]$ and let all its terms $u_n(x)$, and so all its partial sums $s_n(x)$, be continuous; let α be any point in $[a, b]$ and let ε be an arbitrarily small positive number. By virtue of the uniform convergence of (12), we can choose n sufficiently large so that

$$|s_n(x) - s(x)| < \frac{\varepsilon}{3}$$

for all $x \in [a, b]$. Fixing temporarily this value of n and using the continuity of $s_n(x)$ in $[a, b]$ and, in particular, at the point α, we can assert that for all x in a neighborhood U of α we have the inequality

$$|s_n(x) - s_n(\alpha)| < \frac{\varepsilon}{3}.$$

Then, for each $x \in \dot{U}$ we have

$$|s(x) - s(\alpha)| \leq |s(x) - s_n(x)| + |s_n(x) - s_n(\alpha)| + |s_n(\alpha) - s(\alpha)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which shows that $s(x)$ is continuous at $x = \alpha$.

The converse of the theorem just proved is not true. The sum of a series may in some cases be a continuous function even though the series does not converge uniformly. This can be observed in the example given on page 89, where the series is non-uniformly convergent while its sum is equal to zero in the whole interval under consideration. Thus, the uniform convergence of a series of continuous functions, though it guarantees the continuity of its sum, is not a *necessary* prerequisite for this continuity. A theory somewhat more fully developed than the considerations we have so far presented makes it possible to determine a type of convergence which constitutes both a necessary and sufficient condition for the continuity of the sum. In practice, however, we nearly always deduce the continuity of the sum from the uniform convergence of the series.

Another problem in which the idea of uniform convergence finds application is the problem of *term by term integration* of a series of functions. Let us assume that all the terms of the series (12) are continuous (and thus integrable) functions on a closed interval $[a, b]$. Can we assert that the sum $s(x)$ of the series is also integrable over this interval and that

$$\int_a^b s(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx \tag{13}$$

as in the case of finite sums? It is easy to see that this is true in all the examples we have considered so far. The relation (13) is obviously equivalent to either of the relations

$$\lim_{n\to\infty} \int_a^b s_n(x)\,dx = \int_a^b s(x)\,dx$$

or

$$\lim_{n\to\infty} \int_a^b r_n(x)\,dx = 0.$$

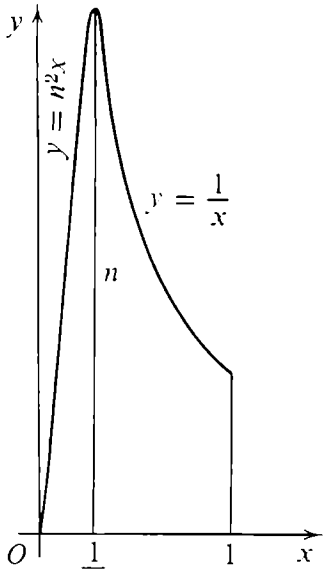

Fig. 17

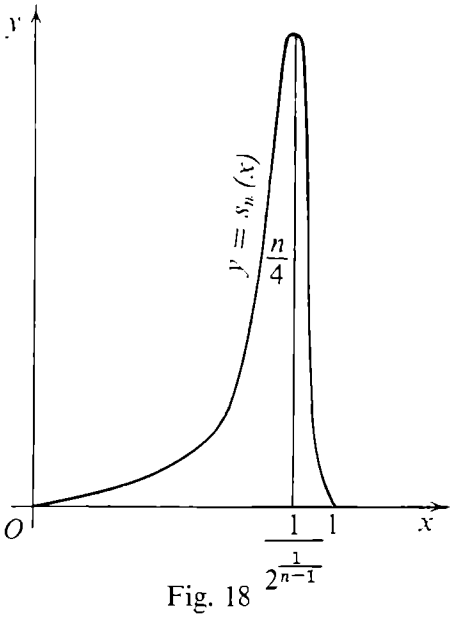

Fig. 18

It is possible, however, to show that these relations are not always valid. First of all, it may turn out that $s(x)$ is not integrable over $[a, b]$. To show this it is sufficient to take the function (Fig. 17)

$$s_n(x) = \begin{cases} n^2x & \text{if } 0 \leq x \leq \frac{1}{n}, \\ \frac{1}{x} & \text{if } \frac{1}{n} \leq x \leq 1, \end{cases}$$

(you can easily find, if you wish, the corresponding expressions for $u_n(x)$). Here the sum

$$s(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$$

is, as you know, nonintegrable over [0, 1] since

$$\int_a^1 s(x)\,dx = \ln \frac{1}{a} \to \infty \text{ as } a \to 0.$$

Secondly, it may happen that $s(x)$ is integrable over $[a, b]$, yet (13) turns out to be false. For example (Fig. 18), let

$$s_n(x) = nx^{n-1}(1 - x^{n-1}) \qquad (n \geq 1,\ 0 \leq x \leq 1).^1$$

[1] Here again, $s_1(0) = 1$.

Since for $0 \leq x < 1$ we have $\lim_{n\to\infty} nx^{n-1} = 0$,[1] it follows that $s(x) = 0 (0 \leq x \leq 1)$; hence

$$\int_0^1 s(x)\,dx = 0.$$

At the same time (as can easily be computed), we have

$$\int_0^1 s_n(x)\,dx = n\int_0^1 (x^{n-1} - x^{2n-2})\,dx = \frac{1}{2} - \frac{1}{2(2n-1)} \to \frac{1}{2},$$

and so (13) does not hold.

But it is easy to prove the following:

THEOREM 10. *For every uniformly convergent series of continuous functions, the relation* (13) *is valid.*

Proof. First of all, we know that in this case $s(x)$ is continuous and therefore integrable over $[a, b]$. Further, for every $\varepsilon > 0$ there exists an integer n_0 such that for $n \geq n_0$ and any $x \in [a, b]$ we have

$$|r_n(x)| < \varepsilon.$$

Applying the well-known theorem from the integral calculus by which the inequality

$$|f(x)| \leq \varphi(x) \qquad (a \leq x \leq b)$$

implies the inequality

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b \varphi(x)\,dx,$$

we have

$$\left|\int_a^b r_n(x)\,dx\right| \leq \varepsilon(b-a), \text{ for } n \geq n_0,$$

[1] An elementary proof of this property is as follows:

Let $0 < x < 1$, $\frac{1}{x} = z$, and $z - 1 = y > 0$. Then

$$z^n = (1+y)^n > 1 + ny + \frac{n(n-1)}{2}y^2 > \frac{n(n-1)}{2}y^2,$$

from which

$$nx^n = \frac{n}{z^n} < \frac{2}{(n-1)y^2} \to 0, \text{ as } n \to \infty.$$

and this, in view of the arbitrary choice of ε, means that

$$\int_a^b r_n(x)\,dx \to 0 \quad \text{as} \quad n \to \infty,$$

which is equivalent to (13).

Although uniform convergence is a sufficient condition for the term by term integrability of a series of continuous functions, it is, as in the problem of continuity of the sum, not a necessary condition. To show this, it is enough to consider again the example on p. 89. Here the convergence is nonuniform, but at the same time we have

$$\int_0^1 r_n(x)\,dx = \int_0^1 x^{2n}\,dx - \int_0^1 x^n\,dx = \frac{1}{2n+1} - \frac{1}{n+1} \to 0$$

as $n \to \infty$, that is, (13) holds true.

The uniform convergence of a given series is most frequently established with the aid of the following simple test:

THEOREM 11. *If* $\sum_{n=1}^{\infty} \alpha_n$ *is a convergent series with positive terms and if for n sufficiently large and all* $x \in M$ *we have*

$$|u_n(x)| \leq \alpha_n,$$

then the series (12) *converges uniformly on M.*

Proof. Under the conditions of our hypothesis, we have

$$\left|\sum_{i=n+1}^{n+k} u_i(x)\right| \leq \sum_{i=n+1}^{n+k} \left|u_i(x)\right| \leq \sum_{i=n+1}^{n+k} \alpha_i < \varepsilon$$

for a sufficiently large n, any $k > 0$, and all x in M. (The last of this string of inequalities follows from the convergence of $\sum_{n=1}^{\infty} \alpha_n$ and Cauchy's condition.) Applying Cauchy's condition, it follows that the series (12) converges. Letting $k \to \infty$ in the inequality $\left|\sum_{i=n+1}^{n+k} u_i(x)\right| < \varepsilon$, we see that for sufficiently large n and for all $x \in M$ we have

$$|r_n(x)| \leq \varepsilon.$$

Hence the convergence is uniform.

30. POWER SERIES

Without a doubt, the most important special class of series of functions consists of the so-called *power series,* that is, series of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots, \tag{14}$$

where $a_0, a_1, a_2, \ldots, a_n, \ldots$ are given real numbers. We shall now examine some of the most important properties of these series.

To begin with, for any given power series there always exists an open interval $(-r, r)$, the so-called *interval of convergence,* within which the given series converges and outside of which it diverges (with the possible exception of the points $-r$ and r). The *radius of convergence* r may have any value from 0 to ∞, not excluding either of these limiting values (for example, $r = \infty$ when $a_n = \frac{1}{n!}$ and $r = 0$ when $a_n = n!$). When the radius of convergence is given, the interval of convergence is determined only up to its two end points and may be either closed, open, or half-open (i.e. it may include one of its end points, but not the other). For example, each of the three series with the respective coefficients $a_n = 1$, $a_n = \frac{1}{n+1}$, and $a_n = \frac{1}{(n+1)^2}$ has a radius of convergence equal to 1. But the first of these series diverges for $x = 1$ and $x = -1$, that is, for both end points of the interval of convergence (this series is a geometric progression with the ratio x); the second series reduces to the harmonic series for $x = 1$ and to Leibniz's series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} + \cdots$$

for $x = -1$, and thus it is divergent at $x = 1$ and conditionally convergent at $x = -1$; and the third series is absolutely convergent at both end points of the interval of convergence.

We shall recall briefly the proof of our assertions concerning the domain of convergence of a power series. It is based on the following remarkable theorem.

THEOREM 12. *If a power series is convergent at $x = \alpha$, where $\alpha \neq 0$, then it is absolutely convergent at every value of x for which $|x| < |\alpha|$.*

Proof. From the convergence of the series (14) for $x = \alpha$ it follows that $a_n\alpha^n \to 0$ as $n \to \infty$. This implies that there exists a number $c > 0$ such that $|a_n\alpha^n| < c$ $(n = 1, 2, \ldots)$. Let us now take $|x| < |\alpha|$; then

$$\sum_{k=1}^{n} \left|a_k x^k\right| = \sum_{k=1}^{n} \left|a_k\alpha^k\right| \cdot \left|\frac{x}{\alpha}\right|^k$$

$$< c \sum_{k=1}^{n} \left|\frac{x}{\alpha}\right|^k < c \sum_{k=1}^{\infty} \left|\frac{x}{\alpha}\right|^k = \frac{c}{1 - \left|\frac{x}{\alpha}\right|}.$$

Since the right-hand side does not depend on n, the left-hand side remains bounded as $n \to \infty$, and thus our series is absolutely convergent.

We obtain at once from this theorem the special form of the domain of convergence for a power series, since, given any point x in this domain, all the points lying closer to zero also belong to the domain. Another corollary of this theorem is that at every *interior* point of the interval of convergence the series is *absolutely* convergent. As to uniform convergence, we cannot assert that it holds for the whole interval of convergence. This is apparent from the example of the geometric series ($a_n = 1$): no matter how large n may be, the remainder $r_n(x) = \sum_{k=n+1}^{\infty} x^k$ becomes arbitrarily large when x is sufficiently close to 1. It is easy, however, to show the following:

THEOREM 13. *A power series is uniformly convergent in every closed interval whose end points are interior to the interval of convergence.*

Proof. To show this, let r be the radius of convergence of the series (14) and let $0 < r' < r$; for all x satisfying the inequality $|x| \leq r'$ and for any n we have

$$|a_n x^n| \leq |a_n| r'^n.$$

Since the series $\sum_{n=1}^{\infty} |a_n| r'^n$ is a convergent series with positive terms, it follows (in view of Theorem 11 on page 95) that (14) is uniformly convergent in the interval $[-r', r']$.

From this there follows the highly important corollary.

COROLLARY. *The sum of a power series is continuous at every point within the interval of convergence.*

ABEL'S THEOREM. *If a power series* (14) *is convergent at* $x = r > 0$, *then it is uniformly convergent in the interval* $[0, r]$.

This remarkable theorem plays an important role in the theory of functions. To prove it, we first establish the following lemma, which is likewise due to Abel:

LEMMA.

$$\sum_{n=n_1}^{n_2} a_n(b_n - b_{n-1}) = \sum_{n=n_1}^{n_2} b_n(a_n - a_{n+1}) - a_{n_1}b_{n_2-1} + a_{n_2+1}b_{n_2},$$

where $0 < n_1 < n_2$ *and the sequences* a_k *and* b_k *are arbitrary.*

Proof of Lemma.

$$\begin{aligned}
&\sum_{n=n_1}^{n_2} a_n(b_n - b_{n-1}) \\
&= a_{n_1}(b_{n_1} - b_{n_1-1}) + a_{n_1+1}(b_{n_1+1} - b_{n_1}) + a_{n_1+2}(b_{n_1+2} - b_{n_1+1}) \\
&\quad + \cdots + a_{n_2-1}(b_{n_2-1} - b_{n_2-2}) + a_{n_2}(b_{n_2} - b_{n_2-1}) \\
&= -a_{n_1}b_{n_1-1} + b_{n_1}(a_{n_1} - a_{n_1+1}) + b_{n_1+1}(a_{n_1+1} - a_{n_1+2}) \\
&\quad + \cdots + b_{n_2-1}(a_{n_2-1} - a_{n_2}) + b_{n_2}a_{n_2} \\
&= \sum_{n=n_1}^{n_2} b_n(a_n - a_{n+1}) - a_{n_1}b_{n_1-1} + a_{n_2+1}b_{n_2}.
\end{aligned}$$

We now turn to the proof of the theorem itself.

Proof of Abel's Theorem. Let ε be any positive number. By virtue of Cauchy's condition, we have

$$\left| \sum_{i=n+1}^{n+k} a_i r^i \right| < \varepsilon$$

for all sufficiently large n and any $k > 0$. For brevity, let us set $\sum_{i=n+1}^{n+k} a_i r^i = \sigma_k$ and $\sigma_0 = 0$, so that $|\sigma_k| < \varepsilon$ $(k = 0, 1, 2, \ldots)$. Then, for $0 \leq x \leq r$ we have

$$\begin{aligned}
\sum_{i=n+1}^{n+k} a_i x^i &= \sum_{i=n+1}^{n+k} a_i r^i \left(\frac{x}{r}\right)^i \\
&= \sum_{i=1}^{k} a_{i+n} r^{i+n} \left(\frac{x}{r}\right)^{i+n} = \sum_{i=1}^{k} (\sigma_i - \sigma_{i-1}) \left(\frac{x}{r}\right)^{i+n}.
\end{aligned}$$

Therefore, by the lemma just proved (and the fact that $\sigma_0 = 0$) we have

$$\begin{aligned}\left|\sum_{i=n+1}^{n+k} a_i x^i\right| &= \left|\sum_{i=1}^{k} \sigma_i\left\{\left(\frac{x}{r}\right)^{n+i} - \left(\frac{x}{r}\right)^{n+i+1}\right\} + \sigma_k\left(\frac{x}{r}\right)^{n+k+1}\right| \\ &\leq \sum_{i=1}^{k} |\sigma_i|\left(\frac{x}{r}\right)^{n+i}\left(1 - \frac{x}{r}\right) + |\sigma_k|\left(\frac{x}{r}\right)^{n+k+1} \\ &\leq \varepsilon\left\{\left(1 - \frac{x}{r}\right)\sum_{i=1}^{k}\left(\frac{x}{r}\right)^{n+i} + \left(\frac{x}{r}\right)^{n+k+1}\right\} \\ &= \varepsilon\left(\frac{x}{r}\right)^{n+1} \\ &\leq \varepsilon.\end{aligned}$$

It follows, by Cauchy's condition, that the series (14) converges, and passing to the limit as $k \to \infty$ we see from the last inequality that for sufficiently large n

$$|r_n(x)| < \varepsilon \qquad (0 \leq x \leq r),$$

which means, precisely, that the series (14) converges uniformly in the closed interval $[0, r]$.

It should be stressed, however, that from the convergence of (14) at $x = r$ it would be false to conclude that the series converges uniformly in the interval $(-r, r)$. For example, the series

$$1 - \frac{x}{1} + \frac{x^2}{2} - \frac{x^3}{3} + \cdots$$

converges at $x = 1$. If this series were uniformly convergent in the open interval $(-1, 1)$, then for any $\varepsilon > 0$ we would have

$$\left|\sum_{n=m}^{m+k} \frac{(-1)^n x^n}{n}\right| < \varepsilon$$

for sufficiently large m, all $k > 0$, and all x such that $-1 < x < 1$. Letting $x \to -1$, we obtain $\left|\sum_{n=m}^{m+k} \frac{1}{n}\right| \leq \varepsilon$ for m sufficiently large and all $k > 0$. But this contradicts the divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

It is immediately clear that one of the most important problems in connection with power series consists in determining the radius of convergence r from the given coefficients a_n. As you prob-

ably know, the elementary criteria for convergence of numerical series with positive terms make the solution of this problem possible under certain hypotheses. For example, if there exists $\lim_{n\to\infty} \sqrt[n]{|a_n|} = l$, then $r = \frac{1}{l}$, $+\infty$, or 0 according to whether $0 < l < +\infty$, $l = 0$, or $l = +\infty$ respectively. A more complete solution to the problem is given in the following theorem.

THEOREM 14. *Let* $l = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$. *Then*

$$r = \begin{cases} \frac{1}{l} & \text{if } 0 < l < +\infty, \\ 0 & \text{if } l = +\infty, \\ +\infty & \text{if } l = 0 \end{cases}$$

The indicated upper limit (finite or infinite) exists for every series, and thus we have a solution of the given problem without any restrictive hypotheses on the coefficients. Before taking up the proof of the theorem, however, let us recall that if the number l is the upper limit of a sequence, then for any positive ε all the terms of that sequence are smaller than $l + \varepsilon$ from a certain point on; on the other hand, arbitrarily far in the given sequence we may find terms greater than $l - \varepsilon$.

Proof of Theorem 14. Let $0 < l < +\infty$. We set $r = \frac{1}{l}$ and we must prove that the series (14) converges for $0 < x < r$ and diverges for $x > r$.

Consider $0 < x < r$, so that $lx < 1$. Clearly, it is possible to find a positive number ε so small that $(l + \varepsilon)x < 1$. From the definition of l, it follows that for n sufficiently large we have

$$\sqrt[n]{|a_n|} < l + \varepsilon, \quad \text{or} \quad |a_n| < (l + \varepsilon)^n,$$

and therefore

$$|a_n| x^n < [(l + \varepsilon)x]^n.$$

Since $(l + \varepsilon)x < 1$, the terms of the series (14) for sufficiently large n do not exceed in their absolute value the corresponding terms of a convergent geometric progression. From this there follows the convergence of (14).

Now consider $x > r$, so that $lx > 1$. For a sufficiently small positive ε we shall have $(l - \varepsilon)x > 1$. From the definition of l it follows that there exist arbitrarily large values of n for which $\sqrt[n]{|a_n|} > l - \varepsilon$, from which it follows that $|a_n| > (l - \varepsilon)^n$, and therefore

$$|a_n| x^n > [(l - \varepsilon)x]^n > 1.$$

Hence the series (14) contains infinitely many terms whose absolute value exceeds one. Therefore, the general term of the series cannot be infinitely small and the series is divergent.

Let $l = 0$. We have to prove that the series (14) converges for every positive value of x. From the definition of l, it follows that for n sufficiently large we have the inequalities

$$\sqrt[n]{|a_n|} < \frac{1}{2x}, \quad |a_n| < \frac{1}{2^n x^n}, \text{ and } |a_n| x^n < \frac{1}{2^n};$$

that is, the nth term of our series for n sufficiently large is smaller in absolute value than the nth term of a convergent geometric series. Therefore the series (14) converges.

Let $l = +\infty$. We have to prove that the series diverges for every positive value of x. From the definition of l it follows that for n arbitrarily large there occurs the inequality

$$\sqrt[n]{|a_n|} > \frac{1}{x} \quad \text{or} \quad |a_n| x^n > 1.$$

Again, the general term of the series fails to approach zero as $n \to \infty$ and the series is divergent. Thus the theorem is established.

5. The Derivative

31. THE DERIVATIVE AND DERIVATES

Up to now, we have been preoccupied either with auxiliary analytic constructions or with the analysis of fundamental concepts. We now pass to the central edifice of mathematical analysis, the theory of differentiation and integration.

Let $y = f(x)$ be defined in a neighborhood of the point $x = \alpha$. If from that point we pass to the point $\alpha + h$, the function y will acquire an *increment* $f(\alpha + h) - f(\alpha)$. If we wish to get an idea of how fast y changes as x changes, that is, to what degree the function $f(x)$ is *sensitive* to a variation of its independent variable, we must compare in some manner the increment of y with the change h in x. For this purpose, it is most natural to consider the ratio

$$\frac{f(\alpha + h) - f(\alpha)}{h}, \tag{1}$$

which gives us the average change of y per unit of change of x. This computation must, however, be performed for a particular value of h and, in general, will give different results for different h. In order to obtain a unique solution for the given problem, we would have to agree to choose the quantity h according to some uniform rule.

If our purpose is to investigate the behavior of $f(x)$ in the immediate vicinity of the point α, then it is obvious that the smaller we choose $|h|$ the better the quantity (1) will serve as a measure of the *variability* of $f(x)$ at this point. For the quantity (1) characterizes the *average variability* of the function in the interval $[\alpha, \alpha + h]$, and the smaller the value of h, the more closely does this interval adhere to the point α. Since we are now well acquainted with the concept of passage to a limit, it is only a step further to the following realization. We shall obtain the best solution of our problem if

we choose as the characteristic of variability in the immediate neighborhood of α the limit

$$f'(\alpha) = \lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h}$$

(assuming, of course, that this limit exists).

DEFINITION. *This limit is called the* derivative *of* $f(x)$ *at the point* α, *or at* $x = \alpha$.[1]

Thus, the derivative of a function at a given point is used to characterize the relative rate of change of the function in the immediate neighborhood of that point: the greater the absolute value of $f'(\alpha)$, the more sensitive is $f(x)$ to small variations of x from its initial value α. The sign of $f'(\alpha)$ characterizes the direction of the change: the quantity $f'(\alpha)$ is positive or negative depending upon whether $f(x)$ increases or decreases when x acquires small positive increments to its initial value α. If we represent the function $y = f(x)$ graphically, then the rate of change we are considering will be expressed by the steepness of ascent or descent of the curve traced as x passes through the value α. In more precise terms, the derivative is equal to the slope of the tangent to the curve $y = f(x)$ at $x = \alpha$.

The derivative finds its simplest concrete interpretation when the independent variable x denotes *time*. The quantity (1) then represents the average rate of change of y during the time interval $[\alpha, \alpha + h]$, and the derivative $f'(\alpha)$ represents the rate of this change at the moment α. In particular, if $y = f(x)$ denotes the distance traversed by a moving point in the time interval from a certain fixed moment a to the moment x, then the concept of the derivative is identical to the concept of instantaneous velocity in mechanics.

In the mathematical treatment of the natural sciences and in other applications of analysis, the role played by the derivative is of great importance. It describes the local behavior of a phenomenon in a highly important respect: it measures the variability of one of two related quantities with respect to the other.

[1] Another generally accepted system of notation is:

$$h = \Delta x, f(\alpha + h) - f(\alpha) = \Delta y, \text{ and } f'(\alpha) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

The whole computation, together with all the reasoning which we have applied to a single point α (arbitrarily chosen) in the interval $[a, b]$, may be repeated for any point x in this interval (provided, of course, that in each case the limit in question exists). It is the function $f'(x)$ obtained in this way that is called the *derived function of* f. It should be clear that what we have just said does not in the least alter the fundamental fact that the derivative is a *local* characteristic of the given function, computed separately for each individual point, and used to describe the behavior of this function not in the whole interval but solely in the immediate neighborhood of its individual points.

DEFINITION. *We call a function* $f(x)$ differentiable *at* $x = \alpha$ *if this function has a derivative at the point* α; *the function is* differentiable in the interval $[a, b]$ *if the derivative exists at each point interior to the interval.*

It is obvious that differentiability, like continuity, is a local property of the function. As you know, differentiability of a function at $x = \alpha$ implies its continuity at this point. This is evident from the fact that, as the denominator of the fraction (1) tends to zero it is necessary, if the limit is to exist, that the numerator also tend to zero. This at once implies the continuity of $f(x)$ at α. You no doubt also know that the converse is not true; a continuous function need not have a derivative. For example, the function $f(x)$ which is represented graphically in Figure 19 (for which there

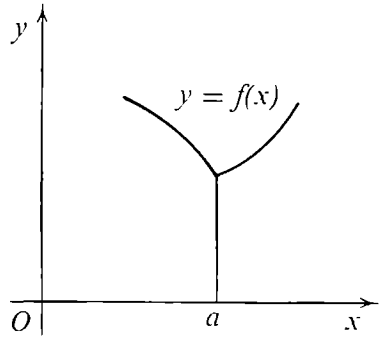

Fig. 19

is no need to give an analytical expression) is obviously continuous. Yet, at $x = \alpha$ the two limits for (1) as $h \to +0$ and $h \to -0$ are unequal. (Geometrically, this is expressed by the fact that at $x = \alpha$ the curve does not have a unique tangent.) Consequently, no limit exists as $h \to 0$.

The limits of (1) as $h \to +0$ and $h \to -0$, if they exist, are called, respectively, the *right-hand* and *left-hand derivatives* of $f(x)$ at $x = \alpha$. For a function to be differentiable at a point α, it is evidently necessary and sufficient that the right-hand and left-hand derivatives both exist and be equal. Figure 19 gives us the simplest example of the failure of a continuous function to have a derivative at some point: however, both the right-hand and left-hand derivatives exist. It is natural to ask whether we always have such a situation, and it is easy to see that there are cases where the function is nondifferentiable in a much deeper sense. Figure 20 represents

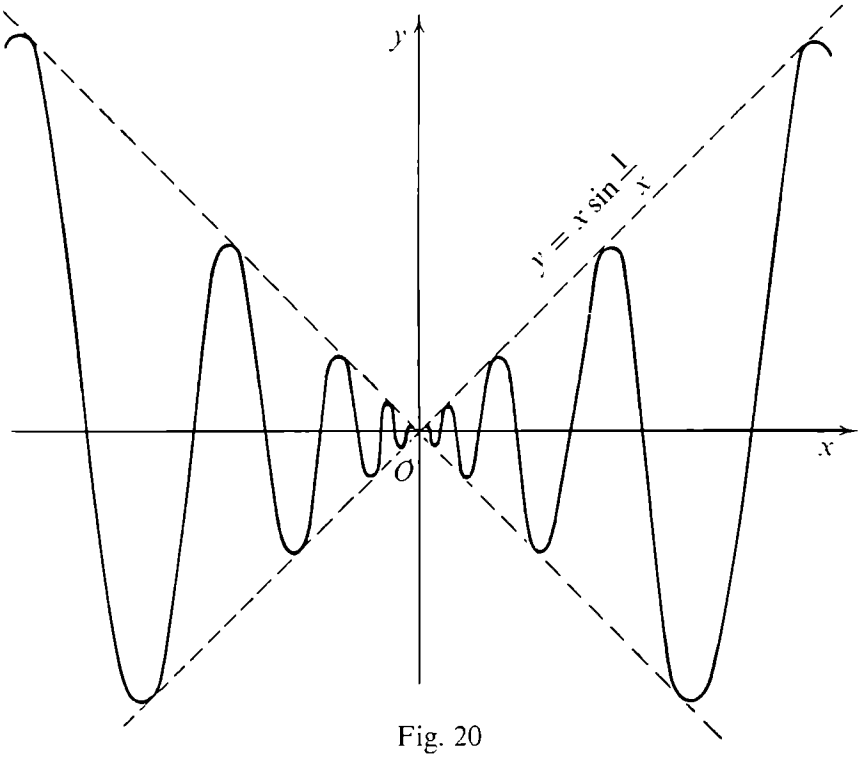

Fig. 20

graphically the behavior of the function $y = x \sin \frac{1}{x}$ in the vicinity of $x = 0$. Since we have

$$\left| x \sin \frac{1}{x} \right| < |x| \to 0$$

as $x \to 0$, it follows (setting $y = 0$ at $x = 0$) that the function y is continuous at the point 0. As x tends to zero (say, from the right) $\frac{1}{x} \to +\infty$ and, consequently, $\sin \frac{1}{x}$ oscillates infinitely many times

between $+1$ and -1. As a result of this, $x \sin \frac{1}{x}$ oscillates infinitely many times between the lines $y = x$ and $y = -x$. The expression (1), which in the present case is equal to $\sin \frac{1}{x}$ (for $\alpha = 0$ and $h = x$), has as partial limits all the numbers in the interval $[-1, +1]$; its upper limit is equal to $+1$, its lower limit to -1. And precisely the same thing happens as $x \to 0$ from the left. Hence, we have in this case neither a right-hand nor a left-hand derivative at $x = 0$.

DEFINITION. *The* derivates (*or derived numbers*) *of the function at the given point are defined as the four limits: an upper limit and a lower limit of the expression* (1) *as* $h \to +0$ *and as* $h \to -0$. *Each limit may be either a number or* $\pm\infty$.

Thus, every function has at each point (interior to a neighborhood in which it is defined) four derivates: right upper, right lower, left upper, and left lower. If the two right (left) derivates coincide, then the function has, at the given point, a right-hand (left-hand) derivative. When all four of the derivates are finite and equal (and only in this case), the function is differentiable at the given point. An example in which all four derivates are infinite is given for the point $x = 0$ by the function

$$y = \begin{cases} \sqrt{|x|} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

as you can easily verify.

If we consider that a function may have totally different types of differentiability at different points, we can see how complicated this aspect of a function's behavior can become. Indeed, phenomena similar to those indicated in Figure 20 are not necessarily confined to individual isolated points: there exist continuous functions having equally complicated structure in the vicinity of *every* point (and which are, consequently, nowhere differentiable). Unfortunately, the scope of these lectures does not permit us to dwell on any of the large number of examples of this kind which have been constructed.

Instead, let us consider an example of the application of derivates. The reader knows, of course, that the behavior of differentiable functions is closely connected with the sign of the deriva-

tive. In particular, if for all the points in an interval $[a, b]$ we have $f'(x) \geq 0$ (or ≤ 0), then the function $f(x)$ is nondecreasing (or nonincreasing) in this interval. These criteria certainly leave nothing to be desired where the given function has a derivative everywhere in the interval. But in the case of nondifferentiable functions they, of course, give us no information. On closer examination, however, we find that a criterion of this kind exists even for completely nondifferentiable functions.

THEOREM 1. *Let $f(x)$ be continuous in the interval $[a, b]$ and let one of the four derivates, we shall call it $Df(x)$, be nonnegative for all x in this interval, then $f(b) \geq f(a)$.*

It is obvious that this criterion is substantially stronger than the one given above, since it is applicable to any continuous, and in general nondifferentiable, function.

Proof. Let us assume, contrary to the assertion of the theorem, that $f(b) < f(a)$, and let ε be a number such that $\frac{f(a) - f(b)}{b - a} > \varepsilon > 0$; for definiteness, let $Df(x)$ be the right upper derivate of $f(x)$. We set $\varphi(x) = f(x) - f(a) + \varepsilon(x - a)$; clearly, $\varphi(a) = 0$ and

$$\varphi(b) = (b - a)\left(\varepsilon - \frac{f(a) - f(b)}{b - a}\right) < 0. \qquad (2)$$

Let M be the set of points in $[a, b]$ at which $\varphi(x) = 0$, and let α be the l.u.b. of this set. If we had $\varphi(\alpha) > 0$ (or $\varphi(\alpha) < 0$), the point α would be contained in a neighborhood U where $\varphi(x) > 0$ for all x in U (or, correspondingly, $\varphi(x) < 0$ for all x in U). On the other hand, by the definition of least upper bound, every neighborhood of α contains points at which $\varphi(x) = 0$. This contradiction shows that $\varphi(\alpha) = 0$. Now, since $\varphi(b) < 0$, we clearly have $\varphi(x) < 0$ for $\alpha < x < b$, and for any such value of x we have

$$\frac{\varphi(x) - \varphi(\alpha)}{x - \alpha} < 0.$$

It follows that $D\varphi(\alpha) \leq 0$. But

$$D\varphi(\alpha) = Df(\alpha) + \varepsilon,$$

whence

$$Df(\alpha) = D\varphi(\alpha) - \varepsilon < 0,$$

which contradicts our assumption. The theorem is thus proved.

32. THE DIFFERENTIAL

The next fundamental notion in the theory of differentiation is that of the *differential.* Today we consider the differential a secondary concept defined by means of the derivative, but it was not always so. At the period of inception of the infinitesimal calculus and for a long time thereafter, it was the differential which was considered the primary notion in analysis. The derivative was defined as the ratio of differentials, that is to say, as a secondary concept. In so doing, the notion of the differential was often left without a precise definition. It even contained within itself contradictory features, since, at that time, mathematical thought, though capable of grasping as a single object of thought a variable quantity in its process of change, had not yet attained sufficient development.

You know, of course, the formal definition of the differential:

DEFINITION. *The differential of* $y = f(x)$ *is the quantity*

$$dy = f'(x)\,\Delta x \tag{3}$$

where Δx *is the increment of the independent variable.*

The differential of $f(x)$ is thus a function of two variables, the quantity x and its increment Δx. The values of these two variables are not connected in any way and can be chosen independently of each other.

Taking, in particular, the function $y = x$ we see that $dx = \Delta x$, i.e., for the independent variable the differential and the increment always coincide. Substituting dx for Δx in (3), we find that

$$dy = f'(x)\,dx.$$

This gives

$$y' = f'(x) = \frac{dy}{dx},$$

that is, the derivative is equal to the ratio of the differential of the function and the differential of the independent variable.

However, all these purely formal considerations fail to bring out the tremendous significance which the notion of the differential has for analysis and its applications. To get a better understanding of this matter, it is necessary to look more closely into the essence of this notion. Perhaps it will be most enlightening if we begin by considering the special case in which x denotes time and $y = f(x)$

denotes the distance traversed by a moving point in the time interval from 0 to x. In this case, as we know, the derivative $f'(x)$ signifies the instantaneous velocity of the moving point at the moment x. It follows that $dy = f'(x)\,\Delta x$ is the length of the path which the moving point would cover in the time interval $[x, x + \Delta x]$ if during that time it moved uniformly and with the same velocity it had at the moment x. The actual distance covered during that time interval will, in general, be different, since the velocity does not remain constant.

