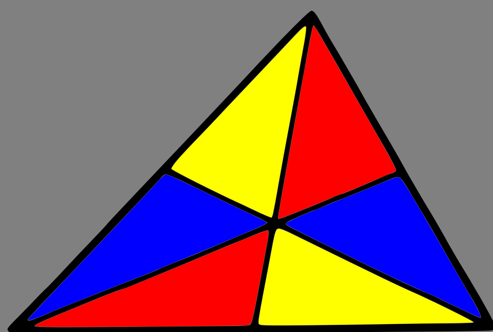
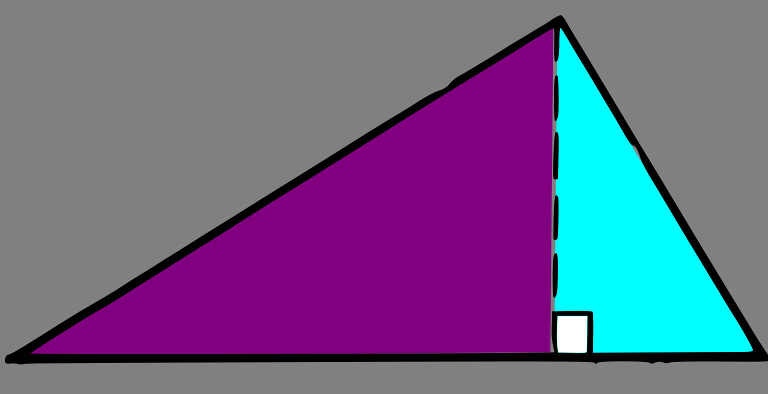
A.G. Tsypkin, A.I.Pinsky

Methods of Solving Problems in High-School Mathematics





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А. Г. ЦЫПКИН, А. И. ПИНСКИЙ

СПРАВОЧНОЕ ПОСОБИЕ ПО МЕТОДАМ РЕШЕНИЯ ЗАДАЧ ПО МАТЕМАТИКЕ

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Methods of Solving Problems in High-School Mathematics

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Contents

From the Editor

In this educational aid, intended for high-school students, an attempt has been made to classify the problems encountered in high-school mathematics by their solution methods.

It was rather difficult to attain the aim the authors set for themselves. On one hand, a detailed classification of problems by methods of solution would require the consideration of a large number of concrete problems and, on the other hand, a schematic classification would not yield a useful aid for solving different kinds of problems. Therefore, alongside a large number of worked problems, the book includes many problems (about 2500) for the reader to solve.

In addition to the traditional problems from the course of high-school mathematics, the book includes methods for solving simple differential and integral calculus problems as well as problems which require the use of coordinates and vector algebra. These sections only include problems whose solutions require knowledge that is beyond the scope of high-school mathematics.

Some problems in the book can only be solved by a combined application of the knowledge from the traditional and new divisions of mathematics. These include, for instance, problems connected with the calculation of limits, derivatives and antiderivatives of functions which must first be simplified by means of identity transformations.

The authors consider all the most frequent methods of solving problems from the high-school course of mathematics. The fact that many problems are not followed by their solutions makes it possible to use the book for preparing for the entrance examinations to higher educational establishments.

V. I. Blagodatskikh

From the Authors

Our aim when writing this book was to help students to systematize their knowledge of problem solving. The structure of the book follows from this aim, hence each section begins with some theory (definitions, principal theorems and formulas) which must be known in order to cope with the subsequent problems without resorting to textbooks. Next the method of solving a specific kind of problem is indicated, followed by an example showing how to use the method. The remaining problems are left for the reader to solve.

We feel that this form of presentation is the best way of helping the student actively master the methods of problem solving. In a number of cases, the solution presented is not the briefest or most elegant. This is due, first of all, to our wish to give the most visual application to the method being suggested, rather than to demonstrate non-standard approaches.

We arranged the problems not followed by answers or hints in order of increasing difficulty, fully realizing that every reader may want to change the sequence of the problems according to his knowledge or inclination. Plane geometry and solid geometry, the traditional divisions of school mathematics, are, in the main, represented by problems on calculations.

The material on solid geometry was not structured strictly in the way we explained above since, as distinct from problems in plane geometry, for which the solution methods can be neatly classified, any non-trivial problem in solid geometry may require the use of several different methods. Thus the problems considered in Chapter 12 are followed by quite detailed solutions, in which the techniques which reduce the original problem to simpler ones are emphasized.

Chapters 6-9 contain problems in mathematical analysis. Many of these should be solved using traditional high-school mathematics. Chapter 13 includes some very difficult problems in geometry whose solution can be considerably simplified with the use of vectors and the method of coordinates.

Since the theory of relativity and the related divisions of mathematics have drawn considerable attention lately, we deemed it necessary to include combinatorics and elements of relativity theory. We have taken into account that this material is absolutely new to the majority of readers and so the theory part of the chapter is somewhat larger than those in the other chapters.

A double numeration system was adopted for both the problems and examples. The first digit indicates the number of the section and the other digit, the number of the problem or example within the section. An asterisk indicates a difficult problem and two asterisks mean that the problem is followed by a complete solution (given in "Answers and Hints" at the end of the book).

In conclusion, we wish to express our gratitude to all those who helped us to improve the structure and content of the book by their advice and remarks, especially to N. V. Reveruk who took part in writing Chapter 13 and who prepared some of the problems for it. Our thanks also go to K. K. Andreev who thoroughly studied the chapters on mathematical analysis and made valuable remarks. The authors are especially grateful to V. I. Blagodatskikh whose cordial attention was felt at every stage of writing this book.

Chapter 1

Algebraic Equations and Systems of Equations

An *identity* is an equation satisfied for all possible values of the variables involved. (The numerical values of the variables are said to be permissible if all the operations on the variables appearing in the equation can be performed.)

An *equation* is an equality which can be satisfied only for certain values of the variables entering into it. By the hypothesis, the variables in the equation can be nonequivalent: some of them can assume all their permissible values and are called *coefficients* (less frequently, *parameters*) *of the equation*; other variables whose values must be found are called *unknowns* (they are usually denoted by the last letters of the alphabet, x, y, z, or by those variables with indices, $x_1, x_2, \ldots, x_n$ or $y_1, y_2, \ldots, y_k$)*.

In a general form, an equation in n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$F(x_1, x_2, \ldots, x_n) = 0, \tag{1}$$

where $F(x_1, x_2, \ldots, x_n)$ is a function of the indicated variables. According to the number of the unknowns an equation is said to be that in one unknown, in two unknowns, and so on.

The values of the unknowns which turn an equation into an identity are called *solutions* or *roots of the equation*.

The domain of definition of the function $F(x_1, x_2, \ldots, x_n)$ is the *domain* (set) *of the permissible values* of the unknowns entering into equation (1).

If all the roots of the equation $F = 0$ are roots of the equation $G = 0$, then the equation $G = 0$ is said to be a consequence of the equation $F = 0$, and the notation is

$$F=0 \Rightarrow G=0.$$

Two equations $F = 0$ and $G = 0$ are said to be *equivalent* if each of them is a consequence of the other one, and then the notation is

$$F=0 \Leftrightarrow G=0.$$

Thus two equations are equivalent if the sets of their roots coincide.

The equation $F = 0$ is considered to be equivalent to two (or several) equations $F_1 = 0$, $F_2 = 0$ if the set of the roots of equation

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* Unless otherwise specified, the unknowns are considered to assume real values.

$F = 0$ coincides with the union of the sets of the roots of the equations $F_1 = 0$ and $F_2 = 0$.

Some equivalent equations.

(1) The equation $F + G = G$ is equivalent to the equation $F = 0$ considered on the set of permissible values of the initial equation.

(2) The equation $F/G = 0$ is equivalent to the equation $F = 0$ considered on the set of permissible values of the initial equation.

(3) The equation $F \cdot G = 0$ is equivalent to two equations, $F = 0$ and $G = 0$, each of which is considered on the set of permissible values of the initial equation.

(4) The equation $F^n = 0$ is equivalent to the equation $F = 0$.

(5) The equation $F^n = G^n$ is equivalent to the equation $F = G$ for odd n and to two equations, $F = G$ and $F = -G$ for even n.

An *algebraic equation* is an equation which can be reduced to the form

$$P_n = 0,$$

where P_n is a polynomial of degree n of one or several variables. The number n is a *degree* of the equation.

1. Rational Equations in One Unknown

An equation of the form

$$ax + b = 0 \tag{2}$$

is called a *linear* equation. A linear equation has a single root $x = -b/a$. An equation of the form

$$ax^2 + bx + c = 0 \tag{3}$$

is called a *quadratic* equation. The expression $b^2 - 4ac = D$ is a *discriminant* of the quadratic equation. In the case when $D > 0$, equation (3) has two real roots:

$$x_1 = \frac{-b + \sqrt{D}}{2a}, \qquad x_2 = \frac{-b - \sqrt{D}}{2a}.$$

If $D = 0$, equation (3) has one real root of multiplicity 2: $x = -\frac{b}{2a}$. If $D < 0$, then equation (3) has no real roots.

A solution of many rational equations consists in reducing them, by some technique, to equations of form (2) or (3). An introduction of an auxiliary unknown is one of those techniques.

Example 1.1. Solve the equation

$$\frac{1}{x(x+2)} - \frac{1}{(x+1)^2} = \frac{1}{12}.$$

Solution. Designating $z = x^2 + 2x$, we write the initial equation in the form

$$\frac{1}{z} - \frac{1}{z+1} = \frac{1}{12}. \tag{*}$$

Simple transformations reduce equation (*) to the equation

$$\frac{z^2+z-12}{12z(z+1)}=0, \qquad (**)$$

which is equivalent to the equation $z^2 + z - 12 = 0$. The equivalence of these equations follows from the fact that the roots $z = -4$ and $z = 3$ of the last equation belong to the set of permissible values of equation (**). Thus the initial equation is equivalent to two quadratic equations, $x^2 + 2x - 3 = 0$ and $x^2 + 2x + 4 = 0$. The roots of the first equation are $x_1 = 1$ and $x_2 = -3$. The second equation has no real roots.

Answer. $x = 1$, $x = -3$.

Solve the following equations.

1.1. $\frac{21}{x^2-4x+10}-x^2+4x=6.$ 1.2. $\frac{4}{x^2+4}+\frac{5}{x^2+5}=2.$

1.3. $(x^2-6x)^2-2(x-3)^2=81.$

1.4. $\frac{24}{x^2+2x-8}-\frac{15}{x^2+2x-3}=2.$

1.5. $7\left(x+\frac{1}{x}\right)-2\left(x^2+\frac{1}{x^2}\right)=9.$

1.6. $\frac{x^2+2x+1}{x^2+2x+2}+\frac{x^2+2x+2}{x^2+2x+3}=\frac{7}{6}.$

1.7. $20\left(\frac{x-2}{x+1}\right)^2-5\left(\frac{x+2}{x-1}\right)^2+48\frac{x^2-4}{x^2-1}=0.$

1.8. $(x^2 + 2x)^2 - (x + 1)^2 = 55.$

1.9. $(x^2 + x + 1)(x^2 + x + 2) - 12 = 0.$

1.10*. $(x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 0.$

1.11*. $(x - 2)(x + 1)(x + 4)(x + 7) = 19.$

1.12*. $(2x^2 + 3x - 2)(5 - 6x - 4x^2) = -5(2x^2 + 3x + 2).$

1.13. $x^4 - 13x^2 + 36 = 0.$ 1.14. $2x^8 + x^4 - 15 = 0.$

1.15. $(2x - 1)^6 + 3(2x - 1)^3 = 10.$ 1.16*. $(1 + x)^8 + (1 + x^2)^4 = 2x^4.$

1.17. $(x - 2)^6 - 19(x - 2)^3 = 216.$

One of the ways to solve an equation of a degree exceeding two is to factor the polynomial appearing on the left-hand side of the equation and thus reduce the solution of the initial equation to the solution of several equations of lower degrees. This method is based on the following *property of the roots of an nth-degree polynomial.* If the number c is a root of the polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n,$$

then the polynomial can be written in the form

$$P(x) = (x - c)\,Q(x), \qquad (4)$$

where $Q(x)$ is a polynomial of degree $n - 1$ (i.e. the polynomial $P(x)$ can be divided by the polynomial $x - c$).

Factoring a polynomial is equivalent to finding its roots. Finding the roots of a polynomial is a difficult problem and in a general case it is impossible to indicate a universal method of seeking the roots of an nth-degree polynomial with real coefficients. There is a theorem, however, which makes it possible to find rational roots of an nth-degree polynomial with integral coefficients.

The *rational roots* of the polynomial

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n,$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are integers, can only be numbers m/p (m is integral and p is natural), the number $|m|$ being the divisor of the number $|a_n|$ and the number p, the divisor of the number $|a_0|$.

Example 1.2. Find the roots of the equation

$$4x^4 + 8x^3 - 3x^2 - 7x + 3 = 0.$$

Solution. The numbers 1, 3 are the divisors of the number 3, and 1, 2, 4 are the divisors of the number 4. The set $\{1, -1, 3, -3\}$ is the set of values of m, and the set $\{1, 2, 4\}$ is the set of values of p. The set $\{\pm 1, \pm 3, \pm 1/2, \pm 1/4, \pm 3/2, \pm 3/4\}$ is the set of various distinct rational numbers of the form m/p. Substituting these numbers into the equation, we find that the numbers $x_1 = 1/2$ and $x_2 = -3/2$ are its roots. According to (4), this means that the given polynomial can be divided by the linear polynomials $(x - 1/2)$ and $(x + 3/2)$ and, consequently, it can also be divided by their product

$$\left(x-\frac{1}{2}\right)\left(x+\frac{3}{2}\right)=x^2+x-\frac{3}{4}.$$

Performing a division, we find the polynomial of the quotient: $4x^2 + 4x - 4$. Solving the quadratic equation

$$4x^2 + 4x - 4 = 0,$$

we get two real roots: $x_3=\dfrac{-1+\sqrt{5}}{2}$ and $x_4=\dfrac{-1-\sqrt{5}}{2}$.

We have thus completed the solution of the problem, i.e. have found the four roots of the initial equation:

$$x_1=\frac{1}{2}, \quad x_2=-\frac{3}{2}, \quad x_3=\frac{-1+\sqrt{5}}{2}, \quad x_4=\frac{-1-\sqrt{5}}{2}.$$

Answer. $\left\{\dfrac{1}{2}, -\dfrac{3}{2}, \dfrac{-1+\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}\right\}$.

Solve the following equations.

1.18. $8x^4 + 6x^3 - 13x^2 - x + 3 = 0.$

1.19. $x^3 + 6x + 4x^2 + 3 = 0.$

1.20. $2x^4 - x^3 - 9x^2 + 13x - 5 = 0.$

1.21*. $(x - 1)^3 + (2x + 3)^3 = 27x^3 + 8.$

1.22. $x^3 - (2a + 1)x^2 + (a^2 + a)x - (a^2 - a) = 0.$

1.23. $x^4 - 4x^3 - 19x^2 + 106x - 120 = 0.$

Some equations of a special form. Equations of the fourth degree whose left-hand side is a product of quadratic trinomials, which differ by a constant term, and the right-hand side is a number, can be reduced to quadratic equations by introducing an auxiliary unknown which is equal to the common part of the two factors.

Example 1.3. Solve the equation

$$x(x+1)(x+2)(x+3) = 0.5625 \qquad (*)$$

Solution. Multiplying separately $x(x+3)$ and $(x+1)(x+2)$, we obtain

$$(x^2+3x)(x^2+3x+2) = 0.5625$$

Introducing an auxiliary unknown $y = x^2 + 3x$, we get, after simple transformations, a quadratic equation

$$y^2 + 2y - 0.5625 = 0,$$

whose roots are $y_1 = 0.25$ and $y_2 = -2.25$. Returning to the initial unknown, we infer that (*) is equivalent to two equations

$$x^2 + 3x - 0.25 = 0 \text{ and } x^2 + 3x + 2.25 = 0.$$

The first equation has two different roots, $x = \frac{-3+\sqrt{10}}{2}$ and $x = \frac{-3-\sqrt{10}}{2}$, and the second equation has one root of multiplicity 2, $x = -\frac{3}{2}$.

Answer. $x = \frac{-3+\sqrt{10}}{2}$, $x = \frac{-3-\sqrt{10}}{2}$, $x = -\frac{3}{2}$.

Find the roots of the following equations.

1.24. $(x+a)(x+2a)(x-3a)(x-4a) = b^4$.
1.25. $(x-4)(x-5)(x-6)(x-7) = 1680$.
1.26. $(6x+5)^2(3x+2)(x+1) = 35$.
1.27. $x^4 - 2x^3 + x - 132 = 0$.
1.28. $(x-1)(x+1)(x+2)x = 24$.
1.29. $(x-4)(x+2)(x+8)(x+14) = 354$.
1.30*. $(x^2+x+1)(2x^2+2x+3) = 3(1-x-x^2)$.

The equation of degree n

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0 \qquad (5)$$

is said to be *symmetric* if $a_k = a_{n-k}$ for all $k = 0, \ldots, n$. If $n = 2l$, then we can divide both parts of equation (5) by x^l and pass to an equivalent equation

$$a_0x^l + a_1x^{l-1} + \ldots + a_{n-1}\frac{1}{x^{l-1}} + a_n\frac{1}{x^l} = 0,$$

and equation (5) can be reduced to an equation of degree l by introducing a new unknown $z = x + \frac{1}{x}$. If $n = 2l + 1$, then a direct verification shows that $x = -1$ is a root of the equation. The division by $(x + 1)$ reduces equation (5) to a symmetric equation of degree $n = 2l$.

Equation (5), where $a_k = (-1)^k a_{n-k}$ is called *skew-symmetric*. The arguments presented above are applicable to it with due account of the following changes: for $n = 2l$, it is necessary to introduce $z = x - 1/x$ as a new unknown, and for $n = 2l + 1$, $x = 1$ is one of the roots of the equation.

Example 1.4. Solve the equation

$$x^7 + 2x^6 - 5x^5 - 13x^4 - 13x^3 - 5x^2 + 2x + 1 = 0. \qquad (*)$$

Solution. The given equation is symmetric and $n = 7$ and $x = -1$ are its roots. Consequently, equation (*) can be represented as

$$(x + 1)(x^6 + x^5 - 6x^4 - 7x^3 - 6x^2 + x + 1) = 0,$$

and its solution reduces to the solution of an equation of an even degree

$$x^6 + x^5 - 6x^4 - 7x^3 - 6x^2 + x + 1 = 0.$$

Dividing both sides of the equation by x^3, we get

$$\left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right) - 6\left(x + \frac{1}{x}\right) - 7 = 0.$$

We introduce the designation $z = x + 1/x$ and, taking into account that

$$x^2 + \frac{1}{x^2} = z^2 - 2, \quad x^3 + \frac{1}{x^3} = z^3 - 3z,$$

we get an equation $z^3 + z^2 - 9z - 9 = 0$ which is equivalent to the equation

$$(z + 1)(z^2 - 9) = 0.$$

Consequently, the solution of the initial equation reduces to the solution of the following three equations:

$$x + \frac{1}{x} = -1, \quad x + \frac{1}{x} = 3, \quad x + \frac{1}{x} = -3,$$

which are respectively equivalent to the quadratic equations

$$x^2 + x + 1 = 0, \; x^2 - 3x + 1 = 0, \; x^2 + 3x + 1 = 0.$$

The first equation has no real roots, and the roots of the second and third equations can be calculated by the formula for the roots of a buadratic equation.

Answer. $x = -1$, $x = \frac{3 + \sqrt{5}}{2}$, $x = \frac{3 - \sqrt{5}}{2}$, $x = \frac{-3 + \sqrt{5}}{2}$, $x = \frac{-3 - \sqrt{5}}{2}$.

Solve the following equations.

1.31. $x^4 + 5x^3 + 2x^2 + 5x + 1 = 0.$
1.32. $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0.$
1.33. $15x^5 + 34x^4 + 15x^3 - 15x^2 - 34x - 15 = 0.$
1.34. $6x^3 - x^2 - 20x + 12 = 0.$
1.35. $x^4 + 1 = 2(1 + x)^4.$

Hint. To solve equations 1.34 and 1.35, it is first necessary to make a change $y = ax + b$. As a result of a requisite choice of the numbers a and b, the equations become symmetric (or skew-symmetric) with respect to the unknown y.

A rational algebraic equation of the form

$$\frac{P(x)}{Q(x)} = 0 \qquad (6)$$

is equivalent to the equation $P(x) = 0$ which can be solved on the set of the permissible values of equation (6).

Example 1.5. Solve the equation

$$\frac{9-x}{x-4} = \frac{5}{x-4} - 3.$$

Solution. The initial equation is equivalent to the equation

$$9 - x - 5 + 3(x - 4) = 0$$

provided that $x - 4 \neq 0$. Solving the equation obtained, we find that $x = 4$. But since $x = 4$ does not belong to the set of permissible values of the unknown, the given equation has no solutions.

Find the roots of the following equations.

1.36. $\dfrac{12x+1}{6x-2} - \dfrac{9x-5}{3x+1} = \dfrac{108x-36x^2-9}{4(9x^2-1)}.$

1.37. $\dfrac{1}{2x+3} - \dfrac{1}{x^2-16} + \dfrac{1}{2x^2+11x+12} - \dfrac{x-8}{2x^3+3x^2-32x-48} = 0.$

1.38. $\dfrac{x^2+x+1}{x^2-x+1} = \dfrac{7}{9}\,\dfrac{x+1}{x-1}.$

1.39. $\dfrac{x+1}{2(x-1)} = \dfrac{9}{2(x+4)} + \dfrac{1}{x-1}.$

1.40. $\dfrac{x^2}{x^2-4} + \dfrac{x+1}{2(x-2)} = \dfrac{1}{2-x} - \dfrac{1}{x+2}.$

1.41. $\dfrac{4x^2+29x+45-(x+1)(2x+15)}{(2(x-1))^2-2(x+1)(x-2)} = \dfrac{(x+1)(x+5)}{(x-1)(x-2)}.$

1.42.* $\dfrac{(a-x)^4+(x-b)^4}{(a+b-2x)^2} = \dfrac{a^4+b^4}{(a+b)^2}.$

1.43. $\dfrac{1}{x^2-2x+2} + \dfrac{2}{x^2-2x+3} = \dfrac{6}{x^2-2x+4}.$

Equations with an unknown under the sign of an absolute value. If certain expressions containing an unknown are under the sign of an absolute value, then, to remove the signs of an absolute value, it is necessary to consider the initial equation separately on each interval of the constancy of sign of those expressions.

Example 1.6. Solve the equation

$$|2x - 5| = x - 1.$$

Solution. The expression $2x - 5$ under the sign of an absolute value is nonnegative for $x \geqslant 5/2$ and negative for $x < 5/2$. Let us consider the initial equation separately on each of the intervals. Assume $2x - 5 \geqslant 0$, i.e. $x \geqslant 5/2$. Then, by the definition of an absolute value, $|2x - 5| = 2x - 5$, and the given equation reduces to the form

$$2x - 5 = x - 1.$$

Solving this equation, we find that $x = 4$. Since the number 4 satisfies the assumption we have made ($2 \times 4 - 5 \geqslant 0$), $x = 4$ is a solution of the initial equation.

Let us now assume that $2x - 5 < 0$. Then, by the definition of an absolute value, $|2x - 5| = -(2x - 5)$, and the given equation assumes the form

$$-(2x - 5) = x - 1.$$

Solving this equation, we find that $x = 2$. Since the number 2 satisfies the assumption ($2 \times 2 - 5 < 0$), $x = 2$ is a solution of the initial equation.

Answer. $x = 2$, $x = 4$.

Example 1.7. Solve the equation

$$|x - 1| - 2|x - 2| + 3|x - 3| = 4.$$

Solution. The given equation is equivalent to the following equations:

(1) $1 - x + 2(x - 2) - 3(x - 3) = 4$ for $x \leqslant 1$;
(2) $x - 1 + 2(x - 2) - 3(x - 3) = 4$ for $1 < x \leqslant 2$;
(3) $x - 1 - 2(x - 2) - 3(x - 3) = 4$ for $2 < x \leqslant 3$;
(4) $x - 1 - 2(x - 2) + 3(x - 3) = 4$ for $x > 3$.

The first equation has a solution $x = 1$; the second equation turns into an identity for all values of x which satisfy the inequality $1 < x \leqslant 2$, the third equation has no solutions; the fourth equation has a solution $x = 5$.

Answer. $x \in [1, 2]$, $x = 5$.

Solve the following equations.

1.44. $|x| = x + 2$. **1.45.** $|-x + 2| = 2x + 1$.

1.46. $|x - 1| + |x - 2| = 1$. **1.47.** $|x - 1| + |x + 2| - |x - 3| = 4$.

1.48. $|2 - |1 - |x||| = 1$. **1.49.** $\left|\frac{x+1}{x-1}\right| = 1$.

1.50. $|5x - x^2 - 6| = x^2 - 5x + 6$. **1.51.** $|x^2 - 1| = -|x| + 1$.

1.52. $\left|\frac{1}{2}x^2 - 2x + \frac{3}{2}\right| + \left|\frac{1}{2}x^2 - 3x + 4\right| = \frac{3}{4}$.

2. Irrational Equations

The *irrational equation* is an equation in which the unknown quantity is under the radical sign. The domain of permissible values of an irrational equation consists of the values of the unknown for which all the expressions under the radical signs of an even degree are nonnegative.

One of the ways of solving an irrational equation is to raise both sides of the equation successively to a power which is the least common multiple of the exponents of all the radicals entering into the given equation. If the power to which the equation is raised is even, the resulting consequence of the initial equation can have extraneous roots. In that case the roots must be verified.

Example 2.1. Solve the equation

$$\sqrt{3x+4}+\sqrt{x-4}=2\sqrt{x}. \qquad (*)$$

Solution. We square both sides of the equation:

$$3x+4+2\sqrt{(3x+4)(x-4)}+x-4=4x. \qquad (**)$$

Collecting terms, we get an equation

$$2\sqrt{(3x+4)(x-4)}=0,$$

whose roots are $x = -4/3$ and $x = 4$. One of the roots obtained, namely, $x = -4/3$, does not satisfy the initial equation since it does not belong to the set of its permissible values. Verification shows that for $x = 4$ the initial equation turns into an identity.

Answer. $x = 4$.

Solve the following equations.

2.1. $\sqrt{x+1}=8-\sqrt{3x+1}$.

2.2. $\sqrt{x+\sqrt{x+11}}+\sqrt{x-\sqrt{x+11}}=4$.

2.3. $\sqrt{17+x}-\sqrt{17-x}=2$. 2.4. $\sqrt{3x+7}-\sqrt{x+1}=2$.

2.5. $\sqrt{25-x}=2-\sqrt{9+x}$. 2.6. $\sqrt{x^2+1}+\sqrt{x^2-2x+3}=3$.