In general, the derivative $f'(x)$ is considered as a measure of the relative variability of the quantities y and x at a given value of x. The differential $dy = f'(x)\,\Delta x$ can, therefore, be considered as the increment which y would receive by changing x to the value $x + \Delta x$, if, at all points of the interval $[x, x + \Delta x]$, the relative variability continued to be the same as at the point x. This idea is represented graphically in Figure 21: dy is the increment which the

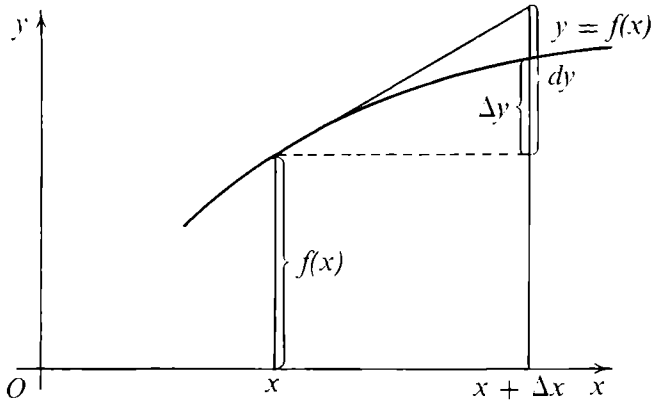

Fig. 21

ordinate of the curve $y = f(x)$ would have received between x and $x + \Delta x$ if the slope of the curve in that whole interval were the same as at the point x. In other words, if we replaced the curve by the tangent at the point whose abscissa is equal to x.

Many of you undoubtedly know that in the applied sciences we frequently do not distinguish between the increment Δy and the differential dy of a function for small values of Δx. Sometimes this is the source of a wrong and harmful idea that the differential is an *infinitely small increment,* when, in reality, the differential is neither an increment nor an infinitely small quantity.

It will be useful for us to try to answer two questions which naturally arise in connection with this substitution of the differential for the increment: one, to what degree is such a substitution justified? and two, what advantage might it offer? To answer the first question, we shall start with the relation

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Let us denote by α the difference $\frac{\Delta y}{\Delta x} - f'(x)$, which by virtue of the above relationship is infinitesimal as $\Delta x \to 0$. This gives

$$\Delta y = f'(x)\,\Delta x + \alpha \Delta x = dy + \alpha \Delta x. \qquad (3')$$

Since $\alpha \to 0$ as $\Delta x \to 0$, the product $\alpha \Delta x$ is an infinitesimal of a higher order than Δx. But, for a fixed x, the ratio $\frac{dy}{dx} = f'(x)$ is a constant quantity, and the ratio $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ has this constant as its limit. Consequently, provided only that $f'(x) \neq 0$, all three quantities Δx, Δy, and dy are infinitesimals of the same order. Therefore, $\alpha \Delta x$, being an infinitesimal of higher order than Δx, is at the same time an infinitesimal of higher order than either of the quantities Δy and dy. Thus the relation (3′) shows that if $f'(x) \neq 0$, *the difference between the increment and the differential of the function as* $\Delta x \to 0$ *is an infinitesimal of higher order than either of these quantities by itself.* In other words, by replacing the increment with the differential (or vice versa) we incur only an infinitely small *relative* error.

The relation (3′) is obviously equivalent, when $f'(x) \neq 0$, to the relation

$$\frac{\Delta y}{dy} = 1 + \frac{\alpha}{f'(x)},$$

from which it follows immediately that $\frac{\Delta y}{dy} \to 1$ as $\Delta x \to 0$, and, thus, Δy and dy are equivalent infinitesimals. These results justify the possibility of replacing a small increment Δy of a function by the differential of the function, as an approximation to the increment.

To answer the second question, what advantage is obtained by replacing the increment of a function by the differential, we observe that the computation of the differential is theoretically simpler and in practice more convenient than the computation of the increment. The differential dy is a *linear* function of the quantity Δx, the character of its variation as Δx changes is exceptionally simple, and its use requires nothing more than the computation of $f'(x)$ at *a single point.* Clearly, nothing like this holds for the quantity Δy. Imagine, for example, that we want to construct a table of the values of the function $y = \sin x$ for values very close to 60°, e.g., 60°01′, 60°02′, and so on, but that we have no means for the precise computation of these quantities. That is, we know that for $x = 60°$ we have $y = \frac{\sqrt{3}}{2}$, but, on passing to other, nearby, values of x, we do not have a method of finding the corresponding increments Δy of the quantity y. Using the fact that the increments Δx in this problem are small, let us replace the increments Δy by the differentials dy. Since $y' = \cos 60° = \frac{1}{2}$, we have $dy = \frac{1}{2}\Delta x$ where Δx is expressed in radian measure $\left(1' = \frac{\pi}{(180)(60)} \text{ radians}\right)$. From this we obtain at once

$$\sin 60°01' \approx \frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\pi}{(180)(60)}$$

$$\sin 60°02' \approx \frac{\sqrt{3}}{2} + \frac{1}{2}\frac{2\pi}{(180)(60)}$$

$$\sin 60°03' \approx \frac{\sqrt{3}}{2} + \frac{1}{2}\frac{3\pi}{(180)(60)}$$

. . . . . .

As we have seen, the differential dy has the following two remarkable properties: one, it is a linear function of Δx; and two, it differs from Δy by an infinitesimal of a higher order than Δx. We shall now show that these two properties completely characterize the differential, so that one could begin the study of the differential by defining it as a function of x and Δx having these two properties.

For this purpose, let $dy = a\Delta x + b$, where a and b are constants independent of Δx, and let $\Delta y - dy = \alpha\Delta x$, where $\alpha \to 0$ as $\Delta x \to 0$. Hence, $\Delta y = dy + \alpha\Delta x = (a + \alpha)\,\Delta x + b$. If the func-

tion y is continuous (which we assume), then $\Delta y \to 0$ as $\Delta x \to 0$, whence, necessarily, $b = 0$ and $\Delta y = (a + \alpha)\,\Delta x$. Accordingly,

$$\frac{\Delta y}{\Delta x} = a + \alpha, \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = a, \quad \text{and} \quad dy = f'(x)\,\Delta x.$$

One of the important properties of the differential (though many textbooks do not stress it sufficiently) is its so-called *invariance* with respect to a transformation of the independent variable: If x is an independent variable then, as we have seen,

$$dy = f'(x)\,\Delta x = f'(x)\,dx. \tag{4}$$

Let us now consider x as a function $x = \varphi(t)$ of a new variable t. It is clear that now *both* of the relationships (4) cannot simultaneously be valid, since now (in general)

$$dx = \varphi'(t)\,\Delta t \neq \Delta x.$$

It is worth noting that the second relation, that is,

$$dy = f'(x)\,dx,$$

remains valid after any such change of the independent variable. For, as a result of such a change, y becomes a function of t:

$$y = \psi(t) = f[\varphi(t)].$$

Using the rule for differentiation of composite functions, we have

$$\psi'(t) = f'[\varphi(t)]\,\varphi'(t),$$
$$dy = \psi'(t)\,dt = f'[\varphi(t)]\,\varphi'(t)\,dt,$$

and from $\varphi(t) = x$ and $\varphi'(t)\,dt = dx$ it follows that $dy = f'(x)\,dx$, precisely as if x were the independent variable. Hence, it is immaterial whether the variable x in the expression for the derivative $y' = \dfrac{dy}{dx}$ is independent or is a function of another variable.

We have just employed the rule for the differentiation of a composite function. Generally speaking, we do not intend in these lectures to discuss elementary rules for differentiation, but this particular rule deserves attention. In the majority of textbooks, its proof is either incorrect or unnecessarily complicated. Here is a simple yet rigorous proof:[1]

[1] Communicated by M. A. Kreines.

Let $y = f(x)$, $x = \varphi(t)$, and $y = \psi(t) = f[\varphi(t)]$. We assume, of course, that the functions f and φ are differentiable, so that in (3′) we have $\alpha \to 0$ as $\Delta x \to 0$. It follows from (3′) that

$$\frac{\Delta y}{\Delta t} = f'(x)\frac{\Delta x}{\Delta t} + \alpha\frac{\Delta x}{\Delta t}.$$

But as $\Delta t \to 0$, we have $\Delta x \to 0$ and, consequently, $\alpha \to 0$, $\frac{\Delta x}{\Delta t} \to \varphi'(t)$, and $\frac{\Delta y}{\Delta t} \to \psi'(t)$. Passing to the limit, we have

$$\psi'(t) = f'(x)\,\varphi'(t) = f'[\varphi(t)]\,\varphi'(t),$$

which is precisely the rule that we wished to derive.

33. LAGRANGE'S THEOREM (FIRST MEAN VALUE THEOREM)

The construction of the differential calculus is based, to a considerable degree, upon Lagrange's theorem. It represents the first example of the so-called *mean value theorems*, to which, in view of their exceptional importance in all fields of analysis, we shall have to give considerable attention.

THEOREM 2 (Lagrange). *If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable within that interval, then there exists a point c, $a < c < b$, such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. It is obvious that the function

$$\varphi(x) = (b - a)[f(x) - f(a)] - (x - a)[f(b) - f(a)]$$

is differentiable within the interval $[a, b]$. Since $\varphi(a) = \varphi(b) = 0$, either $\varphi(x) = 0$ for all $x \in [a, b]$ whereupon we have $\varphi'(x) = 0$ identically, or else $\varphi(x)$ assumes, within the interval, values different from zero. Among these values there exists either a maximum or a minimum (since the function $\varphi(x)$ is continuous). Let us suppose, for definiteness, that the function $\varphi(x)$ attains its maximum value at $x = c$, where $a < c < b$. Necessarily, $\varphi'(c) = 0$, for if $\varphi'(c) > 0$ we would have, for sufficiently small positive h,

$$\frac{\varphi(c + h) - \varphi(c)}{h} > 0.$$

Consequently, $\varphi(c + h) > \varphi(c)$ contrary to our definition of the point c. If $\varphi'(c) < 0$, we obtain a similar contradiction by assigning negative values to h. From $\varphi'(c) = 0$ the desired relation follows, and the theorem is proved.

Frequently we formulate this theorem using another system of notation, letting the end points of the given interval be x and $x + \Delta x$, where $\Delta x > 0$. Then we speak of the existence of a number θ $(0 < \theta < 1)$ such that

$$f'(x + \theta\,\Delta x)\,\Delta x = f(x + \Delta x) - f(x).$$

For functions having derivatives of higher order, there exists an important generalization of Lagrange's theorem:

THEOREM 3. *If $y = f(x)$ has a continuous derivative of order $n - 1$ throughout the closed interval $[x, x + n\,\Delta x]$ and has, within this interval, a derivative of order n,*[1] *then there exists a number θ $(0 < \theta < 1)$ such that*

$$\Delta^n y = f^{(n)}(x + \theta n\,\Delta x)(\Delta x)^n.$$

The symbol $\Delta^n y$ denotes here the so-called *nth order difference* of $y = f(x)$. The first order difference is simply the increment $\Delta y = f(x + \Delta x) - f(x)$. We define the difference of the $(n + 1)$th order as the first order difference of the nth order difference, so that

$$\begin{aligned}\Delta^2 y &= \Delta(\Delta y) = [f(x + \Delta x + \Delta x) - f(x + \Delta x)] \\ &\quad - [f(x + \Delta x) - f(x)] = f(x + 2\Delta x) - 2f(x + \Delta x) + f(x), \\ \Delta^3 y &= f(x + 3\Delta x) - 3f(x + 2\Delta x) + 3f(x + \Delta x) - f(x),\end{aligned}$$

and so on. In general, as is easy to prove by mathematical induction,

$$\begin{aligned}\Delta^n y = f(x + n\,\Delta x) &- nf[x + (n - 1)\,\Delta x] \\ &+ \frac{n(n-1)}{2} f[x + (n - 2)\,\Delta x] - \cdots \\ &+ (-1)^{n-1} nf(x + \Delta x) + (-1)^n f(x).\end{aligned}$$

Proof. For $n = 1$, the theorem to be proved coincides with Lagrange's theorem. Let us now assume that the theorem is true for differences of order n. We then show it is true for differences of

[1] At the end points of the interval it suffices that there exist one-sided derivatives of the order $n - 1$.

order $n + 1$. Thus we assume that $y = f(x)$ has a continuous derivative of the nth order throughout the closed interval $[x, x + (n + 1)\,\Delta x]$ and a derivative of order $n + 1$ in its interior. Consequently, the function $y = f(x + \Delta x) - f(x)$ has a derivative of the nth order throughout the interval $[x, x + n\,\Delta x]$. We can therefore apply our inductive hypothesis to its nth order difference, obtaining

$$\Delta^n(\Delta y) = (\Delta x)^n\{f^{(n)}(x + \Delta x + \theta_1 n\,\Delta x) - f^{(n)}(x + \theta_1 n\,\Delta x)\},$$

where $0 < \theta_1 < 1$. But it follows from our assumptions that $f^{(n)}(x)$ is continuous on the closed interval $[x + \theta_1 n\,\Delta x, x + (1 + \theta_1 n)\,\Delta x]$ and differentiable in its interior. Therefore, applying Lagrange's theorem to the expression in braces in the above equality, we find

$$\begin{aligned}\Delta^{n+1}y &= (\Delta x)^{n+1}f^{(n+1)}(x + \theta_1 n\,\Delta x + \theta_2\,\Delta x)\\ &= (\Delta x)^{n+1}f^{(n+1)}(x + \theta\,[n + 1]\,\Delta x),\end{aligned}$$ q.e.d.

Another important generalization of Lagrange's theorem is contained in the remarkable *Cauchy's formula:*

CAUCHY'S FORMULA. *Given two functions $f_1(x)$ and $f_2(x)$ continuous on the closed interval $[a, b]$ and differentiable in its interior, and supposing that $f_2'(x)$ never assumes the value zero inside the interval, then there exists a point c in the interior of $[a, b]$ such that*

$$\frac{f_1(b) - f_1(a)}{f_2(b) - f_2(a)} = \frac{f_1'(c)}{f_2'(c)}.$$

The proof of this general formula (which for $f_2(x) = x$ reduces to Lagrange's theorem) is carried out in the same manner as that of Lagrange's theorem.

Proof. Setting $\varphi(x) = f_1(x)[f_2(b) - f_2(a)] - f_2(x)[f_1(b) - f_1(a)]$, we find that $\varphi(a) = \varphi(b)$. Hence, the function $\varphi(x)$ is either constant on $[a, b]$ or attains in the interior of this interval a maximum or minimum value. Therefore, by precisely the same reasoning as before, we conclude that, at some point c interior to $[a, b]$, we must have $\varphi'(c) = 0$, from which the desired formula follows.

Let us further note that the left side of Cauchy's formula cannot turn out to be meaningless, since if $f_2(a) = f_2(b)$, then, by virtue of Theorem 2, we would have $f_2'(x) = 0$ at some interior point of $[a, b]$; this is excluded by the hypothesis of the theorem.

34. DERIVATIVES AND DIFFERENTIALS OF HIGHER ORDER

Derivatives and differentials of higher order are defined, as you know, by induction: the derivative of order n is obtained by differentiating the derivative of order $n-1$, and similarly for differentials.

The higher order derivatives and differentials of a function $y = f(x)$ are connected by relations of the form

$$d^n y = f^{(n)}(x)\, dx^n \tag{5}$$

(where $d^n y$ denotes the differential of $d^{n-1}y$ and dx^n means the nth power of the *first* differential of the independent variable), which are also proved by induction. For if x is the independent variable, then $dx = \Delta x$ does not depend on x and, therefore, by differentiating the relation (5) we find (the prime mark $'$ indicating differentiation with respect to x):

$$d^{n+1}y = d(d^n y) = (d^n y)'\, dx = f^{(n+1)}(x)\, dx^{n+1}.$$

As we have seen earlier, for $n = 1$ the relation (5) remains true when x (and hence y also) is a differentiable function of an independent variable t: $x = \varphi(t)$. It is easy to see that for $n = 2$ the situation is already different: differentials of higher order are not invariant with respect to a transformation of the independent variable. Denoting the second differential of $y = f(x)$ by $(d^2y)_x$ or $(d^2y)_t$, depending on the choice of x or t as the independent variable, we have

$$(d^2x)_x = f''(x)\, dx^2,$$

while

$$\begin{aligned}(d^2y)_t &= \frac{d^2 f[\varphi(t)]}{dt^2}\, dt^2 \\ &= \frac{d\{f'[\varphi(t)]\,\varphi'(t)\}}{dt}\, dt^2 \\ &= \{f''[\varphi(t)]\,\varphi'^2(t) + f'[\varphi(t)]\,\varphi''(t)\}\, dt^2 \\ &= f''(x)\, dx^2 + f'[\varphi(t)]\,\varphi''(t)\, dt^2.\end{aligned}$$

We see that these two expressions are different: the second one contains an additional term $f'[\varphi(t)]\,\varphi''(t)\, dt^2$. This term is absent in the first expression and is identically zero only if x is a *linear* function of the variable t, $\varphi(t) = at + b$. Hence the second differential, in contrast to the first, remains invariant only with respect to linear transformations of the independent variable.

It is also possible to define derivatives of higher order in another way, connecting them with differences of corresponding order. It is possible to show that

$$\lim_{\Delta x \to 0} \frac{\Delta^n y}{(\Delta x)^n} = f^{(n)}(x). \tag{6}$$

This is similar to the method used in defining the first derivative as the limit of the ratio $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$. If $f^{(n)}(x)$ is assumed to be continuous, then (6) follows directly from Theorem 3, but it is important to prove that the validity of (6) does not depend upon this assumption.

Since (6) is valid for $n = 1$, it is only necessary to show that, assuming this relation is valid for n, it must also be valid for $n + 1$ (presupposing, of course, that $y = f(x)$ has a derivative of order $n + 1$). Let us then assume the relation (6) for a given n (and for any function y differentiable n times). Since the function $\Delta y = f(x + \Delta x) - f(x)$ has a derivative of order n, we may apply Theorem 3 to it, obtaining (as we have already seen on p. 115)

$$\begin{aligned} \Delta^n(\Delta y) &= (\Delta x)^n (\Delta y)^{(n)}_{x+\theta n \Delta x} \\ &= (\Delta x)^n \{f^{(n)}(x + \theta n\, \Delta x + \Delta x) - f^{(n)}(x + \theta n\, \Delta x)\}, \end{aligned} \tag{7}$$

where $0 < \theta < 1$. But because of the existence of the derivative $f^{(n+1)}(x)$, we have

$$f^{(n)}(x + \theta n\, \Delta x + \Delta x) - f^{(n)}(x) = (1 + n\theta)\, \Delta x \{f^{(n+1)}(x) + \alpha_1\}$$

and

$$f^{(n)}(x + \theta n\, \Delta x) - f^{(n)}(x) = n\theta\, \Delta x\, \{f^{(n+1)}(x) + \alpha_2\},$$

where $\alpha_1 \to 0$ and $\alpha_2 \to 0$ as $\Delta x \to 0$. Consequently,

$$\begin{aligned} f^{(n)}(x + \theta n\, \Delta x + \Delta x) - f^{(n)}(x + \theta n\, \Delta x) \\ = \Delta x f^{(n+1)}(x) + \alpha_1(1 + n\theta)\, \Delta x - \alpha_2 n\theta\, \Delta x. \end{aligned}$$

The relation (7) thus gives us

$$\frac{\Delta^{n+1} y}{(\Delta x)^{n+1}} = f^{(n+1)}(x) + \alpha_1(1 + n\theta) - \alpha_2 n\theta,$$

and letting $\Delta x \to 0$, we obtain

$$\lim_{\Delta x \to 0} \frac{\Delta^{n+1} y}{(\Delta x)^{n+1}} = f^{(n+1)}(x).$$

35. LIMITS OF RATIOS OF INFINITELY SMALL AND INFINITELY LARGE QUANTITIES

From among the applications of the formulas of Lagrange and Cauchy let us first of all consider the important problem which often appears in courses in analysis under the meaningless title of *the evaluation of indeterminate expressions.* The reference here is to the application of the methods of differential calculus to computing the limits of ratios of two infinitely small or two infinitely large quantities.

Let us assume that $f_1(a) = f_2(a) = 0$, that in some neighborhood of the point a both functions are differentiable, and that $f_2'(x) \neq 0$ for $x \neq a$ in some neighborhood of a. Since for $f_1(a) = f_2(a) = 0$ the Cauchy formula gives

$$\frac{f_1(a+h)}{f_2(a+h)} = \frac{f_1'(a+\theta h)}{f_2'(a+\theta h)} \qquad (0 < \theta < 1),$$

we can state the following proposition.

L'HOSPITAL'S RULE. *If* $f_1(a) = f_2(a) = 0$ *and the ratio* $\dfrac{f_1'(x)}{f_2'(x)}$ *tends to a limit as* $x \to a$, *then the ratio* $\dfrac{f_1(x)}{f_2(x)}$ *tends to the same limit.*

If $f_1'(a) = f_2'(a) = 0$, then applying this rule to the ratio $\dfrac{f_1'(x)}{f_2'(x)}$ (and assuming, of course, the existence of the second derivatives of the functions $f_1(x)$ and $f_2(x)$ in some neighborhood of a), we have: if $f_1(a) = f_2(a) = f_1'(a) = f_2'(a) = 0$, then from the relation $\lim\limits_{x \to a} \dfrac{f_1''(x)}{f_2''(x)} = l$ it follows that we also have $\lim\limits_{x \to a} \dfrac{f_1(x)}{f_2(x)} = l$. And, in general: if $f_1(a) = f_2(a) = f_1'(a) = f_2'(a) = \cdots = f_1^{(n-1)}(a) = f_2^{(n-1)}(a) = 0$, and if the functions $f_1(x)$ and $f_2(x)$ have derivatives of order n in some neighborhood of a, then from the relation[1]

$$\frac{f_1^{(n)}(x)}{f_2^{(n)}(x)} \to l \text{ as } x \to a$$

[1] This implies that $f_2^{(n)}(x) \neq 0$ for $x \neq a$ in some neighborhood of a. It then follows by Lagrange's formula that the same thing also holds for $f_2^{(n-1)}(x), f_2^{(n-2)}(x), \ldots, f_2'(x)$, allowing us to apply repeatedly Cauchy's formula and thus prove that

$$\frac{f_1(x)}{f_2(x)} \to l.$$

it follows that

$$\frac{f_1(x)}{f_2(x)} \to l \text{ as } x \to a.$$

Thus if $f_1^{(n)}(x)$ and $f_2^{(n)}(x)$ are continuous at a and if $f_2^{(n)}(a) \neq 0$, then

$$\lim_{x \to a} \frac{f_1(x)}{f_2(x)} = \frac{f_1^{(n)}(a)}{f_2^{(n)}(a)}. \tag{8}$$

L'Hospital's rule serves as a very efficient tool for computing limits, and in many cases allows us to find these limits very easily when application of the elementary methods would involve great difficulties. For example, if $f_1(x) = \tan x - \sin x$ and $f_2(x) = x^3$, we have $f_1(0) = f_1'(0) = f_1''(0) = 0, f_1'''(0) = 3, f_2(0) = f_2'(0) = f_2''(0) = 0$, and $f_2'''(0) = 6$. By (8), we obtain at once

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \frac{f_1'''(0)}{f_2'''(0)} = \frac{1}{2}.$$

L'Hospital's rule remains valid even when $a = \pm\infty$. For

$$\lim_{x \to +\infty} \frac{f_1(x)}{f_2(x)} = \lim_{y \to +0} \frac{f_1\left(\frac{1}{y}\right)}{f_2\left(\frac{1}{y}\right)},$$

and by virtue of what we have already proved, the last limit coincides with

$$\lim_{y \to +0} \frac{-\frac{1}{y^2} f_1'\left(\frac{1}{y}\right)}{-\frac{1}{y^2} f_2'\left(\frac{1}{y}\right)} = \lim_{x \to +\infty} \frac{f_1'(x)}{f_2'(x)},$$

if this limit exists. Of course, a necessary prerequisite here for the validity of the rule is the condition

$$\lim_{x \to +\infty} f_1(x) = \lim_{x \to +\infty} f_2(x) = 0.$$

Moreover, L'Hospital's rule can also be applied to finding the limit of a ratio of infinitely large quantities, though here the proof is somewhat more complicated. Suppose, to be definite, that $f_1(x) \to +\infty$ and $f_2(x) \to +\infty$ as $x \to a$. In addition, we assume, as

before, that $f_2'(x)$ never vanishes in some neighborhood of a. Let us take in this neighborhood two arbitrary points x and α such

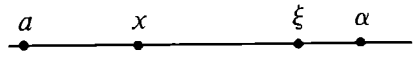

Fig. 22

that x lies between a and α, and suppose, for definiteness, that $a < x < \alpha$ (Fig. 22). By Cauchy's formula, we have

$$\frac{f_1(x) - f_1(\alpha)}{f_2(x) - f_2(\alpha)} = \frac{f_1'(\xi)}{f_2'(\xi)}, \tag{9}$$

where ξ is some interior point of the interval $[x, \alpha]$. Since, on the other hand,

$$\frac{f_1(x) - f_1(\alpha)}{f_2(x) - f_2(\alpha)} = \frac{f_1(x)}{f_2(x)} \cdot \frac{1 - \dfrac{f_1(\alpha)}{f_1(x)}}{1 - \dfrac{f_2(\alpha)}{f_2(x)}},$$

the relation (9) gives us

$$\frac{f_1(x)}{f_2(x)} = \frac{f_1'(\xi)}{f_2'(\xi)} \cdot \frac{1 - \dfrac{f_2(\alpha)}{f_2(x)}}{1 - \dfrac{f_1(\alpha)}{f_1(x)}}. \tag{10}$$

We shall now be more specific about the choice of the points x and α. If, as we assume,

$$\lim_{x \to a} \frac{f_1'(x)}{f_2'(x)} = l$$

exists, then, for any arbitrarily small $\varepsilon > 0$, there is a neighborhood U of a such that for $x \in U$ we have

$$l - \varepsilon < \frac{f_1'(x)}{f_2'(x)} < l + \varepsilon.$$

Let α (and thus x also) belong to this neighborhood. Then ξ also belongs to U, and, therefore,

$$l - \varepsilon < \frac{f_1'(\xi)}{f_2'(\xi)} < l + \varepsilon.$$

Let us now hold α fixed and let x approach a. Since, in this case, $f_1(x) \to +\infty$ and $f_2(x) \to +\infty$, the second factor on the right side

of (10) tends to unity. Therefore, for all x in a neighborhood V of the point a this factor will be contained between $1 - \varepsilon$ and $1 + \varepsilon$. Thus, for all $x \in V$ formula (10) gives

$$(l - \varepsilon)(1 - \varepsilon) < \frac{f_1(x)}{f_2(x)} < (l + \varepsilon)(1 + \varepsilon),$$

and since ε is arbitrarily small, we have

$$\lim_{x \to a} \frac{f_1(x)}{f_2(x)} = l.$$

This rule, like the rule for the ratio of two infinitely small quantities, also remains valid when $a = \pm\infty$.

EXAMPLES:

(*a*) $f_1(x) = \ln x$ and $f_2(x) = \frac{1}{x}$. If $x \to 0$, then $f_1(x) \to -\infty$ and $f_2(x) \to +\infty$. But $\frac{f_1'(x)}{f_2'(x)} = -x \to 0$, whence

$$\frac{f_1(x)}{f_2(x)} = x \ln x \to 0 \qquad \text{as} \qquad x \to 0.$$

(*b*) Suppose $f_2(x) = x$. Then, as $x \to \infty$, we have $f_1(x) = \ln x \to +\infty$ and $\frac{f_1'(x)}{f_2'(x)} = \frac{1}{x} \to 0$. Hence,

$$\frac{f_1(x)}{f_2(x)} = \frac{\ln x}{x} \to 0 \qquad \text{as} \qquad x \to \infty.$$

Substituting $x = e^y$ and replacing y again by x, we find $xe^{-x} \to 0$ as $x \to \infty$.

36. TAYLOR'S FORMULA

We shall now take up Taylor's formula which, as you undoubtedly know, is one of the most important tools in analysis, as well as in its applications. This formula, although it is discussed in detail in all courses in analysis, nevertheless requires our attention. The derivation of the formula, as it is usually presented, is uninspired and formal, leaving its essential content completely unilluminated. Thus, the importance which the formula later acquires is unexpected and sometimes remains a riddle forever to many students.

As we have already stressed more than once, the basic relation

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)$$

may be written in the equivalent form

$$f(a + h) = f(a) + hf'(a) + \alpha h, \tag{11}$$

where $\alpha \to 0$ as $h \to 0$. Consequently, αh is here an infinitesimal of higher order than h, that is, a quantity whose ratio to h tends to zero as $h \to .0$.

In general, let us agree to denote by $o(x)$ any quantity whose ratio to the quantity x tends to zero during a given process of change. We then may write the product αh in the form $o(h)$. And since, by our definition, we have for any $o(h)$

$$\frac{o(h)}{h} = \alpha \to 0, \text{ as } h \to 0,$$

we also have the converse: every $o(h)$ may be represented in the form αh where $\alpha \to 0$ as $h \to 0$. Consequently, the relation (11) can be written in the form

$$f(a + h) = f(a) + hf'(a) + o(h).$$

As we see, this formula is valid in all cases where $f'(a)$ exists. We have already had occasion to apply it more than once, and the reason for its usefulness is immediately clear: If h is so small that we can neglect quantities of the form $o(h)$, then the approximation formula

$$f(a + h) \approx f(a) + hf'(a)$$

permits us to replace the complicated function $f(a + h)$ with a linear function. The substantial advantage which this gives us is obvious.

It will now be readily understood why we might desire to extend this useful procedure somewhat further. If we wish to obtain greater precision, it is sometimes impossible to be satisfied with approximate formulas whose error is small only in comparison with h. For example, we may have to take into consideration quantities of order h^2, so that we can neglect only quantities of the form $o(h^2)$, that is, quantities infinitesimal in comparison with h^2.

Then, of course, we must set ourselves to the problem of finding a polynomial of the second degree $a_0 + a_1h + a_2h^2$ such that, for small values of h, we shall have the relation

$$f(a + h) = a_0 + a_1h + a_2h + o(h^2).$$

In general, if we want to take into consideration quantities of the order h^n, but agree to neglect quantities of the form $o(h^n)$, then we shall, of course, look for a polynomial $a_0 + a_1h + \cdots + a_nh^n = P_n(h)$ which will satisfy the relationship

$$f(a + h) = P_n(h) + o(h^n).$$

If we succeed in solving this problem, we shall then be able to use a simple polynomial of degree n, instead of the (in general) complicated function $f(a + h)$ in all problems where quantities of the form $o(h^n)$ may be neglected.

For $n = 1$ we have already found the solution, assuming that the function $f(x)$ has a first derivative at $x = a$. In the general case we shall, of course, assume that $f^{(n)}(a)$ exists. This obviously requires that $f(x)$ have derivatives of all preceding orders in some neighborhood of the point a.

The celebrated Taylor's theorem gives us at once the solution to the problem we have just described. That is, it proves the existence of the desired polynomial $P_n(h)$ and gives the formula for its coefficients.

TAYLOR'S THEOREM. *When $f^{(n)}(a)$ exists the polynomial $P_n(h)$ is uniquely*[1] *defined by the formula*

$$P_n(h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a).$$

In other words,

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + o(h^n). \quad (12)$$

[1] The uniqueness of $P_n(h)$ follows immediately from the relation $f(a + h) = P_n(h) + o(h^n)$. For, if two such polynomials $P_n(h) = \sum_{k=1}^{n} a_kh^k$ and $Q_n(h) = \sum_{k=1}^{n} b_kh^k$, it would follow that $\sum_{k=1}^{n} (a_k - b_k)h^k = o(h^n)$. Dividing repeatedly by h and letting $h \to 0$, we obtain successively $a_0 - b_0 = 0, a_1 - b_1 = 0, \ldots, a_n - b_n = 0$.

Proof. Let us set

$$f(a + h) - P_n(h) = \varphi(h);$$

then, to prove our assertion, it will suffice to show that

$$\frac{\varphi(h)}{h^n} \to 0 \text{ as } h \to 0. \tag{12'}$$

For this purpose we note that

$$\varphi(h) = f(a + h) - f(a) - hf'(a) - \cdots - \frac{h^n}{n!}f^{(n)}(a),$$

$$\varphi'(h) = f'(a + h) - f'(a) - hf''(a) - \cdots - \frac{h^{n-1}}{(n-1)!}f^{(n)}(a),$$

. . . . . .

$$\varphi^{(n-2)}(h) = f^{(n-2)}(a + h) - f^{(n-2)}(a) - hf^{(n-1)}(a) - \frac{h^2}{2!}f^{(n)}(a),$$

$$\varphi^{(n-1)}(h) = f^{(n-1)}(a + h) - f^{(n-1)}(a) - hf^{(n)}(a),$$

from which

$$\varphi(0) = \varphi'(0) = \cdots = \varphi^{(n-2)}(0) = 0.$$

Since the derivatives of the function h^n up to (and including) order $n - 2$ also take the value zero for $h = 0$ and the derivative of order $n - 1$ is equal to $n!h$, we have, by L'Hospital's rule,

$$\lim_{h\to 0} \frac{\varphi(h)}{h^n} = \lim_{h\to 0} \frac{\varphi^{(n-1)}(h)}{n!h},$$

assuming that the limit on the right side exists. But

$$\lim_{h\to 0} \frac{\varphi^{(n-1)}(h)}{n!h} = \frac{1}{n!} \lim_{h\to 0} \left\{ \frac{f^{(n-1)}(a + h) - f^{(n-1)}(a)}{h} - f^{(n)}(a) \right\} = 0,$$

since $f^{(n)}(a)$, by our assumption, exists. This proves relation (12') and the derivation of Taylor's formula (12) is completed.

As we proceed further, the question arises of finding convenient expressions for the *remainder* term $o(h^n)$ in formula (12), so that, when the need arises, we may estimate it as accurately as possible. We shall present here the most frequently used forms of such expressions, without giving their proofs, which can be found in any unabridged text in analysis.

Let us denote by $R_n(h)$ the quantity which, in formula (12), was loosely denoted by $o(h^n)$, and which we have also called $\varphi(h)$. On the mere assumption that $f^{(n+1)}(a)$ exists, it can be shown that

$$\lim_{h \to 0} \frac{R_n(h)}{h^{n+1}} = \frac{f^{(n+1)}(a)}{(n+1)!}.$$

If we assume that $f^{(n+1)}(x)$ exists, not only at $x = a$, but throughout the interval $[a, a + h]$, we can obtain for $R_n(h)$ an expression known as *Schlömilch's form of the remainder:*

$$R_n(h) = \frac{h^{n+1}(1-\theta)^{n+1-p}}{n!p} f^{(n+1)}(a + \theta h),$$

where $0 < \theta < 1$ and p is *any* positive number not exceeding $n + 1$. From this very general formula we obtain, by particular choice of the number p, a number of special forms of the remainder. Those used most frequently are:

(*a*) For $p = n + 1$, we obtain *Lagrange's form of the remainder*

$$R_n(h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(a + \theta h).$$

(*b*) For $p = 1$, we obtain *Cauchy's form of the remainder,*

$$R_n(h) = \frac{h^{n+1}(1-\theta)^n}{n!} f^{(n+1)}(a + \theta h).$$

In these individual forms, as well as in the general form, we assume that $f^{n+1}(x)$ exists throughout the interval $[a, a + h]$.