2.7. $\sqrt{x^2+x-5}+\sqrt{x^2+8x-4}=5$.

2.8. $\sqrt{x^2+x+1}=\sqrt{x^2-x+1}+1$.

2.9. $(x^2-4)\sqrt{x+1}=0$. 2.10. $\sqrt{4x-3}+\sqrt{5x+1}=\sqrt{15x+4}$.

2.11. $\sqrt{x+5}+\sqrt{x+3}=\sqrt{2x+7}$. 2.12. $\sqrt{4-x}+\sqrt{5+x}=3$.

2.13. $\sqrt{4x+2}+\sqrt{4x-2}=4$.

2.14. $\sqrt{x-\sqrt{x-2}}+\sqrt{x+\sqrt{x-2}}=2$.

2.15. $\sqrt{x+7}-x+3=0$. 2.16. $\sqrt[3]{x+34}-\sqrt[3]{x-3}=1$.

2.17. $\sqrt{2x+5}-\sqrt{3x-5}=2$. 2.18. $\sqrt[3]{x}+\sqrt[3]{x-16}=\sqrt[3]{x-8}$.

2.19. $\sqrt[3]{x+5}+\sqrt[3]{x+6}=\sqrt[3]{2x+11}$.

2.20. $\sqrt[3]{x+1}+\sqrt[3]{3x+1}=\sqrt[3]{x-1}$.

2.21. $\sqrt[3]{x+1}+\sqrt[3]{x+2}+\sqrt[3]{x+3}=0.$

2.22. $\sqrt[3]{1+\sqrt{x}}+\sqrt[3]{1-\sqrt{x}}=2.$

Some special methods of solving irrational equations. In some cases we can rationalize an equation by multiplying its both sides by a certain expression which does not assume the value zero.

Example 2.2. Solve the equation

$$\sqrt{3x^2+5x+8}-\sqrt{3x^2+5x+1}=1. \qquad (*)$$

Solution. Let us multiply both sides of the equation by the expression $\sqrt{3x^2+5x+8}+\sqrt{3x^2+5x+1}$, conjugate with respect to the left-hand side of equation (*). After collecting terms, we get an equation

$$7=\sqrt{3x^2+5x+8}+\sqrt{3x^2+5x+1}, \qquad (**)$$

which is equivalent to the initial equation since the equation

$$\sqrt{3x^2+5x+8}+\sqrt{3x^2+5x+1}=0$$

has no real roots. Summing up equations (*) and (**), we get

$$\sqrt{3x^2+5x+8}=4.$$

Squaring the last equation, we get a quadratic equation

$$3x^2+5x-8=0,$$

whose roots are $x=-8/3$ and $x=1$. Verification shows that these are the roots of the initial equation.

Answer. $x=1$, $x=-8/3$.

Solve the following equations.

2.23. $\sqrt{3x^2-2x+15}+\sqrt{3x^2-2x+8}=7.$

2.24. $\sqrt{x^2+9}-\sqrt{x^2-7}=2.$ 2.25. $\sqrt{15-x}+\sqrt{3-x}=6.$

2.26. $\sqrt{Ax^2+Bx+C}+\sqrt{Ax^2+Bx+C_1}=p.$

2.27. $\dfrac{\sqrt{21+x}+\sqrt{21-x}}{\sqrt{21+x}-\sqrt{21-x}}=\dfrac{21}{x}, \quad x\neq 0.$

In some cases, an introduction of auxiliary unknowns makes it possible to pass from an irrational equation to a system of rational equations which is its consequence.

Example 2.3. Solve the equation

$$x^2-4x-6=\sqrt{2x^2-8x+12}.$$

Solution. Introducing the designation $\sqrt{2x^2-8x+12}=y$, we get the following system of equations:

$$y^2 = 2x^2 - 8x + 12, \qquad (*)$$
$$y = x^2 - 4x - 6.$$

Eliminating the unknown x in system $(*)$, we get an equation

$$y^2 - 2y - 24 = 0.$$

The roots of this equation are $y_1 = 6$ and $y_2 = -4$. Since y denotes an arithmetic root, we choose a positive root from the two roots of the equation we have found. Substituting it into the second equation of system $(*)$, we get for x an equation

$$x^2 - 4x - 12 = 0,$$

whose roots are $x_1 = -2$, $x_2 = 6$. Verification shows that these are the roots of the initial equation.

Answer. $x = 6$, $x = -2$.

Solve the following equations.

2.28. $\sqrt[5]{(7x-3)^3}+8\sqrt[5]{(3-7x)^{-3}}=7$.

2.29. $\sqrt{x-2}+\sqrt{4-x}=x^2-6x+11$.

2.30. $\sqrt[4]{47-2x}+\sqrt[4]{35+2x}=4$.

2.31. $(x+4)(x+1)-3\sqrt{x^2+5x+2}=6$.

2.32. $\sqrt[5]{\dfrac{16z}{z-1}}+\sqrt[5]{\dfrac{z-1}{16z}}=2.5$.

2.33. $\sqrt{x^2+32}-2\sqrt[4]{x^2+32}=3$.

2.34.* $\dfrac{\sqrt{x+4}+\sqrt{x-4}}{2}=x+\sqrt{x^2-16}-6$.

2.35.* $\sqrt[7]{\dfrac{5-x}{x+3}}+\sqrt[7]{\dfrac{x+3}{5-x}}=2$.

2.36. $\sqrt{x\sqrt[5]{x}}-\sqrt[5]{x\sqrt{x}}=56$.

2.37. $\dfrac{(5-x)\sqrt{5-x}-(x-3)\sqrt{x-3}}{\sqrt{5-x}+\sqrt{x-3}}=2$.

2.38. $x\sqrt[3]{x}-4\sqrt[3]{x^2}+4=0$.

2.39. $x^2+3x-18+4\sqrt{x^2+3x-6}=0$.

2.40. $\sqrt{3y^2+6y+16}+\sqrt{y^2+2y}=2\sqrt{y^2+2y+4}$.

2.41.* $\sqrt{\dfrac{\sqrt{x^2+66^2}+x}{x}}-\sqrt{x\sqrt{x^2+66^2}-x^2}=5$.

2.42. $\dfrac{3(x-2)+4\sqrt{2x^2-3x+1}}{2(x^2-1)}=1$.

2.43. $\sqrt{x-2+\sqrt{2x-5}}+\sqrt{x+2+3\sqrt{2x-5}}=7\sqrt{2}$.

2.44.* $(x-3)^2+3x-22=\sqrt{x^2-3x+7}$.

2.45. $\dfrac{3+x}{3x}=\sqrt{\dfrac{1}{9}+\dfrac{1}{x}\sqrt{\dfrac{4}{9}+\dfrac{2}{x^2}}}$.

Example 2.4. Solve the equation

$$\sqrt[3]{x+1}=\sqrt{x-3}.$$

Solution. We introduce the designation $\sqrt[3]{x+1}=u$, $\sqrt{x-3}=v$. Eliminating x in the equations $u^3=x+1$ and $v^2=x-3$, we arrive at a system of equations

$$u=v,$$
$$u^3-v^2=4.$$

Its solution reduces to that of the equation

$$v^3-v^2-4=0,$$

whose only real root is $v=2$. Returning to the initial unknown, we get a linear equation $4=x-3$, whose root is the only root of the initial equation.

Answer. $x=7$.

Solve the following equations.

2.46. $\sqrt[3]{5x+7}-\sqrt[3]{5x-12}=1$.

2.47. $\sqrt[3]{9-\sqrt{x+1}}+\sqrt[3]{7+\sqrt{x+1}}=4$.

2.48. $\sqrt[3]{24+\sqrt{x}}-\sqrt[3]{5+\sqrt{x}}=1$.

Equations 2.49-2.56 can be solved by the method of isolating the perfect square in the radicands which makes it possible to simplify the process of solution.

Example 2.5. Solve the equation

$$\sqrt{x-1+2\sqrt{x-2}}-\sqrt{x-1-2\sqrt{x-2}}=1.$$

Solution. We designate $\sqrt{x-2}=t$; then the initial equation assumes the form

$$\sqrt{t^2+2t+1}-\sqrt{t^2-2t+1}=1. \qquad (*)$$

Since the radicands on the left-hand side of equation (*) are perfect squares, it can be represented in the following equivalent form:

$$|t+1|-|t-1|=1.$$

The only root of this equation is $t = 0.5$. Returning to the unitial unknown, we get an equation

$$\sqrt{x-2} = 0.5,$$

whose root is $x = 2.25$.

Answer. $x = 2.25$.

Solve the following equations.

2.49. $\sqrt{x+2+2\sqrt{x+1}}+\sqrt{x+2-2\sqrt{x+1}}=2.$

2.50. $\sqrt{x+5-4\sqrt{x+1}}+\sqrt{x+2-2\sqrt{x+1}}=1.$

2.51. $\sqrt{x+8+2\sqrt{x+7}}+\sqrt{x+1-\sqrt{x+7}}=4.$

2.52. $\sqrt{x^2+2\sqrt{x^2-1}}-\sqrt{x^2-2\sqrt{x^2-1}}=1.$

2.53. $\sqrt{x+2\sqrt{x-1}}-\sqrt{x-2\sqrt{x-1}}=3.$

2.54. $\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}=x-1.$

2.55. $\sqrt{2x-2\sqrt{2x-1}}-2\sqrt{2x+3-4\sqrt{2x-1}}$
$+3\sqrt{2x+8-6\sqrt{2x-1}}=4.$

2.56. $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1.$

Applying the methods presented above, solve the following equations.

2.57. $\dfrac{1}{x-\sqrt{x^2-x}}-\dfrac{1}{x+\sqrt{x^2-x}}=\sqrt{3}.$

2.58. $\sqrt{12-\dfrac{12}{x^2}}+\sqrt{x^2-\dfrac{12}{x^2}}=x^2.$

2.59. $2\sqrt{5\sqrt[4]{x+1}+4}-\sqrt{2\sqrt[4]{x+1}-1}=\sqrt{20\sqrt[4]{x+1}+5}.$

2.60. $\sqrt{2x^2-9x+4}+3\sqrt{2x-1}=\sqrt{2x^2+21x-11}.$

2.61. $\sqrt{4x^2+9x+5}-\sqrt{2x^2+x-1}=\sqrt{x^2-1}.$

2.62*. $\sqrt[3]{4-4x+x^2}+\sqrt[3]{49+14x+x^2}=3+\sqrt[3]{14-5x-x^2}.$

2.63*. $\sqrt{2x^2+8x+6}+\sqrt{x^2-1}=2x+2.$

2.64. $\sqrt{x-2}+\sqrt{1-x}=2.$

2.65. $\frac{1}{4}x=(\sqrt{1+x}-1)(\sqrt{1-x}+1).$

2.66. $\dfrac{1}{\sqrt{x+2\sqrt{x-1}}}+\dfrac{1}{\sqrt{x-2\sqrt{x-1}}}=\dfrac{2}{2-x}.$

2.67. $\sqrt[3]{(a+x)^2}+4\sqrt[3]{(a-x)^2}=5\sqrt[3]{a^2-x^2}.$

2.68. $\sqrt[n]{(x+1)^2}+\sqrt[n]{(x-1)^2}=4\sqrt[n]{x^2-1}.$

2.69*. $\dfrac{\sqrt{x^2+8x}}{\sqrt{x+1}}+\sqrt{x+7}=\dfrac{7}{\sqrt{x+1}}.$

3. Systems of Algebraic Equations

Several equations

$$F_1(x_1, x_2, \ldots, x_n) = 0,$$
$$F_2(x_1, x_2, \ldots, x_n) = 0, \ldots,$$
$$F_k(x_1, x_2, \ldots, x_n) = 0$$

considered simultaneously constitute a *system of equations*. A *solution* of that system is an ordered collection of the values of the unknowns which turns all the equations of the system into identities.

A system of equations which has solutions is said to be *consistent*, and that which has no solutions is said to be *inconsistent*.

A *linear equation in n unknowns* is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n, b$ are some numbers.

A system of equations is *linear* if all its equations are linear. A consistent system is said to *possess a unique solution* if there is a unique collection of numbers $k_1, \ldots, k_n$ turning all the equations of the system into identities. A consistent system of equations may possess *several different solutions*. Two consistent systems of equations are *equivalent* if all their solutions coincide. In what follows (especially when solving systems of linear equations) we shall often have to subtract one equation of the system from another, both sides of the former equation being multiplied by the same number. The resulting system of equations is equivalent to the initial system.

When seeking solutions of a system of m linear equations in n unknowns, it is convenient to use the *Gaussian algorithm* which consists in reducing the given system to a triangular or trapezoidal form. Here is an example illustrating the idea of the Gauss method.

Example 3.1. Solve the system

$$x + 2y + 3z = 8,$$
$$3x + y + z = 6,$$
$$2x + y + 2z = 6.$$

Solution. Multiplying both sides of the first equation by -3 and adding it to the second equation, we obtain $-5y - 8z = -18$ or, what is the same,

$$5y + 8z = 18.$$

Multiplying both sides of the first equation by -2 and adding it to the third equation of the system, we get an equation $-3y - 4z = -10$, or, what is the same,

$$3y + 4z = 10.$$

Consequently, the given system can be written as an equivalent system in which the second and third equations do not contain the unknown x:

$$\begin{aligned} x + 2y + 3z &= 8, \\ 5y + 8z &= 18, \\ 3y + 4z &= 10. \end{aligned} \qquad (*)$$

Multiplying both sides of the second equation by 3 and those of the third equation by —5 and adding these equations together, we get an equation $4z = 4$; system (*) can be written in the form of an equivalent system

$$\begin{aligned} x + 2y + 3z &= 8, \\ 5y + 8z &= 18, \\ z &= 1. \end{aligned}$$

We have thus reduced the initial system to a triangular form. Substituting $z = 1$ into the second equation of the system, we find $y = 2$. Substituting $z = 1$ and $y = 2$ into the first equation, we find $x = 1$.

Answer. $x = 1$, $y = 2$, $z = 1$.

Use the Gaussian algorithm to solve the following systems of linear equations.

3.1. $\begin{aligned} 2x + y + z &= 7, \\ x + 2y + z &= 8, \\ x + y + 2z &= 9. \end{aligned}$

3.2. $\begin{aligned} 3x - 4y + 5z &= 18, \\ 2x + 4y - 3z &= 26, \\ x - 6y + 8z &= 0. \end{aligned}$

3.3. $\begin{aligned} 10x - 9z &= 19, \\ 8x - y &= 10, \\ y - 12z &= 10. \end{aligned}$

3.4. $\begin{aligned} x + 2y + z + 7 &= 0, \\ 2x + y - z - 1 &= 0, \\ 3x - y + 2z - 2 &= 0. \end{aligned}$

3.5. $\begin{aligned} \frac{x}{a^3} - \frac{y}{a^2} + \frac{z}{a} &= 1, \\ \frac{x}{b^3} - \frac{y}{b^2} + \frac{z}{b} &= 1, \\ \frac{x}{c^3} - \frac{y}{c^2} + \frac{z}{c} &= 1. \end{aligned}$

3.6. $\begin{aligned} x + a^2y + b^2z &= 0, \\ x + ay + bz &= 0, \\ x + y + z &= 1. \end{aligned}$

Solution and investigation of systems of two linear equations in two unknowns. Let us consider a system of two linear equations in two unknowns:

$$\begin{aligned} a_{11}x + a_{12}y &= b_1, \\ a_{21}x + a_{22}y &= b_2 \end{aligned} \qquad (*)$$

under the condition that at least one coefficient in each equation of the system is nonzero. Assume that Δ, Δ_x, and Δ_y are determinants of system (*):

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

$$\Delta_x = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1a_{22} - b_2a_{12},$$

$$\Delta_y = \begin{vmatrix} a_{11} & b_1 \\ a_{12} & b_2 \end{vmatrix} = a_{11}b_2 - a_{12}b_1.$$

For $\Delta \neq 0$ the system is *consistent and has a unique solution*. The solution of this system is $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$.

For $\Delta = 0$

(1) system (*) is *inconsistent* (i.e. has no solutions) if at least one of the determinants, Δ_x or Δ_y, is nonzero;

(2) system (*) is *consistent and possesses several different solutions* if $\Delta_x = \Delta_y = 0$.

Each equation of system (*) puts the variables x and y into a linear correspondence. In a rectangular system of coordinates every linear correspondence between the variables x and y defines a certain straight line. In the case when the system has a unique solution, the straight lines defined by the first and the second equation intersect. If the system has an infinite number of solutions, the straight lines coincide; if the system is inconsistent, the straight lines are parallel.

Example 3.2. Solve and investigate the behaviour of the system

$$ax + y = 2,$$
$$x + ay = 2a.$$

Solution. Let us calculate the determinants of the system:

$$\Delta = \begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} = a^2 - 1,$$

$$\Delta_x = \begin{vmatrix} 2 & 1 \\ 2a & a \end{vmatrix} = 2a - 2a = 0,$$

$$\Delta_y = \begin{vmatrix} a & 2 \\ 1 & 2a \end{vmatrix} = 2a^2 - 2.$$

(1) Assume $\Delta = a^2 - 1 \neq 0$, i.e. $a \neq \pm 1$. In this case the system has a unique solution

$$x = \frac{\Delta_x}{\Delta} = \frac{0}{a^2 - 1} = 0, \quad y = \frac{\Delta_y}{\Delta} = \frac{2a^2 - 2}{a^2 - 1} = 2.$$

(2) Assume $\Delta = a^2 - 1 = 0$, i.e. $a = \pm 1$. In this case $\Delta = \Delta_x = \Delta_y = 0$, i.e. the system is consistent and has several different solutions.

For $a = 1$, the system assumes the form

$$x + y = 2, \quad x + y = 2,$$

and any pair of numbers $(x; y)$ related as $x + y = 2$ is its solution.

For $a = -1$, we have

$$\begin{matrix} -x + y = 2, \\ x - y = -2, \end{matrix} \Leftrightarrow \begin{matrix} x - y = -2, \\ x - y = -2, \end{matrix}$$

and any pair of numbers (x, y) related as $x - y = -2$ is its solutions.

Answer. For $a \neq \pm 1$ the system possesses a unique solution $x = 0$, $y = 2$;

for $a = 1$ any pair of numbers (x, y) related as $x + y = 2$ is its solutions;

for $a = -1$ any pair of numbers (x, y) related as $x - y = -2$ is its solutions.

Solve and investigate the following systems of equations.

3.7. $x + ay - 1 = 0,$ $ax - 3ay - (2a + 3) = 0.$ 3.8. $3x + ay = 5a^2,$ $3x - ay = a^2.$

3.9. $(a + 5)\,x + (2a + 3)\,y - (3a + 2) = 0,$
$(3a + 10)\,x + (5a + 6)\,y - (2a + 4) = 0.$

3.10. $a\,(a - 1)\,x + (a + 1)\,ay = a^3 + 2,$ $(a^2 - 1)\,x + (a^3 + 1)\,y = a^4 - 1.$ 3.11. $ax - y = b,$ $bx + y = a.$

3.12. $(a^2 + b^2)\,x + (a^2 - b^2)\,y = a^2,$
$(a + b)\,x + (a - b)\,y = a.$

3.13. Find the values of the parameters m and p such that the system

$$(3m - 5p + b)\,x + (8m - 3p - a)\,y = 1,$$
$$(2m - 3p + b)\,x + (4m - p)\,y = 2$$

should possess several different solutions.

3.14. Are the equations

$$x + ay = b + c,$$
$$x + by = c + a,$$
$$x + cy = a + b,$$

where a, b and c are real numbers such that $a^2 + b^2 + c^2 = 1$, consistent?

3.15. The numbers a and b are such that the system

$$a^2x - ay = 1 - a,$$
$$bx + (3 - 2b)\,y = 3 + a$$

possesses a unique solution $x = 1$, $y = 1$. Find the numbers a and b.

3.16. For what values of a and b does the system

$$a^2x - by = a^2 - b,$$
$$bx - b^2y = 2 + 4b$$

possess an infinite number of solutions?

3.17. For what values of a does the system

$$a^2x + (2 - a)\,y = 4 + a^2,$$
$$ax + (2a - 1)\,y = a^5 - 2$$

possess no solutions?

3.18. The numbers a, b, and c are such that the system

$$ax - by = 2a - b,$$
$$(c + 1)\,x + cy = 10 - a + 3b$$

has infinitely many solutions, and $x = 1$, $y = 3$ is one of the solutions. Find a, b, and c.

3.19. For what values of the parameter a does the system of equations

$$ax - 4y = a + 1,$$
$$2x + (a + 6)\,y = a + 3$$

possess no solutions?

3.20. For what values of the parameter a does the system

$$2x + ay = a + 2,$$
$$(a + 1)\,x + 2ay = 2a + 4$$

possess infinitely many solutions?

If one of the equations of a system of two equations in two unknowns is linear and the other equation is nonlinear, then the system can be solved as follows. One of the unknowns of a linear equation is expressed in terms of the other and is substituted into the second equation which turns into an algebraic equation in one unknown.

Solve the following systems of equations.

3.21. $(x - y)(x^2 - y^2) = 45,$
$x + y = 5.$

3.22. $(x + 0.2)^2 + (y + 0.3)^2 = 1,$
$x + y = 0.9.$

3.23. $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{13}{6},$
$x + y = 5.$

3.24. $(x+y)^4 + 4(x+y)^2 - 117 = 0,$
$x - y = 25.$

3.25. $x^2 + y^2 = 2(xy + 2),$
$x + y = 6.$

3.26. $x^2 + y^2 + 10x - 10y = 2xy - 21,$
$x + y = 5.$

Equations in two unknowns

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_kx^{n-k}y^k + \ldots + a_ny^n = 0$$

are called *homogeneous equations*. In a homogeneous equation each term contains a product of the powers of x and y the sum of whose exponents is constant. When one of the two equations of a non-linear system is homogeneous, it is possible to use that equation to express linearly one unknown of the system in terms of the other.

Example 3.3. Solve the system

$$x^2 - 5xy + 6y^2 = 0, \qquad (*)$$
$$x^2 + y^2 = 10.$$

Solution. Dividing the first equation by y^2, we get a quadratic equation

$$t^2 - 5t + 6 = 0,$$

whose roots are $t_1 = 2$, $t_2 = 3$, with respect to the unknown $t = x/y$. Returning to the initial unknowns, we get the following linear relationships between the unknowns appearing in the initial system (*):

$$x = 2y, \quad x = 3y. \qquad (**)$$

Substituting successively $x = 3y$ and $x = 2y$ to the second equation of the given system, we obtain quadratic equations $y^2 = 1$ and

$y^2 = 2$, whose roots are $y_{1,2} = \pm 1$, $y_{3,4} = \pm\sqrt{2}$, with respect to the unknown y. Equations (**) can be used to find the requisite values $x_1, \ldots, x_4$.

Answer. (3, 1), (−3, −1), $(2\sqrt{2}, \sqrt{2})$, $(-2\sqrt{2}, -\sqrt{2})$.

Solve the following systems of equations.

3.27. $x^2y^3 + x^3y^2 = 12$, $x^2y^3 - x^3y^2 = 4$.

3.28. $x^3 + y^3 = 65$, $x^2y + xy^2 = 20$.

3.29. $x^4 - y^4 = 15$, $x^3y - xy^3 = 6$.

A system of equations in n unknowns $x_1, x_2, \ldots, x_n$ is said to be *symmetric* if it does not change upon a permutation of the unknowns. If there are two unknowns (x and y), then the solution of such systems can often be found by introducing new unknowns $u = x + y$ and $v = xy$. In that case, it is convenient to use the following equations:

$$x^2 + y^2 = (x + y)^2 - 2xy = u^2 - 2v,$$

$$x^3 + y^3 = (x + y)^3 - 3xy(x + y) = u^3 - 3uv,$$

$$x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 = [(x + y)^2 - 2xy]^2 - 2x^2y^2 = [u^2 - 2v]^2 - 2v^2,$$

which make it possible to express the combinations of the unknowns $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$ in terms of the unknowns u and v.

Example 3.4. Solve the system of equations

$$x^2 + y^2 = 2(xy + 2),$$
$$x + y = 6.$$

Solution. We designate $v = xy$, $u = x + y$ and then, make use of the equation

$$x^2 + y^2 = (x + y)^2 - 2xy,$$

to obtain a system

$$u^2 - 2v = 2v + 4,$$
$$u = 6,$$

with respect to the new unknowns, whose only solution is $u = 6$, $v = 8$. Returning to the initial unknowns, we find that the solution of the initial system reduces to that of a simpler system

$$x + y = 6, \quad xy = 8,$$

whose roots can be found by the use of the Vieta theorem, for instance.

Answer. (2, 4), (4, 2).

Solve the following systems of equations.

3.30. $x^2y + y^2x = 20$, $\frac{1}{x} + \frac{1}{y} = \frac{5}{4}$.

3.31. $x^2 + y^2 = a$, $\frac{1}{x^2} + \frac{1}{y^2} = b$.

3.32. $x^2y + xy^2 = 6$, $xy + x + y = 5$.

3.33. $x^4 + y^4 = 82$, $x + y = 4$.

3.34. $x^3 + y^3 = 9,$ $xy = 2.$ 3.35. $x^3 + y^3 = 2,$ $xy(x + y) = 2.$

3.36. $(x^2 + y^2)xy = 78,$ $x^4 + y^4 = 97.$

3.37. $5(x^4 + y^4) = 41(x^2 + y^2),$ $x^2 + y^2 + xy = 13.$

3.38. $x^4 + y^4 = 97,$ $xy = 6.$

Solve the following systems of equations combining the methods presented above.