As you already know, the name *Maclaurin's formula* designates the (in no way particularly remarkable) special case of Taylor's formula for $a = 0$:

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^n}{n!} f^{(n)}(0) + o(h^n).$$

37. MAXIMA AND MINIMA

An important application of Taylor's formula in elementary differential calculus is the theory of maxima and minima, the fundamentals of which are already known to you. Suppose we are looking for a point in the interval $[a, b]$ at which $y = f(x)$, differentiable in this interval, attains its greatest value. This *absolute*

maximum may be attained either at an end point of the interval or at an interior point c. In the latter case, the point c is also a point at which the function attains a *relative* maximum, which means that $f(c) \geq f(x)$ for all x in a neighborhood of c. Thus, the search for the maximum of a function in a given interval is reduced to finding all its relative maxima in that interval. And it is to this problem that we apply the methods of differential calculus.

You know, of course, that a necessary condition for a differentiable function $f(x)$ to reach a relative maximum (or minimum) at the point $x \quad (a < x < b)$ is the relation

$$f'(x) = 0. \tag{13}$$

Thus, the first step toward the solution of the problem in question consists in finding all the real roots of the equation (13), the so-called *critical values* of x. After this is done, each critical value must be tested individually to establish whether or not the function attains a maximum or minimum there. We shall turn our attention momentarily to this testing process.

Let α be one of the critical values of $f(x)$, that is, let $a < \alpha < b$ and $f'(\alpha) = 0$. And let us assume that $f^{(i)}(\alpha) = 0$ for $1 \leq i < n$, but that $f^{(n)}(\alpha) \neq 0$. In such a case, Taylor's formula (12) gives

$$f(\alpha + h) - f(\alpha) = h^n \frac{f^{(n)}(\alpha)}{n!} + o(h^n). \tag{14}$$

It is clear that the sign of the right-hand side of (14), whose second term is infinitesimal in comparison to the first, coincides with the sign of this first term for sufficiently small $|h|$, that is, coincides with the sign of the product $h^n f^{(n)}(\alpha)$. If n is odd, then h^n, and hence the whole product, changes sign when h (which may be positive or negative) changes sign. In this case the sign of the difference $f(\alpha + h) - f(\alpha)$ will also, by virtue of (14), change sign as the increment h changes sign. This clearly implies that $f(x)$ has neither a maximum nor minimum at $x = \alpha$. Now suppose that n is even, so that $h^n > 0$ for any $h \neq 0$. The sign of the difference $f(\alpha + h) - f(\alpha)$ for all sufficiently small $|h|$ clearly coincides with the sign of $f^{(n)}(\alpha)$, which does not depend on h. If $f^{(n)}(\alpha) > 0$, then for all sufficiently small $|h|$ we have

$$f(\alpha + h) > f(\alpha),$$

that is, $f(x)$ has a relative *minimum* at $x = \alpha$. And similarly, if $f^{(n)}(\alpha) < 0$, then $f(x)$ has a relative *maximum* at $x = \alpha$. We thus arrive at the following rule:

Let n be the order of the first nonvanishing derivative of $f(x)$ at $x = \alpha$. If n is even, $f(x)$ has a maximum or a minimum at $x = \alpha$ according to whether $f^{(n)}(\alpha) < 0$ or $f^{(n)}(\alpha) > 0$ respectively. But if n is odd, then $f(x)$ has neither a maximum nor a minimum at $x = \alpha$.

We can see that this rule solves completely the problem of determining the nature of each critical point, provided only that the function is differentiable at this point a sufficient number of times and that not all of its derivatives are equal to zero at this point.

38. PARTIAL DERIVATIVES

We must now direct our attention to functions of several variables. But for simplicity we shall restrict ourselves to functions of only two independent variables x and y.

As you know, the *partial derivative* $f_x'(x, y)$ or $\dfrac{\partial f(x, y)}{\partial x}$ of $f(x, y)$ with respect to x is defined as

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

that is, as the derivative in the ordinary sense of the function of x obtained from $f(x, y)$ by assigning to y a constant value. $\dfrac{\partial f}{\partial y} = f_y'(x, y)$ is defined analogously. And each of these partial derivatives is, in turn, a function of x and y. For convenience and clarity, we shall frequently replace the words *at $x = a$ and $y = b$* by the expression *at the point (a, b)*, meaning the point in the (x, y) plane having the coordinates $x = a$ and $y = b$.

First of all, we must define the concept of *differentiability* for a function $z = f(x, y)$ at a point (a, b). For a function of one variable, the existence of a derivative was equivalent to the existence of a differential. And, as we have seen, the differential dy could be defined for $y = f(x)$ as a linear function of the increment Δx which differs from Δy by an infinitesimal of higher order as $\Delta x \to 0$.

Let us now consider the situation for a function of two variables $z = f(x, y)$. Here again, we agree to define a differential dz of the function z as a linear combination $A\,\Delta x + B\,\Delta y + C$ of the increments Δx and Δy which, as $\Delta x \to 0$ and $\Delta y \to 0$, differs from Δz by an infinitesimal of higher order. In the one-dimensional case, we could (assuming $f'(x) \neq 0$) accept as the fundamental infinitesimal any one of the quantities Δx, Δy, or dy (their order of smallness being identical). In the two-dimensional case, it is convenient (though by no means necessary) to take as the fundamental infinitesimal the quantity $\rho = \sqrt{\Delta x^2 + \Delta y^2}$, which expresses the distance of a *displaced* point $(x + \Delta x, y + \Delta y)$ from the *initial* point (x, y). If Δx and Δy are infinitesimals of the same order, then the quantity ρ is clearly of this same order.

Thus, the expression $A\,\Delta x + B\,\Delta y + C$ is a differential dz of a function $z = f(x, y)$ if, as $\Delta x \to 0$ and $\Delta y \to 0$, we have the relation

$$\Delta z - (A\,\Delta x + B\,\Delta y + C) = o(\rho). \tag{15}$$

We shall now show that if dz exists, then the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ also exist and $A = \frac{\partial z}{\partial x}$, $B = \frac{\partial z}{\partial y}$, and $C = 0$, so that

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y. \tag{16}$$

First of all, because of the continuity of the function z (which we assume, of course) the relation (15), by a passage to the limit, gives $C = 0$. Having established this, let us set $\Delta y = 0$ in formula (15). According to our assumption, this formula must be valid for *any* way in which the quantities Δx and Δy tend to zero. This gives $(\Delta z)_{\Delta y = 0} = A\,\Delta x + o(\Delta x)$, and hence,

$$A = \lim_{\Delta x \to 0} \frac{(\Delta z)_{\Delta y = 0}}{\Delta x} = \frac{\partial z}{\partial x}.$$

Similarly, we may establish that $B = \frac{\partial z}{\partial y}$. The existence of the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ is, therefore, a consequence of the existence of the differential dz.

So far everything runs in complete analogy with the one-dimensional case. Unlike the one-dimensional case, however, the

existence at a given point of the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ does not guarantee the existence of the differential dz. For example, considering the function

$$z = \begin{cases} \dfrac{2xy}{\sqrt{x^2 + y^2}} & \text{if } x^2 + y^2 \neq 0, \\ 0 & \text{if } x = y = 0, \end{cases}$$

at (0, 0), we obtain

$$(\Delta z)_{\Delta x=0} = (\Delta z)_{\Delta y=0} = 0,$$

and, hence,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0.$$

If the differential dz existed at $x = y = 0$ then, by (16), it would be identically zero, so that, from (15), we would obtain $\Delta z = o(\rho)$. But this is contradictory, since for $\Delta x = \Delta y \neq 0$, we have

$$\Delta z = \sqrt{2}\,\Delta x,\ \rho = \sqrt{2}\,\Delta x, \text{ and } \Delta z = \rho.$$

Let us observe, however, that if the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are *continuous* at the point (x, y), then the expression (16) is the differential of $z = f(x, y)$ at that point. For,

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]. \end{aligned}$$

The first difference on the right-hand side has, by Lagrange's mean value theorem, the form

$$\Delta x f_x'(x + \theta\,\Delta x, y + \Delta y) \qquad (0 < \theta < 1).$$

From the assumed continuity of $f_x'(x, y)$ at the point (x, y), it follows that, as $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$f_x'(x + \theta\,\Delta x, y + \Delta y) - f_x'(x, y) \to 0.$$

Hence, the first of the two differences mentioned above differs from $f_x'(x, y)\,\Delta x$ by a quantity of the form $o(\Delta x) = o(\rho)$:

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = f_x'(x, y)\,\Delta x + o(\rho).$$

On the other hand, from the very definition of partial derivatives, we have

$$f(x, y + \Delta y) - f(x, y) = f_y'(x, y)\,\Delta y + o(\Delta y) = f_y'(x, y)\,\Delta y + o(\rho).$$

It follows that

$$\Delta z = f_x'(x, y)\,\Delta x + f_y'(x, y)\,\Delta y + o(\rho),$$

which is exactly what we set out to prove. Our argument even shows that it is sufficient to assume the continuity of only one of the two partial derivatives at the point (x, y).

We have seen, in the general case, that partial derivatives may exist even when the differential does not. Hence, in defining differentiability for a function of two variables, we must make a choice: should we base the definition on the existence of the partial derivatives, or on the existence of the differential? By general agreement, *we call a function differentiable at a given point if it has a differential at that point.* Such a definition, requiring of a differentiable function something more than the mere existence of the partial derivatives, is more convenient because of the closer analogy to the one-dimensional case. As further development of the theory reveals, only functions having differentials show in their properties some close resemblance to differentiable functions of a single independent variable.

To obtain the partial derivative $\frac{\partial z}{\partial x}$, we add an increment to only one variable, x, leaving the variable y unchanged. Speaking geometrically, we displace the point $P(x, y)$ parallel to the x-axis to the point $P_1(x + \Delta x, y)$ (Fig. 23).

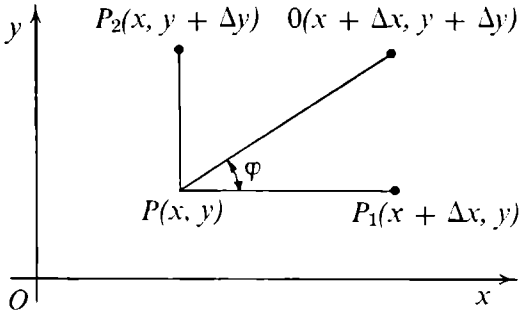

Fig. 23

The partial derivative $\frac{\partial z}{\partial x}$ is defined as the limit of the ratio

$$\frac{f(P_1) - f(P)}{\overline{PP}_1}$$

as $P_1 \to P$, where $\overline{PP}_1 = \Delta x$ is the distance from P to P_1, $f(P) = f(x, y)$, and $f(P_1) = f(x + \Delta x, y)$. Similarly,

$$\frac{\partial z}{\partial y} = \lim_{P_2 \to P} \frac{f(P_2) - f(P)}{\overline{PP}_2},$$

where P_2 is the point with coordinates $(x, y + \Delta y)$ which is made to approach the point P along a line parallel to the y-axis. We can thus say that, for the function z at the point P, the derivative $\frac{\partial z}{\partial x}$ is the derivative *in the direction* Ox and $\frac{\partial z}{\partial y}$ is the derivative *in the direction* Oy. But it is clear that, in a completely analogous way, we can define a partial derivative in any other direction; say the direction making angle φ with the positive x-axis. For this purpose, we need only displace the point P to a point Q where the vector $\overline{PQ}$ forms the angle φ with the direction Ox, and then make Q approach P along this vector. It will be natural to call the quantity

$$\lim_{Q \to P} \frac{f(Q) - f(P)}{\overline{PQ}}$$

(if it exists) *the derivative of* $z = f(x, y)$ *in the direction* φ. It is clear that specifying such a mode of approach is equivalent to fixing a linear dependence $\Delta y = \tan \varphi \, \Delta x$ between the increments Δx and Δy as they simultaneously approach zero.

If we agree to denote the derivative of $z = f(x, y)$ in the direction φ by $D_\varphi(z)$, we may then write

$$\frac{\partial z}{\partial x} = D_0(z) \quad \text{and} \quad \frac{\partial z}{\partial y} = D_{\frac{\pi}{2}}(z).$$

We shall now show that *a function* $z = f(x, y)$ *differentiable at a point* (x, y) *has, at this point, a derivative in any direction, and we have*

$$D_\varphi(z) = \frac{\partial z}{\partial x} \cos \varphi + \frac{\partial z}{\partial y} \sin \varphi.$$

Since corresponding to the displacement of the point (x, y) in the direction φ through a distance ρ we have

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + o(\rho),$$

where

$$\Delta x = \rho \cos \varphi \quad \text{and} \quad \Delta y = \rho \sin \varphi,$$

it follows that

$$\frac{\Delta z}{\rho} = \frac{\partial z}{\partial x}\cos \varphi + \frac{\partial z}{\partial y}\sin \varphi + \frac{o(\rho)}{\rho}.$$

That is,

$$\lim_{\rho \to 0} \frac{\Delta z}{\rho} = D_{\varphi}(z) = \frac{\partial z}{\partial x}\cos \varphi + \frac{\partial z}{\partial y}\sin \varphi.$$

39. DIFFERENTIATING IMPLICIT FUNCTIONS

The further development of the theory of partial differentiation proceeds in rather close analogy with the differential calculus of the one-dimensional case, and, for lack of time, we shall not deal with it. Instead, we have still to consider some important problems in the one-dimensional case for whose solution we utilize partial derivatives. Under this heading belong, first of all, theorems on the existence and differentiability of *implicit* functions. (This generally accepted name is rather unsatisfactory. It would be more proper to speak of functions *given implicitly* or *defined implicitly,* since it is not a question of a special kind of functions, but only of a special way of defining them.)

It has, of course, been brought to your attention more than once that the equation

$$F(x, y) = 0 \tag{17}$$

determines y as an implicit function of x. This means that there exists a function $y = f(x)$ such that

$$F(x, f(x)) = 0. \tag{18}$$

It is clear that the conditions which guarantee the existence of such a function, its properties, and also the range of values x for which equation (18) is valid, all require special study. Here we shall prove only the fundamental theorem in this area. The further

development of the theory of implicit functions, based upon this theorem, may be found in any unabridged course in analysis and presents nothing new in principle.

THEOREM 4. *Let* $F(x, y)$ *be:* (i) *continuous in a neighborhood of the point* (x_0, y_0), (ii) *differentiable at* (x_0, y_0), *and,* (iii) *equal to zero at that point, while at the same time* $F_y'(x_0, y_0) \neq 0$. *Then there exists, in some neighborhood of the point* x_0, *a function* $y = f(x)$ *for which* $f(x_0) = y_0$ *and the equality* (18) *is an identity. This function* $f(x)$ *is differentiable at* x_0 *and*

$$f'(x_0) = -\frac{F_x'(x_0, y_0)}{F_y'(x_0, y_0)}.$$

Proof. To be definite, let us assume that $F_y'(x_0, y_0) > 0$. Then, for a sufficiently small positive β, we shall have $F(x_0, y_0 - \beta) < 0$ and $F(x_0, y_0 + \beta) > 0$ (Fig. 24), with $F(x, y)$ continuous at (x_0, y) for all y in $[y_0 - \beta, y_0 + \beta]$ and nonzero there except at (x_0, y_0). Therefore, for $\alpha > 0$ sufficiently small, the inequalities

$$F(x, y_0 - \beta) < 0 \qquad \text{and} \qquad F(x_0, y_0 + \beta) > 0$$

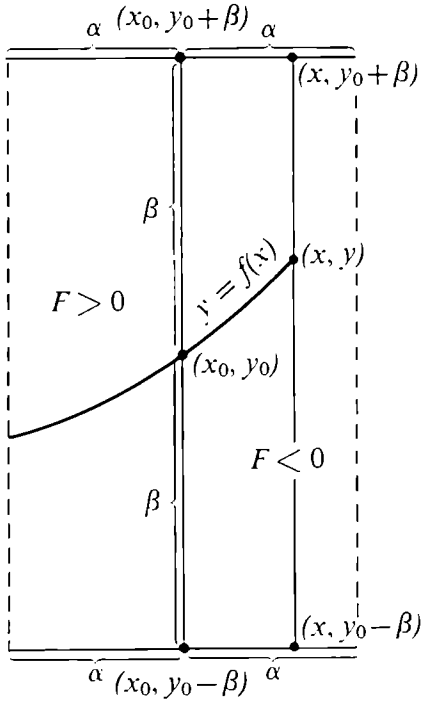

Fig. 24

will be valid for all points x in the interval $[x_0 - \alpha, x_0 + \alpha]$. Utilizing Theorem 4 in Lecture 3 (p. 57), we can find, by virtue of the continuity of F, for each such x a value y ($|y - y_0| < \beta$) such that $F(x, y) = 0$. This value of y, dependent on x, (or one of them if there are several[1]) we shall denote by $f(x)$, so that

$$F(x, f(x)) = 0 \tag{18}$$

for $x_0 - \alpha \leq x \leq x_0 + \alpha$. Clearly, at $x = x_0$ the function $f(x)$ has only one possible value y_0, so that $f(x_0) = y_0$.

It remains only to prove the existence of the derivative $f'(x_0)$ and to find its value. Let $|\Delta x| < \alpha$ and $\Delta y = f(x_0 + \Delta x) - f(x_0)$. Then, by (18), we shall have $F(x_0 + \Delta x, y_0 + \Delta y) = F(x_0 + \Delta x, f(x_0 + \Delta x)) = 0$. By the differentiability of $F(x, y)$ at the point (x_0, y_0), we obtain

$$\begin{aligned} 0 &= F(x_0 + \Delta x, y_0 + \Delta y) - F(x_0, y_0) = \Delta F \\ &= F_y'(x_0, y_0)\,\Delta y + F_x'(x_0, y_0)\,\Delta x + o(\rho), \end{aligned}$$

where

$$\rho = \sqrt{\Delta x^2 + \Delta y^2} \leq |\Delta x| + |\Delta y|.$$

It follows from this that

$$F_y'(x_0, y_0)\,\Delta y + F_x'(x_0, y_0)\,\Delta x = \lambda(|\Delta x| + |\Delta y|),$$

where $\lambda \to 0$ as $\rho \to 0$; or, in another form,

$$[F_y'(x_0, y_0) \pm \lambda]\,\Delta y + [F_x'(x_0, y_0) \pm \lambda]\,\Delta x = 0.^2$$

Since $\gamma = F_y'(x_0, y_0) > 0$, we have for sufficiently small α and β (which implies sufficiently small Δx, Δy, and ρ),

$$|\lambda| < \frac{1}{2}\gamma \quad \text{and} \quad F_y'(x_0, y_0) \pm \lambda = \gamma \pm \lambda > \frac{1}{2}\gamma,$$

and, consequently,

$$\frac{\Delta y}{\Delta x} = -\frac{F_x'(x_0, y_0) \pm \lambda}{F_y'(x_0, y_0) \pm \lambda} \quad \text{and} \quad \left|\frac{\Delta y}{\Delta x}\right| < \frac{F_x'(x_0, y_0) + \frac{1}{2}\gamma}{\frac{1}{2}\gamma}.$$

[1] For instance, we may take this value of y to be the g.l.b. of all y in the interval $[y_0 - \beta, y_0 + \beta]$ for which $F(x, y) = 0$. It will then follow from the continuity of F that $F(x, f(x)) = 0$.

[2] The signs in front of λ need not be in agreement. For example, if $\Delta x > 0$ and $\Delta y < 0$, we have $+\lambda$ in the first term and $-\lambda$ in the second term.

It is at once evident from this inequality, that as $\Delta x \to 0$ we have $\Delta y \to 0$, and hence $\rho \to 0$ and $\lambda \to 0$ also. Consequently,

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{F_x'(x_0, y_0)}{F_y'(x_0, y_0)}.$$

Let us further note that if the derivative F_y' exists and $F_y' \neq 0$ not only at the point (x_0, y_0), but also in some neighborhood of that point, then the solution of equation (17) which we have found is unique. For, if $F(x, y_1) = 0$ and $F(x, y_2) = 0$ where $y_1 < y_2$, we would have by Lagrange's theorem

$$F(x, y_2) - F(x, y_1) = 0 = (y_2 - y_1)\, F_y'(x, y),$$

where $y_1 < y < y_2$. But from this $F_y'(x, y) = 0$, which is impossible by our assumptions.

Let us now apply this rule for differentiating an implicit function to the simplest case of the so-called conditional maxima and minima problem. The generalization of this problem to the case of a larger number of variables constitutes the important and interesting theory of conditional extrema, which regrettably we cannot fully develop here.

Suppose that $F(x, y)$ is differentiable in some domain. Furthermore, suppose that x is an independent variable while y is defined in $[a, b]$ by the relation

$$\Phi(x, y) = 0, \tag{19}$$

where $\Phi(x, y)$ is also differentiable. Thus, $F(x, y)$ is actually a function of one independent variable x, defined in the complicated manner just described. Both in analysis and in its applications we frequently have to consider functions defined in such a way.

We wish to find the extrema of the given function in the interval $[a, b]$. According to the general theory, the derivative of the function must assume the value zero at these extrema (relative maxima and minima). We must therefore equate to zero the derivative of $z = F(x, y)$ with respect to x, considering y as a function of x defined by (19); that is, we must equate to zero the so-called *total* derivative $\frac{dz}{dx}$ of the given function with respect to x:

$$\frac{dz}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0. \tag{20}$$

In this equation $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are both functions of x and y. To determine the quantity $\frac{dy}{dx}$ we must, of course, apply the rule for the differentiation of implicit functions. From equation (19), which defines the function y, we obtain

$$\frac{dy}{dx} = -\frac{\frac{\partial \Phi}{\partial x}}{\frac{\partial \Phi}{\partial y}}$$

and, consequently, (20) can be written in the form

$$\frac{\partial F}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial \Phi}{\partial x}\frac{\partial F}{\partial y} = 0.$$

This equation, together with (19), permits us to determine all the *critical* pairs of values (x, y), that is, all the points at which $\frac{dz}{dx} = 0$.

6. The Integral

40. INTRODUCTION

The integral calculus was initially developed independently of the theory of differentiation. Only toward the end of the seventeenth century, after both disciplines had already achieved a considerable degree of development and had succeeded in solving, each by its own methods, a great number of problems in geometry and mechanics, was the profound connection between them fully revealed. It then became clear that their fundamental problems are mutually inverse, and integration and differentiation of functions are to each other as addition and subtraction of numbers.

This historical moment is usually considered to be the birth of the science which is now called *mathematical analysis.* And from this moment onward, with the idea of their unbreakable theoretical bond serving as the main propellant of both subjects, their development took on extraordinary speed. In the integral calculus, in particular, the theory passed from the solution of individual unrelated problems to the creation of quite powerful and general methods.

The historical development of the two basic branches of analysis is still reflected in the way they are presented in textbooks. While some authors, for the sake of greater logical coherence, define the integration of functions as an operation inverse to differentiation, others, reproducing to a certain degree the historical development, prefer to define the two operations independently of each other, and only later establish their mutual connection. On the *formal* side it is, of course, irrelevant which of these paths we follow, and it is quite difficult, perhaps impossible, to maintain that one of these methods is preferable to the other. The point is, that here, as almost everywhere, much depends on the needs and interests of the student. One for whom a logical and theoretical interest in mathematics outweighs the interest in the applied and the practical will welcome the introduction of integration as an operation inverse to differentiation. Following the habit of

mathematicians, he himself might have wondered, while learning differentiation, what the inverse of such an operation would be like. To the student who is interested in practical applications, this approach may appear artificial: studying an inverse operation as such, when one does not yet see in what concrete problems it can be useful, may seem to lack justification.

41. DEFINITION OF THE INTEGRAL

The concept of the integral arose and gradually entrenched itself in a position of importance when a whole series of problems in geometry and mechanics led to the necessity of performing the same analytical operation on a variety of functions. The essence of this operation was a certain passage to the limit, the nature of which is well known to you.

We are given a function $f(x)$ on a closed interval $[a, b]$. Let us subdivide this interval into n parts, denoting the points of partition by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

In each of the subintervals $[x_{k-1}, x_k]$ (where $k = 1, 2, \ldots, n$) we choose an arbitrary point ξ_k. We multiply the value of the function $f(x)$ at the point ξ_k by the length $x_k - x_{k-1}$ of the corresponding subinterval, and form the sum of all these products,

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}). \tag{1}$$

Let us now imagine that we have before us the class of all such partitions and of all possible choices of the points ξ_k (remembering that the number n of subintervals may be changed arbitrarily), and, hence, also the set of all possible values of the sum (1). Let us denote by l the length of the largest subinterval $[x_{k-1}, x_k]$ in a given partition. If there exists a number I such that the sum (1) tends to I as $l \to 0$, regardless of how the partitions are constructed and the points ξ_k are selected, then we call that number the integral of $f(x)$ over $[a, b]$ and denote it by

$$\int_a^b f(x)\, dx.$$

A somewhat more precise formulation is as follows:

DEFINITION. *The number I is called the integral of $f(x)$ over the interval $[a, b]$ if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition of $[a, b]$ satisfying the condition $l < \delta$ and for any choice of the points ξ_k, we have the inequality*

$$\left| I - \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \right| < \varepsilon.$$

In each case we see that we are dealing with a peculiar and quite complicated passage to the limit. The underlying process of this passage to the limit is difficult to describe in the usual manner, in terms of the behavior of some independent variable. We could take l as such a variable (and describe the process with the help of the symbol $l \to 0$), but we have then to remember that the sum (1), whose limit we are discussing, is not a single-valued function of the quantity l. It is easy to see that to a given value l there corresponds an infinite number of different partitions and that to each of these partitions there corresponds an infinite number of possible choices of the points ξ_k. Thus for the limit I to exist we must be able to make the sum (1) differ arbitrarily little from I merely by taking l sufficiently small, irrespective of how we make these choices.

This complexity of the process underlying integration leads to certain inconveniences. In general theoretical constructions, as well as in concrete practical cases, it is possible to prove rigorously that the limit I is independent of the nature of the partition and the choice of the points ξ_k. These proofs, however, with their cumbersome formalism, frequently make the reasoning very awkward. For this reason, many modern presentations resort to defining the integral in a somewhat different manner so as to avoid introducing, at the beginning, any passage to the limit. We shall now turn to this other formulation (which, of course, is formally equivalent to the previous one).

Let $f(x)$ be an arbitrary function bounded on the closed interval $[a, b]$, and let us denote its bounds by M (least upper) and m (greatest lower). For any partition T of $[a, b]$, let us denote by M_k and m_k the least upper and greatest lower bounds, respectively, of $f(x)$ in the interval $[x_{k-1}, x_k]$ and write

$$x_k - x_{k-1} = \Delta_k \qquad (1 \leq k \leq n).$$

Also let

$$S_T = \sum_{k=1}^{n} M_k \Delta_k$$

and

$$s_T = \sum_{k=1}^{n} m_k \Delta_k.$$

Thus, to each partition T there corresponds a definite *upper sum* S_T and a definite *lower sum* s_T. Since $m \leq m_k \leq M_k \leq M$ $(1 \leq k \leq n)$, we have, for any partition T, the inequalities $m(b - a) \leq s_T \leq S_T \leq M(b - a)$. The set of values of the upper sum S_T, as well as of the lower sum s_T for all possible different partitions of $[a, b]$, is therefore, a bounded set.

We call the greatest lower bound of the set of all upper sums S_T the *upper integral* of $f(x)$ over $[a, b]$ (or between the limits a and b) and we denote it by the symbol

$$\overline{I} = \overline{\int_a^b} f(x)\, dx.$$

In the same way, we call the least upper bound of all lower sums s_T the *lower integral* of $f(x)$ over $[a, b]$ (or between the limits a and b) and denote it by

$$\underline{I} = \underline{\int_a^b} f(x)\, dx.$$

Thus, every bounded function defined on a closed interval has an upper and a lower integral over the given interval. These two integrals are defined as the bounds of certain sets and, as you see, without any appeal to the concept of a limit.

DEFINITION. *If the upper and the lower integral of* $f(x)$ *over* $[a, b]$ *coincide, we call their common value the* integral *of* $f(x)$ *over* $[a, b]$ *(or between the limits* a *and* b*) and denote it by*

$$I = \int_a^b f(x)\, dx.$$

The function $f(x)$ *is then said to be* integrable *over* $[a, b]$.

To draw the needed conclusions from this definition, we shall prove some elementary lemmas.

We shall say that a partition T' of the interval $[a, b]$ is a *refinement* of the partition T if all the points of partition of T are also points of partition of T'. (In general, T' also contains new points of partition which do not belong to T.)

LEMMA 1. *If T' is a refinement of the partition T, then*

$$S_{T'} \leq S_T \quad \text{and} \quad s_{T'} \geq s_T.$$

Proof. In passing from the partition T to its refinement T', each term $M_k\Delta_k$ of the sum S_T is replaced by a group of terms

$$\sum_{r=1}^{s} M_{k_r}\Delta_{k_r},$$

where $\Delta_{k_1}, \Delta_{k_2}, \ldots, \Delta_{k_s}$ are the lengths of the subintervals formed from the interval $[x_{k-1}, x_k]$ in passing from T to T', and M_{k_r} is the least upper bound of $f(x)$ in Δ_{k_r}. Since $M_{k_r} \leq M_k$ $(1 \leq r \leq s)$, we have

$$\sum_{r=1}^{s} M_{k_r}\Delta_{k_r} \leq M_k \sum_{r=1}^{s} \Delta_{k_r} = M_k\Delta_k;$$

that is, the group of terms which replaces $M_k\Delta_k$ has a sum not greater than $M_k\Delta_k$. And as this is true for all k, we have $S_{T'} \leq S_T$. (The second inequality is proved in a precisely analogous manner.)

LEMMA 2. *For any two partitions T_1 and T_2 of the interval $[a, b]$ we have the inequality*

$$S_{T_1} \geq s_{T_2}.$$

Proof. The set of all points of partition of T_1 and T_2 determines a third partition T, which is clearly a refinement of both T_1 and T_2. For the partition T, we clearly have $S_T \geq s_T$. Hence, by Lemma 1

$$S_{T_1} \geq S_T \geq s_T \geq s_{T_2}, \qquad \text{q.e.d.}$$

A direct corollary of Lemma 2 is the following:

LEMMA 3. *For every (bounded) function we have the inequality $\overline{I} \geq \underline{I}$.*

Proof. Since none of the sums s_T exceeds any of the sums S_T, the least upper bound $\underline{I}$ of the set of all s_T cannot exceed the greatest lower bound $\overline{I}$ of the set of all S_T.

Let us now denote by l_T the length of the largest subinterval of the partition T.

LEMMA 4. *For any $\varepsilon > 0$ there exists a $\lambda > 0$ such that for all partitions in which $l_T < \lambda$, we have $S_T < \bar{I} + \varepsilon$ and $s_T > \underline{I} - \varepsilon$.*

Sometimes this lemma is expressed more briefly by saying that $S_T \to \bar{I}$ and $s_T \to \underline{I}$ as $l_T \to 0$. Such a formulation cannot cause any misunderstanding, if we simply remember that S_T and s_T are not single-valued functions of the length l_T.

Proof. Without loss of generality, we may assume that $f(x) \geq 0$ for $a \leq x \leq b$. We can always achieve this by adding to $f(x)$ a sufficiently large number A, whereupon all the integrals and sums will increase by $A(b-a)$. By the definition of the greatest lower bound, there exists a partition T_0 such that $S_{T_0} < \bar{I} + \frac{\varepsilon}{2}$. Let us denote by $x_1, x_2, \ldots, x_n$ the interior points of partition of T_0, and let M be an upper bound for $|f(x)|$ in $[a, b]$. Finally, let us introduce $\lambda = \frac{\varepsilon}{4nM}$, and let T be any partition of $[a, b]$ for which $l_T < \lambda$. We shall show that $S_T < \bar{I} + \varepsilon$.

For this purpose we shall divide the intervals formed from $[a, b]$ by the partition T into two groups: in the first group (I) we include every interval which is entirely contained in one of the intervals $[x_k - \lambda, x_k + \lambda]$, $1 \leq k \leq n$, and in the second group (II) the remaining intervals (Fig. 25). Clearly, every interval in the second

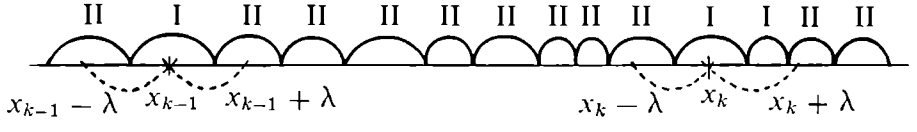

Fig. 25

group is entirely contained in one of the subintervals of the partition T_0. Thus, the sum S_T breaks up into the sums S_T^{I} and S_T^{II}. In the sum S_T^{I}, all the terms have first factors not greater than M, while the sum of the lengths of the intervals which constitute the second factors is not greater than $2n\lambda = \frac{\varepsilon}{2M}$. Therefore,

$$S_T^{\mathrm{I}} \leq M \frac{\varepsilon}{2M} = \frac{\varepsilon}{2}.$$

On the other hand, if we consider $S_T{}^{II}$, the portion of the sum contributed by the intervals of the partition T which are contained within some $[x_{k-1}, x_k]$ (Fig. 25), we see that its first factors are not greater than M_k and that the sum of its second factors is not greater than $x_k - x_{k-1} = \Delta_k$. Therefore, the value of this portion of $S_T{}^{II}$ is not greater than $M_k\Delta_k$. Consequently,

$$S_T{}^{II} \leq \sum_{k=1}^{n} M_k\Delta_k = S_{T_0}.$$

We thus have

$$S_T = S_T{}^{I} + S_T{}^{II} \leq S_T{}^{I} + S_{T_0} \leq \frac{\varepsilon}{2} + \bar{I} + \frac{\varepsilon}{2} = \bar{I} + \varepsilon,$$

which is what we set out to prove. And the second inequality of our lemma is proved in an analogous manner.

THEOREM 1. *Let $f(x)$ be integrable over $[a, b]$. Then for any $\varepsilon > 0$ there exists a $\lambda > 0$ such that for any partition T in which $l_T < \lambda$, and for any choice of the points ξ_k ($x_{k-1} \leq \xi_k \leq x_k$ and $1 \leq k \leq n$), we have the inequality*

$$\left| \sum_{k=1}^{n} f(\xi_k)\,\Delta_k - I \right| < \varepsilon,$$

where $I = \int_a^b f(x)\,dx$.

Proof. Since $m_k \leq f(\xi_k) \leq M_k$ $(1 \leq k \leq n)$ for any choice of the points ξ_k in the corresponding subintervals, we have

$$s_T \leq \sum_{k=1}^{n} f(\xi_k)\,\Delta_k \leq S_T.$$

On the other hand, by Lemma 4 we have

$$I - \varepsilon = \underline{I} - \varepsilon < s_T \leq S_T < \bar{I} + \varepsilon = I + \varepsilon$$

for all partitions with sufficiently small l_T. Combining these inequalities with those immediately above, we find that for all partitions with sufficiently small l_T, we have

$$I - \varepsilon < \sum_{k=1}^{n} f(\xi_k)\,\Delta_k < I + \varepsilon.$$

The converse theorem is also true, as we can readily see.