3.39*. $(x^2-x+1)(y^2-y+1)=3,$ $(x+1)(y+1)=6.$ 3.40*. $\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{10}{3},$ $x^2+y^2=5.$

3.41*. $2x^3y^2 - y^3x^2 = 36,$ $2x^2y - y^2x = 6.$ 3.42. $xy - x + y = 1,$ $x^2y - y^2x = 30.$

3.43. $xy + x - y = 3,$ $x^2y - xy^2 = 2.$ 3.44. $x^2 + xy + x = 10,$ $y^2 + xy + y = 20.$

3.45. $x^2 + xy + 2y^2 = 37,$ $2x^2 + 2xy + y^2 = 26.$ 3.46*. $x^2 - xy + y^2 = 19,$ $x^4 + x^2y^2 + y^4 = 931.$

3.47. $(x^2 + 1)(y^2 + 1) = 10,$ $(x + y)(xy - 1) = 3.$ 3.48*. $x^5 - y^5 = 3093,$ $x - y = 3.$

Symmetric systems of three equations in three unknowns x, y and z are usually solved by introducing new unknowns:

$$u = x + y + z, \quad v = xy + yz + xz, \quad w = xyz.$$

In this case, it is convenient to use the following equations:

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx) = u^2 - 2v,$$
$$x^3 + y^3 + z^3 = (x + y + z)^3 - 3(x + y + z)(xy + yz + zx) + 3xyz = u^3 - 3uv + 3w.$$

Example 3.5. Solve the system of equations

$$x + y + z = 1,$$
$$xy + yz + zx = -4,$$
$$x^3 + y^3 + z^3 = 1.$$

Solution. The system is symmetric. Introducing auxiliary unknowns

$$x + y + z = u, \ xy + yz + xz = v, \ xyz = w,$$

and using the equation

$$x^3 + y^3 + z^3 = u^3 - 3uv + 3w,$$

we obtain a system

$$u = 1, \ v = -4, \ u^3 - 3uv + 3w = 1,$$

or, returning to the old unknowns, we get a system

$$\begin{aligned} x + y + z &= 1, \\ xy + yz + xz &= -4, \\ xyz &= -4. \end{aligned} \qquad (*)$$

The given system can be solved by using the Vieta theorem for a cubic polynomial: the roots t_1, t_2, t_3 of the cubic polynomial $t^3 + at^2 + bt + c$ are related as

$$\begin{aligned} t_1 + t_2 + t_3 &= -a, \\ t_1t_2 + t_1t_3 + t_2t_3 &= b, \\ t_1t_2t_3 &= -c. \end{aligned}$$

It is easy to notice that for $a = -1$, $b = -4$, $c = 4$ the roots of the cubic equation

$$t^3 - t^2 - 4t + 4 = 0$$

are in the same relationship as the unknowns x, y, z in system (*) and, consequently, the triple of the values of the unknowns

$$x = t_1, \; y = t_2, \; z = t_3$$

is a solution of system (*). By virtue of the symmetry of the system, in addition to this triple of the values of the unknowns, the following triples of the values of the unknowns are also solutions of the system:

$$\begin{aligned} &x = t_2, \quad y = t_1, \quad z = t_3, \\ &x = t_3, \quad y = t_2, \quad z = t_1, \\ &x = t_1, \quad y = t_3, \quad z = t_2, \\ &x = t_2, \quad y = t_3, \quad z = t_1, \\ &x = t_3, \quad y = t_1, \quad z = t_2. \end{aligned}$$

Thus the solution of the given system has reduced to finding the roots of the cubic equation

$$t^3 - t^2 - 4t + 4 = 0,$$

whose roots are

$$t_1 = 2, \quad t_2 = -2, \quad t_3 = 1.$$

Consequently, the following ordered triples of numbers are solutions of the given system:
(2, −2, 1), (−2, 2, 1), (1, 2, −2), (−2, 1, 2), (1, −2, 2), (2, 1, −2).

Solve the following systems of equations.

3.49. $$\begin{aligned} x + y + z &= 0, \\ x^2 + y^2 + z^2 &= x^3 + y^3 + z^3, \\ xyz &= 2. \end{aligned}$$

3.50. $$\begin{aligned} x + y + z &= 1, \\ x^2 + y^2 + z^2 &= 1, \\ x^3 + y^3 + z^3 &= 1. \end{aligned}$$

Systems of three equations in three unknowns are sometimes solved by an introduction of auxiliary unknowns.

Example 3.6. Solve the system of equations

$$\frac{3xy}{x+y}=5, \quad \frac{2xz}{x+z}=3, \quad \frac{yz}{y+z}=4.$$

Solution. The given system is equivalent to the system

$$\frac{x+y}{xy}=\frac{3}{5}, \quad \frac{x+z}{xz}=\frac{2}{3}, \quad \frac{y+z}{yz}=\frac{1}{4}.$$

Performing a term-by-term division on the left-hand sides of the equations, we reduce the system to the form

$$\frac{1}{y}+\frac{1}{x}=\frac{3}{5}, \quad \frac{1}{x}+\frac{1}{z}=\frac{2}{3}, \quad \frac{1}{y}+\frac{1}{z}=\frac{1}{4}.$$

Designating $\frac{1}{x}=u$, $\frac{1}{y}=v$, $\frac{1}{z}=w$, we get a linear system with respect to the unknowns we have introduced:

$$u+v=\frac{3}{5}, \quad u+w=\frac{2}{3}, \quad w+v=\frac{1}{4}.$$

The triple of numbers $\left(\frac{61}{120}, \frac{11}{120}, \frac{19}{120}\right)$ is its solution. Consequently, the triple of numbers $\left(\frac{120}{61}, \frac{120}{11}, \frac{120}{19}\right)$ is a solution of the initial system.

Solve the following systems.

3.51. $3xy-\frac{16}{xz}=-5,$
$xy+\frac{8}{yz}=-5,$
$yz-\frac{3}{yx}=1.$

3.52. $x+y=\frac{xy}{1+xy},$
$x+z=\frac{xz}{1+xz},$
$y+z=\frac{yz}{1+yz}.$

3.53. $x^2-(y-z)^2=a,$
$y^2-(z-x)^2=b,$
$z^2-(x-y)^2=c.$

3.54. $\frac{xyz}{y+z}=a,$
$\frac{xyz}{z+x}=b,$
$\frac{xyz}{x+y}=c.$

3.55. $x+y+z=13,$
$x^2+y^2+z^2=91,$
$y^2=xz.$

3.56. $x+y+z=0,$
$x^2+y^2+z^2=2(y-x-z)-2,$
$x^3+y^3+z^3=3(x^2-y^2+z^2).$

3.57. $x + y + z = 1,$
$4x^2 + y^2 + z^2 - 5x = x^3 + y^3 + z^3 - 2,$
$xyz = 2 + yz.$

3.58. $xy + yz + zx = 11,$
$x^2 + y^2 + z^2 = 14,$
$xyz = 6.$

3.59. $2(x + y) = xy,$
$xy + yz + zx = 108,$
$xyz = 180.$

3.60. $x(x + y + z) = a,$
$y(x + y + z) = b,$
$z(x + y + z) = c.$

3.61. $x^2 + y^2 = axyz,$
$y^2 + z^2 = bxyz,$
$z^2 + x^2 = cxyz.$

3.62. $x^2y = x + y - z,$
$z^2x = x - y + z,$
$y^2x = y - x + z.$

3.63. $4xy + x^2 + y^2 = 1,$
$8xz + x^2 + 4z^2 = -2,$
$8yz + y^2 + 4z^2 = 1.$

3.64. $2(x^2 + y^2) = xyz,$
$10(y^2 + z^2) = 29xyz,$
$5(z^2 + x^2) = 13xyz.$

3.65. $xyz^2 = -y - 2x,$
$2x^2yz = -y - z,$
$3xy^2z = 2x - z.$

3.66. $xy + x + y = 7,$
$yz + y + z = -3,$
$xz + x + z = -5.$

If a system contains irrational equations, we usually rationalize them when solving the system. In that case we use methods which are usually used in solving irrational equations.

Example 3.7. Solve the system of equations

$$\sqrt[4]{1+5x}+\sqrt[4]{5-y}=3,$$
$$5x-y=11.$$

Solution. Introducing the designations $\sqrt[4]{1 + 5x} = u$ and $\sqrt[4]{5 - y} = v$, we get a system of equations

$$u + v = 3, \quad u^4 + v^4 = 17, \qquad (*)$$

which is a symmetric system of nonlinear equations whose solutions are $u = 1, v = 2$, and $u = 2, v = 1$. Returning to the initial unknowns, we obtain the following systems of linear equations:

$$1 + 5x = 16, \quad 1 + 5x = 1,$$
$$5 - y = 1, \quad 5 - y = 16.$$

Answer. (3, 4), (0, −11).

Solve the following systems of equations.

3.67. $\sqrt{2x+y-1}-\sqrt{x+y}=1,$
$3x+2y=4.$

3.68. $\sqrt[3]{x+2y}+\sqrt[3]{x-y+2}=3,$
$2x+y=7.$

3.69. $\sqrt{x^2-xy}-\sqrt{xy-y^2}=3(x-y),$
$x^2-y^2=41.$

3.70. $\sqrt{x^2+y^2}+\sqrt{2xy}=8\sqrt{2},$
$\sqrt{x}+\sqrt{y}=4.$

3.71. $\sqrt{x^2+5}+\sqrt{y^2-5}=5,$
$x^2+y^2=13.$

3.72. $\sqrt{x}+\sqrt{y}=2,$
$x-2y+1=0.$

3.73. $\sqrt{x}+\sqrt{y}=8,$
$x+y-\sqrt{x}+\sqrt{y}-2\sqrt{xy}=2.$

3.74. $xy+\sqrt{x^2y^2-y^4}=8(\sqrt{x+y}+\sqrt{x-y}),$
$(x+y)^{3/2}-(x-y)^{3/2}=26.$

Chapter 2

Logarithms. Exponential and Logarithmic Equations

1. Identity Transformations of Exponential and Logarithmic Expressions

Let a be a positive number different from unity and M any positive number. The *logarithm of the number M to the base a* is a number, designated $\log_a M$, such that

$$a^{\log_a M} = M.$$

The principal properties of logarithms.

$$\log_a bc = \log_a b + \log_a c, \quad a \neq 1, \ a > 0, \ b > 0, \ c > 0, \qquad (1)$$

$$\log_a \frac{b}{c} = \log_a b - \log_a c, \quad a \neq 1, \quad a > 0, \quad b > 0, \quad c > 0, \qquad (2)$$

$$\log_{a^p} b^q = \frac{q}{p} \log_a b, \quad a \neq 1, \quad b > 0, \quad a > 0, \quad p \neq 0, \qquad (3)$$

$$\log_a b = \frac{\log_c b}{\log_c a}, \quad a \neq 1, \quad c \neq 1, \quad a > 0, \quad b > 0, \quad c > 0, \qquad (4)$$

$$\log_a b = \frac{1}{\log_b a}, \quad a \neq 1, \quad b \neq 1, \quad a > 0, \quad b > 0. \qquad (5)$$

Example 1.1. Simplify the expression

$$\frac{\log_a \sqrt{a^2-1}\, \log^2_{1/a} \sqrt{a^2-1}}{\log_{a^2}(a^2-1)\, \log_{\sqrt[3]{a}} \sqrt[6]{a^2-1}}.$$

Solution. According to (3), we have

$$(\log_{1/a} \sqrt{a^2-1})^2 = (-\log_a \sqrt{a^2-1})^2 = (\log_a \sqrt{a^2-1})^2, \qquad (*)$$

$$\log_{\sqrt[3]{a}} \sqrt[6]{a^2-1} = \log_{(\sqrt[3]{a})^3} (\sqrt[6]{a^2-1})^3 = \log_a \sqrt{a^2-1}, \qquad (**)$$

$$\log_{a^2}(a^2-1) = \log_{(a^2)^{1/2}} (a^2-1)^{1/2} \log_a \sqrt{a^2-1}. \qquad (***)$$

Substituting the right-hand sides of expressions (*) to (***) into the initial fraction, we obtain

$$\frac{\log_a \sqrt{a^2-1}\, \log^2_a \sqrt{a^2-1}}{\log_a \sqrt{a^2-1}\, \log_a \sqrt{a^2-1}} = \log_a \sqrt{a_2-1}.$$

Answer. $\log_a \sqrt{a^2-1}$.

Example 1.2. Calculate

$$81^{1/\log_5 3}+27^{\log_9 36}+3^{4/\log_7 9}.$$

Solution. According to (5) we have

$$81^{1/\log_5 3}=81^{\log_3 5}.$$

Using the properties of powers, we obtain

$$81^{1/\log_3 5}=(3^4)^{\log_3 5}=(3^{\log_3 5})^4.$$

According to the definition of a logarithm, we have $3^{\log_3 5}=5$. Thus we have

$$81^{1/\log_5 3}=5^4=625.$$

Similarly,

$$3^{4/\log_7 9}=3^{4\log_9 7}=(3^2)^{2\log_9 7}=(9^{\log_9 7})^2=7^2=49,$$

$$27^{\log_9 36}=27^{\log_3 6}=3^{3\log_3 6}=(3^{\log_3 6})^3=216.$$

Adding the resulting values together, we get the required number.
Answer. 890.

Simplify the following expressions.

1.1. $\dfrac{81^{1/\log_5 9}+3^{3/\log_{\sqrt{6}} 3}}{409}\left((\sqrt{7})^{2/\log_{25} 7}-125^{\log_{25} 6}\right)$

1.2. $a^{2/\log_b a+1}b-2a^{\log_a b+1}b^{\log_b a+1}+ab^{2/\log_a b+1}.$

1.3. $\left(2^{\log_{4\sqrt{2}} a}-3^{\log_{27}(a^2+1)^3}-2a\right):\left(7^{4\log_{49} a}-a-1\right).$

1.4*. $\log_3 2\log_4 3\log_5 4\log_6 5\log_7 6\log_8 7.$

1.5. $\log_2 2x^2+\log_2 x\, x^{\log_x(\log_2 x+1)}+\frac{1}{2}\log_4^2 x^4+2^{-3\log_{1/2}\log_2 x}.$

1.6. $\dfrac{\log_a b+\log_a\left(b^{\frac{1}{2\log_b a^2}}\right)}{\log_a b-\log_{ab} b}\;\dfrac{\log_{ab} b\log_a b}{b^{2\log_b\log_a b}-1}$

1.7. $5^{\log_{1/5}\left(\frac{1}{2}\right)}+\log_{\sqrt{2}}\dfrac{4}{\sqrt{7}+\sqrt{3}}+\log_{1/2}\dfrac{1}{10+2\sqrt{21}}.$

When calculating the values of one logarithmic or exponential expression from some other, known, logarithmic or exponential expressions, we usually factor all the numbers appearing in the given expressions into primes.

Example 1.3. Find $\log_{30} 8$ if it is known that $\log 5=a$ and $\log 3=b$.

Solution. We represent $\log_{30} 8$ as

$$\log_{30} 8=\frac{\log 8}{\log 30}.$$

Using prime factorization of the numbers 30 and 8 and the properties of logarithms, we obtain

$$\log_{30} 8 = \frac{3 \log 2}{\log 5 + \log 3 + \log 2}.$$

Taking into account that

$$\log 2 = \log \frac{10}{5} = 1 - \log 5,$$

we get

$$\log_{30} 8 = \frac{3(1-a)}{b+1}.$$

Answer. $\frac{3(1-a)}{b+1}$.

1.8. Calculate

$$\frac{\log_3 135}{\log_{15} 3} - \frac{\log_3 5}{\log_{405} 3}$$

without resort to tables.

1.9. Knowing that $\log 2 = a$, $\log_2 7 = b$, find $\log 56$.

1.10. Knowing that $\log 3 = a$, $\log 2 = b$, find $\log_5 6$.

1.11. It is known that $\log_3 7 = a$, $\log_7 5 = b$, $\log_5 4 = a$. Find $\log_3 12$.

1.12. Knowing that $b = 8^{1/(1-\log_8 a)}$ and $c = 8^{1/(1-\log_8 b)}$, express $\log_8 a$ in terms of $\log_8 c$.

1.13. It is known that

$$\log_a x = \alpha,\ \log_b x = \beta,\ \log_c x = \gamma,\ \log_d x = \delta;$$
$$x \neq 1.$$

Find $\log_{abcd} x$.

To prove the identity of two logarithmic expressions when certain conditions are fulfilled, it is sometimes convenient to transform the given conditions and then to take their logarithms.

Example 1.4. Prove that

$$\log \frac{a+b}{3} = \frac{1}{2} (\log a + \log b), \qquad (*)$$

if $a^2 + b^2 = 7ab$, $a > 0$, $b > 0$.

Solution. Let us transform the conditions by isolating the perfect square:

$$a^2 + b^2 + 2ab = 9ab,$$

i.e.

$$(a + b)^2 = 9ab.$$

Taking logarithms of this equality to the base 10 and collecting terms, we obtain

$$2 \log (a + b) - 2 \log 3 = \log a + \log b.$$

Dividing both sides of the equation by 2 and using property (2) of logarithms, we get (*).

1.14. Show that if $x > 0$, $y > 0$, then it follows from the equation $x^2 + 4y^2 = 12xy$ that

$$\log(x+2y) - 2\log 2 = \frac{1}{2}(\log x + \log y).$$

1.15. Prove that

$$\log_{a+b} m + \log_{a-b} m = 2 \log_{a+b} m \log_{a-b} m$$

if it is known that $m^2 = a^2 - b^2$.

1.16. Prove that if a, b, c are successive (positive) terms of a geometric progression, then

$$\frac{\log_a N - \log_b N}{\log_b N - \log_c N} = \frac{\log_a N}{\log_c N}.$$

1.17. Prove that if

$$(ac)^{\log_a b} = c^2,$$

then the numbers $\log_a N$, $\log_b N$, $\log_c N$ are three successive terms of an arithmetic progression for any positive value of N.

When proving identities (i.e. verifying the validity of some equalities on the whole domain of definition of the functions appearing in them), use should be made of the same techniques as those employed in simplifying logarithmic and exponential expressions.

Example 1.5. Prove that

$$\log_p \log_p \underbrace{\sqrt[p]{\sqrt[p]{\ldots \sqrt[p]{p}}}}_{n} = -n$$

for $p > 1$.

Solution. Transforming the irrational expression under the second sign of logarithm, we obtain

$$\underbrace{\sqrt[p]{\sqrt[p]{\ldots \sqrt[p]{p}}}}_{n} = p^{1/p^n}.$$

Taking logarithms in this equation to the base p, we get

$$\log_p p^{1/p^n} = \frac{1}{p^n}.$$

Taking logarithms in this expression to the base p once again, we get the required identity.

1.18. Prove that the equality

$$\frac{1}{\log_a N} + \frac{1}{\log_{a^2} N} + \frac{1}{\log_{a^3} N} + \frac{1}{\log_{a^4} N} = 10 \log_N a,$$

holds for any permissible positive numbers a and N.

1.19. Prove that

$$2\left(\sqrt{\log_a \sqrt[4]{ab}+\log_b \sqrt[4]{ab}}-\sqrt{\log_a \sqrt[4]{\frac{b}{a}}+\log_b \sqrt[4]{\frac{a}{b}}}\right)\sqrt{\log_a b} = \begin{cases} 2, & b \geqslant a > 1, \\ 2\log_a b, & 1 < b < a. \end{cases}$$

1.20. Prove that

$$\log_a N \log_b N + \log_b N \log_c N + \log_c N \log_a N = \frac{\log_a N \log_b N \log_c N}{\log_{abc} N}.$$

1.21. Prove the identity

$$\log_{a/b} x = \frac{\log_a x \log_b x}{\log_b x - \log_a x}.$$

2. Exponential Equations

Exponential equations are the equations in which the unknown enters only into the exponents, the bases being constant. The elementary exponential equation is an equation of the form

$$a^x = b. \tag{1}$$

For $a > 0$ and $b > 0$, $a \neq 1$, its solution is

$$x = \log_a b.$$

If some function $f(x)$ is in the exponent instead of x, i.e. an equation has the form

$$a^{f(x)} = b,\ a > 0,\ b \neq 1,\ b > 0, \tag{2}$$

then taking logarithms of both sides of the equation, we arrive at an equivalent equation

$$f(x) = \log_a b.$$

Some exponential equations can be reduced to form (1) or (2) by means of the equations

$$a^{x+y} = a^x \cdot a^y,$$
$$(a^x)^y = a^{xy},$$
$$\frac{a^x}{a^y} = a^{x-y},$$
$$(a \cdot b)^x = a^x \cdot b^x,$$
$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}.$$

Example 2.1. Solve the equation

$$6^{2x+4} = (3^{3x}) \cdot (2^{x+8}).$$

Solution. We rewrite the given equation as

$$(3^{2x+4})(2^{2x+4}) = (3^{3x})(2^{x+8}).$$

Using the property of the members of a proportion, we get

$$\frac{3^{2x+4}}{3^{3x}}=\frac{2^{x+8}}{2^{2x+4}}$$

or, after simplification, $3^{4-x} = 2^{4-x}$. We reduce this equation to the form

$$\left(\frac{2}{3}\right)^{4-x}=1$$

and get $4 - x = 0$, whence $x = 4$.

Answer. $x = 4$.

Solve the following equations.

2.1. $\sqrt{3^x}\cdot\sqrt{5^x}=225$. **2.2.** $(2^{3x})\cdot(5^x)=1600$.

2.3. $(9^{3-5x})(7^{5x-3})=1$. **2.4.** $(3^{2x-1})(5^{3x+2})=\frac{9}{5}(5^{2x})(3^{3x})$.

2.5. $3\cdot 4^x+\frac{1}{3}\cdot 9^{x+2}=6\cdot 4^{x+1}-\frac{1}{2}\cdot 9^{x+1}$.

2.6. $7^{\frac{2x^2-5x-9}{2}}=(\sqrt{2})^{3\log_2 7}$. **2.7.** $4\cdot 3^{x+2}+5\cdot 3^x-7\cdot 3^{x+1}=40$.

2.8. $5\left(\frac{1}{25}\right)^{\sin^2 x}+4\cdot 5^{\cos 2x}=25^{\frac{1}{2}\sin 2x}$.

2.9. $16^{\frac{x+5}{x-7}}=512\cdot 64^{\frac{x+17}{x-3}}$. **2.10.** $5^{|4x-6|}=25^{3x-4}$.

2.11. $\sqrt{3}\cdot(3^{\frac{x}{1+\sqrt{x}}})\cdot\left(\frac{1}{3}\right)^{\frac{2+\sqrt{x}+x}{2(1+\sqrt{x})}}=81$.

2.12. $(8^{\frac{x-3}{3x-7}})\cdot\sqrt{\sqrt{0.25^{\frac{3x-1}{x-1}}}}=1$

2.13. $0.6^x\cdot\left(\frac{25}{9}\right)^{x^2-12}=\left(\frac{27}{125}\right)^3$.

2.14. $(2.4)^{1-\sin x}\cdot[0.41(6)]^{\sin x+1/2}=\left(\frac{12}{5}\right)^{1/2}$.

2.15. Find the solution of the equation

$$3^{x^2+4x}=\frac{1}{25}$$

which satisfies the condition $x > -3$.

If an exponential equation has the form

$$g\left(a^{f(x)}\right) = 0, \qquad (3)$$

then its solution by the substitution $y = a^{f(x)}$ reduces to the solution of an equation of the form

$$a^{f(x)} = y_i,$$

where y_i are roots of the equation $g(y) = 0$.

Example 2.2. Solve the equation

$$4^{\sqrt{x^2-2}+x} - 5 \cdot 2^{x-1+\sqrt{x^2-2}} = 6.$$

Solution. Designating $2^{\sqrt{x^2-2}+x} = y$, we get a quadratic equation

$$y^2 - \frac{5}{2} y - 6 = 0,$$

whose roots are $y_1 = 4$ and $y_2 = -3/2$. Thus we have reduced the solution of the given equation to that of the equations

$$2^{x+\sqrt{x^2-2}} = 4, \quad 2^{x+\sqrt{x^2-2}} = -3/2.$$

The second equation has no solutions since $2^{x+\sqrt{x^2-2}} > 0$ for all permissible values of x. From the first equation we get

$$x + \sqrt{x^2-2} = 2.$$

Separating out the radical and squaring both sides of the equation, we obtain

$$x^2 - 2 = 4 - 4x + x^2.$$

Collecting terms, we get the only root $x = 3/2$. Verification shows that this root satisfies the initial equation.

Answer. $x = 3/2$.

Solve the following equations.

2.16. $2^{x+1} + 3 \cdot 2^{x-3} = 76$. **2.17.** $3\sqrt[x]{81} - 10\sqrt[x]{9} + 3 = 0$.

2.18. $3^{1-x} - 3^{1+x} + 9^x + 9^{-x} = 6$. **2.19.** $64^{1/x} - 2^{3+3/x} + 12 = 0$.

2.20. $4^{\log_9 x} - 6 \cdot 2^{\log_9 x} + 2^{\log_3 27} = 0$.

2.21. $4^{\sqrt{3x^2-2x}+1} + 2 = 9 \cdot 2^{\sqrt{3x^2-2x}}$.

Exponential equations whose power bases are successive terms of a geometric progression and the exponents are the same can be reduced to equations of form (3) by means of a division by any of the end terms.

Example 2.3. Solve the equation

$$6 \cdot 4^x - 13 \cdot 6^x + 6 \cdot 9^x = 0.$$

Solution. We divide both sides of the equation by 9^x and get

$$6\left(\frac{4}{9}\right)^x - 13\left(\frac{6}{9}\right)^x + 6 = 0.$$

Designating $\left(\frac{2}{3}\right)^x = y$, we get an equation

$$6y^2 - 13y + 6 = 0,$$

whose roots are $y_1 = 3/2$ and $y_2 = 2/3$. Thus the solution of the equation reduces to the solution of two elementary exponential equations

$$\left(\frac{3}{2}\right)^x = \frac{3}{2}, \quad \left(\frac{3}{2}\right)^x = \frac{2}{3}.$$

Answer. $x = 1$, $x = -1$.

Solve the following equations.

2.22. $7 \cdot 4^{x^2} - 9 \cdot 14^{x^2} + 2 \cdot 49^{x^2} = 0$. **2.23.** $3 \cdot 16^x + 36^x = 2 \cdot 81^x$.

2.24. $8^x + 18^x = 2 \cdot 27^x$. **2.25.** $6\sqrt[x]{9} - 13\sqrt[x]{6} + 6\sqrt[x]{4} = 0$.

2.26. $16^x - 5 \cdot 8^x + 6 \cdot 4^x = 0$. **2.27.** $2^{3x-3} - 5 + 6 \cdot 2^{3-3x} = 0$.

2.28. $27^x + 12^x = 2 \cdot 8^x$. **2.29.** $(4 + \sqrt{15})^x + (4 - \sqrt{15})^x = 62$.

2.30. $\left(\sqrt{5 + 2\sqrt{6}}\right)^x + \left(\sqrt{5 - 2\sqrt{6}}\right)^x = 10$.

2.31. Solve the equation

$$\left(\sqrt{a + \sqrt{a^2 - 1}}\right)^x \left(\sqrt{a - \sqrt{a^2 - 1}}\right)^x = 2a,$$

by substituting the value $\sqrt{2}$, $\sqrt{3}$, 3, 4, 5, 6, 7 for a.

2.32. $5^{1+x^3} - 5^{1-x^3} = 24$. **2.33.** $5^{x-1} + 5 \cdot 0.2^{x-2} = 26$.

2.34. $10^{2/x} + 25^{1/x} = 4.25 \cdot 50^{1/x}$.