THEOREM 2. *If there exists a number I such that for any partition T with sufficiently small l_T and with any choice of the points ξ_k, the sum $\sum_{k=1}^{n} f(\xi_k)\,\Delta_k$ differs from I by arbitrarily little, then the function $f(x)$ is integrable over $[a, b]$.*

Proof. Given any $\varepsilon > 0$ choose T with l_T so small that any sum $Y = \sum_{1}^{n} f(\xi_k)\,\Delta_k < I + \frac{\varepsilon}{2}$. We may choose ξ_k in the interval $[x_{k-1}, x_k]$ so $M_k - f(\xi_k) < \frac{\varepsilon}{2} \cdot \Delta_k$ (since M_k is the *least* upper bound of f in this interval). Then $S_T - Y = \sum_{1}^{n} (M_k - f(\xi_k))\,\Delta_k < \frac{\varepsilon}{2}$. Thus $S_T < Y + \frac{\varepsilon}{2} < I + \varepsilon$, and since $\overline{I} \leq S_T$, $\overline{I} < I + \varepsilon$. Since ε is arbitrary, $\overline{I} \leq I$. We may show similarly that $I \leq \underline{I}$, so that (using Lemma 3) $\underline{I} \leq \overline{I} \leq I \leq \underline{I}$ and equality must hold throughout.

Thus, the new definition of the integral is equivalent to the original definition.

The following is clearly a corollary of Lemma 4:

COROLLARY. *If, by $\omega_k = M_k - m_k$ we denote the oscillation of $f(x)$ in the interval $[x_{k-1}, x_k]$, then a necessary and sufficient condition for the integrability of $f(x)$ over $[a, b]$ is that the sum*

$$\sum_{k=1}^{n} \omega_k\,\Delta_k = S_T - s_T \tag{2}$$

be arbitrarily small for all partitions T with sufficiently small l_T.

42. CRITERIA FOR INTEGRABILITY

The necessary and sufficient condition for integrability which was given in the foregoing corollary is usually not applied directly to any given function, as in most cases it is not easy to learn the behavior of the sum (2). But with the help of this condition, it is very easy to establish general criteria for integrability which apply to more or less wide classes of functions.

First of all, we shall show that *every function $f(x)$ continuous on $[a, b]$ is integrable over this interval.* For, by the property of uniform continuity (Theorem 4 in Lecture 3, p. 57), it follows that for each $\varepsilon > 0$, there exists a $\lambda > 0$ such that the oscillation of the function

$f(x)$ will be less than ε in every interval of length less than λ. Then, for any partition T for which $l_T < \lambda$, all ω_k in the sum (2) will be smaller than ε and, consequently,

$$\sum_{k=1}^{n} \omega_k \Delta_k < \varepsilon \sum_{k=1}^{n} \Delta_k = \varepsilon (b - a);$$

that is, the sum (2) becomes arbitrarily small for sufficiently small l_T. The integrability of $f(x)$ over $[a, b]$ follows.

Can a noncontinuous function be integrable? It is easy to show by examples that this is possible, and we can even establish certain general laws concerning this question. We shall show, for example, that *a bounded function $f(x)$ which has only one point c of discontinuity in the interval $[a, b]$ is integrable over this interval* (regardless of the nature of the discontinuity).

For this purpose, let us denote by μ an upper bound for $|f(x)|$ in $[a, b]$ and choose any $\varepsilon > 0$. The function $f(x)$ is continuous in the intervals $\left[a, c - \frac{\varepsilon}{2\mu}\right]$ and $\left[c + \frac{\varepsilon}{2\mu}, b\right]$. Therefore, we may again find a $\lambda > 0$ such that the oscillation of the function in any interval of length less than λ and wholly contained either in $\left[a, c - \frac{\varepsilon}{2\mu}\right]$ or in $\left[c + \frac{\varepsilon}{2\mu}, b\right]$ will be less than ε. Now let T be any partition of $[a, b]$ for which $l_T < \lambda$. In general, among the intervals Δ_k there will be some that are wholly contained in one of the two above mentioned intervals, and some that will overlap the interval $\left[c - \frac{\varepsilon}{2\mu}, c + \frac{\varepsilon}{2\mu}\right]$ (Fig. 26).

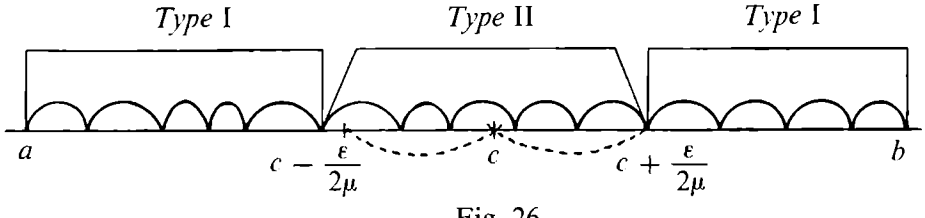

Fig. 26

For the intervals Δ_k of the first type, the oscillations ω_k in the sum (2) are less than ε, and, consequently, the part of the sum (2) contributed by these intervals of the first type is less than $\varepsilon(b - a)$. For the intervals Δ_k of the second type, we can say with regard to the oscillations ω_k only that each of them is not greater than 2μ. But since the intervals of the second type are all wholly con-

tained in the interval $\left[c - \frac{\varepsilon}{2\mu} - \lambda, c + \frac{\varepsilon}{2\mu} + \lambda\right]$, the sum of their lengths does not exceed $\frac{\varepsilon}{\mu} + 2\lambda$; hence the corresponding part of the sum (2) is not greater than $2\mu\left(\frac{\varepsilon}{\mu} + 2\lambda\right) = 2\varepsilon + 4\lambda\mu$. Combining this with our estimates of the first part of (2), we find that, for $l_T < \lambda$,

$$\sum_{k=1}^{n} \omega_k \Delta_k < \varepsilon(b - a) + 2\varepsilon + 4\lambda\mu.$$

As ε and λ are arbitrarily small, the integrability of $f(x)$ over $[a, b]$ is established.

We have deliberately carried out this simple reasoning in the fullest detail. It is important here to grasp the basic idea of the proof: the sum (2) turns out to be arbitrarily small because, for the intervals in which the function is continuous, the *first* factors ω_k are arbitrarily small, and in the intervals near the point of discontinuity the *second* factors yield an arbitrarily small sum. Therefore, both parts of the sum (2) are arbitrarily small, and hence the whole sum is also. It is now easy to understand (and prove rigorously) that the existence of any finite number of points of discontinuity of $f(x)$ in $[a, b]$ does not interfere with its integrability, provided the function remains bounded. On the other hand, if the points of discontinuity constitute too large a part of the interval $[a, b]$, the function may prove to be nonintegrable. For example, the Dirichlet function (Lecture 3, p. 46) which is everywhere discontinuous, is not integrable over any interval. This is so because for any partition we have $\omega_k = 1$ for all k. Consequently, for any interval $[a, b]$ we obtain

$$\sum_{k=1}^{n} \omega_k \Delta_k = \sum_{k=1}^{n} \Delta_k = b - a.$$

The question concerning the number of discontinuities which a bounded function may have in an interval $[a, b]$ and still remain integrable finds its complete solution in the following proposition.

THEOREM 3. *A necessary and sufficient condition that a bounded function $f(x)$ be integrable over the interval $[a, b]$ is the following: for any $\varepsilon > 0$, all the points of $[a, b]$ at which the oscillation of $f(x)$ exceeds ε can be included in a finite number of intervals, the sum of whose lengths is less than ε.*

Proof. (i) Suppose the condition stated in the theorem is satisfied. Let us denote by $\delta_1, \delta_2, \ldots, \delta_n$ a family of intervals containing in their interiors all the points at which the oscillation of $f(x)$ exceeds ε and such that $\sum_{i=1}^{n} |\delta_i| < \varepsilon$, where $|\delta_i|$ denotes the length of the interval δ_i. We shall denote by $d_1, d_2, \ldots, d_n$ the family of complementary intervals (that is, the family of intervals obtained from $[a, b]$ by removing the intervals $\delta_1, \delta_2, \ldots, \delta_n$). Since at each point in any of the intervals d_i the oscillation of $f(x)$ does not exceed ε, it follows, from Theorem 9 on page 65 in Lecture 3, that the oscillation of the function will be less than 2ε in any interval of sufficiently small length contained wholly in one of the intervals d_i. Now we have only to imagine any sufficiently fine partition (that is, having sufficiently small l_T) of the interval $[a, b]$ into subintervals Δ_k, and argue along the lines indicated on page 142. Divide the sum (2) into two parts according to whether the interval Δ_k is wholly contained in one of the intervals d_i or whether it overlaps, even partially, one of the intervals δ_i. We readily find that the first part of the sum (2) is less than $2\varepsilon(b - a)$, and that the second part of the sum (2) does not exceed $2\mu(\varepsilon + 2nl_T)$, where μ is the least upper bound of $|f(x)|$ on $[a, b]$ and n is the number of intervals δ_i. Taking ε and l_T sufficiently small, we can make the sum (2) as small as we please. We have thus proved that the condition of the theorem is sufficient.

(ii) To prove that the condition is necessary, suppose $f(x)$ to be integrable over $[a, b]$ and let ε be any positive number. We can choose a partition of $[a, b]$ such that the corresponding sum (2) will be less than ε^2. We now observe that all the terms of the sum (2) are nonnegative. Therefore, we shall not increase this sum if we retain in it only those terms in which $\omega_k > \varepsilon$ and discard all the remaining ones. Thus,

$$\varepsilon^2 > \sum_{k=1}^{n} \omega_k \Delta_k \geq \sum_{\omega_K > \varepsilon} \omega_k \Delta_k \geq \varepsilon \sum_{\omega_K > \varepsilon} \Delta_k,$$

and, consequently,

$$\sum_{\omega_K > \varepsilon} \Delta_k < \varepsilon.$$

But the intervals Δ_k included in this last sum clearly contain all the points of $[a, b]$ at which the oscillation of the function exceeds

ε (since such a point cannot belong to an interval Δ_k for which $\omega_k \leq \varepsilon$). Thus, the criterion for integrability formulated in our theorem is seen to be necessary.

From the theorem just proved we have the following:

COROLLARY. *Every function monotonic on an interval* $[a, b]$ *is integrable over this interval.*

Proof. As we have seen in Lecture 3, every such function must be bounded in the given interval. Furthermore, no matter how small the positive number ε, the number of points in $[a, b]$ at which the oscillation of the function is greater than ε must be finite. As it is always possible to contain a finite number of points in a finite family of intervals such that the sum of their lengths is arbitrarily small, the criterion of integrability in the theorem above is satisfied by every monotonic function.

43. GEOMETRIC AND PHYSICAL APPLICATIONS

The concept of the integral considered above is closely connected (through the notion of limit or bound) with sums of a certain type. Although, in itself, it is totally unrelated to the concepts of the differential calculus, it has, as you know, many geometric and physical applications. We shall not dwell on these applications as such, nor shall we even stop to enumerate the most important of these. It is, however, very important for us to notice a characteristic and rather subtle logical situation, frequently encountered in these applications, which is usually passed over too lightly in textbooks.

For example, let us consider the computation of the area of a *curvilinear trapezoid* (Fig. 27) bounded above by the graph of a

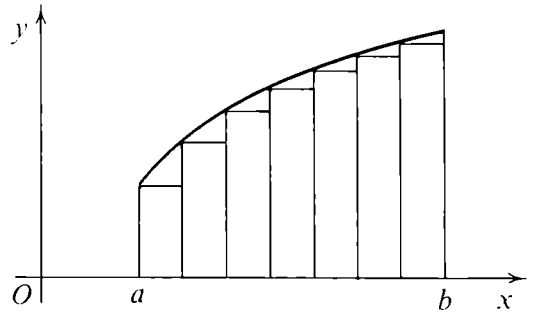

Fig. 27

function $y = f(x)$, which for simplicity we assume to be positive and continuous. You are, of course, familiar with this problem as well as with its solution by means of the integral calculus. But let us try to penetrate more deeply into the logical situation which we find here. The computation of the area of any particular figure makes sense only when the concept of area itself is exactly defined. But at this moment, when we are looking for a method which would permit us to compute the area of the curvilinear trapezoid of Figure 27, do we yet have such a definition? Can we precisely define that quantity which we plan to compute? Obviously not; we know the definition of area only for rectilinear figures (polygons) and certain parts of the circle. However, the curve $y = f(x)$ which bounds our trapezoid has, in general, nothing in common with the circumference of a circle.

How then, without knowing what we mean by the required area, can we proceed to its computation? And what is more amazing, how is it that we succeed in performing this computation, although we do not know logically what it is that we are computing? The fact is that we both state and solve a problem, rather than carry out an ordinary computation: with the concept of the integral, we simultaneously *define* the area for our curvilinear trapezoid and find a method for computing that area. For, when we state that the area of our figure is equal to the limit of areas of rectilinear step-shaped figures, one of which is represented in Figure 27, such an assertion is not a theorem, but a *definition* of the area of the figure. It would make no sense, therefore, to try to prove that statement. On the other hand, the assertion that the above mentioned limit *exists* under such and such conditions, for instance the continuity of the function $f(x)$, is a *theorem* which can and should be proved. And precisely the same may be said concerning the logical nature of every geometric and physical problem of similar type. Whether we plan to compute the length of an arc, a volume, an area of revolution, or the work of a given force over a given portion of path, we are concerned with finding a quantitative measure for a concept which has previously been defined only for the simplest particular cases. The problem is to formulate, for this concept, a suitable general definition which will at the same time embody a method for computing the corresponding quantity.

What are the common features of the problems of geometry and physics mentioned above (and many others) which make it

possible to solve all these problems analytically in the same way? We wish to use the integral as the tool which defines the desired quantity and simultaneously gives its value. Two basic features can be observed. In all cases, the quantity we seek depends on the interval $[a, b]$ over which it *extends,* and it will vary as this interval is changed. Thus, in Figure 27 we obtain a different area if we replace the segment $[a, b]$ by another segment (of course, leaving the function unchanged). The work of a force is different over various parts of the path described by the point in motion, and so on. On the other hand, the quantity under consideration in each particular problem depends on some function $f(x)$. In Figure 27, it is the ordinate of the point on the upper boundary of the trapezoid whose abscissa is equal to x; in the problem of computing work, it is the value of the force acting at the distance x from the origin; and so on.

Thus in formulating a problem of this type it is necessary first of all to set up a function $f(x)$ and an interval $a \leq x \leq b$ to which our problem will refer. We can say that the quantity whose definition and value we seek is a function $V(f; a, b)$ of three elements which can be chosen independently of one another: the function $f(x)$ and the numbers a and b. It is easy to see that the application of the integral as a method of solving all the above-mentioned problems depends basically upon the following properties of this function $V(f; a, b)$.

Property 1. As a function of an interval $[a, b]$, the quantity V is *additive,* that is, for $a < c < b$ we have

$$V(f; a, b) = V(f; a, c) + V(f; c, b).$$

Indeed, under any reasonable definition, the area of Figure 27 should be equal to the sum of the areas of the curvilinear trapezoids into which that area breaks up when we subdivide the interval $[a, b]$ into two intervals. The volume of a solid of revolution is equal to the sum of the volumes of the solids generated about the separate segments into which we may divide the axis of revolution; the length of an arc is equal to the sum of the lengths of its parts; the work of a force over a given path is equal to the sum of the work performed by the force on each separate portion of that path; and so on.

Property 2. If the function $f(x)$ is constant on $[a, b]$, that is, $f(x) = C$, then

$$V(f; a, b) = C(b - a).$$

Indeed, if $f(x) = C$ $(a \leq x \leq b)$, then the Figure 27 is a rectangle whose area is equal to $C(b - a)$. In the case of a solid of revolution, if $f(x)$ represents the area of its cross section, and if $f(x) = C$ $(a \leq x \leq b)$, then we have a cylinder, whose volume is equal to $C(b - a)$; if the force acting on a point remains constant over the path $a \leq x \leq b$, the work of this force over the given path will be equal to $C(b - a)$; and so on.

It is now readily seen that in all cases in which the dependence of the desired quantity V upon the given elements $f(x)$, a, and b has the characteristics Properties 1 and 2, it is natural to expect that the solution of the problem will be the integral

$$V = \int_a^b f(x)\, dx.$$

For, if we subdivide the given interval, as usual, by the points of partition

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

then, by Property 1,

$$V(f;\, a,\, b) = \sum_{k=1}^{n} V(f;\, x_{k-1},\, x_k).$$

And, if the function $f(x)$ had for all x in the interval $[x_{k-1},\, x_k]$ the constant value C_k, then, by Property 2, we would have

$$V(f;\, x_{k-1},\, x_k) = C_k(x_k - x_{k-1}).$$

In general, the function $f(x)$ is not constant on $[x_{k-1},\, x_k]$. If the function is continuous, however, and if the interval $[x_{k-1},\, x_k]$ is very small, then values of $f(x)$ in this interval differ very little from one another. Taking one of these, $f(\xi_k)$, we may say that in the whole interval $[x_{k-1},\, x_k]$, the function $f(x)$ is approximately equal to $f(\xi_k)$, and, consequently, it is natural to consider the quantity $V(f;\, x_{k-1},\, x_k)$ (we must not forget that this quantity is not yet determined) as approximately equal to $f(\xi_k)(x_k - x_{k-1})$. Thus, we have the approximation

$$V(f;\, a,\, b) \approx \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}),$$

where ξ_k is an arbitrary point in $[x_{k-1},\, x_k]$. We assume, furthermore, that the error of this approximation is infinitesimal as

the intervals of partition become smaller. We are thus led, in a natural manner, to define the precise value of the quantity V as

$$\lim_{l_T \to 0} \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \int_a^b f(x)\, dx.$$

44. RELATION OF INTEGRATION TO DIFFERENTIATION

In the first stage of the development of the integral calculus, when the enormous importance of the connection between integration and differentiation had not yet sufficiently permeated mathematical thought, problems of the type described above were solved by direct computation of the integral as the limit of a sum. By the choice of particularly convenient partitions and especially selected values of ξ_k, mathematicians tried in each case, that is, for each particular function $f(x)$, to make this computation, in general very cumbersome, as easy and simple as possible.

A number of problems had already been solved in this manner in ancient times; to these were later added a series of new achievements. Yet, in spite of this, until the relationship between integration and differentiation became the basic method of computing integrals, all results remained scattered and each new problem required an essentially new method of attack. Integral calculus developed its general and most valuable methods only in close collaboration with the theory of differentiation.

You know, of course, the nature of this connection between integration and differentiation:

THEOREM 4. *At each point x at which the function $f(x)$ is continuous (for a continuous function, in particular, this means everywhere), the derivative of the integral*

$$F(x) = \int_a^x f(u)\, du$$

is the integrand $f(x)$.

Proof. The proof is very simple. If $|h|$ is sufficiently small, then, for

$$x - |h| \leq u \leq x + |h|,$$

we have

$$f(x) - \varepsilon \leq f(u) \leq f(x) + \varepsilon,$$

where ε is an arbitrarily small number chosen beforehand.

Consequently,

$$hf(x) - \varepsilon|h| \leq \int_x^{x+h} f(u)\,du \leq hf(x) + \varepsilon|h|,$$

and, hence,

$$f(x) - \varepsilon \leq \frac{1}{h}\int_x^{x+h} f(u)\,du = \frac{F(x+h) - F(x)}{h} \leq f(x) + \varepsilon.$$

Since ε is arbitrarily small, it follows that

$$F'(x) = f(x).$$

By virtue of this relation, every differentiation formula yields a certain integration formula when read, so to speak, from right to left. But more important than this, many general rules for differentiation may be at least partially reversed, and thus lead to the most important general methods of integration. For example, the rule for the differentiation of an algebraic sum is completely reversible and yields the formula for the integration of an algebraic sum (which, of course, may also be obtained directly from the definition of the integral). The differentiation formula for a product leads to *integration by parts*, with whose effectiveness you are familiar, while the reverse of the chain rule for composite functions is the likewise familiar and even more powerful method of *integration by substitution.*

The application of this entire group of methods makes it possible to integrate a great number of elementary functions, in particular, *all* rational functions. And if, nevertheless, we are unable to find the integrals of many other elementary functions, it is not because of the insufficient power of the methods we employ. It is for another reason, of incomparably greater theoretical significance. While the differentiation of elementary functions always leads to other elementary functions, the situation is completely different when we are integrating elementary functions. It very often happens that the integral of such a function, although it exists, is nevertheless not an elementary function and, therefore, cannot be expressed by any elementary formula. This is the case, for instance, with the integrals of such simple functions as $\frac{1}{\ln x}$ and $\frac{1}{\sqrt{x^3+1}}$. Such a new function has to be studied without any other tool (at least initially) except the integral defining it.

45. MEAN VALUE THEOREMS FOR INTEGRALS

In the differential calculus the term *mean value theorem* is commonly applied to Lagrange's theorem (Section 33) which states: *If the function $f(x)$ is continuous on the interval $[a, b]$ and differentiable within it, there exists an interior point c of that interval such that*

$$f(b) - f(a) = f'(c)(b - a).$$

In general, there is characteristically present in the formulation of *mean value theorems* a certain number c (the *mean value* of the quantity x between a and b), about which we know only that it lies inside the interval $[a, b]$, and nothing more. In this sense, Cauchy's formula (Lecture 5) as well as Taylor's formula with the various forms of the remainder, also represent mean value theorems. Very often, theorems of this kind are formulated using somewhat different notation: one speaks of the interval $[a, a + h]$, and the unspecified interior point of that interval is denoted by $a + \theta h$, where the double inequality $0 < \theta < 1$ is all that these theorems assert concerning the number θ. The presence in the formulation of the theorem of such an unspecified number satisfying the inequalities $0 < \theta < 1$ may be considered as a typical feature of mean value theorems.

The role of mean value theorems in the integral calculus is no smaller than their role in the differential calculus. One of them, called *the first mean value theorem*, is undoubtedly known to you. In the simplest case this theorem states that: If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = (b - a)f(c), \tag{3}$$

where c is an interior point of $[a, b]$. We can prove this theorem by the direct application of Lagrange's theorem to the function $F(x) = \int_a^x f(u)\,du$, $a \leq x \leq b$.

More general is the relation

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a) \quad (a < b), \tag{4}$$

where m and M are, respectively, the lower and upper bound of $f(x)$ on $[a, b]$. This relation holds for any bounded, integrable

function. From the point of view of estimating the integral, the equality (3) does not give us anything more than the inequality (4), as the location of the point c is not known.

A more general formulation is the following:

FIRST MEAN VALUE THEOREM. *If $f(x)$ and $\varphi(x)$ are continuous on $[a, b]$ and $\varphi(x) > 0$ there, then*

$$\int_a^b f(x)\ \varphi(x)\ dx = f(c)\int_a^b \varphi(x)\ dx, \tag{5}$$

where c again denotes some (unspecified) interior point of $[a, b]$.

Proof. To prove formula (5), which in the particular case where $\varphi(x) = 1$ coincides with formula (3), it is enough to apply Cauchy's formula in the interval $[a, b]$ to the functions

$$F(x) = \int_a^x f(u)\ \varphi(u)\ du$$

and

$$\Phi(x) = \int_a^x \varphi(u)\ du.$$

This gives

$$\frac{\int_a^b f(x)\ \varphi(x)\ dx}{\int_a^b \varphi(x)\ dx} = \frac{F(b) - F(a)}{\Phi(b) - \Phi(a)}$$

$$= \frac{F'(c)}{\Phi'(c)} = \frac{f(c)\ \varphi(c)}{\varphi(c)} = f(c),$$

that is, formula (5).

A considerably sharper analytical tool is provided by the so-called *second mean value theorem,* which we shall now discuss. This theorem also concerns the integral

$$\int_a^b f(x)\ \varphi(x)\ dx, \tag{6}$$

but only where one of the two factors in the integrand is monotonic in the interval $[a, b]$.

Suppose $\varphi(x)$ is nonnegative and nonincreasing for $a < x < b$, and, as usual, let T denote a partition of $[a, b]$. We shall first show that the integral (6) is the limit as $l_T \to 0$ of sums of the form

$$S = \sum_{k=1}^{n} \varphi(\xi_k) \int_{x_{k-1}}^{x_k} f(x)\, dx. \tag{7}$$

Thus let μ denote the least upper bound of $|f(x)|$ in $[a, b]$, and Δ the difference between the sum (7) and the integral (6). From the assumption that $\varphi(x)$ is nonincreasing, it follows that

$$\begin{aligned} |\Delta| &= \left| \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} [\varphi(\xi_k) - \varphi(x)] f(x)\, dx \right| \\ &\leq \sum_{k=1}^{n} [\varphi(x_{k-1}) - \varphi(x_k)]\, \mu(x_k - x_{k-1}) \\ &\leq \mu l_T [\varphi(a) - \varphi(b)], \end{aligned}$$

and, consequently,

$$\Delta \to 0 \quad \text{as} \quad l_T \to 0.$$

Having made this observation, let us now transform the sum (7), using Abel's lemma (Lecture 4, p. 98). Setting

$$A_k = \int_a^{x_k} f(x)\, dx \qquad (k = 0, 1, 2, \ldots, n),$$

we find

$$\left.\begin{aligned} S &= \sum_{k=1}^{n} \varphi(\xi_k) \int_{x_{k-1}}^{x_k} f(x)\, dx \\ &= \sum_{k=1}^{n} \varphi(\xi_k)(A_k - A_{k-1}) \\ &= \sum_{k=1}^{n-1} A_k[\varphi(\xi_k) - \varphi(\xi_{k+1})] + A_n \varphi(\xi_n). \end{aligned}\right\} \tag{8}$$

Now let M denote an upper bound and m a lower bound of the function

$$\int_a^x f(u)\, du \tag{9}$$

in the interval $a \leq x \leq b$. It is obvious that $m \leq A_k \leq M$ $(0 \leq k \leq n)$, and, since by the assumed properties of $\varphi(x)$ all A_k on

the right side of formula (8) are multiplied by nonnegative factors, this formula gives

$$m\varphi(\xi_1) \leq S \leq M\varphi(\xi_1).$$

But as $l_T \to 0$ we have

$$\varphi(\xi_1) \to \varphi(a + 0) \quad \text{and} \quad S \to \int_a^b f(x)\, \varphi(x)\, dx.$$

Consequently,

$$m\varphi(a + 0) \leq \int_a^b f(x)\, \varphi(x)\, dx \leq M\varphi(a + 0).$$

The case where $\varphi(a + 0) = 0$ is, of course, trivial; when $\varphi(a + 0) \neq 0$ the foregoing is equivalent to

$$m \leq \frac{1}{\varphi(a + 0)} \int_a^b f(x)\, \varphi(x)\, dx \leq M.$$

Since the function (9) is continuous, it must attain at some interior[1] point ξ of the interval $[a, b]$ a value (intermediate between its upper and lower bounds) equal to the middle member of the inequality above; whence,

$$\int_a^b f(x)\, \varphi(x)\, dx = \varphi(a + 0) \int_a^\xi f(x)\, dx \qquad (a < \xi < b). \tag{10}$$

This formula expresses the content of the second mean value theorem under the condition that $\varphi(x)$ is nonnegative and nonincreasing and that $f(x)$ is any integrable function. The *mean value* ξ appears here in the role of one of the limits of integration.

If we assume that $\varphi(x)$ is again nonnegative, but this time nondecreasing, then, setting $x = b - y$ and $\varphi(b - y) = \psi(y)$, we have

$$\int_a^b f(x)\, \varphi(x)\, dx = \int_0^{b-a} f(b - y)\, \psi(y)\, dy.$$

[1] That we actually have $a < \xi < b$ rather than $a \leq \xi \leq b$ needs further argument, which is left to the reader. It is assumed, of course, that $\varphi(x)$ is not constant. If it were, we could no longer assert that ξ is an *interior* point of $[a, b]$. To see this, we merely need to consider the case in which $f(x) > 0$ on $[a, b]$.

Since the function $\psi(y)$ is nonnegative and nonincreasing for $0 < y < b - a$, we may apply formula (10) to the last integral above:

$$\int_0^{b-a} f(b-y)\,\psi(y)\,dy = \psi(+0)\int_0^{\eta} f(b-y)\,dy = \varphi(b-0)\int_{\xi}^{b} f(x)\,dx,$$

where $0 < \eta < b - a$ and, consequently, $a < \xi = b - \eta < b$. Thus, the second mean value theorem now takes the form

$$\int_a^b f(x)\,\varphi(x)\,dx = \varphi(b-0)\int_{\xi}^{b} f(x)\,dx.$$

Finally, let us assume that $\varphi(x)$ is monotonic, for example nonincreasing, but let us set no restrictions upon its sign. Since the function $\varphi(x) - \varphi(b-0)$ is nonnegative and nonincreasing on the interval $[a, b]$, applying formula (10) we obtain

$$\int_a^b f(x)[\varphi(x) - \varphi(b-0)]\,dx = [\varphi(a+0) - \varphi(b-0)]\int_a^{\xi} f(x)\,dx,$$

from which we obtain the

Second mean value theorem. Let $\varphi(x)$ be monotonic for $a < x < b$, and let $f(x)$ be an integrable function. Then $\int_a^b f(x)\varphi(x)\,dx = \varphi(a+0)\int_a^{\xi} f(x)\,dx + \varphi(b-0)\int_{\xi}^{b} f(x)\,dx$ for some ξ such that $a < \xi < b$. If, in particular, the function $\varphi(x)$ is continuous at the points a and b, then $\int_a^b f(x)\,\varphi(x)\,dx = \varphi(a)\int_a^{\xi} f(x)\,dx + \varphi(b)\int_{\xi}^{b} f(x)\,dx$ for some ξ such that $a < \xi < b$.

46. IMPROPER INTEGRALS

We shall now direct our attention to two extremely important generalizations of the original notion of an integral: integrals with infinite limits of integration and integrals of unbounded functions. In both cases, we have an actual extension of the theory, and not simply an application of the idea under new conditions; to the original structure of the integral there is added a supplementary passage to the limit. This generalized integral no longer represents

the limit or bound of a particular kind of sum, but the limit of integrals.

As you know, the symbols

$$\int_a^{+\infty}, \quad \int_{-\infty}^b, \quad \text{and} \quad \int_{-\infty}^{+\infty}$$

are, respectively, defined as,

$$\lim_{b\to+\infty}\int_a^b, \quad \lim_{a\to-\infty}\int_a^b, \quad \text{and} \quad \lim_{\substack{a\to-\infty\\ b\to+\infty}}\int_a^b.$$

In order to study better the underlying idea of this generalization, we shall consider a simple case. Let $f(x)$ be positive and non-increasing for $x \geq a$ (Fig. 28). We can then represent the integral

$$\int_a^b f(x)\,dx$$

geometrically by the shaded area in Figure 28. As b increases, this

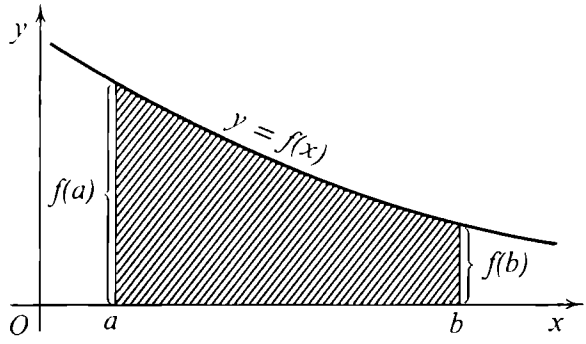

Fig. 28

area increases; as $b \to +\infty$, the area may increase indefinitely, or it may remain bounded and, consequently, tend to a limit. It is thi limit which is designated by the *improper* integral

$$\int_a^{+\infty} f(x)\,dx.$$

As its geometric illustration, we may use the area lying to the right of the line $x = a$ and bound by the x-axis and the curve $y = f(x)$. This area may be finite, even though the corresponding figure extends indefinitely.

The simplest case of the second generalization (the integral of an unbounded function), although it appears to be the solution of

a completely different problem, basically differs very little from the case just discussed. This is seen immediately if we observe that its geometric representation may be obtained simply by the reflection of Figure 28 about the bisector of the angle which determines the first quadrant. If, for example, the function $f(x)$ becomes unbounded in the neighborhood of the point a, we define the integral

$$\int_a^b f(x)\,dx$$

as the limit of the ordinary integral $\int_{a+\varepsilon}^{b} f(x)\,dx$ as $\varepsilon \to +0$ (Fig. 29).

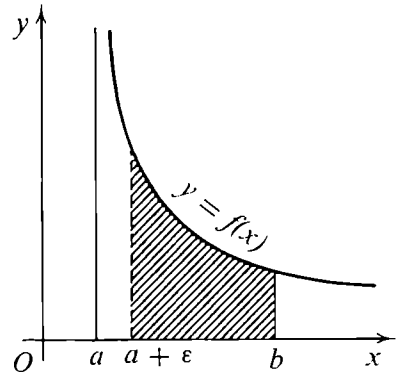

Fig. 29

You can see that here again the problem is the possibility of ascribing a definite value to the area of a figure which extends to infinity. This figure is of the same type as the first figure, only differently situated. That the approximating areas increase differently in this case has no essential significance.

In the general case, that is, when the behavior of the function $f(x)$ is unrestricted,[1] the definitions of the improper integrals do not change, although such a simple geometric interpretation is no longer available. When the corresponding limit exists, we say that the given improper integral *exists*, or that it *has a meaning*, or that it *converges*. Thus, the question of the convergence of an improper integral of the first or second kind is always reduced to the problem of the existence of a limit of some given function. Hence, all general propositions in the theory of limits can be applied to improper integrals. In particular, *Cauchy's condition* is valid in this connection, taking the form:

[1] We still require, of course, that the unboundedness of the integrand occurs in the neighborhood of only *one* point.

CAUCHY'S CONDITION.[1] *If $f(x)$ is integrable over $[a, b]$ for any finite $b > a$, then, for the convergence of the improper integral*

$$f(x)\,dx,$$

it is necessary and sufficient that given any $\varepsilon > 0$ we have the inequality

$$\left|\int_{b_1}^{b_2} f(x)\,dx\right| < \varepsilon$$

for all sufficiently large b_1 and b_2.

In exactly the same way, we have the following two important theorems, presenting, as you will see at once, a complete analogy to the corresponding theorems in the theory of series:

THEOREM 5. *If, for $a < x < +\infty$ we have $0 \leq f(x) \leq \varphi(x)$, where $f(x)$ and $\varphi(x)$ are integrable over any interval $[a, b]$ $(b > a)$, then, from the convergence of the integral*

$$\int_a^{+\infty} \varphi(x)\,dx,$$

there follows the convergence of the integral

$$\int_a^{+\infty} f(x)\,dx.$$

This theorem may be called the *comparison test for integrals.* It is entirely analogous to the *comparison test for series* (Lecture 4, p. 76), and the proof can be carried out easily following the method used there.

THEOREM 6. *From the convergence of the integral*

$$\int_a^{+\infty} |f(x)|\,dx, \tag{11}$$

there follows the convergence of the integral

$$\int_a^{+\infty} f(x)\,dx. \tag{12}$$

[1] For brevity, here and in what follows, only one type of improper integral is employed. But all that is said applies with suitable modifications (which you can easily formulate), to the other type as well.

This theorem is completely analogous to the corresponding theorem in Lecture 4 (p. 81), and can be proved in the same manner. Here, too, we say that the integral (12) is absolutely convergent if the integral (11) is convergent.