Equations of the form

$$[a(x)]^{b(x)} = [a(x)]^{c(x)}$$

with the set of permissible values defined by the condition $a(x) > 0$, can be reduced to the equivalent equation

$$b(x) \log_d [a(x)] = c(x) \log_d [a(x)]$$

by taking logarithms of its both sides. The last equation is equivalent to two equations

$$\log_d [a(x)] = 0, \quad b(x) = c(x).$$

Example 2.4. Solve the equation

$$|x - 2|^{10x^2-1} = |x - 2|^{3x}.$$

Solution. The set of permissible values of the unknown of the given equation is $x \neq 2$. Taking logarithms of both sides of the equation to the base 2 and collecting terms, we get an equation

$$(10x^2 - 3x - 1) \log_2 |x - 2| = 0,$$

which is equivalent to two equations

$$\log_2 |x - 2| = 0, \quad 10x^2 - 3x - 1 = 0.$$

The roots of the first equation are $x_1 = 1$, $x_2 = 3$, the roots of the second equation are $x_3 = 1/2$, $x_4 = -1/5$.

Answer. $-1/5$, $1/2$, 1, 3.

Solve the following equations.

2.35. $\sqrt[4]{|x-3|^{x+1}} = \sqrt[3]{|x-3|^{x-2}}$. 2.36. $|x-3|^{3x^2-10x+3} = 1$.

2.37. $x^{\log_a x} = (a^{\pi})^{\log_a^3 x}$. 2.38. $(\sqrt{x})^{\log_5 x - 1} = 5$.

2.39. $x^{\log x + 7} = 10^{(\log x + 1)\cdot 4}$.

Some equations which contain an unknown in the exponent can be solved by investigating the behaviour of the functions entering into their left-hand and right-hand sides.

Example 2.5. Solve the equation

$$7^{6-x} = x + 2.$$

Solution. By means of selection we find the root $x = 5$. The equation has no other solutions since the function $f(x) = 7^{6-x}$ decreases monotonically and $g(x) = x + 2$ increases monotonically and, consequently, the graphs of these functions intersect not more than once.

Answer. $x = 5$.

Solve the following equations.

2.40*. $\left(\sqrt{2+\sqrt{3}}\right)^x + \left(\sqrt{2-\sqrt{3}}\right)^x = 2^x$.

2.41*. $3^{x-1} + 5^{x-1} = 34$.

2.42*. $2^{3x^2-2x^3} = \dfrac{x^2+1}{x}$.

2.43*. $4^x + (x-1)\,2^x = 6 - 2x$.

2.44*. $(x+1)\,9^{x-3} + 4x\cdot 3^{x-3} - 16 = 0$.

2.45*. $x^2 - x + 1 = 2\cdot 2^{x-1} - 4^{x-1}$.

3. Logarithmic Equations

The *logarithmic* equation is an equation which contains an unknown quantity under the sign of a logarithm. The elementary logarithmic equation

$$\log_a x = b, \; a > 0, \; a \neq 1 \qquad (4)$$

with the set of permissible values $x > 0$ has a solution $x = a^b$.

A logarithmic equation in which a certain function $f(x)$ is under the sign of the logarithm,

$$\log_a f(x) = b, \; a > 0, \; a \neq 1, \qquad (5)$$

has a set of permissible values of x defined by the inequality $f(x) > 0$ and is equivalent to the equation

$$f(x) = a^b.$$

Example 3.1. Solve the equation

$$2 - x + 3\log_5 2 = \log_5 (3^x - 5^{2-x}),$$

Solution. Let us transfer the logarithm appearing on the left-hand side into the right-hand side and, making use of the properties of logarithms, write the equation in the form

$$2-x=\log_5\left(\frac{3^x-5^{2-x}}{8}\right).$$

By the definition of a logarithm, the given equation is equivalent to the equation

$$\frac{3^x-5^{2-x}}{8}=5^{2-x}$$

which can be written in the form

$$3^x = 9\cdot 5^{2-x} \text{ or } 3^{x-2} = 5^{2-x} \text{ or } 15^{x-2} = 1.$$

The resulting exponential equation is equivalent to the equation $x - 2 = 0$ whose solution is $x = 2$.

The set of permissible values of x of the given equation can be found as the solution of the inequality

$$3^x - 5^{2-x} > 0.$$

For $x = 2$ the given equation holds true and, consequently, $x = 2$ is a solution of the initial logarithmic equation.

Answer. $x = 2$.

Solve the following equations.

3.1. $\log_5[2+\log_3(3+x)]=0$.

3.2. $\log(5-x)-\frac{1}{3}\log(35-x^3)=0$.

3.3. $\log_3(3^x - 8) = 2 - x$.

3.4. $\log_{\sqrt{5}}(4^x-6)-\log_{\sqrt{5}}(2^x-2)=2$.

3.5. $\log(3x^2 + 12x + 19) - \log(3x + 4) = 1$.

If the logarithmic equation has the form

$$f(\log_a x) = 0,$$

where f is a certain function, then by the substitution $y = \log_a x$ it can be reduced to equations of form (4):

$$\log_a x = y_i,$$

where y_i are roots of the equation $f(y) = 0$.

Example 3.2. Solve the equation

$$(\log_2 x)^2 - 5(\log_2 x) + 6 = 0.$$

Solution. Designating $\log_2 x = y$, we get an equation

$$y^2 - 5y + 6 = 0$$

whose roots are $y_1 = 2$, $y_2 = 3$. Thus the initial equation is equivalent to two equations of form (4):

$$\log_2 x = 2,\ \log_2 x = 3,$$

whose solutions are $x = 4$ and $x = 8$.

Answer. $x = 4$, $x = 8$.

Solve the following equations.

3.6. $\log^3 x - \log^2 x - 6 \log x = 0$. 3.7*. $x^{2 + \log_3 x} = 3^8$.

3.8*. $9x^{\log x} + 91x^{-\log x} = 60$. 3.9*. $x^{2 \log^2 x} = 10x^3$.

3.10*. $x^{(2 \log^3 x - 1.5 \log x)} = \sqrt{10}$. 3.11. $4 - \log x = 3 \sqrt{\log x}$

3.12. $3 \sqrt{\log x} + 2 \log \sqrt{1/x} = 2$. 3.13*. $\sqrt{\log(-x)} = \log \sqrt{x^2}$

3.14. $\log_3(3^x - 1) \log_3(3^{x+1} - 3) = 6$. 3.15*. $15^{\log_5 3} x^{\log_5 9x+1} = 1$.

3.16*. $x^{\log^2 x + \log x^3 + 3} = \dfrac{2}{\dfrac{1}{\sqrt{x+1}-1} - \dfrac{1}{\sqrt{x+1}+1}}$.

A logarithmic equation of the form

$$\log_{a(x)} \cdot f(x) = \log_{a}(x)\, g(x)$$

is equivalent to the equation

$$f(x) = g(x)$$

which is considered on the set of the permissible values of x defined by a system of inequalities

$$f(x) > 0, \; g(x) > 0, \; a(x) > 0, \; a(x) \neq 1.$$

If the given equation includes logarithms to different bases, it is first necessary to reduce all the logarithms to the same base.

Example 3.3. Solve the equation

$$\log \sqrt{x-1} + \frac{1}{2} \log(2x + 15) = 1. \qquad (*)$$

Solution. The set of permissible values of the unknown x can be found as a solution of the system

$$x - 1 > 0, \; 2x + 15 > 0$$

and is an interval $(1, +\infty)$. Using the properties of logarithms, we reduce the given logarithmic equation to the form

$$\log(\sqrt{(x-1)} \cdot \sqrt{2x+15}) = 1.$$

By the definition of a logarithm, from the last equation we get an irrational equation

$$\sqrt{(x-1)(2x+15)} = 10,$$

whose solutions are $x_1 = 5$, $x_2 = -23/2$. The set of permissible values of x of the initial equation includes only the root $x_1 = 5$ which is indeed a solution of the initial equation.

Answer. $x = 5$.

Solve the following equations.

3.17. $2 \log_3(x-2) + \log_3(x-4)^2 = 0$.

3.18. $\log_5\left(\dfrac{2+x}{10}\right) = \log_5\left(\dfrac{2}{x+1}\right)$.

3.19. $\frac{1}{2}\log(x^2-10x+25)+\log(x^2-6x+3)=$

$2\log(x-5)+\frac{1}{2}\log 25.$

3.20. $\frac{1}{10}\log\sqrt[3]{x^2-4x+4}-\frac{1}{2}\log x-\log\frac{1}{\sqrt{x}}=0.$

3.21. $\log(x(x+9))+\log\frac{x+9}{x}=0.$

3.22. $\log_2(2x^2-2)=\log_2(5x-4).$ 3.23. $\frac{\log(35-x^3)}{\log(5-x)}=3.$

3.24. $\log_{x+1}(x-0.5)=\log_{x-0.5}(x+1).$

3.25. $\log_{x^3+2x^2-3x+5}(x^3+3x^2+2x-1)=\log_{2x}x+\log_{2x}2.$

3.26. $\log_{1+x}(2x^3+2x^2-3x+1)=3.$

3.27. $\log_{x+1}(x^3-9x+8)\log_{x-1}(x+1)=3.$

3.28. $\log_{x+1}(x^2+x-6)^2=4.$

3.29. $\log_{(3-4x^2)}(9-16x^4)=2+\frac{1}{\log_2(3-4x^2)}.$

3.30. $\log_{3x}\frac{3}{x}+\log_3^2 x=1.$ 3.31. $\log_{x^2}16+\log_{2x}64=3.$

3.32. $20\log_{4x}\sqrt{x}+7\log_{16x}x^3-3\log_{x/2}x^2=0.$

3.33. $\log_x 2-\log_4 x+\frac{7}{6}=0.$

3.34. $2-\log_{b^2}(1+x)=3\log_b\sqrt{x-1}-\log_{b^4}(x^2-1)^2.$

3.35. $3\log_x 4+2\log_{4x}4+3\log_{16x}4=0.$

Some logarithmic equations can be solved by investigating the behaviour of the functions appearing on their left-hand and right-hand sides.

Example 3.4. Solve the equation

$$\log_7(x+2)=6-x.$$

Solution. We make sure by selection that $x=5$ is a solution of the equation. It has no other solutions since the function appearing on the left-hand side increases and that on the right-hand side decreases and, consequently, the graphs of these functions cannot have more than one intersection.

Answer. $x=5$.

Solve the following equations.

3.36*. $(x+1)\log_3^2 x+4x\log_3 x-16=0.$

3.37*. $3x^2-2x^3=\log_2(x^2+1)-\log_2 x.$

3.38*. $3^x=10-\log_2 x.$

3.39*. $\log_2^2 x+(x-1)\log_2 x=6-2x.$

4. Systems of Exponential and Logarithmic Equations

Systems containing exponential or logarithmic equations are usually solved by reducing the exponential (or logarithmic) equation to an algebraic equation and solving the resulting algebraic system.

Example 4.1. Solve the system of equations

$$8(\sqrt{2})^{x-y} = 0.5^{y-3},$$
$$\log_3(x-2y) + \log_3(3x+2y) = 3.$$

Solution. The set of permissible values of the unknowns x and y is defined by the system of inequalities

$$x - 2y > 0,\ 3x + 2y > 0. \qquad (*)$$

From the exponential equation of the initial system written in the form

$$(\sqrt{2})^{x-y+6} = (\sqrt{2})^{6-2y},$$

we get an equation

$$x - y + 6 = 6 - 2y,$$

from the logarithmic equation written in the form

$$\log_3[(x-2y)(3x+2y)] = 3$$

we get an equation

$$(x-2y)(3x+2y) = 27.$$

Thus we have reduced the solution of the initial system to that of the system of equations

$$x - y + 6 = 6 - 2y, \qquad (**)$$
$$(x-2y)(3x+2y) = 27$$

considered on the set of permissible values of the unknowns which is defined by system (*). Finding the expression for y from the first equation of system (**) and substituting $y = -x$ into the second equation, we get an equation

$$3x^2 = 27,$$

whose solutions are $x_1 = 3$, $x_2 = -3$. From the first equation of the system we find $y_1 = -3$, $y_2 = 3$. From the two pairs of numbers $(3, -3)$ and $(-3, 3)$ we have found for the solutions of system (**), only the pair $(3, -3)$ satisfies the system of inequalities (*).

Answer. $(3, -3)$.

Solve the following systems of equations.

4.1. $\log_y x + \log_x y = 2,\quad x^2 - y = 20.$

4.2. $\log_4 x - \log_2 y = 0,\quad x^2 - 2y^2 - 8 = 0.$

4.3. $4^{x+y}=2^{y-x}$,
$4^{\log_{\sqrt{2}} x}=y^4-5$.

4.4. $\frac{x^2}{y}+\frac{y^2}{x}=12$,
$2^{-\log_2 x}+5^{\log_5 \frac{1}{y}}=\frac{1}{3}$.

4.5. $x+y=12$,
$2\,(2\log_{y^2}x-\log_{1/x}y)=5$.

4.6*. $4^x-7\cdot 2^{x-y/2}=2^{3-y}$,
$y-x=3$.

4.7. $3\left(\frac{2}{3}\right)^{2x-y}+7\left(\frac{2}{3}\right)^{(2x-y)/2}-6=0$,
$\log(3x-y)+\log(x+y)-4\log 2=0$.

4.8. $4^{x/y+y/x}=32$,
$\log_3(x-y)=1-\log_3(x+y)$.

4.9. $9^{\sqrt[4]{xy^2}}-27\cdot 3^{\sqrt{y}}=0$,
$\frac{1}{4}\log x+\frac{1}{2}\log y=\log\,(4-\sqrt[4]{x})$.

4.10. $3^{-x}\cdot 2^y=1152$,
$\log_{\sqrt{5}}(x+y)=2$.

4.11. $2^x\cdot 3^y=6$,
$3^x\cdot 4^y=12$.