With the help of the above theorems, it is easy to establish the convergence of a great number of frequently encountered integrals: Thus, from the obvious convergence of the integral

$$\int_1^{+\infty} \frac{dx}{x^2}$$

there follows, by Theorem 5, the convergence of the integral

$$\int_1^{+\infty} \frac{|\sin x|}{x^2}\, dx,$$

and from this, by Theorem 6, the convergence (which is absolute) of the integral

$$\int_1^{+\infty} \frac{\sin x}{x^2}\, dx.$$

If, in Theorem 5, we take for $\varphi(x)$ various positive functions whose integrals are known to converge, the theorem will immediately yield a whole series of convergence tests for integrals with positive integrands. Then, with the help of Theorem 6, we shall obtain convergence tests for integrands of arbitrary sign. Some of these tests are undoubtedly known to you and we shall not dwell on them. Instead, we shall examine two criteria less widely known, but more delicate, as they concern integrals which may not be absolutely convergent (from which it follows that these criteria cannot be derived from the first theorem above).

Criterion 1. Let $\varphi(x)$ *be a monotonic function such that* $\lim_{x \to \infty} \varphi(x) = 0$, *and let the integral*

$$\int_a^x f(u)\, du = F(x)$$

remain bounded as $x \to \infty$. *Then, the integral*

$$\int_a^{+\infty} \varphi(x) f(x)\, dx \tag{13}$$

converges.

Proof. By the second mean value theorem, for $a < b_1 < b_2 < +\infty$ there exists a point β within the interval $[b_1, b_2]$ such that

$$\int_{b_1}^{b_2} \varphi(x) f(x)\, dx = \varphi(b_1 + 0) \int_{b_1}^{\beta} f(x)\, dx + \varphi(b_2 - 0) \int_{\beta}^{b_2} f(x)\, dx$$

$$= \varphi(b_1 + 0)[F(\beta) - F(b_1)] + \varphi(b_2 - 0)[F(b_2) - F(\beta)].$$

Since for $b_1 \to +\infty$ and $b_2 \to +\infty$ the quantities $\varphi(b_1 + 0)$ and $\varphi(b_2 - 0)$ tend, by hypothesis, to zero, while the values of the other factors remain bounded; the right side, and consequently the left side also, of the last equality tends to zero. It follows from Cauchy's condition that the integral (13) converges.

EXAMPLE 1. The integral

$$\int_1^{+\infty} \frac{\sin x}{x}\, dx \tag{14}$$

is convergent. This follows directly from Criterion 1 if we substitute

$$\varphi(x) = x^{-1}, f(x) = \sin x \quad \text{and} \quad F(x) = \cos 1 - \cos x.$$

It is easy to show, however, that the convergence of the integral (14) is not absolute.

Criterion 2. If $\varphi(x)$ is monotonic and bounded, then the convergence of the integral (12) *implies the convergence of the integral* (13).

Proof. To prove this, we apply once more the second mean value theorem. In the first equality of Criterion 1, the quantities $\varphi(b_1 + 0)$ and $\varphi(b_2 - 0)$ remain bounded as $b_1 \to +\infty$ and $b_2 \to +\infty$, while both integrals tend to zero by virtue of Cauchy's condition and the assumed convergence of the integral (12). Thus, the left side also tends to zero, from which, by Cauchy's condition again, we conclude that the integral (13) converges.

EXAMPLE 2. Setting

$$f(x) = \frac{\sin x}{x} \quad \text{and} \quad \varphi(x) = \arctan x,$$

we conclude from the proven convergence of the integral (14) that the integral

$$\int_1^{+\infty} \frac{\sin x \arctan x}{x}\, dx$$

is also convergent.

47. DOUBLE INTEGRALS

The special process of summation underlying the integral calculus can also be applied successfully to functions of several variables. In many fields of application, particularly in mechanics and physics, such *multidimensional integrals* or, as they are more often called, *multiple* integrals play an important role. The theory of these integrals, although it contains almost nothing new in principle as compared with the theory of the ordinary integral, is very cumbersome in its formal aspect. In what follows, limiting ourselves to the two-dimensional case, we shall briefly demonstrate how the ideas presented at the beginning of this lecture can be applied, in practically unchanged form, to the definition of multiple integrals and the establishment of a number of their properties.

Let $z = f(x, y)$ be a bounded function of two variables defined on a bounded closed[1] region D in the xy coordinate plane. Let M and m denote, respectively, the upper and lower bounds of the function $f(x, y)$ in the region D, whose area we agree to denote by $|D|$. As in the one-dimensional case, here also we shall consider various partitions T of the region D. Of course, we have here a much more complicated situation than before. In the one-dimensional case, both the basic region and the parts into which it was subdivided had one form: they were line segments. In the present case, the region D as well as the parts $\Delta_1, \Delta_2, \ldots, \Delta_n$ into which we subdivide it may have a great variety of shapes.

For our theory, it is desirable to impose as few restrictions as possible on the shape of these regions. All that is required is that each of them have a definite area and that the area of the common part (that is, the region of overlap) of any two different subregions Δ_l and Δ_k be equal to zero. Beyond this, we shall not impose, for the time being, any special conditions on the partition T.

As in the one-dimensional case, let M_k and m_k designate, respectively, upper and lower bounds of $f(x, y)$ in the subregion Δ_k. Let us set

$$S_T = \sum_{k=1}^{n} M_k |\Delta_k| \quad \text{and} \quad s_T = \sum_{k=1}^{n} m_k |\Delta_k|,$$

where $|\Delta_k|$ denotes here the area of Δ_k. As before, we have

$$m|D| \leq s_T \leq S_T \leq M|D|,$$

[1] That is, the boundary points are included in the region.

so that the sums S_T and s_T are bounded above and below. Let $\overline{I}$ and $\underline{I}$ denote, respectively, the g.l.b. of all sums S_T and the l.u.b. of all sums s_T. Here also we shall call these numbers the upper and lower integrals, respectively, of $f(x, y)$ over D.

If $\overline{I} = \underline{I}$, then $f(x, y)$ is said to be *integrable* over D and its *integral* is

$$\overline{I} = \underline{I} = I = \int\int_D f(x, y)\, dxdy.$$

Underlying the theory of such *double* integrals is a series of theorems exactly analogous to those we have already established for the one-dimensional case. We shall now briefly consider them, dwelling on the proofs only when they differ from the proofs given earlier.

For lack of a better term, let us agree to use the name *cells* for the subregions $\Delta_1, \Delta_2, \ldots, \Delta_n$ into which we subdivide the basic region D. We shall understand by the *diameter* of a given cell the least upper bound of the distances between all possible pairs of points belonging to that cell. For a given partition T, we denote by d_T the greatest of the diameters of the cells $\Delta_1, \Delta_2, \ldots, \Delta_n$. It is clear that this quantity must play the same role here as the quantity l_T in the one-dimensional case. Finally, we shall say that the partition T' is a refinement of the partition T if every cell of T' is entirely contained in one of the cells of T.

We shall now establish four lemmas corresponding exactly to the four lemmas which we established in the one-dimensional case (Section 41).

LEMMA 1′. *If the partition T' is a refinement of the partition T, then*

$$S_{T'} \leq S_T \quad \text{and} \quad s_{T'} \geq s_T.$$

The proof follows that for the one-dimensional case verbatim.

LEMMA 2′. *For any two partitions T_1 and T_2 we have*

$$S_{T_1} \geq s_{T_2}.$$

The proof which we gave in the one-dimensional case applies here also. Some explanation is needed, however, in regard to constructing a partition T which will serve, simultaneously, as a refinement of each of the two given partitions T_1 and T_2. We simply

take as a cell of the partition T, any set consisting of all points belonging simultaneously to one of the cells of the partition T_1 and to one of the cells of the partition T_2. Combining all such possible pairs, we obtain all the cells of the partition T. From the very definition of a refinement, it is obvious that T is simultaneously a refinement of T_1 and T_2.

LEMMA 3′. $\bar{I} \geq \underline{I}$.

This proposition, as in the one-dimensional case, is a direct corollary of Lemma 2.

LEMMA 4′. *For any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that, for any partition* T *satisfying the condition* $d_T < \delta$, *we have*

$$S_T < \bar{I} + \varepsilon \quad \text{and} \quad s_T > \underline{I} - \varepsilon.$$

In short, as $d_T \to 0$ the sums S_T tend to their lower bound and the sums s_T tend to their upper bound.

The proof of this lemma does not differ in principle from that given in the one-dimensional case. Here, however, in view of the more complicated structure of the cells, a detailed presentation is necessary to make the proof formally irreproachable.

Proof. First of all, since $\bar{I}$ was defined as the greatest lower bound of all the sums S_T, there exists a partition T_0 for which $S_{T_0} < \bar{I} + \frac{\varepsilon}{2}$. Let Δ_k designate any cell of this partition T_0. The set of all points of the plane whose distance from the perimeter of this cell does not exceed δ forms a kind of *band* along that perimeter. (In Fig. 30 the boundaries of this band are marked by dotted

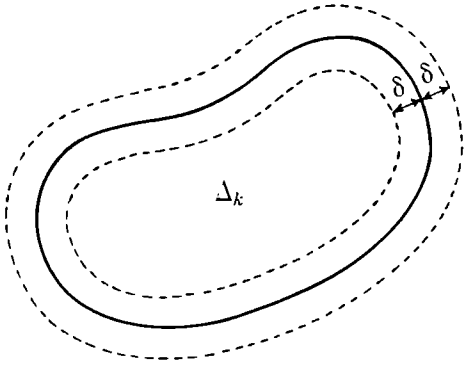

Fig. 30

lines.) It is not difficult to see that, for sufficiently small δ, the area of such a *band* is equal to $2\delta l_k$, where l_k is the length of the boundary of the cell.[1] Let us set

$$\sum_{k=1}^{n} l_k = L.$$

Then the area of the union D_1 of all the *bands* of the type described will not be greater than $2\delta L$. Now let T be any partition for which $d_T < \delta$. We shall divide the cells of this partition into two groups, including in the first group each cell wholly contained in D_1, and in the second, all the remaining cells. We denote by $S_T^{(I)}$ and $S_T^{(II)}$ the parts of the sum S_T which correspond to these two groups. We shall now estimate each of these sums separately. And, just as in the one-dimensional case, without restricting the generality of the argument we may assume that $f(x, y) \geq 0$ in the whole region D.

Since the first factors in the terms of the sum $S_T^{(I)}$ do not exceed M, and the sum of the areas of the cells included in the sum $S_T^{(I)}$ does not exceed the area of D_1 (which, as we have seen, is not greater than $2\delta L$), we have

$$S_T^{(I)} \leq 2ML\delta. \tag{15}$$

As for the cells belonging to the sum $S_T^{(II)}$, each of them is contained in one of the cells Δ_k of the partition T_0. For any cell Δ of the second group which did not satisfy this condition would have to contain a point P of the boundary of one of the cells Δ_k. Being a cell of the second group, Δ cannot be entirely contained in the *band* surrounding this boundary; hence it must contain a point Q outside that *band*. But then the distance between the points P and Q, and hence the diameter of the cell Δ, will exceed δ, which is impossible.

We shall not decrease the sum $S_T^{(II)}$ if, in each of its terms $M_\alpha |\Delta_\alpha|$ (corresponding to the cell Δ_α of the partition T), we replace the first factor M_α by the quantity M_k (corresponding to the cell Δ_k of the partition T_0 containing the cell Δ_α). On the other hand,

[1] We are now tacitly imposing on the cells of our partitions more restrictive conditions than before. For example, we assume that each cell is a simply connected region whose perimeter is of finite length. Unfortunately, within the limits of these lectures we have no opportunity to discuss these problems in greater detail.

since the function $f(x, y)$ is nonnegative, the sum $S_T^{(II)}$ in its altered form will still not exceed S_{T_0}. In this way, we obtain the inequality

$$S_T^{(II)} \leq S_{T_0} < \bar{I} + \frac{\varepsilon}{2}.$$

Combining this inequality with the inequality (15), and taking δ to be less than $\frac{\varepsilon}{4ML}$, we obtain

$$S_T = S_T^{(I)} + S_T^{(II)} < 2ML\delta + \bar{I} + \frac{\varepsilon}{2} < \bar{I} + \varepsilon.$$

In precisely the same way, we can prove that if $d_T < \delta$ then $s_T > \underline{I} - \varepsilon$, and this will complete the proof of Lemma 4.

THEOREM 7. *If $f(x, y)$ is integrable over the region D, then, for all partitions T in which the maximum cell diameter d_T is sufficiently small, we have*

$$\left| \sum_{k=1}^{n} f(\xi_k, \eta_k) |\Delta_k| - I \right| < \varepsilon,$$

where ε is any given positive number and (ξ_k, η_k) $(k = 1, 2, \ldots, n)$ is an arbitrary point of the cell Δ_k.

In short, the sum $\sum_{k=0}^{n} f(\xi_k, \eta_k) |\Delta_k|$ tends to the integral I as $d_T \to 0$ for all possible choices of the partition T as well as of the points (ξ_k, η_k).

Proof. For brevity, let us denote this sum simply by Σ_T. Then, from the obvious inequalities

$$m_k \leq f(\xi_k, \eta_k) \leq M_k,$$

we obtain for any partition and any choice of the points (ξ_k, η_k),

$$s_T \leq \Sigma_T \leq S_T.$$

Hence, by Lemma 4, we have for sufficiently small d_T,

$$I - \varepsilon \leq \Sigma_T \leq I + \varepsilon,$$

or

$$|\Sigma_T - I| < \varepsilon.$$

The converse of this theorem is also true, the proof being the same as for the one-dimensional case.

THEOREM 8. *Just as in the one-dimensional case,*

$$S_T - s_T = \sum_{k=1}^{n} \omega_k \Delta_k \to 0 \quad \text{as} \quad d_T \to 0$$

is a necessary and sufficient condition for the existence of the integral.

Proof. (i) If this condition is satisfied, then, by virtue of the inequality $S_T \geq \overline{I} \geq \underline{I} \geq s_T$ (for any partition), we have

$$\overline{I} - \underline{I} \leq S_T - s_T < \varepsilon$$

for sufficiently small d_T, whence $\overline{I} = \underline{I}$, since ε is arbitrary.

(ii) If the function $f(x, y)$ is integrable, then, by Lemma 4 for sufficiently small d_T, we have

$$S_T < I + \varepsilon \quad \text{and} \quad s_T > I - \varepsilon,$$

whence

$$S_T - s_T < 2\varepsilon,$$

which means that $S_T - s_T \to 0$ as $d_T \to 0$.

Finally, we prove the integrability of continuous functions by precisely the same reasoning as in the one-dimensional case. The basis of the proof is that a continuous function is uniformly continuous in a bounded closed region (a bounded region including its boundary points).

48. EVALUATION OF DOUBLE INTEGRALS

In discussing ordinary integrals earlier, we mentioned that the summation process, though fundamental in the definition of an integral, is almost useless for the actual computation of integrals. It is even less to be expected that double integrals could be computed readily by the process of two-dimensional summation described above. We have seen that powerful and general methods for computing one-dimensional integrals are obtained only by exploiting the connection between the integral and the differential calculus. Such relations may also be discovered for double integrals (and for multiple integrals in general). However, the most general and effective method for computing double integrals is obtained by the familiar process of reducing the problem to two successive one-dimensional integrations. Let us consider briefly how this can be done.

Let us assume that the region D, on which the continuous function $f(x, y)$ is defined, is such that each line parallel to one of the coordinate axes intersects the perimeter in not more than two points (Fig. 31). The partitions T, in terms of which the integral I

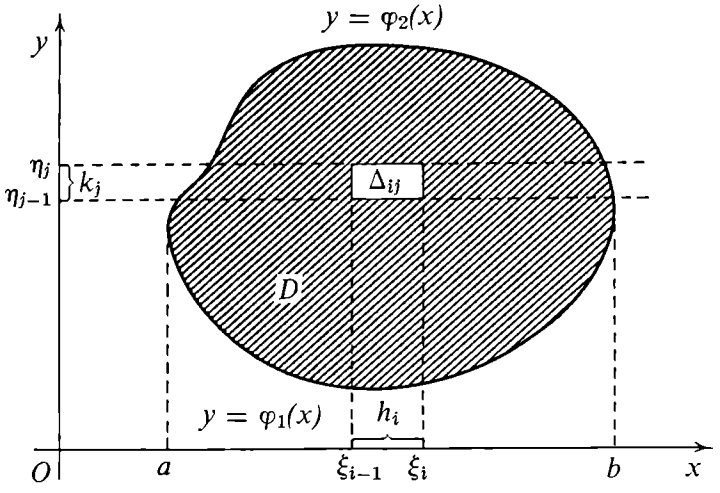

Fig. 31

of $f(x, y)$ is defined, may have cells of any shape, providing only that their diameters become arbitrarily small. We shall now construct a partition by using a network of lines parallel to the coordinate axes, so that the cells will be rectangles (except those lying along the perimeter of D). We shall denote by a and b the greatest lower and the least upper bounds, respectively, of the abscissas of the set of all points in D. Let $a = \xi_1 < \xi_2 < \cdots < \xi_m = b$ denote the abscissas of the partition lines parallel to the y-axis and $\eta_1 < \eta_2 < \cdots$ the ordinates of the lines parallel to the x-axis. Finally, we set $\xi_i - \xi_{i-1} = h_i$ and $\eta_j - \eta_{j-1} = k_j$.

Each cell of our partition either fills a rectangle $\xi_{i-1} \leq x \leq \xi_i$, $\eta_{j-1} \leq y \leq \eta_j$ completely, in which case we denote the cell Δ_{ij}, or is contained in such a rectangle but does not fill it. We shall denote cells of the second type by $\overline{\Delta}_1, \overline{\Delta}_2, \ldots$. We shall also use $|\Delta_{ij}|$ and $|\overline{\Delta}_r|$ to denote the areas of the corresponding cells. We now choose in each cell $\overline{\Delta}_r$, a point $(\overline{x}_r, \overline{y}_r)$, and form the sum

$$S = \sum_{i,j} f(\xi_i, \eta_j)|\Delta_{ij}| + \sum_r f(\overline{x}_r, \overline{y}_r)|\overline{\Delta}_r| = S^{(\mathrm{I})} + S^{(\mathrm{II})}.$$

Let δ denote the largest of the numbers h_i and k_j. Then, for sufficiently small δ, the sum S differs by arbitrarily little from the integral I of $f(x, y)$ over D.

Let us now take a closer look at the troublesome rectangles which contain both points in D and points not in D. The strip included between the lines $x = \xi_{i-1}$ and $x = \xi_i$ may contain such rectangles at the top and at the bottom. In Figure 32, we have

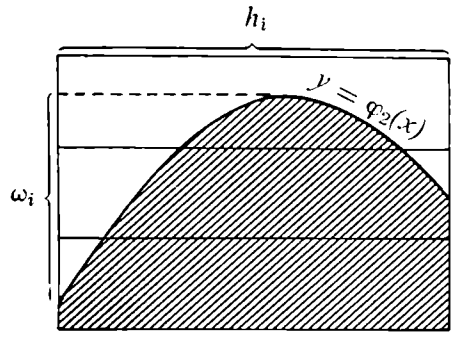

Fig. 32

schematically represented the upper end of such a strip. These rectangles (we have three of them in Fig. 32) are contained in a rectangle whose base is equal to h_i and whose altitude clearly does not exceed $\omega_i + 2\delta$, where ω_i is the oscillation of the function $\varphi_2(x)$ in the interval $[\xi_{i-1}, \xi_i]$. If δ is sufficiently small, then, because of the uniform continuity of the function $\varphi_2(x)$, all ω_i, and hence all $\omega_i + 2\delta$ also, will be less than an arbitrarily small positive number μ. Thus, the sum of the areas of the nonrectangular cells represented in Figure 32 will be less than μh_i, and, consequently, the sum of the areas of all the cells adjacent to the upper half of the perimeter of region D will be less than $\mu \sum_i h_i = \mu(b - a)$. We clearly obtain a similar estimate for the area of the cells adjacent to the lower half of the perimeter, so that taken all together, the *incomplete* cells account for an area not exceeding $2\mu(b - a)$. Hence, $S^{(II)}$ does not exceed in absolute value $2\mu M(b - a)$, where M designates the upper bound of $|f(x, y)|$ in the region D. Thus, for δ sufficiently small, $S^{(I)}$ differs arbitrarily little from S.

We now construct the sum

$$\hat{S} = \sum_{i,j} f(\xi_i, \eta_j)\, h_i k_j$$

$$= \sum_i h_i \sum_j f(\xi_i, \eta_j)\, k_j,$$

where the summation is over all i and j such that (ξ_i, η_j) is in D. The terms of $\hat{S}$ consist of the terms of $S^{(I)}$ plus terms corre-

sponding to rectangles $\xi_{i-1} \leq x \leq \xi_i$, $\eta_{j-1} \leq y \leq \eta_j$ containing the point (ξ_i, η_j) of D, but also containing points not in D. It follows from the argument above that, for sufficiently small δ, $\hat{S}$ differs arbitrarily little from $S^{(I)}$, and, hence, differs arbitrarily little from the integral I of $f(x, y)$ over the region D.

If, while keeping the values $\xi_1, \xi_2, \ldots$ fixed, we decrease indefinitely all the differences $k_j = \eta_j - \eta_{j-1}$, then the inner sum $\sum_j f(\xi_i, \eta_j)\, k_j$ will clearly have as its limit the integral

$$\int_{\varphi_1(\xi_i)}^{\varphi_2(\xi_i)} f(\xi_i, y)\, dy.$$

For brevity, we shall denote this integral by $F(\xi_i)$, agreeing to write in general

$$F(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy.$$

(While integrating, we consider x as a constant, so that we integrate the function $f(x, y)$ as a function of one variable y.) We can take the quantities k_j so small that, for each i, we shall have the inequality

$$\left| \sum_j f(\xi_i, \eta_j)\, k_j - F(\xi_i) \right| < \varepsilon,$$

from which

$$\left| h_i \sum_j f(\xi_i, \eta_j)\, k_j - F(\xi_i)\, h_i \right| < \varepsilon h_i,$$

and

$$\left| \sum_i h_i \sum_j f(\xi_i, \eta_j)\, k_j - \sum_i F(\xi_i)\, h_i \right| = \left| \hat{S} - \sum_i F(\xi_i)\, h_i \right| < \varepsilon \sum_i h_i = \varepsilon(b - a).$$

Finally, for sufficiently small h_i, the sum $\sum_i F(\xi_i)\, h_i$ clearly differs arbitrarily little from the integral

$$\int_a^b F(x)\, dx.$$

We see then that for a suitably chosen partition of D, all three differences $I - \hat{S}$, $\hat{S} - \sum_i F(\xi_i)\, h_i$, and $\sum_i F(\xi_i)\, h_i - \int_a^b F(x)\, dx$

become arbitrarily small; hence, this is also true for their sum $I - \int_a^b F(x)\,dx$. This latter sum, being entirely independent of the choice of partitions, must therefore equal zero. Thus, we obtain

$$\iint_D f(x, y)\,dxdy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy,$$

so that the computation of a double integral is actually reduced to two successive ordinary integrations. The value of the inner integral depends, of course, upon x, but not on y.

Instead of the order of integration we have used, we could also have chosen the reverse order. This fact sometimes has great practical importance, since by changing the order of integration we may obtain functions which are much easier to integrate.

49. THE GENERAL OPERATION OF INTEGRATION

Our lecture has been a long one. Nevertheless, before concluding it, we ought to take a look at the general relationship between differentiation and integration. This will throw additional light on the underlying nature of every process of integration, regardless of whether it involves an ordinary integral, a multiple integral, or an even more general and abstract integral.

First of all, we are dealing with a given region D located in some space, the term *space* being understood here in a very broad sense. The space may be a straight line, or a plane, or our usual three-dimensional space, or it may be any multidimensional space, or even some completely different kind of space. We do not intend here to give a general definition of this term. For our purpose, it is important only that, first, the distance between any two points be defined and, second, the region D and those of its parts (cells) with which we shall be concerned have *extent.* Depending upon the type of space, this extent is usually called length, area, volume, and so on. In general, we shall call it simply the *measure* of the given part of our space.

Now let us imagine a substance distributed over the region D. Don't let the word *substance* frighten you; we shall not attempt to define the concept expressed by it. This substance is for us simply *something of which a definite portion is contained in each part of D.*

It may be mass, electrical charge, heat, or any other measurable quantity which can be distributed in various ways over D. It may be, for instance, the quantity of precipitation falling on a plane region D during a given period of time. It is evident that, for the construction of our mathematical picture, the nature of this substance is absolutely irrelevant. We require only that a definite quantity $F(\Delta)$ of this substance be attributable to each of those parts Δ of the region D with which we are concerned. The total amount of the substance $F(D)$ is the sum of the substance within the individual cells:

$$F(D) = F(\Delta_1) + F(\Delta_2) + \cdots + F(\Delta_n).$$

In all concrete interpretations of the formal scheme we have outlined, an essential role is played by the concept of the *density* of the substance at a given point P. This density, from the point of view of mathematics, can be defined, as we shall now see, by a process of *differentiation.*

If Δ is any part of D having the measure $m(\Delta)$, then it is natural to call the ratio $\dfrac{F(\Delta)}{m(\Delta)}$ the *average density* of our substance in the subregion Δ; it is the amount of substance in Δ per unit of extent (measure). Now let P be any point in D. If we surround this point by a subregion Δ having a small diameter $d(\Delta)$ (the diameter of any region in our space is completely defined, since the distance between any two points is defined), then the average density of the substance in this small region will characterize the density of the substance in the immediate neighborhood of P. And, if this average density tends, as we shall assume, to a limit $f(P)$ as Δ contracts to the point P (that is, as $d(\Delta) \to 0$), then we call this limit the *density of the substance at the point P.* It is in this way that we define the density of all physical substances (such as mass, electricity, etc.). We see that the density $f(P)$ of the substance is a function of the *point* P, while the quantity $F(\Delta)$ is a function of the *set* Δ. We can say that the function $F(\Delta)$ gives a *global* (region-related), and the function $f(P)$ a *local* (point-related), characteristic of the distribution of our substance in the region D. If the function $F(\Delta)$ is given, then we obtain $f(P)$ from it by some differentiation process. The essence of the differentiation process can perhaps be described most conveniently as *the passage from the global description of a phenomenon to its local characterization.*

But let us now suppose, conversely, that at each point P of the region D the density $f(P)$ of the substance is given, and we are to determine the total amount $F(D)$ of the substance contained in this region. This is the problem of integration in its most general form, *the passage from the local to the global characterization of the phenomenon in question.* Let us assume, for simplicity, that the function $f(P)$ is uniformly continuous on D. To solve our problem, we subdivide D into cells $\Delta_1, \Delta_2, \ldots, \Delta_n$ of small diameter,[1] and in each cell Δ_k we choose an arbitrary point P_k. The quantity $f(P_k)$ is the limit of the ratio $\dfrac{F(\Delta)}{m(\Delta)}$ as the subregion Δ_k containing the point P decreases indefinitely in diameter. Therefore, if the cells Δ_k are very small, we have

$$\left| f(P_k) - \frac{F(\Delta_k)}{m(\Delta_k)} \right| < \varepsilon,$$

where ε is any positive number specified in advance. Thus,

$$F(\Delta_k) - \varepsilon m(\Delta_k) < f(P_k)m(\Delta_k) < F(\Delta_k) + \varepsilon m(\Delta_k).$$

Adding these inequalities with respect to k, we find

$$F(D) - \varepsilon m(D) < \sum_{k=1}^{n} f(P_k)m(\Delta_k) < F(D) + \varepsilon m(D),$$

and, as the cell diameters decrease indefinitely, we have in the limit,

$$\lim \sum_{k=1}^{n} f(P_k)m(\Delta_k) = F(D).$$

We see then that given the function $f(P)$ (the density of the substance at each point), we can actually determine the quantity $F(D)$ (the total amount of the substance) by the already familiar process of integration: dividing the region D into small cells Δ_k, selecting the points P_k, summing the products of the form $f(P_k)m(\Delta_k)$ over all the cells and, finally, passing to the limit under the condition that the diameters of the cells tend to zero.

[1] We are restricting ourselves to the case where this can be done; for example, when D is a bounded region in Euclidean space. Here the author is only interested in the general picture and, consequently, makes no attempt to establish a detailed theory with precise hypotheses on D and $f(P)$. The reader who wishes to go further into the subject may consult such texts as S. Saks, *Theory of the Integral* (Warsaw Mathematical Monographs, Vol. 7), translated by L. C. Young with notes by S. Banach, 2d rev. ed. (New York: Hafner Publishing Company, 1937.)

Thus, from the very general point of view which we have adopted, the process of integration in arbitrary spaces appears to be a method allowing us to determine the amount of substance contained in any region when we know the density of distribution of this substance at each particular point. Every procedure of this type, unfailingly, has as its correlate some differentiation process (solving the inverse problem), whose goal is to find the local density of the distributed substance when the amount contained in each subregion is known.

It is entirely possible, with a sufficiently exact formulation of the fundamental assumptions, to construct a theory of integration on this general, abstract base. Ordinary, double, and other special integrals then become particular cases of such a general theory, their most important properties being established once and for all by this general theory, so that they do not require separate proofs in each particular case.

7. Expansion of Functions in Series

50. USE OF SERIES IN THE STUDY OF FUNCTIONS

From the original geometric definitions of the sine and cosine, we can at once deduce a number of very important properties of these functions. The introduction of the *analytic expressions* $\sin x$ and $\cos x$ adds nothing new to our knowledge of these functions, just as we learn nothing new about the properties of the Dirichlet function when we denote it by $f(x)$. When we subsequently obtain, however, new analytic expressions for the trigonometric functions in the form of the power series

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \quad \text{and} \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

we easily discover a number of properties of these functions. Without analytic expressions, these properties could only be derived from the original definitions by the most complicated reasoning (if at all). And above all, these expansions make it possible for us to compute in a simple manner the values of the trigonometric functions for all values of the argument.

In Lecture 3, we pointed out the danger in the excessive worship of analytic expressions. And among the representatives of the applied sciences, we often encounter the harmful consequences of this phenomenon. We cannot deny, however, that among modern mathematicians we occasionally find an attitude at the opposite extreme. This deliberate unconcern with analytical expressions, and a concomitant helplessness in dealing with them, may be no less harmful. If, however, we can approach analytic expressions with a complete mastery, while simultaneously bearing in mind that they are only an instrument for the study of functions, then they can play a role of decisive importance in this study. It is perfectly natural for a mathematician investigating a given function to begin by trying to find an appropriate analytical expression for it. By means of such an apparatus, he will, in most cases, obtain in the most efficient manner interesting and important properties of the function.

Due to their simplicity, flexibility, clearness, and convenience in application, series expansions unquestionably take first place among the various analytical devices capable of serving as instruments in the study of functions. The idea of this extremely important analytical apparatus is very simple: the function to be investigated is represented as the limit of a sequence of other functions (the partial sums of the representing series) which are simpler and more accessible to study. If such a partial sum approximates the function closely in the entire region under consideration, it is reasonable to expect that from the properties of this partial sum we can learn, if only approximately, some properties of the function itself. In particular, if we know how to compute approximately the values of these partial sums for various values of the argument, we have a method for approximating the corresponding values of the functions.

But what functions will most conveniently and usefully serve as elements of the expansion, that is, serve as the terms of the series which is to represent the given function and help us in its study? To this question (as it is natural to expect) there is no unique and universally applicable answer; here almost everything depends on the nature of the function and the character of the problem before us. We must, however, note that there are a few types of series of such demonstrated merit that they are employed very frequently, and this has naturally led to their extensive development. In the first rank among such series belong *power series* (in which the elements of the expansion are integral, primarily nonnegative powers of the independent variable) and *trigonometric series* (with elements of the form $\sin kx$ and $\cos kx$, where $k = 0, 1, 2, \ldots$). In many cases, however, it is convenient to choose as elements of the expansion not these simplest, *universal* functions, but totally different functions. These functions, though not so simple, are by their properties more closely related to the function under study (for example, the so-called *proper functions* in boundary value problems). In general, the guiding principle in selecting the type of expansion to be used should be the absence of any bias; the specific character of the problem before us should be taken into account in each and every instance.

In what follows, we shall touch briefly only the most important questions related to the expansion of functions in power series and trigonometric series.

51. EXPANSION IN POWER SERIES

We know that the domain of convergence of a power series is an (open, closed, or half-closed) interval with end points $-r$ and r. What properties must a function $f(x)$ have to be expressible in a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

convergent in this interval? We know that it is necessary that $f(x)$ be continuous in the open interval $(-r, r)$ (Lecture 4, p. 98), but this is far from sufficient.

THEOREM 1. *If $f(x)$ is to be expressed as a power series convergent in $(-r, r)$, it must have a derivative $f'(x)$ at each point x of $(-r, r)$. Moreover, this $f'(x)$ can be represented by a power series*

$$\sum_{n=1}^{\infty} n a_n x^{n-1}, \tag{1}$$

convergent in the interval $(-r, r)$ and obtained by differentiating the given series term by term.

In proving this, we must keep in mind that neither the existence of the derivative $f'(x)$, nor the convergence of the series (1) are given beforehand, so that both of these facts will have to be established in the course of our argument.

Proof. Let $|x| < r$ and let ρ be any number between $|x|$ and r. For $|h| < \rho - |x|$, we have $|x + h| < \rho < r$, and, consequently,

$$f(x + h) = \sum_{n=0}^{\infty} a_n (x + h)^n.$$

From this,

$$\frac{f(x + h) - f(x)}{h} = \sum_{n=0}^{\infty} a_n \frac{(x + h)^n - x^n}{h} = S_N(h) + R_N(h),$$

where

$$S_N(h) = \sum_{n=0}^{N} a_n \frac{(x + h)^n - x^n}{h} \text{ and } R_N(h) = \sum_{n=N+1}^{\infty} a_n \frac{(x + h)^n - x^n}{h}.$$

Since

$$\left|\frac{(x+h)^n - x^n}{(x+h) - x}\right| = |(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1}|$$

$$< \rho^{n-1} + \rho^{n-2}|x| + \cdots + |x|^{n-1} = \frac{\rho^n - |x|^n}{\rho - |x|} < \frac{\rho^n}{\rho - |x|},$$

we have, for $|h| < \rho - |x|$,

$$|R_N(h)| < \frac{1}{\rho - |x|} \sum_{n=N+1}^{\infty} |a_n| \rho^n.$$

But $\rho < r$, so the sum on the right side of this inequality, being the remainder of a convergent series, becomes smaller than an arbitrarily small positive number ε for sufficiently large N. Thus, if N is sufficiently large, and $|h| < \rho - |x|$, we have

$$|R_N(h)| < \varepsilon.$$

Hence,

$$S_N(h) - \varepsilon < \frac{f(x+h) - f(x)}{h} < S_N(h) + \varepsilon.$$

If now N remains fixed, but h is made to tend to zero, the sum $S_N(h)$ will have as its limit the sum

$$S_N = \sum_{n=1}^{N} n a_n x^{n-1}.$$

The last inequality allows us, therefore, to assert that, as $h \to 0$, the upper and lower limits of the quantity

$$\frac{f(x+h) - f(x)}{h}$$

will differ from S_N by no more than ε, and hence will differ from each other by no more than 2ε. Since ε is arbitrary, it follows that these limits coincide, that is, that the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

exists. Moreover, we have

$$|f'(x) - S_N| \leq \varepsilon$$

under the sole condition that N is sufficiently large. But this means that the series (1) converges, and that its sum is $f'(x)$. Thus, we have proved all our assertions.

We have shown that a function represented by a power series must be not only continuous but also differentiable. And, since $f'(x)$ can be represented by a power series converging for all $|x| < r$, by virtue of the very theorem we have just proved, it follows that the second derivative $f''(x)$ also exists. Repeating this reasoning we come to the following conclusion.

THEOREM 2. *A function which can be expressed in an interval by a power series must have, at every interior point of that interval, derivatives of all orders. Moreover, each of these derivatives may be represented in this (open) interval by a power series obtained by differentiating the given series term by term a corresponding number of times.*

Thus,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and, in general,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n x^{n-k}.$$

Substituting in this formula $x = 0$, we find

$$f^{(k)}(0) = k!a_k,$$

whence,

$$a_k = \frac{f^{(k)}(0)}{k!} \qquad (k = 0, 1, 2, \ldots).$$

Thus, we have simultaneously proved the *uniqueness* of the expansion of a function into a power series and found the formulas expressing the coefficients of this expansion in terms of the derivatives of the given function at $x = 0$. Therefore, if a function $f(x)$ can be expanded in a power series, this expansion must have the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \tag{2}$$

This is the so-called Maclaurin series. Substituting $a + h$ for x and denoting $f(a + h)$ by $\varphi(h)$, it may be shown that $\varphi(h)$ is capable of expression in a series of powers of h, convergent for $|h| < r - |a|$, where $(-r, r)$ is the interval of convergence of (2). We thus have

$$\varphi(h) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} h^n.$$

From this, by returning to our original notation, we arrive at the more general *Taylor series*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n. \tag{3}$$

The situation becomes particularly clear when we compare this with what was said in Lecture 5 about the Taylor and Maclaurin expansions. There we were not concerned with infinite series; we called the quantity

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

the *remainder* of the Maclaurin expansion, and we examined its behavior for infinitely small values of x. It is clear that the behavior of $R_n(x)$ decides the question whether the function $f(x)$ can be expanded in a power series. In Lecture 5 we considered n to be constant and made x tend to zero. Now, on the contrary, the value of x is kept fixed and n is allowed to increase without limit. The condition

$$\lim_{n \to \infty} R_n(x) = 0$$

is quite obviously necessary and sufficient for the validity of formula (2). We usually establish the possibility of expanding a given function in a Maclaurin series by considering one of the many forms in which the remainder $R_n(x)$ may be expressed, some of which we encountered in Lecture 5. Which of these forms is most convenient for this purpose depends entirely on the nature of the function whose expansion we seek to obtain.

As we have seen above, a function which can be represented in a given interval by a power series must have, at every interior point of this interval, derivatives of all orders. Since, conversely, for any function having this property, it is formally possible to write the Taylor series (3) for each point a of this interval, one might be tempted to suppose that this condition is sufficient for developing a function in a power series. This, however, is not the case. First of all, it may happen that the Maclaurin series (2), written formally for the given function, diverges for all $x \neq 0$. Much more interesting, however, is the fact that, even when the Maclaurin series constructed for the given function converges, its sum may not equal the given function. A classical example of such a case is given by

$$\varphi(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It may easily be verified that at $x = 0$, the function itself and its derivatives of all orders have the value zero, and, consequently, all coefficients of the Maclaurin series (and thus its sum for all values of x) are equal to zero.

From this it follows that, while a given function can be developed in a power series in no more than one way, any convergent power series may be regarded as the Maclaurin series not merely of one, but of an infinite set of functions. For, let $f(x)$ be the sum of the given series. The given series is thus the Maclaurin expansion for not only $f(x)$, but also every function of the form $f(x) + \alpha\varphi(x)$, where α is any real number and $\varphi(x)$ is the function defined above. Only one function of this family, however, can be *represented by* the given series (that is, can be the sum of the series).

52. SERIES OF POLYNOMIALS AND THE WEIERSTRASS THEOREM

Since the successive partial sums of a power series are polynomials of continually increasing degree, every function $f(x)$ developable in a power series can be represented by an approximating polynomial with any specified degree of accuracy. We can express this more precisely as follows: If a function $f(x)$ can be expanded in a power series with radius of convergence equal to r, then, for any positive ρ less than r and for any positive ε, there exists a polynomial which differs from $f(x)$ by less than ε for all x in the interval $[-\rho, \rho]$ (we cannot, however, make the same assertion concerning the interval $(-r, r)$, since in this interval, the convergence of the series may be nonuniform). Let us recall that it was to this same problem of the approximation of functions by means of polynomials that we applied the Taylor and Maclaurin formulas in Lecture 5.

But, while it is possible to approximate by polynomials any function capable of expansion in a power series, the converse is not true. That is to say, from the fact that a function $f(x)$ in a given interval $[a, b]$ can be expressed in the form of a polynomial with arbitrarily specified accuracy, it does not follow that it is developable in a power series. This remark is of great theoretical importance. The approximate representation of functions by polynomials is one of the most important tools for studying them; on the other hand, we know that expansion in power series is possi-

ble for only a comparatively narrow class of functions, a class which does not even include all functions having derivatives of all orders (which is itself already a very restrictive requirement). Hence, if the possibility of approximating a function by means of polynomials were to depend upon its capability of expansion in a power series, this very valuable property would indeed be limited to a highly restricted class of functions.

In reality, the case is otherwise. It turns out that *a necessary and sufficient condition for the possibility of approximating, with any required accuracy, a function in a given interval by means of polynomials, is its continuity on the interval.* Thus, even nowhere differentiable functions can have such an approximate representation. The necessity of continuity is obvious: A function which can be represented with any desired accuracy by a polynomial is the limit of a uniformly convergent sequence of polynomials, and, consequently, it is also the sum of a uniformly convergent series of polynomials, that is, of continuous functions. Hence, by a theorem which we have already encountered in Lecture 4 (p. 91), it must itself be a continuous function. That this necessary condition is also sufficient is one of the most profound and important facts of mathematical analysis and constitutes the following celebrated theorem.

WEIERSTRASS' THEOREM. *If a function f is continuous on a closed interval $[a, b]$, then given any $\varepsilon > 0$, there exists a polynomial $P_n(x)$ such that*

$$|f(x) - P_n(x)| < \varepsilon, \qquad a \leq x \leq b.$$

Before proving this,[1] we must consider some rather remote preliminaries.

Let us consider the integral

$$I_n = \int_{-1}^{+1} (1 - u^2)^n \, du,$$

which is clearly greater than zero for every $n = 1, 2, \ldots$. Let us divide the domain of integration into two parts, and consider separately the integral

$$K_n = \int_{-n^{-\frac{1}{3}}}^{+n^{-\frac{1}{3}}} (1 - u^2)^n \, du$$

[1] At present, a considerable variety of different proofs of this theorem are known. We have selected here one of the most interesting from the methodological point of view.

extending over the interval $\left(-\frac{1}{\sqrt[3]{n}}, \frac{1}{\sqrt[3]{n}}\right)$ (which is small for large n), and the complementary integral,

$$L_n = \int_{n^{-\frac{1}{3}} \leq |u| \leq 1} (1 - u^2)^n \, du,$$

which is actually the sum of two integrals extending over the intervals $\left(-1, -\frac{1}{\sqrt[3]{n}}\right)$ and $\left(\frac{1}{\sqrt[3]{n}}, 1\right)$, respectively. As n increases indefinitely, the domain of integration of the integral K_n contracts to the point $u = 0$, while the domain of integration of L_n expands, tending to include the whole interval $[-1, 1]$. Nevertheless, we shall presently see that, for large values of n, the component K_n of the sum $I_n = K_n + L_n$ exhausts almost the entire value of I_n, leaving only a negligible part of this value for the integral L_n. The explanation for this is that, for large n, the integrand $(1 - u^2)^n$ has a perceptible value only for very small values of $|u|$, while for somewhat greater values of $|u|$ it becomes negligibly small. The graph of this function is given in Figure 33.

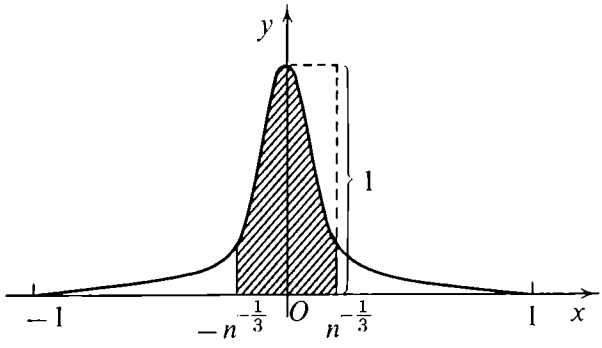

Fig. 33

LEMMA. *As* $n \to \infty$,

$$\frac{K_n}{I_n} \to 1 \quad \textit{and} \quad \frac{L_n}{I_n} \to 0.$$

Proof. Since each of these two relations is evidently a consequence of the other one, it is sufficient to prove either one of them.

The integrand in L_n clearly attains its maximum value when $|u| = n^{-\frac{1}{3}}$. Since the domain of integration is a part of the interval $[-1, 1]$, we have

$$L_n < 2(1 - n^{-\frac{2}{3}})^n.$$

On the other hand, it is evident that

$$I_n > \int_{-\frac{1}{2}n^{-\frac{1}{3}}}^{\frac{1}{2}n^{-\frac{1}{3}}} (1 - u^2)^n \, du > n^{-\frac{1}{3}} \left(1 - \frac{1}{4}n^{-\frac{2}{3}}\right)^n,$$

and consequently

$$\frac{L_n}{I_n} < 2n^{\frac{1}{3}} \left(\frac{1 - n^{-\frac{2}{3}}}{1 - \frac{1}{4}n^{-\frac{2}{3}}}\right)^n.$$

But[1]

$$\frac{1 - n^{-\frac{2}{3}}}{1 - \frac{1}{4}n^{-\frac{2}{3}}} = 1 - \frac{\frac{3}{4}n^{-\frac{2}{3}}}{1 - \frac{1}{4}n^{-\frac{2}{3}}} < 1 - \frac{3}{4}n^{-\frac{2}{3}} < e^{-\frac{3}{4}n^{-\frac{2}{3}}},$$

from which it follows that

$$\frac{L_n}{I_n} < 2n^{\frac{1}{3}} e^{-\frac{3}{4}n^{\frac{1}{3}}}$$

or, setting $\frac{3}{4}n^{\frac{1}{3}} = z$,

$$\frac{L_n}{I_n} < \frac{8}{3} z e^{-z}.$$

But as $n \to \infty$ we have $z \to \infty$; and since, under these circumstances, $ze^{-z} \to 0$, we have[2]

$$\lim_{n \to \infty} \frac{L_n}{I_n} = 0,$$

which proves our lemma.

[1] It is known that $1 + x < e^x$ for all $x \neq 0$; the simplest way to prove this is to find the minimum value of the function $e^x - 1 - x$.

[2] The simplest way to see this is to observe that for $z > 0$, we have

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots > \frac{1}{2}z^2, \text{ and, consequently, } ze^{-z} = \frac{z}{e^z} < \frac{2}{z}.$$

You may wonder what possible significance our detailed examination of the integral I_n could have for the proof of such a general proposition as the Weierstrass theorem. This is by no means the only instance in mathematical analysis where an extremely general theorem is proved by applying a very special analytic tool, such as the integral I_n in the present case. The specific property of this integral which makes it a convenient instrument in establishing the Weierstrass theorem finds its expression in the lemma which we have just proved. Our proof of the Weierstrass theorem can thus serve as an instructive example of the methods used in analysis.

Proof of the Weierstrass theorem. Let $f(x)$ be any continuous function on the interval [0, 1]. The integral

$$P_n(x) = \frac{1}{I_n}\int_0^1 f(v)\{1 - (v - x)^2\}^n \, dv$$

is clearly a polynomial in x of degree $2n$ (since the integrand is such a polynomial). Transforming the variable of integration by setting $v = x + u$, we obtain

$$P_n(x) = \frac{1}{I_n}\int_{-x}^{1-x} f(u + x)(1 - u^2)^n \, du. \tag{3'}$$

Let $0 < \alpha < \beta < 1$ and $\alpha \leq x \leq \beta$; then $-x \leq -\alpha < 0 < 1 - \beta \leq 1 - x$. Consequently, if n is so large that $n^{-\frac{1}{3}}$ is less than both α and $1 - \beta$, then

$$-x < -n^{-\frac{1}{3}} < n^{-\frac{1}{3}} < 1 - x \qquad (\alpha \leq x \leq \beta).$$

In the equality (3'), we can write

$$\int_{-x}^{1-x} = \int_{-x}^{-n^{-\frac{1}{3}}} + \int_{-n^{-\frac{1}{3}}}^{n^{-\frac{1}{3}}} + \int_{n^{-\frac{1}{3}}}^{1-x}.$$

Denoting by M the maximum value of $|f(x)|$ in the interval [0, 1], we clearly have

$$\frac{1}{I_n}\left|\left\{\int_{-x}^{-n^{-\frac{1}{3}}} + \int_{n^{-\frac{1}{3}}}^{1-x}\right\} f(u + x)(1 - u^2)^n \, du\right|$$

$$\leq \frac{M}{I_n}\left\{\int_{-1}^{-n^{-\frac{1}{3}}} + \int_{n^{-\frac{1}{3}}}^{1}\right\}(1 - u^2)^n \, du = \frac{ML_n}{I_n} \to 0$$

as $n \to \infty$, by virtue of our lemma. Since $\frac{ML_n}{I_n}$ does not de-

pend upon x, the left side of this inequality tends to zero uniformly with respect to x, as $n \to \infty$ in the whole interval $[\alpha, \beta]$. Setting

$$\frac{1}{I_n}\left\{\int_{-x}^{-n^{-\frac{1}{3}}} + \int_{n^{-\frac{1}{3}}}^{1-x}\right\} f(u + x)(1 - u^2)^n \, du = R_n(x),$$

we then have

$$P_n(x) = \frac{1}{I_n}\int_{-n^{-\frac{1}{3}}}^{n^{-\frac{1}{3}}} f(u + x)(1 - u^2)^n \, du + R_n(x),$$

where $R_n(x) \to 0$ uniformly in the interval $[\alpha, \beta]$ as $n \to \infty$.

No doubt, you have already guessed that it is the polynomial $P_n(x)$ that will turn out to be the desired approximation to $f(x)$. If so, then you must also have divined the further course of the proof: considering the uniform smallness of the quantity $R_n(x)$ for large values of n, it is only necessary to prove that, for large values of n, the first term on the right side of the last equality differs little from $f(x)$ in the whole interval $[\alpha, \beta]$. But this is almost obvious, since the integrand $f(u + x)$ differs little from $f(x)$ when the values of u belonging to the domain of integration are extremely small. So, replacing $f(u + x)$ by $f(x)$, we thereby replace the whole first term by the expression $f(x)\,\dfrac{K_n}{I_n}$, which, in view of our lemma, tends to $f(x)$ as $n \to \infty$.

To complete the proof, we have only to carry through this reasoning with the necessary formal rigor. For this purpose it will be simplest to use the first mean value theorem (Lecture 6, p. 155), by virtue of which

$$\int_{-n^{-\frac{1}{3}}}^{n^{-\frac{1}{3}}} f(u + x)(1 - u^2)^n \, du = f(x + \theta n^{-\frac{1}{3}})\, K_n,$$

where $-1 < \theta < 1$. We then obtain

$$\begin{aligned} |f(x) - P_n(x)| &= \left| f(x) - f(x + \theta n^{-\frac{1}{3}})\frac{K_n}{I_n} - R_n(x) \right| \\ &\leq \left| f(x) - f(x + \theta n^{-\frac{1}{3}})\frac{K_n}{I_n} \right| + |R_n(x)|. \end{aligned}$$

Let $\varepsilon > 0$ be an arbitrarily small number. As we already know, for sufficiently large n, we have

$$|R_n(x)| < \frac{\varepsilon}{2} \qquad (\alpha \leq x \leq \beta).$$

On the other hand, since $\frac{K_n}{I_n} \to 1$ as $n \to \infty$, while the function $f(x)$ is uniformly continuous in the whole interval [0, 1], it follows that, for sufficiently large n, we have

$$\left| f(x) - f(x + \theta n^{-\frac{1}{3}}) \frac{K_n}{I_n} \right| < \frac{\varepsilon}{2} \qquad (\alpha \leq x \leq \beta).$$

Thus, for sufficiently large n, we obtain

$$|f(x) - P_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad (\alpha \leq x \leq \beta).$$

In other words, as $n \to \infty$ the polynomial $P_n(x)$ tends to $f(x)$ uniformly in $[\alpha, \beta]$.

The proof of the Weierstrass theorem is now essentially complete; we have only to free ourselves of some minor restrictions. First, we have to pass from the particular interval [0, 1] to any interval $[a, b]$. Second, we have to show that the uniform approximation required by the Weierstrass theorem takes place, not only in any interval $[a', b']$ lying entirely in the interior of the interval $[a, b]$ (only this has been shown so far, since the interval $[\alpha, \beta]$ was an interval lying entirely in the interior of [0, 1]), but also in the whole interval $[a, b]$.

To achieve the first of these two goals, let us set

$$\frac{x - a}{b - a} = y, \; x = a + (b - a)y \qquad (a \leq x \leq b),$$

and

$$f(x) = f[a + (b - a)y] = \varphi(y).$$

As x increases from a to b, y increases from 0 to 1; and, since $f(x)$ is continuous on $[a, b]$, $\varphi(y)$ is continuous on [0, 1]. Let $a < a' < b' < b$, and let us set

$$\frac{a' - a}{b - a} = \alpha \quad \text{and} \quad \frac{b' - a}{b - a} = \beta, \qquad 0 < \alpha < \beta < 1.$$

By what we have just proved, we can find, for any positive ε, a polynomial $P_n(y)$ such that

$$|\varphi(y) - P_n(y)| < \varepsilon \qquad (\alpha \leq y \leq \beta),$$

or

$$\left| f(x) - P_n\left(\frac{x - a}{b - a}\right) \right| < \varepsilon \qquad (a' \leq x \leq b').$$

But it is clear that

$$\Pi_n(x) = P_n\left(\frac{x-a}{b-a}\right)$$

is a polynomial in x of the same degree as $P_n(x)$. Thus, a function $f(x)$, continuous in any interval $[a, b]$, can be uniformly approximated by polynomials, with any required accuracy, in every interval $[a', b']$ contained entirely in the interior of the interval $[a, b]$.

Finally, to remove the last restriction, let us assume again that $f(x)$ is continuous on $[a, b]$, and let us set

$$F(x) = \begin{cases} f(a) & \text{if } a - 1 \leq x \leq a, \\ f(x) & \text{if } a \leq x \leq b, \\ f(b) & \text{if } b \leq x \leq b + 1. \end{cases}$$

It is obvious that $F(x)$ is defined and continuous on $[a - 1, b + 1]$, which contains in its interior the whole interval $[a, b]$. By what we have proved, $F(x)$ can be uniformly approximated by polynomials on the interval $[a, b]$, with any required accuracy. Since $F(x) = f(x)$ in this interval, the same thing is true for the function $f(x)$, and the proof of the Weierstrass theorem is now complete.

The Weierstrass theorem can also be formulated as follows: *every continuous function is the sum of a uniformly convergent series of polynomials.* For, if $P_n(x)$ denotes a polynomial differing from $f(x)$ by less than $\frac{1}{n}$ in the interval $[a, b]$, then the series

$$P_1(x) + \{P_2(x) - P_1(x)\} + \cdots + \{P_n(x) - P_{n-1}(x)\} + \cdots,$$

whose terms are polynomials, tends to $f(x)$ uniformly on the interval $[a, b]$.

53. TRIGONOMETRIC SERIES

A trigonometric series is a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{4}$$

Since all the terms of this series are periodic functions with periods which are divisors of 2π, the sum of this series will have a period 2π. Therefore, if a function which we wish to expand into

such a series does not have this periodicity, its expansion in a series of type (4) can be achieved only for intervals whose length does not exceed 2π. And, because of the periodicity, it is sufficient to limit our study of the series (4) to some fixed interval of length 2π, for example, to the interval $[-\pi, \pi]$.

We shall now present three reasons why the expansion of a function in a series of the form (4) may, in some cases, be more appropriate than its expansion in a power series.

1. As we know, expansion in power series is possible only for those functions which have derivatives of all orders (and even this restrictive assumption is, in general, not sufficient). But, as we shall see later, for the development of a function in a series of the form (4) much more modest assumptions will suffice. In particular, such an expansion exists for every function which has a bounded and integrable derivative, and even this assumption is dispensable.

2. The terms of the series (4) are periodic, wave-like functions, and a certain degree of this wave-like form is retained by the partial sums of this series, a characteristic which power series, in general, do not possess. Thus, if the function under consideration shows some tendency even to approximate wave-like form (as often happens in mechanics, physics, biology, and economics), we are justified in expecting that the partial sums of a trigonometric series will imitate the behavior of such a function better than the polynomials which form the partial sums of a power series.

3. Finally, the trigonometric functions which constitute the terms of the series (4) have one remarkable property, which greatly facilitates both the study of and the operations with trigonometric series, and which is completely lacking in the terms of power series. This property is the so-called *orthogonality* of the system of functions

$$1, \quad \sin nx, \quad \text{and} \quad \cos nx \qquad (n = 1, 2, \ldots) \tag{5}$$

over any interval of length 2π. Two different functions are said to be *orthogonal* over an interval if the integral of their product over that interval is equal to zero. You can easily prove that this property applies to the functions in (5) over any interval of length 2π by transforming the integrand into a sum by means of elementary trigonometric identities.

It would be difficult to overestimate the importance of the property of orthogonality. Functions much more complicated than

those in (5) may, if only they form an orthogonal system, serve as the terms of series which turn out to be extremely useful in the investigation of functions. Numerous properties of the trigonometric system (5) are shared by all orthogonal systems, a fact which has led to the construction of an entire theory of such systems. Today this theory is highly developed and includes many important and profound analytical results.

54. FOURIER COEFFICIENTS

In the case of power series, we have seen that the coefficients of the expansion of a given function can be determined easily if we know the values of the function and its derivatives at $x = 0$. It is natural to raise the same question with regard to trigonometric series.

Suppose that $f(x)$ is represented in the interval $[-\pi, \pi]$ by a uniformly convergent series (4). Multiplying both sides of the equality

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

by $\cos kx$, where k is zero or a positive integer, we clearly obtain the expansion of $f(x) \cos kx$ in a series which is uniformly convergent. Therefore, in integrating both sides of this new equality from $-\pi$ to $+\pi$ we may integrate the series on the right side *term by term* (see Lecture 4, p. 94). By virtue of the orthogonality of the system (5), the integrals of all the terms will vanish except for the one integral

$$\int_{-\pi}^{\pi} a_k \cos^2 kx \, dx = a_k \int_{-\pi}^{\pi} \frac{1 + \cos 2kx}{2} dx = \pi a_k. \qquad (6)$$

We have assumed here that $k > 0$; for $k = 0$ the integral differing from zero will be

$$\int_{-\pi}^{\pi} \frac{a_0}{2} dx = \pi a_0,$$

so that the general result (6) remains valid in this case also. (It is with just this end in mind that the first term of the series (4)

is usually written in the form $\frac{a_0}{2}$.) Thus, for any $k \geq 0$ we have

$$\int_{-\pi}^{\pi} f(x) \cos kx \, dx = \pi a_k$$

or

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \qquad (k = 0, 1, 2, \ldots). \tag{7}$$

In a similar manner, we find

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx \qquad (k = 1, 2, \ldots). \tag{8}$$

Formulas (7) and (8) solve our stated problem for the case when the series (4) is uniformly convergent; they express the coefficients of the series by means of the function which this series represents. As you can see, these formulas in themselves do not even require the existence of the first derivative of $f(x)$. The coefficients a_k and b_k, which can thus be constructed for any integrable function, are usually called the *Fourier coefficients* of the given function, and the series (4) constructed with their help is called the *Fourier series* of this function (even though formulas (7) and (8) were first discovered by Euler). Needless to say, it does not follow from this (the logical situation here being completely analogous to that encountered in our study of power series) that the Fourier series constructed for an integrable function must converge. In fact, any number of cases exist where the Fourier series constructed for an integrable function is divergent. And even if it does converge, it does not follow that its sum must coincide with $f(x)$.

All that we have established up to this point can be formulated as follows: *If a function $f(x)$ can be expressed by a trigonometric series converging uniformly in $[-\pi, \pi]$, then this series is the Fourier series of the function;* that is, its coefficients are expressed in terms of the given function by the formulas (7) and (8). In the case of Maclaurin series, we came to an analogous conclusion with only one difference: in that case, there was no need to require explicitly uniform convergence, since the very nature of a power series implies its uniform convergence in every subinterval contained within the interval of convergence.

Obviously, the basic problem of the theory of trigonometric series consists in determining the conditions under which a given function will equal the sum of its Fourier series. This question has occupied the attention of a very large number of investigators, and a voluminous literature has been dedicated to it. In what follows, we shall examine several of the simplest results pertaining to this problem.

55. APPROXIMATION IN THE MEAN

We shall show, first of all, how Fourier coefficients make their appearance quite naturally in connection with the solution of a completely different problem.

Let $f(x)$ be bounded and integrable over $[-\pi, \pi]$. Wishing to find for this function an approximate expression in the form of a *trigonometric polynomial of order n,*

$$\Pi_n(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx),$$

we inquire how we should select the coefficients α_k and β_k so as to make this approximation as accurate as possible. Of course, this formulation of our problem is not yet well defined, since we have not yet stated how the error of approximation is to be measured. It is clear that we have a wide range of choice. For example, we could accept as the error of approximation the least upper bound of the difference

$$|f(x) - \Pi_n(x)|$$

for $-\pi \leq x \leq \pi$. Another possible choice is to evaluate the error of approximation by means of the integral

$$\int_{-\pi}^{\pi} |f(x) - \Pi_n(x)|\, dx.$$

Finally, it is possible to evaluate this error by means of the integral

$$\int_{-\pi}^{\pi} [f(x) - \Pi_n(x)]^2\, dx. \tag{9}$$

This last expression is, from a formal point of view, the most convenient both in theoretical investigations and in practical computation. It does not contain the absolute value sign, the presence

of which very frequently makes analytic operations more difficult. It is this last form of evaluating the error which we shall consider. The approximation corresponding to such a definition of the magnitude of error is usually called the *approximation in the mean.*

We now have before us a well-defined problem: that of choosing the numbers α_k and β_k in such a way that the value of the integral (9) is the smallest possible. It is evident that this integral is a function of $2n + 1$ variables $\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$, so that we have to deal with a multidimensional minimum problem. However, the special character of this problem permits us to solve it without using the methods of the differential calculus.

Let us write the integral (9) in the form

$$\int_{-\pi}^{\pi} f^2(x)\,dx + \int_{-\pi}^{\pi} \Pi_n^2(x)\,dx - 2\int_{-\pi}^{\pi} f(x)\,\Pi_n(x)\,dx.$$

Denoting by a_k and b_k the Fourier coefficients of $f(x)$, by virtue of the formulas (7) and (8) we have

$$\int_{-\pi}^{\pi} f(x)\,\Pi_n(x)\,dx = \pi\left\{\frac{\alpha_0 a_0}{2} + \sum_{k=1}^{n} (\alpha_k a_k + \beta_k b_k)\right\}. \qquad (10)$$

If we further take into consideration that

$$\int_{-\pi}^{\pi} \cos^2 kx\,dx = \int_{-\pi}^{\pi} \sin^2 kx\,dx = \pi,$$

the orthogonality of the system (5) at once gives us the expression

$$\int_{-\pi}^{\pi} \Pi_n^2(x)\,dx = \pi\left\{\frac{\alpha_0^2}{2} + \sum_{k=1}^{n} (\alpha_k^2 + \beta_k^2)\right\}. \qquad (11)$$

Combining (10) and (11) and making use of the elementary relations

$$\alpha_k^2 - 2a_k\alpha_k = (\alpha_k - a_k)^2 - a_k^2,$$
$$\beta_k^2 - 2b_k\beta_k = (\beta_k - b_k)^2 - b_k^2,$$

we obtain

$$\int_{-\pi}^{\pi} \Pi_n^2(x)\,dx - 2\int_{-\pi}^{\pi} f(x)\,\Pi_n(x)\,dx$$
$$= \pi\left\{\frac{(\alpha_0 - a_0)^2}{2} + \sum_{k=1}^{n} [(\alpha_k - a_k)^2 + (\beta_k - b_k)^2]\right\}$$
$$-\pi\left\{\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)\right\}.$$

Consequently,

$$\int_{-\pi}^{\pi}[f(x) - \Pi_n(x)]^2\,dx = \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\left\{\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right\}$$

$$+ \pi\left\{\frac{(\alpha_0 - a_0)^2}{2} + \sum_{k=1}^{n}[(\alpha_k - a_k)^2 + (\beta_k - b_k)^2]\right\}.$$

On the right side, only the last term depends upon the numbers α_k and β_k; therefore, we have to find the minimum of this last component. But all its terms are nonnegative and, therefore, it will attain its minimum value (zero) when all these terms become equal to zero, that is, when

$$\alpha_0 = a_0, \quad \alpha_k = a_k, \quad \text{and} \quad \beta_k = b_k \qquad (k = 1, 2, \ldots, n).$$

Thus, the solution of our problem reads:

THEOREM 3. *Let $f(x)$ be bounded and integrable on $[-\pi, \pi]$. Then, from among all trigonometric polynomials of order n, the best approximation in the mean to $f(x)$ is provided by the trigonometric polynomial whose coefficients are the corresponding Fourier coefficients of this function.*

Denoting this trigonometric polynomial by $P_n(x)$, we have

$$\int_{-\pi}^{\pi}[f(x) - P_n(x)]^2\,dx = \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\left\{\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right\}.$$

Incidentally, this relation proves another very important theorem. Since the left side is obviously nonnegative, for any n we have

$$\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx;$$

and since the right side of this inequality does not depend on n, the left side remains bounded as n increases. This gives us:

THEOREM 4. *The series*

$$\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$

is convergent for any bounded and integrable function $f(x)$. In particular, it follows from this that the Fourier coefficients a_k and b_k of every such function tend to zero as $k \to \infty$.

These results, obtained by an altogether elementary argument, have very great significance for the theory of Fourier series.

56. COMPLETENESS OF THE SYSTEM OF TRIGONOMETRIC FUNCTIONS

As we have seen in the theory of power series, one and the same series can serve as the Maclaurin series for infinitely many different functions. Is the analogous phenomenon possible for Fourier series? This question is, as we shall presently show, closely connected with a certain remarkable property of the system (5) of trigonometric functions, the property of *completeness.*

If a series of the form (4) were the Fourier series of two different functions $f_1(x)$ and $f_2(x)$, both continuous[1] on the interval $[-\pi, \pi]$, then the corresponding Fourier coefficients of these two functions would coincide. The difference

$$f(x) = f_1(x) - f_2(x)$$

of these two functions, though not identically equal to zero on $[-\pi, \pi]$, would have all its Fourier coefficients equal to zero. But, by formulas (7) and (8), this would be equivalent to the statement that $f(x)$ is *orthogonal* to every function of the system (5). In this case, the orthogonal system (5) would not be, as we say, *complete;* that is, we could add to it a new function, not identically equal to zero, such that the enlarged system would remain orthogonal. And, clearly, the argument can be reversed: If the system (5) is not complete, then any function $f_0(x)$ which is orthogonal with respect to all the functions of the system (5) has all its Fourier coefficients equal to zero. But then all functions of the form

$$f(x) + \alpha f_0(x),$$

where α is any real number, have one and the same Fourier series. We shall now prove the following.

THEOREM 5. *The system* (5) *is complete; in other words, any continuous function orthogonal to all functions of* (5) *must be identically equal to zero.*

Proof. Let $f(x)$ be such a function. We then have

$$\int_{-\pi}^{\pi} f(x)\, T(x)\, dx = 0$$

for any trigonometric polynomial $T(x)$. Suppose that $f(x)$ is not

[1] If we omit the condition of continuity, then the answer to our question of uniqueness is trivially negative, since two functions differing from each other at only one point have the same set of Fourier coefficients.

identically equal to zero and, to be definite, suppose that $f(x) > 0$ at $x = \alpha$. Then,[1] as we know from the theory of continuous functions, we can find positive numbers c and δ such that, for $-\pi < \alpha - \delta \leq x \leq \alpha + \delta < \pi$, we have the inequality

$$f(x) > c.$$

Consider now the expression

$$T_n(x) = \left(\frac{1 + \cos(x - \alpha)}{2}\right)^n.$$

Expanding by the binomial formula, we find for the function $T_n(x)$ an expression of the form

$$T_n(x) = \sum_{r=0}^{n} c_r \cos^r(x - \alpha).$$

But, as we know from trigonometry, any power $\cos^r x$ can be expressed as a linear combination of the functions

$$1, \quad \cos x, \quad \cos 2x, \quad \ldots, \quad \cos rx$$

with constant coefficients. (This can be proved very simply by induction.) Thus we obtain

$$T_n(x) = \sum_{r=0}^{n} d_r \cos r(x - \alpha).$$

Finally, taking into consideration that

$$\cos r(x - \alpha) = \cos r\alpha \cos rx + \sin r\alpha \sin rx,$$

we shall arrive at the expression

$$T_n(x) = \frac{\alpha_0}{2} + \sum_{r=0}^{n} (\alpha_r \cos rx + \beta_r \sin rx),$$

where α_r and β_r are constant coefficients. Thus, $T_n(x)$ is a trigonometric polynomial for every n, and, consequently,

$$\int_{-\pi}^{\pi} f(x)\, T_n(x)\, dx = 0 \qquad (n = 1, 2, \ldots). \tag{12}$$

[1] It is obvious that, under our conditions, we can take α to be an interior point of the interval $[-\pi, \pi]$.

Let us now consider the behavior of $T_n(x)$ for large values of n. Since

$$\frac{1 + \cos(x - \alpha)}{2}$$

clearly lies between 0 and 1 in the interval $[-\pi, \pi]$ and is equal to 1 only at $x = \alpha$, it follows that for large values of n, the quantity $T_n(x)$ is nonnegative, equal to 1 at $x = \alpha$, and negligibly small for all values differing appreciably from α. In other words, this function has a graph of the type shown in Figure 33 (p. 185), with the difference that the interval $[-1, 1]$ has to be replaced by the interval $[-\pi, \pi]$ and the maximum has to be transferred from the point $x = 0$ to the point $x = \alpha$.

The outline of our further reasoning is now clear. Since, for large values of n, the function $T_n(x)$ is negligibly small outside the interval $[\alpha - \delta, \alpha + \delta]$, the corresponding parts of the integral (12) are also negligibly small. As a result, its sign is determined by the sign of the integral

$$\int_{\alpha-\delta}^{\alpha-\delta} f(x)\, T_n(x)\, dx. \tag{13}$$

Since in the integrand of (13) $T_n(x) > 0$ and $f(x) > c$, it follows that the integral (12) cannot be equal to zero, we thus arrive at a contradiction. It remains for us, now, to demonstrate that the parts of the integral (12) which we have neglected are small in absolute value in comparison with (13). In order to carry out the computation necessary for this purpose, let us set

$$\int_{\alpha-\delta}^{\alpha+\delta} f(x)\, T_n(x)\, dx = I_1 \text{ and} \int_{-\pi}^{\alpha+\delta} f(x)\, T_n(x)\, dx + \int_{\alpha+\delta}^{\pi} f(x)\, T_n(x)\, dx = I_2,$$

so that, from (12), we get

$$0 = \int_{-\pi}^{\pi} f(x)\, T_n(x)\, dx = I_1 + I_2.$$

Noting that $\frac{1 + \cos(x - \alpha)}{2} = \cos^2 \frac{x - \alpha}{2}$, we have

$$I_1 \geq c \int_{\alpha-\delta}^{\alpha+\delta} \cos^{2n} \frac{x - \alpha}{2}\, dx = c \int_{-\delta}^{\delta} \cos^{2n} \frac{y}{2}\, dy$$

$$= 2c \int_0^{\delta} \left(1 - \sin^2 \frac{y}{2}\right)^n dy.$$

But, for $0 \leq y \leq \delta < \frac{\pi}{2}$, we have

$$0 \leq \sin \frac{y}{2} < 1 \quad \text{and} \quad 0 < \cos \frac{y}{2} \leq 1,$$

from which

$$I_1 > 2c \int_0^{\delta} \left(1 - \sin \frac{y}{2}\right)^n \cos \frac{y}{2}\, dy.$$

Or, if we set $\sin \frac{y}{2} = z$ and $\frac{1}{2} \cos \frac{y}{2}\, dy = dz$,

$$I_1 > 4c \int_0^{\sin \frac{\delta}{2}} (1 - z)^n\, dz = \frac{4c}{n+1} \left\{1 - \left(1 - \sin \frac{\delta}{2}\right)^{n+1}\right\}.$$

Since for sufficiently large n the expression within the braces is greater than $\frac{1}{2}$, we have

$$I_1 > \frac{2c}{n+1}. \tag{14}$$

On the other hand, if we denote by M the maximum value of $f(x)$ in $[-\pi, \pi]$, we note that in the domain of integration of the two integrals which form I_2

$$\cos^2 \frac{x - \alpha}{2} \leq \cos^2 \frac{\delta}{2},$$

and hence

$$T_n(x) \leq \cos^{2n} \frac{\delta}{2}.$$

We now obtain

$$|I_2| \leq M \cos^{2n} \frac{\delta}{2} (\alpha - \delta + \pi + \pi - \alpha - \delta) < 2\pi M \cos^{2n} \frac{\delta}{2}$$
$$= 2\pi M \rho^n, \tag{15}$$

where

$$\rho = \cos^2 \frac{\delta}{2} < 1.$$

Now to complete the proof. If $f(x)$ were orthogonal to the system (5), then we would have

$$\int_{-\pi}^{\pi} f(x)\, T_n(x)\, dx = I_1 + I_2 = 0.$$

Hence, $|I_1| = |I_2|$ and, by (14) and (15), we would have, for n sufficiently large,

$$2\pi M\rho^n > \frac{2c}{n+1} \quad \text{or} \quad (n+1)\,\rho^{n+1} > \frac{c\rho}{\pi M}.$$

But this gives us the required contradiction, for $n\rho^n \to 0$ as $n \to \infty$ and, therefore, $n\rho^n$ cannot remain greater than the constant positive number $\frac{c\rho}{\pi M}$ as n increases.[1]

We have thus proved the completeness of the orthogonal system (5). As we have already seen, it follows from this that a trigonometric series (4) can serve as the Fourier series for no more than one continuous function. In particular, if this series is uniformly convergent, its sum is the only continuous function having this series as its Fourier series.

57. CONVERGENCE OF FOURIER SERIES FOR FUNCTIONS WITH A BOUNDED INTEGRABLE DERIVATIVE

We shall now show that every function $f(x)$ with a period of 2π which has a bounded and integrable derivative can be expanded into a uniformly convergent trigonometric series (which is, therefore, its Fourier series).

Let us agree to designate the Fourier coefficients of $f(x)$ by a_k and b_k, and those of its derivative $f'(x)$ by a_k' and b_k'. Integration by parts, then, gives

$$\pi a_k = \int_{-\pi}^{\pi} f(x) \cos kx\, dx$$

$$= \frac{f(x) \sin kx}{k}\Bigg|_{-\pi}^{\pi} - \frac{1}{k} \int_{-\pi}^{\pi} f'(x) \sin kx\, dx = -\frac{\pi b_k'}{k},$$

[1] Setting $n \ln \frac{1}{\rho} = x$ we have $n\rho^n = \frac{1}{\ln \frac{1}{\rho}}\, xe^{-x} \to 0$ as $x \to \infty$, as we have seen in the

footnote on page 186.

and similarly, $b_k = \frac{a_k'}{k}$. In what follows, it will be convenient to apply the so-called Cauchy-Schwartz inequality:[1]

$$\left[\sum_{k=1}^{n} u_k v_k\right]^2 \leq \sum_{k=1}^{n} u_k^2 \sum_{k=1}^{n} v_k^2,$$

which is valid for any real numbers u_k and v_k and for any n.

Noting that

$$a_k \cos kx + b_k \sin kx = \frac{1}{k}(-b_k' \cos kx + a_k' \sin kx),$$

and utilizing the Cauchy-Schwartz inequality and the inequality

$$(\alpha \cos \varphi + \beta \sin \varphi)^2 = \alpha^2 + \beta^2 - (\alpha \sin \varphi - \beta \cos \varphi)^2 \leq \alpha^2 + \beta^2,$$

we find that for $m > n$

$$\begin{aligned}
&\left|\sum_{k=n}^{m} (a_k \cos kx + b_k \sin kx)\right| \\
&\quad = \left|\sum_{k=n}^{m} \frac{1}{k}(-b_k' \cos kx + a_k' \sin kx)\right| \\
&\quad \leq \sum_{k=n}^{m} \frac{1}{k^2} \sum_{k=n}^{m} (-b_k' \cos kx + a_k' \sin kx)^2 \\
&\quad \leq \sum_{k=n}^{m} \frac{1}{k^2} \sum_{k=n}^{m} (a_k'^2 + b_k'^2).
\end{aligned}$$

But the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, and so is the series $\sum_{k=1}^{\infty} (a_k'^2 + b_k'^2)$, since a_k' and b_k' are the Fourier coefficients of the bounded integrable function $f'(x)$ (see p. 196). Thus, on the right side of the last inequality both factors tend to zero for any $m > n$ as $n \to \infty$ (by the Cauchy condition). This means that

$$\lim_{\substack{n \to \infty \\ m > n}} \sum_{k=n}^{m} (a_k \cos kx + b_k \sin kx) = 0,$$

and that, moreover, the convergence is uniform with respect to x, since the right side of the preceding inequality does not depend on x. It follows by the Cauchy condition that the series (4) is uniformly convergent in the interval $[-\pi, \pi]$. Denoting its sum by

[1] $\sum_{k=1}^{n} (u_k x + v_k)^2$ is, as a function of x, a nonnegative quadratic trinomial; hence, its discriminant

$$4\left\{\left[\sum_{k=1}^{n} u_k v_k\right]^2 - \sum_{k=1}^{n} u_k^2 \sum_{k=1}^{n} v_k^2\right\} \leq 0.$$

$s(x)$, we see that (4) is the Fourier series of both functions $f(x)$ and $s(x)$. But then these functions, being continuous, must coincide, and our theorem is established.

The modern theory of trigonometric series proves the convergence of Fourier series for significantly more extensive classes of functions than the class which we have considered. This extension, however, cannot actually be carried very far. We know that, even among the continuous functions, there exist some whose Fourier series do not converge for some values of x. Here, of course, we shall not be able to discuss these problems in greater detail. Let us simply note that the theory of trigonometric series, to which an extensive literature, including many textbooks, is devoted, includes to this day a great many important problems for which solutions have not yet been found. Hence, the field naturally attracts the efforts of many investigators.

58. EXTENSION TO ARBITRARY INTERVALS

So far we have been considering functions defined on $[-\pi, \pi]$, and only on this interval did we seek the expansion of the given function in a trigonometric series. Moreover, we tacitly assumed that $f(\pi) = f(-\pi)$, since only under this condition can a function $f(x)$ be represented by a series of the form (4) at all points of the closed interval $[-\pi, \pi]$. Now we shall see how we can free ourselves from these restrictions, for it is clear that the theory of trigonometric series can gain wide application only if we can expand functions defined on arbitrary intervals, not restricted by any requirements of periodicity.

First of all, it is obvious that (as we pointed out at the very beginning) nothing of what has been said will be changed if, instead of the interval $[-\pi, \pi]$, we take as the starting point of our discussion any interval $[a, a + 2\pi]$ of length 2π, providing $f(a + 2\pi) = f(a)$. Thus, in our exposition we set requirements only on the length of the interval under consideration, not on its position. And we required of the function $f(x)$ that its values at the end points of the given interval be equal.

Suppose now that we wish to expand in a trigonometric series a function $f(x)$ defined on a completely arbitrary interval $[a, b]$ and not subject to any conditions of periodicity. As before, we shall assume only that $f(x)$ is differentiable at each point of $[a, b]$ and that its derivative is bounded and integrable in this interval.

Let us assume first that $b - a < 2\pi$. Obviously, it is possible to define in infinitely many ways a function $f^*(x)$ on the interval $[a, a + 2\pi]$ having in this interval a bounded and integrable derivative, satisfying the condition $f^*(a + 2\pi) = f^*(a)$, and coinciding with $f(x)$ at all points of $[a, b]$. By what we have already proved, $f^*(x)$ can be expanded in the interval $[a, a + 2\pi]$ into a uniformly convergent trigonometric series. It is obvious that in the interval $[a, b]$ this series represents the function $f(x)$ and thus completely solves our problem.

Now let $b - a$ be greater than 2π. In this case, we shall take, first of all, any number $b' > b$ and define on the interval $[a, b']$ a function $f^*(x)$ in such a way that it will have in this interval a bounded and integrable derivative, it will satisfy the condition $f^*(b') = f^*(a)$, and it will coincide with $f(x)$ at all points of $[a, b]$. If, in addition, we construct $f^*(x)$ so that $f^{*\prime}(b') = f^{*\prime}(a)$ (which, of course, is always possible) and extend $f^*(x)$ as a periodic function beyond the limits of the interval $[a, b']$ in both directions, we shall obtain a periodic function $f^*(x)$ having the period $b' - a = 2l > 2\pi$. This function has a bounded and integrable derivative in any interval and coincides with the function $f(x)$ in the interval $[a, b]$.

Let us now set

$$x = \frac{l}{\pi} y \quad \text{and} \quad f^*(x) = f^*\left(\frac{l}{\pi} y\right) = \varphi(y).$$

Clearly, y increases from $-\pi$ to π as x increases from $-l$ to l, and the function $\varphi(y)$ has a bounded and integrable derivative in the interval $[-\pi, \pi]$, while $\varphi(-\pi) = f^*(-l) = f^*(l) = \varphi(\pi)$. By our theory, the Fourier series of $\varphi(y)$ converges uniformly to this function in the interval $[-\pi, \pi]$:

$$\varphi(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos ny + b_n \sin ny) \qquad (-\pi \leq y \leq \pi).$$

Substituting $y = \frac{\pi}{l} x$, we obtain

$$f^*(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x\right), \qquad (16)$$

this series converging uniformly in the interval $-l \leq x \leq l$. Since the functions $\cos \frac{n\pi}{l} x$ and $\sin \frac{n\pi}{l} x$, like the function $f(x)$, have the period $2l$, the expansion (16) is valid uniformly on the

whole real line. In particular, in the interval $[a, b]$ we shall have uniformly

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x.$$

We see then that $f(x)$ is expressed in the interval $[a, b]$ by a trigonometric series differing from the series (4) only in that the elements of the expansion, instead of being the functions $\cos nx$ and $\sin nx$, are the functions $\cos \frac{n\pi}{l} x$ and $\sin \frac{n\pi}{l} x$, whose period is $2l$ instead of 2π. This result solves our problem in the best possible way, since in view of the inequality $b - a > 2\pi$, the function $f(x)$ in the interval $[a, b]$ cannot, in general, be expressed by a series of the form (4) whose terms have the period 2π.

We have still to see how it is possible to obtain from the function $f(x)$ the coefficients a_n and b_n of the series representing it. For this purpose, we note that by our initial definition

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(y) \cos ky \, dy \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(y) \sin ky \, dy.$$

Substituting $y = \frac{\pi}{l} x$, we find

$$a_k = \frac{1}{l} \int_{-l}^{l} \varphi\left(\frac{\pi}{l} x\right) \cos k \frac{\pi}{l} x \, dx$$

$$= \frac{1}{l} \int_{-l}^{l} f^*(x) \cos k \frac{\pi}{l} x \, dx = \frac{1}{l} \int_{a}^{b} f^*(x) \cos k \frac{\pi}{l} x \, dx$$

and, similarly,

$$b_k = \frac{1}{l} \int_{a}^{b} f^*(x) \sin k \frac{\pi}{l} x \, dx.$$

Since there exist infinitely many functions $f^*(x)$ satisfying the required conditions, the coefficients a_k and b_k are not determined uniquely by the values of $f(x)$ on $[a, b]$. But this should not surprise us: the series (16) is called upon to represent $f(x)$ only in the interval $[a, b]$ whose length is less than the period $2l$ of the elements $\cos \frac{k\pi}{l}$ and $\sin \frac{k\pi}{l}$. Similarly, for any proper subinterval of $[-\pi, \pi]$, there are infinitely many trigonometric series (4) representing $f(x)$ in this subinterval.

8. Differential Equations

59. FUNDAMENTAL CONCEPTS

At the end of Lecture 6, we mentioned that the purpose of every integration process is to obtain from the given *local* characteristics of the phenomenon under consideration its *global* characteristics, that is, the description of the phenomenon as a whole. Problems of this type are extremely numerous in all applied sciences which employ the resources of mathematical analysis. However, the extremely varied requirements set for the mathematical apparatus by problems of this kind cannot always be satisfied with the help of a tool so elementary, from the conceptual point of view, as either the ordinary or the multiple integral. Only in a relatively small number of the most primitive cases does the simple apparatus of integration turn out to be adequate for the solution of the given problem.

More often the situation is such that the known local characteristics of the phenomenon under consideration lead to a system of *differential equations* in which the unknowns are the very functions which describe the global characteristics of this phenomenon. The solution of a given concrete problem is thus reduced to the solution of a system of differential equations, that is, to the determination of the unknown functions appearing in these equations. This mathematical problem is much more complicated than the ordinary integration of functions. In terms of the theory of differential equations, ordinary integration is only the simplest particular case and, moreover, a trivial one. That is, whenever the solution of any type of differential equations can, by one means or another, be reduced to the ordinary integration of functions (or, in the language of the theory of differential equations, to *quadratures*), the general problem is thereupon considered solved.

It will be useful to observe by means of a very simple example how the knowledge of the local characteristics of a phenomenon can give rise to a differential equation whose solution permits us later to describe this phenomenon as a whole.

Let us imagine a container whose capacity is a liters, filled with a salt solution. Let there be a continuous flow into and out of the container such that in one unit of time the flow adds to it b liters of pure water and drains from it the same amount of solution. Let us assume, further, that at a certain initial moment $t = 0$ the container held c kg. of salt. We want to know the number x of kilograms of salt our container will retain after t units of time. (We assume, in addition, that the mixing of the solution is so rapid that we can regard the salt concentration as being at every moment the same in all parts of the container.)

What local characteristics of the phenomenon are given in the above problem? We know that the solution flows out of the container at the rate of b liters per unit of time. At a given moment t the container holds an unknown quantity x kg. of salt and, since the capacity of the container is a liters, every liter of the solution contains $\frac{x}{a}$ kg. of salt. Hence, b liters contains $\frac{bx}{a} = sx$ kg. of salt, where we shall write s for $\frac{b}{a}$. This means that if, during a unit of time starting from the moment t, the concentration of the solution remained constant, the amount of salt in the container would decrease during this unit of time by sx kg. This, therefore, is the magnitude of the rate of decrease of the amount of salt in the container at the moment t. We thus have

$$\frac{dx}{dt} = -sx. \tag{1}$$

This formula, expressing the instantaneous rate of change in the amount of salt in terms of the amount of salt present at a given moment (note $\frac{dx}{dt}$ is negative since x is decreasing), gives us a local characterization of the phenomenon. (Here it would, of course, be more convenient to say a *momentary* or *instantaneous* characterization.) The function $x = x(t)$, which we seek, is the unknown in the equation.

Can this function be found directly by the methods of integral calculus? What is given is a formula expressing its derivative; and finding a function when its derivative is given is precisely the fundamental problem of the integral calculus. Nevertheless, this problem is not altogether ordinary: the derivative of the required

function is expressed not, as we are accustomed, in terms of the independent variables, but in terms of the unknown function itself. The integral calculus does not directly undertake the solution of problems of this kind, and, therefore, we must formally deal with a basically new problem: the solution of the *differential equation* (1). In the present case, of course, the problem reduces in a trivial way to a problem in integral calculus. Writing (1) in the form

$$\frac{dx}{x} = -s\,dt,$$

we, in the customary terminology, *separate* the variables. A simple integration gives $\ln x$ on the left side and $-st$ on the right, so that

$$\ln x = -st + k,$$

where k is a constant. To determine this number, we turn (as is characteristically done when solving differential equations) to the *initial condition:* at $t = 0$ we have $x = c$. This gives us $k = \ln c$, so that, finally,

$$x = ce^{-st}.$$

The given problem is thus completely solved. We see that as the time increases the amount of salt in the container decreases *exponentially*.

Let us now imagine that the liquid flowing out of our container flows into yet another container of the same capacity, initially (that is, at the moment $t = 0$) filled with pure water; and that here also the liquid flows in and out at the rate of b liters per unit of time. It is obvious that salt is then constantly being introduced into and removed from this second container, and it is desired to know the number y of kilograms of salt which will be present in the second container at any given time t.

Here again, we are given only a local (instantaneous) characterization of the phenomenon. It is clear that in one unit of time as much salt flows into the second container as flows out of the first, namely,

$$sx = sce^{-st}.$$

On the other hand, at the time t, every liter of the liquid in the second container holds $\frac{y}{a}$ kg. of salt, so that the b liters which flow out

of it in one unit of time contain sy kg. of salt. Consequently, the total increase of the quantity of salt in the second container in one unit of time at the moment t is

$$\frac{dy}{dt} = sce^{-st} - sy = s(ce^{-st} - y). \tag{2}$$

You see that here again the local description of the phenomenon has found its mathematical expression in a differential equation. Again we have an expression for the derivative $\frac{dy}{dt}$ of the unknown function. But this time the solution of the equation is not so easily found as previously: Equation (2) gives us an expression for the derivative $\frac{dy}{dt}$ which contains the independent variable t as well as the unknown function y, and we cannot immediately *separate the variables* here as we could in equation (1). Thus, we are faced with an essentially new problem, which in the given case can be solved, to be sure, in a relatively simple way, but whose solution in the general case we have no methods of finding.

What is the nature of this *general case?* How do we define the general concept of a differential equation? In our examples, we have been dealing with equations of the form

$$\frac{dy}{dx} = f(x, y), \tag{3}$$

where x is an independent variable and y an unknown function of the variable x; the function $f(x, y)$ is, of course, given. The problem consists in finding a function $y = \varphi(x)$ satisfying equation (3), that is, a function such that on an interval $a \leq x \leq b$ we have identically

$$\varphi'(x) = f[x, \varphi(x)].$$

A somewhat more general type of differential equation is represented by the relation

$$F(x, y, y') = 0,$$

where again x is an independent variable, y is an unknown function of the variable x, and $y' = \frac{dy}{dx}$. Just as before, we are looking for a function $y = \varphi(x)$ identically satisfying the relation

$$F[x, \varphi(x), \varphi'(x)] = 0.$$

In addition, there exist cases in which the local characteristics of the phenomenon under consideration require for their expression not only the first derivative of the unknown function but also derivatives of higher orders, so that the differential equation takes the form

$$F(x, y, y', \ldots, y^{(n)}) = 0. \tag{4}$$

Such an equation is called a differential equation *of order n.* It is the most general type of differential equation for problems which deal with only one unknown function and one independent variable. Such a situation occurs, however, only in the simplest cases; we frequently need to find *several* functions which depend upon *several* independent variables.

Let us turn first to the consideration of an arbitrary number of unknown functions $y_1, y_2, \ldots, y_k$, dependent, however, only on one independent variable x. For the problem to be determinate, the local description must lead us to a *system* of differential equations equal in number to the number k of unknown functions. The general form of such a system of equations is

$$F_i(x, y_1, y_1', \ldots, y_1^{(n_1)}, y_2, y_2', \ldots, y_2^{(n_2)}, \ldots, y_k, y_k', \ldots, y_k^{(n_k)}) = 0$$

where $i = 1, 2, \ldots, k$. Thus, we obtain the most general problem in the theory of so-called *ordinary* differential equations, the term used to denote differential equations containing only one independent variable.

A good example is to be found in the system of equations (with which you are undoubtedly familiar) describing the movement of a point mass:

$$m\frac{d^2x}{dt^2} = X\left(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),$$

$$m\frac{d^2y}{dt^2} = Y\left(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),$$

$$m\frac{d^2z}{dt^2} = Z\left(t, x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right),$$

where the only independent variable is the time t and the unknown functions are the coordinates x, y, and z of the point. Here m denotes the mass of this point and X, Y, and Z are the components of the resultant force arising from all the forces acting on it, dependent in the general case on time and the position and velocity of the moving point. The *initial* values consist of the three

coordinates and three component velocities of this point at a certain specified *initial* moment $t = t_0$.

If there are several independent variables, then we are dealing with an equation (or a system of equations) with *partial derivatives.* The unknown functions are now functions of several variables and the equations naturally contain their partial derivatives with respect to these variables. The theory of partial differential equations is, of course, much more complicated than the theory of ordinary differential equations, and we shall not deal with it at all.

But even the theory of ordinary differential equations does not abound in general methods sufficiently powerful to enable us to find solutions for wide classes of such equations. To some degree, this can be foreseen from the fact that even ordinary integration applied to elementary functions leads, in many cases, to non-elementary functions. All the more is this to be expected with more general and more complicated problems. Even if we take the point of view, customary in the theory of differential equations, that having reduced the solution of a problem to ordinary integrations, we may consider the problem solved, we shall still have progressed little, since such a reduction to quadratures can be performed only for a small number of the simplest (but, from the purely practical point of view, the most important) types of differential equations. We shall not discuss here nor even enumerate these types, as you will find all the material you could wish on the question in any elementary textbook. We shall be concerned, rather, with more basic questions.

60. THE EXISTENCE OF A SOLUTION

We have already mentioned that the fundamental problem of the integral calculus may be considered as the simplest particular case of the solution of a differential equation. Namely, if in the equation

$$\frac{dy}{dx} = f(x, y) \tag{3}$$

the right side does not depend upon y, then we obtain an equation of the form

$$\frac{dy}{dx} = f(x), \tag{3'}$$

whose solution is obviously equivalent to the integration of the function $f(x)$. In connection with this problem, we had occasion to

observe that even when the function $f(x)$ is continuous, we have to prove the existence of the integral, that is, the existence of a solution of the equation (3′). This was done by a special argument, which was based on the uniform continuity of $f(x)$. If the function $f(x)$ is bounded but discontinuous, then, generally speaking, the integral does not exist.

From all this it is evident that the problem of the existence of a solution to equation (3) is very likely to be highly complicated and certainly requires special investigation. And this applies, in an even greater degree, to those more general types of equations and systems of equations which we cited earlier. In order to throw into bold relief the basic features of this fundamental problem of the theory of differential equations, we must, as far as possible, free our exposition from the burden of purely technical complications. We shall therefore limit ourselves in what follows to the consideration of equations of type (3), that is, equations of the first order solved with respect to the derivative of the unknown function.

THEOREM 1. *Suppose that $f(x, y)$ is continuous in a region D of the (x, y) plane. We shall show that for every point (x_0, y_0) within this region, there exists a function $y = \varphi(x)$, satisfying equation* (3) *in a neighborhood of x_0, such that $y_0 = \varphi(x_0)$.*

It is clearly sufficient to prove our theorem for the case where the region D is bounded and closed, and the point (x_0, y_0) is an interior point of D. To carry out the proof, we shall need a lemma which is also of independent interest.

Suppose we are given an infinite family of functions $S = \{F(x)\}$, defined on an interval $[a, b]$. The family of functions S will be said to be *bounded* on $[a, b]$ if there exists a number M such that $|F(x)| < M$ for all $x \in [a, b]$ and for all $F(x)$ in S.

DEFINITION. *We shall call the family S* equicontinuous *on $[a, b]$ if, for any positive ε there exists a $\delta > 0$ such that, for any function $F(x)$ in S and any pair of points x_1 and x_2 in $[a, b]$ satisfying the inequality $|x_1 - x_2| < \delta$, we have*

$$|F(x_1) - F(x_2)| < \varepsilon.$$

It is obvious that if the family S is bounded (or equicontinuous) on $[a, b]$, then each of the functions belonging to it is bounded (or uniformly continuous) on this interval. The converse is, in general,

not true; boundedness and equicontinuity of the family S require, besides the presence of the corresponding property in each function, the further condition that this property be *uniform* for the totality of functions in the given family.

Let us now agree to define the *spread* of a family S of functions in the interval $[a, b]$ as the least upper bound of all quantities $|F_1(x) - F_2(x)|$, where x is any point in $[a, b]$, and F_1 and F_2 are any two functions in S.

LEMMA. *Every infinite family S of functions which is bounded and equicontinuous on an interval I contains a sequence of functions uniformly convergent in this interval.*

We shall carry out the proof in several consecutive stages.

Proof. (i) We shall prove first that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any subinterval J contained in I and of length less than δ, the family S contains an infinite subfamily S', whose spread in J is less than ε.

Let M be a number such that $|F(x)| < M$ for all F in S and all x in I. Because of the equicontinuity of the family S on the interval I, we can find, for any positive integer n, a positive δ such that the oscillation of any function belonging to S on any subinterval of length less than δ will be smaller than $\frac{M}{n}$. Let J be a fixed subinterval of length less than δ, and consider any given function $F(x) \in S$. Let ξ and η denote the minimum and maximum values assumed by $F(x)$ in J, so that $\eta - \xi < \frac{M}{n}$. Since ξ and η are both interior to the interval $[-M, M]$, there exists an integer k, $-(n-1) \leq k \leq n-1$, such that

$$\frac{k-1}{n} M \leq \xi < \eta < \frac{k+1}{n} M.$$

Thus, we see that the values assumed by any function of the family S on the interval J are included in an interval of the form

$$\left[\frac{k-1}{n} M, \frac{k+1}{n} M\right], \quad \text{where } -(n-1) \leq k \leq n-1.$$

Since there is a finite number of these latter intervals, while the family S contains an infinite number of functions, at least one of these intervals will contain all the values assumed by some in-

finite subfamily S' of the family S. It is obvious that the spread of the family S' in the interval J does not exceed $\frac{2M}{n}$ and will be less than ε if we select the number n so that $\frac{2M}{n} < \varepsilon$.

(ii) We shall now show that the family S also contains an infinite subfamily S' whose spread in the entire interval I is less than ε. For this purpose, let us subdivide the original interval I into subintervals $I_1, I_2, \ldots, I_m$, each of length less than δ. Then, by virtue of what we have just established in (i), the family S contains an infinite subfamily S_1 whose spread is less than ε in I_1. For the same reason, the family S_1 contains an infinite subfamily S_2 whose spread is less than ε in I_2, and so on. Finally, we come to an infinite subfamily $S_m = S'$ of S, whose spread is less than ε in the interval I_m, as well as in each of the preceding intervals, since

$$S' = S_m \subset S_{m-1} \subset \cdots \subset S_2 \subset S_1.$$

But this means that the spread of the family S' is less than ε in the whole interval I, and our assertion is proved.

(iii) The proof of our lemma can now be completed quite simply. Let us denote by S_1 an infinite subfamily of S whose spread in I is less than 1. Similarly, let us denote by S_2 an infinite subfamily of S_1 whose spread in I is less than $\frac{1}{2}$, and, in general, let us denote by S_n an infinite subfamily of S_{n-1} whose spread is less than $\frac{1}{n}$. (All these subfamilies exist by virtue of what we have just proved in (ii).) Let $F_1(x)$ be any function belonging to the family S_1; $F_2(x)$ any function belonging to the family S_2, different from $F_1(x)$; and, in general, let $F_n(x)$ be any function belonging to the family S_n and different from $F_{n-1}(x), F_{n-2}(x), \ldots, F_1(x)$. Since $S_{n+p} \subset S_n$, the functions F_n and F_{n+p} are both contained in S_n and, consequently, for every point x in I we have

$$|F_n(x) - F_{n+p}(x)| < \frac{1}{n} \qquad (p = 0, 1, 2, \ldots).$$

It follows, by the Cauchy condition, that the sequence of functions $F_1(x), F_2(x), \ldots, F_n(x)$ converges uniformly in I, and the proof of the lemma is complete.

Let us now remark, for later use, that our lemma is by no means restricted to functions of one variable. Suppose, for example, that S is a family of functions bounded and equicontinuous on the rectangle I. Then, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that the oscillation of any function $F(Q)$ in S is less than ε on any rectangle J contained in I whose diagonal is of length less than δ. The proof of the lemma may now be applied verbatim, and we conclude that there exists a sequence $F_1(Q), F_2(Q), \ldots$, of functions in S converging uniformly in I.

Since the proof of the fundamental theorem, formulated earlier, will be based on an idea which is easiest to grasp in geometrical terms, it will be very useful before beginning the proof to see what form the problem (the finding of a solution for the given equation (3)) assumes geometrically.

Graphical representations of the solutions $y = \varphi(x)$ of equation (3) are usually called *integral curves.* The equation (3) associates with every point (x, y) of the region D a direction given by the slope $\frac{dy}{dx} = f(x, y)$. The aggregate of points of D, together with their corresponding directions, form the so-called *field of linear elements* of the equation (3). An integral curve of this equation is a curve such that the direction of its tangent at each point coincides with the direction of the field at that point. Thus, the problem of finding a solution $y = \varphi(x)$ of equation (3) such that $y = y_0$ for $x = x_0$ becomes, in geometrical terms, the problem of finding an integral curve of the given equation which passes through the point (x_0, y_0). To prove the existence of this solution means, therefore, to prove that through any given point there passes at least one integral curve.

The method that we are about to use for this purpose will consist of constructing auxiliary curves which approximate more and more closely the integral curve. This curve is then obtained by passing to the limit.

Proof of Theorem 1. Let (x_0, y_0) be an interior point of a closed and bounded region D in which the function $f(x, y)$ is continuous, and let M denote an upper bound of $|f(x, y)|$ in this region. If the positive number α is sufficiently small, then the rectangle R, $x_0 - \alpha \leq x \leq x_0 + \alpha$ and $y_0 - M\alpha \leq y \leq y_0 + M\alpha$, will be contained within D. We shall show that there exists a function

$y = \varphi(x)$ satisfying equation (3) for all x in $[x_0 - \alpha, x_0 + \alpha]$ and such that $\varphi(x_0) = y_0$. To make the proof clearer, let us again divide it into separate steps.

(i) Let $x_k = x_0 + \frac{k}{n}\alpha$ $(0 \leq k \leq n)$, so that the points $x_1, x_2, \ldots, x_{n-1}$ divide the interval $[x_0, x_0 + \alpha]$ into n equal parts. We now construct over the interval $[x_0, x_0 + \alpha]$ a broken line $y = \varphi_n(x)$ (Fig. 34) with vertices having abscissas $x_0, x_1, \ldots, x_n$, in such

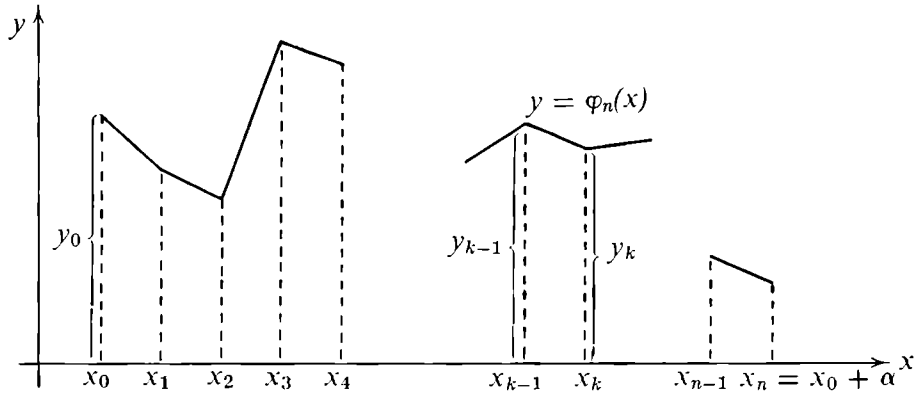

Fig. 34

a way that the slope of each segment coincides with the direction of the field at the left end point of that segment. Clearly, this can be done in successive steps passing from x_{k-1} to x_k. The ordinate $y_k = \varphi_n(x_k)$ will be given by the recursive formula

$$\begin{aligned} y_k &= y_{k-1} + (x_k - x_{k-1}) f(x_{k-1}, y_{k-1}) \\ &= y_{k-1} + \frac{\alpha}{n} f(x_{k-1}, y_{k-1}) \qquad (1 \leq k \leq n), \end{aligned} \tag{5}$$

and the equation of the broken line over each interval $x_{k-1} \leq x \leq x_k$ will be given by the formula

$$\varphi_n(x) = y_{k-1} + (x - x_{k-1}) f(x_{k-1}, y_{k-1}) \qquad (1 \leq k \leq n).$$

To prove that the whole broken line $y = \varphi_n(x)$ from $x = x_0$ to $x = x_0 + \alpha$ is contained in D, it would clearly be sufficient to prove that

$$|y_k - y_0| \leq M\alpha \qquad (1 \leq k \leq n).$$

We shall find it useful, however, to prove the stronger inequality

$$|y_k - y_0| \leq \frac{k}{n} M\alpha \qquad (1 \leq k \leq n).$$

This inequality obviously holds for $k = 0$. If the inequality is true for a particular $k < n$, then the corresponding point (x_k, y_k) is contained in D, and, therefore, $|f(x_k, y_k)| \leq M$. But in this case the recursive formula (5) gives

$$|y_{k+1} - y_0| \leq |y_k - y_0| + \frac{\alpha}{n}|f(x_k, y_k)| \leq \frac{k+1}{n} M\alpha, \quad (6)$$

so that our inequality is also true for the number $k + 1$. It follows that

$$|\varphi_n(x) - y_0| \leq M\alpha \qquad (x_0 \leq x \leq x_0 + \alpha), \quad (7)$$

and the entire broken line $y = \varphi_n(x)$ lies in D. Since this is true for every n, it follows that the family of functions $\varphi_n(x)$ $(n = 1, 2, \ldots)$ is bounded on the interval $[x_0, x_0 + \alpha]$.

(ii) Let x' and x'' be any two points in $[x_0, x_0 + \alpha]$, and, to be definite, let

$$x_{k-1} \leq x' < x_k < \cdots < x_{l-1} < x'' \leq x_l.$$

Then

$$\begin{aligned}\varphi_n(x'') - \varphi_n(x') &= [\varphi_n(x'') - \varphi_n(x_{l-1})] \\ &\quad + [\varphi_n(x_{l-1}) - \varphi_n(x_{l-2})] \\ &\quad + \cdots + [\varphi_n(x_k) - \varphi_n(x')] \\ &= (x'' - x_{l-1}) f(x_{l-1}, y_{l-1}) \\ &\quad + \frac{\alpha}{n}[f(x_{l-2}, y_{l-2}) + \cdots + f(x_k, y_k)] \\ &\quad + (x_k - x') f(x_{k-1}, y_{k-1}). \end{aligned} \quad (8)$$

We are now in a position to show that the family of functions $\varphi_n(x)$ $(n = 1, 2, \ldots)$ is equicontinuous on $[x_0, x_0 + \alpha]$. The relation (8) gives

$$\begin{aligned}|\varphi_n(x'') - \varphi_n(x')| &\leq M[x'' - x_{l-1} + (l - k - 1)\frac{\alpha}{n} + x_k - x'] \\ &= M[x'' - x_{l-1} + (x_{l-1} - x_k) + x_k - x'] \\ &= M(x'' - x'), \end{aligned} \quad (9)$$

from which we see directly that, for sufficiently small $|x'' - x'|$, we have $|\varphi_n(x'') - \varphi_n(x')| < \varepsilon$ for arbitrarily situated points x'' and x' and for any n.

(iii) Since the family $\{\varphi_n(x)\}$ is bounded and equicontinuous on $[x_0, x_0 + \alpha]$, it follows from our lemma that it must contain a

sequence of functions converging uniformly in this interval. Since in what follows we shall have to deal with this sequence only, we can denote it by $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$. Thus, as $n \to \infty$ the functions $\varphi_n(x)$ tend to a limit $\varphi(x)$, uniformly for $x_0 \leq x \leq x_0 + \alpha$. We shall now show that the function $\varphi(x)$, thus defined, satisfies all the conditions of our theorem. Since $\varphi_n(x_0) = y_0$ for every n, it is immediately apparent that $\varphi(x_0) = y_0$.

(iv) We shall now verify that, for $x_0 \leq x \leq x_0 + \alpha$, the function $\varphi(x)$ satisfies the differential equation (3). Let ε be any fixed positive number. Since the function $f(x, y)$ is continuous and the region D is closed and bounded, there exists a $\delta > 0$ such that, for any two points (ξ_1, η_1) and (ξ_2, η_2) in D related by the inequalities

$$|\xi_2 - \xi_1| < \delta \quad \text{and} \quad |\eta_2 - \eta_1| < 2M\delta,$$

the inequality

$$|f(\xi_2, \eta_2) - f(\xi_1, \eta_1)| < \varepsilon$$

is also satisfied.

Let $x' \leq x_i \leq x''$, where x_i is any of the points of subdivision of the interval $[x_0, x_0 + \alpha]$ corresponding to the function $\varphi_n(x)$. Setting, as before, $\varphi_n(x_i) = y_i$, we have from (9) that

$$|y_i - \varphi_n(x')| \leq M(x_i - x') < M(x'' - x');$$

and since, for sufficiently large n, we have

$$|\varphi_n(x') - \varphi(x')| \leq M(x'' - x'),$$

we also have

$$|y_i - \varphi(x')| \leq 2M(x'' - x').$$

Therefore, if $|x'' - x'| < \delta$, then for sufficiently large n and for $x' \leq x_i \leq x''$ we have

$$|x_i - x'| < \delta \quad \text{and} \quad |y_i - y'| < 2M\delta,$$

where $y_i = \varphi_n(x_i)$ and $y' = \varphi(x')$. From this, in accordance with the definition of the number δ, we obtain for $x' \leq x_i \leq x''$ and for sufficiently large n

$$|f(x_i, y_i) - f(x', y')| < \varepsilon,$$

or

$$f(x', y') - \varepsilon \leq f(x_i, y_i) \leq f(x', y') + \varepsilon.$$

If we apply this estimate to every term[1] on the right side of (8), we see that for $0 < x'' - x' < \delta$ and for sufficiently large n we have the inequalities

$$[f(x', y') - \varepsilon](x'' - x') \leq \varphi_n(x'') - \varphi_n(x') \leq [f(x', y') + \varepsilon](x'' - x').$$

Since both the left and the right side of these inequalities are independent of n, we find by passing to the limit that, under the sole condition $|x' - x''| < \delta$, we have

$$[f(x', y') - \varepsilon](x'' - x') \leq \varphi(x'') - \varphi(x') \leq [f(x', y') + \varepsilon](x'' - x');$$

or, what is the same thing,

$$\left| \frac{\varphi(x'') - \varphi(x')}{x'' - x'} - f(x', y') \right| \leq \varepsilon.$$

But this means that

$$\varphi'(x) = f(x, y) \qquad (x_0 \leq x \leq x_0 + \alpha),$$

where, for $x = x_0$, $\varphi'(x)$ denotes the right-hand derivative.

(v) Finally, since we can clearly obtain the same result for the interval $[x_0 - \alpha, x_0]$ in a completely analogous manner, the function $\varphi(x)$ satisfies all the conditions of our theorem, which is therefore proved.

The functions $y = \varphi_n(x)$ which we have constructed have as their graphs broken lines. The direction of each segment of such a line coincides with the direction of the field at the left end point of this segment. The lengths of the segments decrease indefinitely as n increases. Thus the larger the value of n, the denser is the set of points at which the direction of the broken line coincides with the direction of the field. It is therefore quite natural to expect that if these broken lines have a limiting curve (as $n \to \infty$), its direction will coincide at *every* one of its points with the direction of the field; that is, it will be an integral curve. The existence of a limiting function (if not for the whole sequence of functions $\varphi_n(x)$, then at least for some subsequence, which is sufficient for our purpose) has been established in our lemma. Thus, we had only to confirm the correctness of our expectation by a rigorous and precise argument.

[1] That $f(x_{k-1}, y_{k-1})$ in (8) also lies between $f(x', y') - \varepsilon$ and $f(x', y') + \varepsilon$ follows by the same argument, since for sufficiently large n we have $|x' - x_{k-1}| < \delta$.

61. UNIQUENESS OF THE SOLUTION

The importance of Theorem 1 in section 60, guaranteeing a solution for the equation (3) under certain conditions, is self-evident. When in practice we find it necessary to solve a differential equation, we make efforts, sometimes strenuous efforts, to reduce its solution to quadratures (ordinary integrations). But if this fails, we usually apply one or another device of approximate computation. For our activity to make sense, we need the assurance that the object we are seeking actually exists; all our efforts would be wasted if our equation had no solutions at all.

A no less important question is that of the *uniqueness* of the solution. For, even if the problem under consideration leads to the conclusion that the function $\varphi(x)$ must satisfy equation (3) and take the value y_0 at $x = x_0$, we still cannot consider our original problem as solved unless we are assured that there exists no other solution of equation (3) satisfying the same *initial* condition $\varphi(x_0) = y_0$. For, if there were several such solutions, we could not be sure that the particular one we have found is the one which solves our original problem. Even if we had found all the solutions of equation (3), still, generally speaking, we would have no way of knowing which one of them corresponds to our original problem.

It is noteworthy that the condition which permitted us to prove the existence of a solution (namely, the continuity of the function $f(x, y)$ in the given region) cannot assure the uniqueness of this solution. There are cases in which the function $f(x, y)$ is continuous in the region D and where, nevertheless, there exist several solutions of equation (3) which take the value y_0 at $x = x_0$. It is possible, however, to obtain uniqueness of the solution if we require of the function $f(x, y)$ something more than simple continuity. One of the most useful forms of such a strengthened condition, called a *Lipschitz Condition*, states that: *there exists a constant k such that*

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2| \qquad \text{(A)}$$

for any two points (x, y_1) and (x, y_2) in the region D. To be sure, this requirement is not the weakest that might be formulated. But it is fulfilled in the majority of cases encountered in practice, and its simplicity makes it very convenient in application. So, we prove:

THEOREM 2. *If $f(x, y)$ is continuous in D and satisfies condition* (A), *then the solution $y = \varphi(x)$ of equation* (3) *which takes the value y_0 for $x = x_0$ is unique.*

Proof. Suppose that each of the two functions $\varphi_1(x)$ and $\varphi_2(x)$ satisfies equation (3) for $x_0 - \alpha \leq x \leq x_0 + \alpha$ and that $\varphi_1(x_0) = \varphi_2(x_0) = y_0$. Let $\varphi_2(x) - \varphi_1(x) = \omega(x)$, so that $\omega(x_0) = 0$. Then, for $x_0 - \alpha \leq x \leq x_0 + \alpha$, we have

$$\left|\frac{d\omega}{dx}\right| = \left|\frac{d\varphi_2}{dx} - \frac{d\varphi_1}{dx}\right| = \left|f(x, \varphi_2(x)) - f(x, \varphi_1(x))\right|$$
$$\leq k|\varphi_2(x) - \varphi_1(x)| = k|\omega(x)|. \tag{10}$$

We shall denote by μ the maximum value of $|\omega(x)|$ in the smaller of the two intervals $\left[x_0 - \frac{1}{2k}, x_0 + \frac{1}{2k}\right]$ and $[x_0 - \alpha, x_0 + \alpha]$. Let us denote this interval by $[x_0 - r, x_0 + r]$, where r denotes the lesser of the numbers α and $\frac{1}{2k}$. Because of the continuity of $|\omega(x)|$, its maximum value on this interval is attained at some definite point $x = x_1$, that is, $|\omega(x_1)| = \mu$. Applying relation (10) and the first mean value theorem, we find[1]

$$\mu = |\omega(x_1)| = |\omega(x_1) - \omega(x_0)| = \left|\int_{x_0}^{x_1} \frac{d\omega}{dx}\, dx\right|$$
$$\leq \left|\int_{x_0}^{x_1} \left|\frac{d\omega}{dx}\right| dx\right| \leq k\left|\int_{x_0}^{x_1} |\omega(x)|\, dx\right|$$
$$\leq k\mu|x_1 - x_0| \leq k\mu\frac{1}{2k} = \frac{\mu}{2},$$

whence $\mu = 0$. This means that $\omega(x) = 0$ for $x_0 - r \leq x \leq x_0 + r$. If $r = \alpha$ then the theorem is proved. If, however, $r = \frac{1}{2k} < \alpha$, then, at any rate,

$$\omega\left(x_0 - \frac{1}{2k}\right) = \omega\left(x_0 + \frac{1}{2k}\right) = 0,$$

and we can then repeat the reasoning taking in succession as initial points, instead of $x = x_0$, the points $x_0 - \frac{1}{2k}$ and $x_0 + \frac{1}{2k}$. This will permit us to assert that $\omega(x) = 0$ in the lesser of the two intervals $\left[x_0 - 2\frac{1}{2k}, x_0 + 2\frac{1}{2k}\right]$ and $[x_0 - \alpha, x_0 + \alpha]$. Repeating this process a sufficient number of times, we shall clearly extend the interval in which we know $\omega(x) = 0$ until it coincides with $[x_0 - \alpha, x_0 + \alpha]$, thus proving the theorem.

[1] We take the absolute value of the integral in view of the possibility that $x_1 < x_0$.

62. DEPENDENCE OF THE SOLUTION ON PARAMETERS

When a differential equation arises out of some concrete problem, this equation usually contains several parameters, representing the values of the constants which determine the specific conditions of the phenomenon under consideration. Thus, in the problem which we used as an example at the beginning of this lecture, the container capacity a, the velocity b of fluid flow, and the initial amount c of salt in the first container are such quantities. All three of these quantities enter naturally into the equation which we constructed, and, of course, every solution of this equation will depend upon all of these parameters. Thus, if we wish to stress this dependence, we have to write equation (3) in the form

$$\frac{dy}{dx} = f(x, y, p_1, p_2, \ldots, p_r),$$

and its solution in the form

$$y = \varphi(x, p_1, p_2, \ldots, p_r),$$

where $p_1, p_2, \ldots, p_r$ are the parameters of the problem.

It is not hard to see that in applied mathematics the nature of the dependence of the solutions of the differential equation on the parameters is of essential importance, and that the establishment of the *continuity* of this dependence is particularly important. This is especially so, since the values of the parameters, as well as the values of x, are usually obtained as the result of one or another kind of physical measurement, and are usually not given with absolute accuracy, but only as an approximation, with some error, no matter how small. Therefore, if for very small increments of the parameters and of x, the values of $\varphi(x, p_1, p_2, \ldots, p_r)$ could change considerably, the resulting solutions would be completely useless in practice. The only solutions useful for practical purposes are those for which, knowing the approximate values of the parameters and of x, we may also find approximate values of the function. The precise mathematical expression of this property is nothing more than the continuity of the function $\varphi(x, p_1, p_2, \ldots, p_r)$ with respect to $x, p_1, p_2, \ldots, p_r$. Now, we prove:

THEOREM 3. *If $f(x, y, p_1, p_2, \ldots, p_r)$ is continuous in a region of the (x, y) plane, and is also continuous with respect to each of the parameters $p_1, p_2, \ldots, p_r$, then the fulfillment of the conditions under which we proved the existence and uniqueness of a solution of* (3) *will insure that this solution is a continuous function of $x, p_1, p_2, \ldots, p_r$.*

Since we can treat continuity with respect to each parameter separately, we may confine ourselves, for the sake of simplicity, to the case where the function f (and thus the function φ also) depends upon only one parameter p.

Proof. We may confine our attention to a closed and bounded neighborhood D of (x_0, y_0). Then on D and with p restricted to the interval $d = [\lambda_1, \lambda_2]$, f is *uniformly* continuous. By our assumption (A) we have

$$|f(x, y_1, p) - f(x, y_2, p)| \leq k\,|y_1 - y_2|, \tag{A'}$$

for (x, y_1) and (x, y_2) in D and $p \in d$. We assert that in this case the unique solution $y = \varphi(x, p)$ of the equation

$$\frac{dy}{dx} = f(x, y, p), \tag{3''}$$

which takes the value y_0 at $x = x_0$, is continuous with respect to x and p for (x, p) in the rectangle $x_0 - \alpha \leq x \leq x_0 + \alpha$ and $\lambda_1 \leq p \leq \lambda_2$.

To prove this, we shall have to return to the construction used earlier to prove the existence of a solution for equation (3). We investigate the nature of the dependence on the parameter p of the functions $\varphi_n(x)$ which we constructed there (and which we shall now write in the form $\varphi_n(x, p)$). Since the quantities y_i, defined by the recursive formula (5), also depend on p (that is, $y_i = y_i(p)$), formula (5) takes the form

$$y_i(p) = y_{i-1}(p) + \frac{\alpha}{n} f[x_{i-1}, y_{i-1}(p), p]. \tag{11}$$

By virtue of the uniform continuity of $f(x, y, p)$, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x, y, p + h) - f(x, y, p)| < \varepsilon, \tag{12}$$

whenever $|h| < \delta$, $p \in d$, $p + h \in d$, and $(x, y) \in D$. From (11) we have

$$\begin{aligned} y_i(p + h) - y_i(p) &= y_{i-1}(p + h) - y_{i-1}(p) \\ &\quad + \frac{\alpha}{n} \{f[x_{i-1}, y_{i-1}(p + h), p + h] - f[x_{i-1}, y_{i-1}(p), p]\} \end{aligned}$$

where $1 \leq i \leq n$.

Let us write, for brevity,

$$y_i(p + h) - y_i(p) = \Delta_i \qquad (0 \leq i \leq n),$$

so that

$$\Delta_i = \Delta_{i-1} + \frac{\alpha}{n}\{f[x_{i-1}, y_{i-1}(p + h), p + h] - f[x_{i-1}, y_{i-1}(p), p]\} \quad (13)$$

where $1 \leq i \leq n$. The expression within the braces may be written in the form

$$f[x_{i-1}, y_{i-1}(p + h), p + h] - f[x_{i-1}, y_{i-1}(p), p + h] + f[x_{i-1}, y_{i-1}(p), p + h] - f[x_{i-1}, y_{i-1}(p), p].$$

By virtue of (A′), the absolute value of the first of these differences does not exceed $k|\Delta_{i-1}|$, and by (12), the absolute value of the second difference is less than ε. Thus, we obtain

$$|\Delta_i| < |\Delta_{i-1}| + \frac{\alpha}{n}\{k|\Delta_{i-1}| + \varepsilon\} = \frac{\varepsilon\alpha}{n} + |\Delta_{i-1}|\left(1 + \frac{\alpha k}{n}\right)(1 \leq i \leq n).$$

Applying the same estimate to the quantity $|\Delta_{i-1}|$ appearing on the right side, and repeating this process, we arrive at the relation

$$|\Delta_i| < \frac{\varepsilon\alpha}{n} + \frac{\varepsilon\alpha}{n}\left(1 + \frac{\alpha k}{n}\right) + \frac{\varepsilon\alpha}{n}\left(1 + \frac{\alpha k}{n}\right)^2 + \cdots + \frac{\varepsilon\alpha}{n}\left(1 + \frac{\alpha k}{n}\right)^{i-1}$$

$$= \frac{\varepsilon}{k}\left[\left(1 + \frac{\alpha k}{n}\right)^i - 1\right] \leq \frac{\varepsilon}{k}\left[\left(1 + \frac{\alpha k}{n}\right)^n - 1\right] < \frac{\varepsilon}{k}(e^{\alpha k} - 1)$$

where $1 \leq i \leq n$. Hence, for $|h| < \delta$ we have

$$|y_i(p + h) - y_i(p)| < \frac{\varepsilon}{k}(e^{\alpha k} - 1)$$

or, what is the same thing,

$$|\varphi_n(x_i, p + h) - \varphi_n(x_i, p)| < \frac{\varepsilon}{k}(e^{\alpha k} - 1) \ (n = 1, 2, \ldots; 1 \leq i \leq n).$$

But the functions $\varphi_n(x, p)$ and $\varphi_n(x, p + h)$ are linear on each of the subintervals $[x_{i-1}, x_i]$. Therefore, the inequality

$$|\varphi_n(x, p + h) - \varphi_n(x, p)| < \frac{\varepsilon}{k}(e^{\alpha k} - 1),$$

which is satisfied (as we have just seen) for $|h| < \delta$ at the end points of every subinterval, must necessarily be satisfied within each subinterval, and thus in the entire interval $[x_0 - \alpha, x_0 + \alpha]$. In

view of the arbitrariness of ε, this means that the family of functions $\varphi_n(x, p)$ is equicontinuous on the rectangle $x_0 - \alpha \leq x \leq x + \alpha$ and $\lambda_1 \leq p \leq \lambda_2$. Let us denote this family by S. By the lemma on page 213 (see also the remark on page 215) the sequence S contains a subsequence S' which converges uniformly in the rectangle $x_0 - \alpha \leq x \leq x_0 + \alpha$ and $\lambda_1 \leq p \leq \lambda_2$ to the solution $\varphi(x, p)$. Since the functions in the sequence are continuous with respect to x and p, it follows from the proposition on the continuity of the limit of a uniformly converging sequence of continuous functions (of any number of variables) that the same holds true for $\varphi(x, p)$, which is what we sought to prove.

As we know, every solution $\varphi(x)$ of equation (3) is, under our assumptions, uniquely determined by the value y_0 which it assumes at $x = x_0$. The function $\varphi(x)$ changes with any change of the numbers x_0 and y_0, so that, in essence, it is a function $\varphi(x, x_0, y_0)$ of three independent variables. For the same reasons which we mentioned earlier, the nature of the dependence of the solution $\varphi(x, x_0, y_0)$ on these *initial* conditions x_0 and y_0 has an essential importance. At first glance, it may appear that this is a new problem, one which cannot be reduced to the problem considered above (the dependence of the solution on parameters), since the numbers x_0 and y_0 clearly do not appear as arguments of the function $f(x, y)$. But such a reduction is actually possible. If in equation (3) we transform the independent variable, as well as the unknown function, with the aid of the relationships

$$x = x_0 + x^* \quad \text{and} \quad y = y_0 + y^*,$$

where x^* is a new independent variable and y^* a new unknown function, equation (3) takes the form

$$\frac{dy^*}{dx^*} = f(x_0 + x^*, y_0 + y^*). \tag{3'''}$$

It will then be necessary to find the solution of the equation which takes the value $y^* = 0$ at $x^* = 0$. Here, x_0 and y_0 explicitly appear as parameters on the right side of the equation. Let

$$y^* = \varphi^*(x^*, x_0, y_0) \tag{14}$$

be the solution of equation (3'''). It is continuous with respect to x_0 and y_0 by virtue of the theorem which we have just proved, since the function $f(x_0 + x^*, y_0 + y^*)$, being continuous with respect to

each of its two arguments, is automatically continuous with respect to x_0 and y_0. But, in order to obtain the required solution of equation (3), we clearly have only to pass from the new variables to the old ones in expression (14), which gives us

$$y = \varphi(x, x_0, y_0) = y_0 + \varphi^*(x - x_0, x_0, y_0),$$

from which it follows directly that the solution is continuous with respect to the initial values x_0 and y_0.

63. CHANGE OF VARIABLES

As you know, one of the most effective methods for simplifying integration problems is the transformation of the variable of integration (the so-called *method of integration by substitution*). This method also constitutes one of our most important devices for solving differential equations, and its flexibility is considerably increased by the fact that both the independent variable and the unknown function can be subjected to transformation. As we saw at the beginning of this lecture, the solution of a differential equation can easily be reduced to quadratures if the variables can be *separated,* that is, if by multiplication and division the equation is reducible to the form $M(y)\,dy + N(x)\,dx = 0$, where each of the two terms contains only one of the variables x and y. The transformation of variables is often useful in this regard. By transforming either the independent variable x or the unknown function y, or both simultaneously, we are frequently able to replace an equation in which the variables could not be separated by a new equation in which it has become possible to separate them. Although the class of differential equations which can be reduced to quadratures by this means is very limited, nevertheless it contains quite a number of the simplest types, which are, by that very token, the most frequently encountered in practice. As a consequence, the method of transformation of variables acquires great practical importance.

In the class of equations to which the method of transformation is applicable there are, first of all, all *linear* first-order differential equations, that is, equations in which the unknown function y, as well as its derivative y', appear in the first power only. The general form of such an equation is

$$y' + f_1(x)y + f_2(x) = 0, \tag{15}$$

where $f_1(x)$ and $f_2(x)$ are given continuous functions of the variable

x. All equations of this type can be reduced by a single method to a form in which the variables are separable.

To see this, let us consider first the equation

$$z' + f_1(x)z = 0, \tag{16}$$

where z denotes the unknown function. This is an equation of the same form as (15), except that it is without a *free term* $f_2(x)$. The variables in this equation can be separated immediately:

$$\frac{dz}{z} + f_1(x)\,dx = 0,$$

and integration gives

$$\ln\left|\frac{z}{z_0}\right| + \int_{x_0}^{x} f_1(u)\,du = 0.$$

For our purposes, it is enough to have any one solution of equation (16). We therefore set $z_0 = 1$, obtaining

$$\ln|z| + \int_{x_0}^{x} f_1(u)\,du = 0,$$

whence,

$$z = e^{-\int_{x_0}^{x} f_1(u)\,du} = \varphi(x). \tag{17}$$

Thus, we find a solution of equation (16) with the aid of one quadrature.

To reduce the general equation (15) to quadratures, we shall transform the unknown function y by the substitution $y = \varphi(x)y^*$, where y^* is a new unknown function and $\varphi(x)$ is the solution of (16) given by formula (17). Equation (15) then takes the form

$$\varphi(x)y^{*\prime} + \varphi'(x)y^* + f_1(x)\varphi(x)y^* + f_2(x) = 0,$$

or

$$\varphi(x)y^{*\prime} + f_2(x) = 0,$$

since, from equation (16), we have

$$\varphi'(x) + f_1(x)\varphi(x) = 0.$$

Hence,

$$y^{*\prime} = -\frac{f_2(x)}{\varphi(x)} = -f_2(x)e^{\int_{x_0}^{x} f_1(u)\,du},$$

and, consequently,

$$y^* = -\int_{x_0}^{x} f_2(v)e^{\int_{x_0}^{v} f_1(u)\,du}\,dv + C,$$

where C is a constant of integration.

Finally,

$$y = \varphi(x)y^* = -e^{-\int_{x_0}^{x} f_1(u)\,du}\left[\int_{x_0}^{x} f_2(v)e^{\int_{x_0}^{v} f_1(u)\,du}\,dv + C\right].$$

This is the general solution of equation (15), from which, by appropriate choice of the constant C, we can obtain all its particular solutions. If, for example, we wish to have $y = y_0$ for $x = x_0$, then, from the general solution, we find that $y_0 = -C$ and the particular solution takes the form

$$y = e^{-\int_{x_0}^{x} f_1(u)\,du}\left[y_0 - \int_{x_0}^{x} f_2(v)e^{\int_{x_0}^{v} f_1(u)\,du}\,dv\right]. \tag{18}$$

We see, then, that with the aid of an appropriately selected transformation of the unknown function, the solution of equation (15) is reduced to two successive quadratures. As an example, we shall carry out completely the solution of the problem which we considered at the beginning of this lecture. At that time we obtained the equation

$$\frac{dy}{dt} + sy - sce^{-st} = 0.$$

Here, $f_1(t) = s$ and $f_2(t) = -sce^{-st}$. Moreover, $t_0 = 0$ and $y_0 = 0$, since, at the moment $t = 0$, the second vessel does not contain any salt. Thus, we have

$$\int_0^t f_1(u)\,du = st,$$

and formula (18) gives

$$y = e^{-st}\left[0 + \int_0^t sce^{-sv}e^{sv}dv\right] = scte^{-st}. \tag{19}$$

This formula completely solves the given problem. Upon examining formula (19), we readily see that the amount of salt in the second container at first increases and then, from the moment $t = \frac{a}{b}$, begins to decrease, tending to zero as $t \to \infty$. At the moment $t = \frac{a}{b}$, the amount of salt in this container is at its maximum $\frac{c}{e}$. Remarkably, this maximum amount does not depend on either

a or b. On the other hand, the time interval required to reach the maximum salt concentration in the second container depends on a and b, but not at all on c.

All these and many other characteristics of the phenomenon under consideration become evident from the study of the function (19). We see that the solution of the differential equation actually permits us to obtain all the necessary information about the course of the process as a whole, while the differential equation itself gives us only instantaneous (local) relations among the quantities taking part in this process.

Another frequently encountered type of equation, which can be reduced by a simple transformation of the unknown function to an equation with separable variables, is the so-called *homogeneous* differential equation of the general form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \tag{20}$$

To this type belong, in particular, the frequently encountered equations of the form

$$P(x, y)\, dy = Q(x, y)\, dx,$$

where $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials of the same degree n. For, after division by x^n, such a homogeneous polynomial becomes a polynomial with respect to the variable $\frac{y}{x}$. Consequently, the numerator and the denominator on the right side of the relation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} = \frac{Q(x, y)/x^n}{P(x, y)/x^n}$$

are polynomials with respect to $\frac{y}{x}$, which means that the whole right side is a rational function of this ratio.

Transforming the unknown function in equation (20) by means of the substitution $y = xy^*$, we can write the equation in the form

$$y^* + xy^{*\prime} = f(y^*),$$

whence,

$$\frac{dy^*}{dx} = \frac{f(y^*) - y^*}{x} \quad \text{or} \quad \frac{dy^*}{f(y^*) - y^*} = \frac{dx}{x}.$$

The variables have now been separated. Integrating, we easily find an expression for x in terms of y^*. If this expression allows us to express y^* as a single-valued function of x, then we also obtain an expression for y in terms of x. In the general case, this relation determines y^*, and, hence, also y, as an *implicit* function of x.

64. SYSTEMS OF EQUATIONS OF HIGHER ORDERS

In conclusion, we shall touch briefly on systems of first-order differential equations and on equations of higher order.

Suppose that we have a system of n first-order differential equations containing n unknown functions $y_1, y_2, \ldots, y_n$ of one independent variable x. Clearly, we may regard as a *solution* of such a system any system of functions $y_i = \varphi_i(x)$ $(1 \leq i \leq n)$ satisfying all the given equations. Let us assume that the given system of equations has been solved for the derivatives $\frac{dy_i}{dx}$ $(1 \leq i \leq n)$, and therefore has the form

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \ldots, y_n) \qquad (1 \leq i \leq n). \tag{21}$$

Here again, it will be convenient to call a set of corresponding values of the variables $x, y_1, y_2, \ldots, y_n$ a *point*. This is, of course, a point in a space of $n + 1$ dimensions.

Throughout our exposition, we shall assume the continuity of all functions f_i in a region D of this space. The geometrical representation of any system of functions $y_i = \varphi_i(x)$ $(1 \leq i \leq n)$ in this space is a curve; and if the functions in question constitute a solution of a system of equations of the form (21), we may call the curve representing them an *integral curve* of this system.

This geometrical terminology is extremely advantageous here, not only because of its inherently descriptive nature, but also (and chiefly) because with its aid many formulations and arguments can be carried out and presented in the same form and terminology as would be used for a single equation.

Thus, we can formulate the fundamental theorem on the existence of solutions quite concisely as follows:

THEOREM 4. *Through every point within the region D there passes at least one integral curve.*

Analytically this means that for any system of $n + 1$ numbers $x_0, y_1^{(0)}, y_2^{(0)}, \ldots, y_n^{(0)}$ within D there exists a system of functions $y_i = \varphi_i(x)$ $(1 \leq i \leq n)$ satisfying the system of equations (21) in a certain interval $x_0 - \alpha \leq x \leq x_0 + \alpha$, as well as satisfying the condition $\varphi_i(x_0) = y_i^{(0)}$. The proof of this theorem is only slightly more complicated than in the case of one equation. It is based on a lemma which is completely analogous to the one we employed earlier. This lemma may either be proved directly or be deduced as a corollary from our previous lemma. The latter method is particularly simple. Here, the elements of the family are not single functions, but systems $\{F_1(x), F_2(x), \ldots, F_n(x)\} = S$ of n functions; and we have to prove that, under the assumptions of boundedness and equicontinuity for the collection of all functions belonging to any system of the given family, we can select a sequence of systems $S_1, S_2, \ldots$, uniformly convergent in the given interval. This means, setting

$$S_k = \{F_{1k}(x), F_{2k}(x), \ldots, F_{nk}(x)\}$$

$(k = 1, 2, \ldots)$, that we shall obtain n sequences of functions

$$F_{ik}(x) \quad (k = 1, 2, \ldots; 1 \leq i \leq n),$$

each of which is uniformly convergent in the given interval.

Proof of the extended lemma. We construct the proof in the following way: first, on the basis of our previous lemma, we can find a sequence of systems S_k such that the sequence $F_{1k}(x)$ converges uniformly in the given interval. From this sequence of systems we can, again by our previous lemma, extract a subsequence in which the sequence of second functions $F_{2k}(x)$ also converges uniformly in the given interval. Repeating this procedure n times, we clearly arrive at a sequence of systems in which all n sequences $F_{ik}(x)$ $(1 \leq i \leq n)$ converge uniformly in the given interval, and thus the proof of our new, extended lemma is completed.

Proof of Theorem 4. This is now carried out in strict analogy with our previous reasoning, and it is most useful to adopt our geometrical approach as a guide. Again, we construct a set of broken lines with segments of arbitrarily small lengths. From this set, using the lemma just proved, we select a sequence converging uniformly to a certain curve, which we can then prove by the same method as before (with minor and self-evident modifications of a

purely technical nature) to be an integral curve of the system of equations (21). This curve passes through the given point $(x_0, y_1^{(0)}, y_2^{(0)}, \ldots, y_n^{(0)})$, since all the broken lines pass through this point.

The uniqueness of the integral curve passing through a given point can be proved, as before, only after imposing certain additional conditions on the functions f_i. The simplest form of these conditions is completely analogous to condition (A) on page 220 and reduces to the requirement that the inequalities

$$|f_i(x, y_1^{(1)}, y_2^{(1)}, \ldots, y_n^{(1)}) - f_i(x, y_1^{(2)}, y_2^{(2)}, \ldots, y_n^{(2)}| \leq k \sum_{i=1}^{n} |y_i^{(1)} - y_i^{(2)}| \qquad (1 \leq i \leq n),$$

where k is some positive constant, be satisfied in the entire region D.

We have no time left to consider equations of higher order systematically. We shall only show that the solution of any equation

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{22}$$

of order n can be reduced to the solution of a system of first-order equations.

Let us consider a system of n first-order equations with unknown functions $y_1, y_2, \ldots, y_n$ of the following form:

$$\begin{aligned} F(x, y_1, y_2, \ldots, y_{n-1}, y_n, y_n') &= 0, \\ y_1' &= y_2, \\ y_2' &= y_3, \\ &\cdots\cdots\cdots, \\ y_{n-1}' &= y_n. \end{aligned} \tag{23}$$

And let us suppose that we have found a solution

$$y_i = \varphi_i(x) \qquad (1 \leq i \leq n)$$

of this system of equations. By equations (23), we shall then have

$$\begin{aligned} \varphi_1'(x) &= \varphi_2(x) = y_2, \\ \varphi_1''(x) &= \varphi_2'(x) = \varphi_3(x) = y_3, \\ \varphi_1'''(x) &= \varphi_2''(x) = \varphi_3'(x) = \varphi_4(x) = y_4, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \varphi_1^{(n-1)}(x) &= \varphi_2^{(n-2)}(x) = \cdots = \varphi_n(x) = y_n, \\ \varphi_1^{(n)}(x) &= y_n'. \end{aligned}$$

Consequently, the first of the equations (23) yields

$$F[x, \varphi_1(x), \varphi_1'(x), \ldots, \varphi_1^{(n)}(x)] = 0,$$

that is, $y = \varphi_1(x)$ is a solution of equation (22).

Conversely, let the function $y = \varphi(x)$ satisfy the equation (22). Setting

$$y_1 = \varphi(x), y_2 = \varphi'(x), \ldots, y_n = \varphi^{(n-1)}(x),$$

we see, immediately, that the system of functions $(y_1, y_2, \ldots, y_n)$ constitutes a solution of the system of equations (23). Thus, the problems of finding all the solutions of equation (22) and all those of the system of equations (23) are completely reducible to each other.

EIGHT LECTURES ON MATHEMATICAL ANALYSIS is a translation and adaptation of a book by the outstanding Russian mathematician A. Ya. Khinchin. It is based on a series of lectures delivered at the University of Moscow by Professor Khinchin to improve the mathematical qualifications of engineers.

In this book, the reader will find a masterful outline of the fundamental ideas of mathematical analysis. Inessential details have purposely been omitted, and the resulting exposition is clear and easy to follow. The book should be accessible to anyone who has had even a sketchy introduction to the material. And yet, because it is a concise, lucid exposition of the most important concepts of mathematical analysis, the book should be of value to the student enrolled in a university course in analysis.

A. YA. KHINCHIN, until his death in 1959, was a professor at Moscow State University, a corresponding member of the Academy of Sciences, and a member of the Academy of Pedagogical Sciences of the RSFSR. The author of more than one hundred fifty mathematical research papers and books, he will be remembered as a world-renowned authority in mathematical analysis, probability theory, number theory, and mathematical statistics.