4.12. $(0.48^{x^2+2})^{2x-y}=1$,
$\log(x+y)-1=\log 6-\log(x+2y)$.

4.13. $x^{x^2-y^2-16}=1$,
$x-y=2$.

If the power base in the exponential equation of the system is the function of the unknowns, then the system can be reduced to a system of rational equations by taking the logarithm of that function to a certain base as one of the unknowns.

Example 4.2. Solve the system of equations

$$x^{2y^2-1}=5,$$
$$x^{y^2+2}=125.$$

Solution. Taking logarithms in both equations of the system to the base 5, we get a system of equations which is equivalent to the initial system:

$$(2y^2-1)\log_5 x=1,$$
$$(y^2+2)\log_5 x=3.$$

Designating $\log_5 x=z$, we get a system of rational equations

$$(2y^2-1)\,z=1,$$
$$(y^2+2)\,z=3.$$

Finding the expression for z from the first equation and substituting it into the second equation, we get an equation

$$\frac{y^2+2}{2y^2-1}=3,$$

whose solutions are $y_1 = 1$, $y_2 = -1$. From the first equation of the system we find (both for $y = 1$ and for $y = -1$) the unknown $z = 1$. From the equation

$$\log_5 x = 1$$

we find $x = 5$. Thus the solutions of the initial system are two pairs of numbers (5, 1), (5, —1).

Answer. (5, 1), (5, —1).

Solve the following systems of equations.

4.14. $y = 1 + \log_4 x$, $x^y = 4^6$.

4.15. $y - \log_3 x = 1$, $x^y = 3^{12}$.

4.16. $(x+y)^{3y-x} = \frac{5}{27}$, $3\log_5(x+y) = x-y$.

4.17. $x^{x-2y} = 36$, $4(x-2y) + \log_6 x = 9$.

4.18. $(x+y)\cdot 2^{y-2x} = 6.25$, $(x+y)^{1/(2x-y)} = 5$.

Some systems of logarithmic or exponential equations can be reduced to systems of rational equations by a direct replacement of the logarithms appearing in them (or of the powers respectively) by new unknowns.

Example 4.3. Solve the system of equations

$$5^{\sqrt[3]{x}} \cdot 2^{\sqrt{y}} = 200,$$
$$5^{2\sqrt[3]{x}} + 2^{2\sqrt{y}} = 689.$$

Solution. Introducing the designations $z = 5^{\sqrt[3]{x}}$ and $u = 2^{\sqrt{y}}$, we get a system of rational equations

$$zu = 200,$$
$$z^2 + u^2 = 689$$

which is equivalent to two systems

$$zu = 200, \quad zu = 200,$$
$$z + u = 33, \quad z + u = -33.$$

The pairs of numbers (25, 8), (8, 25) or (—25, —8), (—8, —25) are solutions of these systems. The two last pairs are extraneous solutions of the initial system since $z > 0$ and $u > 0$. Returning to the initial unknowns, we get the following systems of equations:

$$5^{\sqrt[3]{x}} = 25, \quad 5^{\sqrt[3]{x}} = 8,$$
$$2^{\sqrt{y}} = 8, \quad 2^{\sqrt{y}} = 25,$$

whose solutions are $x = 8$, $y = 9$ and $x = (\log_5 8)^3$, $y = (\log_2 25)^2$.

Answer. (8, 9), $(27 \log_5^3 2,\ 4 \log_2^2 5)$.

Solve the following systems of equations.

4.19*. $2^{\sqrt{xy}-2}+4^{\sqrt{xy}-1}=5,\quad \dfrac{3(x+y)}{x-y}+\dfrac{5(x-y)}{x+y}=8.$

4.20*. $(x^2+y)\,2^{y-x^2}=1,\quad 9(x^2+y)=6^{x^2-y}.$

4.21. $11^{xz}-2\cdot 5^y=71,\quad 11^z+2\cdot 5^{y/2}=21,\quad 11^{(x-1)z}+5^{y/2}=16.$

5. Miscellaneous Problems

Solve the following equations.

5.1. $12^{2x+4}=3^{3x}\cdot 4^{x+8}$. 5.2. $2^x+4^{(x+1)/2}=8\cdot 3^{x/3}$.

5.3. $2^x-3^{x/2}=1$. 5.4. $(\sin 1)^x+(\cos 1)^x=1$.

5.5. $5^{\sqrt{x}}+12^{\sqrt{x}}=13^{\sqrt{x}}$. 5.6. $10^{x^2}=2\cdot 100^x$.

5.7. $x^{\sqrt{x}}=(\sqrt{x})^x$. 5.8**. $5^x \sqrt[x+1]{8^x}=100$.

5.9. $9^{1/x}+12^{1/x}=16^{1/x}$.

5.10. $11^{3x-2}+13^{3x-2}=13^{3x-1}-11^{3x-1}$.

5.11*. $10^{(x+1)(3x+4)}-2\cdot 10^{(x+1)(x+2)}=10^{1-x-x^2}$.

5.12*. $\left(\dfrac{4}{3}\right)^x=-2x^2+6x-9$. 5.13. $3^{x^2}+4^{x^2}=5^{x^2}$.

5.14. $x^{\sqrt[3]{x^2}}=(\sqrt{x})^x$.

5.15. Find all solutions of the equation

$$x^2\cdot 2^{x+1}+2^{|x-3|+2}=x^2\cdot 2^{|x-3|+4}+2^{x-1}.$$

5.16. $9^x-5\cdot 12^x+6\cdot 16^x=0$.

Solve the following logarithmic equations.

5.17. $\log_x 5+2\log_{25} x=-y^2-y+2.75$.

5.18. $\log^2 x^3-20\log\sqrt{x}+1=0$.

5.19. $|1-\log_{1/6} x|+2=|3-\log_{1/6} x|$.

5.20. $\sqrt{1+\log_x \sqrt{27}}\,\log_3 x+1=0$.

5.21. $\log_{\sqrt{x}}(x+|x-2|)=\log_x(5x-6+5|x-2|)$.

5.22. $\log_4(2x^2+x+1)-\log_2(2x-1)=1$.

5.23. $2\log_2\log_2 x+\log_{1/2}\log_2(2\sqrt{2}x)=1$.

5.24. $\log_{x+1}(x^2+x-6)^2=4$.

5.25. $\dfrac{1+\log_2(x-4)}{\log_{\sqrt{2}}(\sqrt{x+3}-\sqrt{x-3})}=1$.

5.26. $\log_5\left[(2+\sqrt{5})^x-(\sqrt{5}-2)^x\right]=\dfrac{1}{2}-3\log_{1/5}2$.

5.27*. $y x^{\log_y x} = x^{2.5}$,
$\log_3 y \log_y (y - 2x) = 1.$

5.28. $\log_2 x + \log_4 y + \log_4 z = 2,$
$\log_3 y + \log_9 z + \log_9 x = 2,$
$\log_4 z + \log_{16} x + \log_{16} y = 2.$

5.29. $10^{\log \frac{1}{2}(x^2+y^2)+1.5} = 100\sqrt{10},$

$$\frac{\sqrt{x^2+10y}}{3} = \frac{6}{2\sqrt{x^2+10y}-9}.$$

5.30*. $x^{\sqrt[4]{x}+\sqrt{y}} = y^2\sqrt[3]{y^2},$
$y^{\sqrt[4]{x}+\sqrt{y}} = \sqrt[3]{x^2}.$

5.31*. $\log_y |\log_y x| = \log_x |\log_x y|$
$\log^2 x + \log^2 y = 8.$

5.32. $\log_{12} x \left(\frac{1}{\log_x 2} + \log_2 y\right) = \log_2 x,$

$$\log_2 x [\log_3 (x+y)] = 3 \log_3 x.$$

5.33. $z^x = x,\ z^y = y,\ y^y = x.$

5.34. $\log_a x \log_a (xyz) = 48,$
$\log_a y \log_a (xyz) = 12,\quad a > 0,\quad a \neq 1.$
$\log_a z \log_a (xyz) = 84.$

Chapter 3

Inequalities. Equations and Inequalities with Parameters

Assume that $f(x)$ is a number function of one or several variables (arguments). *To solve the inequality*

$$f(x) < 0 \tag{1}$$

is to find all the values of the argument (arguments) of the function f for which inequality (1) holds true. The set of all values of the argument (arguments) of the function f for which inequality (1) holds true is called the *set of solutions* of the inequality or simply the solution of the inequality.

The set of solutions of the nonstrict inequality

$$f(x) \leqslant 0 \tag{2}$$

is a union of the sets of solutions of inequality (1) and the set of solutions of the equation $f(x) = 0$.

Two inequalities are considered to be *equivalent* if the sets of their solutions coincide.

The *set of permissible values* of the unknowns appearing in the inequality is the domain of definition of the function $f(x)$.

Separate inequalities of form (1) or (2) formed for various functions $f_i(x)$ can be reduced to a system of inequalities. *To solve a system of inequalities* is to find the set of all values of the arguments of the functions $f_i(x)$ for which all the inequalities of the system simultaneously hold true.

Systems of inequalities are said to be *equivalent* if the sets of their solutions coincide.

1. Algebraic Inequalities

Linear inequalities (strict and nonstrict) are inequalities of the form $ax + b > 0$, $ax + b < 0$, $ax + b \geqslant 0$, $ax + b \leqslant 0$, $a \neq 0$, whose solutions for $a > 0$ are, respectively, the values

$$x \in \left(-\frac{b}{a};\ +\infty\right), \quad x \in \left(-\infty;\ -\frac{b}{a}\right),$$

$$x \in \left[-\frac{b}{a};\ +\infty\right), \quad x \in \left(-\infty;\ -\frac{b}{a}\right],$$

and for $a < 0$ the values

$$x \in \left(-\infty; -\frac{b}{a}\right), \quad x \in \left(-\frac{b}{a}; +\infty\right),$$

$$x \in \left(-\infty; -\frac{b}{a}\right], \quad x \in \left[-\frac{b}{a}; +\infty\right).$$

Quadratic inequalities (strict and nonstrict) are inequalities of the form

$$ax^2 + bx + c > 0, \quad ax^2 + bx + c < 0,$$
$$ax^2 + bx + c \geqslant 0, \quad ax^2 + bx + c \leqslant 0,$$

where a, b, and c are certain real numbers and $a \neq 0$.

Depending on the values of its coefficients a, b, and c, the quadratic inequality $ax^2 + bx + c > 0$ has the following solutions:

(1) for $a > 0$ and $D = b^2 - 4ac \geqslant 0$,

$$x \in \left(-\infty; \frac{-b-\sqrt{D}}{2a}\right) \cup \left(\frac{-b+\sqrt{D}}{2a}, \quad +\infty\right);$$

(2) for $a > 0$ and $D < 0$, $x \in \mathbf{R}$;

(3) for $a < 0$ and $D \geqslant 0$,

$$x \in \left(\frac{-b+\sqrt{D}}{2a}, \quad \frac{-b-\sqrt{D}}{2a}\right);$$

(4) for $a < 0$ and $D < 0$, $x = \varnothing$ (i.e. there are no solutions).

The solution of the inequality $ax^2 + bx + c < 0$ reduces to the solution of the inequality considered above if we multiply both sides of the inequality by -1.

The set of solutions of the nonstrict inequalities $ax^2 + bx + c \geqslant 0$ and $ax^2 + bx + c \leqslant 0$ are found as the union of the sets of solutions of the corresponding strict inequalities and the equation $ax^2 + bx + c = 0$.

Fractional linear inequalities are inequalities which can be reduced to the form

$$\frac{ax+b}{cx+d} > k, \tag{3}$$

where a, b, c, d, k are certain real numbers and $c \neq 0$ (if $c = 0$, then the fractional linear inequality turns into a linear inequality). Inequalities of form (3), where the signs $<$, $\geqslant$, $\leqslant$ stand instead of the sign $>$, are also fractional linear. The solution of a fractional linear inequality can be reduced to that of a quadratic inequality by multiplying both sides of inequality (3) by the expression $(cx + d)^2$, which is positive for all $x \in \mathbf{R}$ and $x \neq -d/c$.

The method of intervals. Suppose $P(x)$ is an nth-degree polynomial with real coefficients, and $c_1, c_2, \ldots, c_l$ are all real roots of the polynomial with multiplicities $k_1, k_2, \ldots, k_l$ respectively, with $c_1 > c_2 > \ldots > c_l$. The polynomial $P(x)$ can be represented in the form

$$P(x) = (x - c_1)^{k_1} (x - c_2)^{k_2} \dots (x - c_l)^{k_l} Q(x), \qquad (4)$$

where the polynomial $Q(x)$ has no real roots and is either positive or negative for all $x \in \mathbf{R}$. Let us assume for definiteness that $Q(x) > 0$. Then all factors in expansion (4) are positive for $x > c_1$, and $P(x) > 0$. If c_1 is a root of odd multiplicity (k_1 is odd), then all factors in expansion (4), except for the first one, are positive for $c_2 < x < c_1$ and $P(x) < 0$. In that case we say that the polynomial $P(x)$ *changes sign* when passing through the root c_1. Now if c_1 is a root of even multiplicity (k_1 is even), then all factors (the first one inclusive) are positive for $c_2 < x < c_1$ and, consequently, $P(x) > 0$ for $x \in (c_2; c_1)$. In that case we say that the polynomial $P(x)$ *does not change sign* when passing through the root c_1.

In a similar way, using expansion (4), we can easily verify that passing through the root c_2 the polynomial $P(x)$ changes sign if k_2 is odd and does not change sign if k_2 is even. This property of polynomials can be used to solve inequalities by the method of intervals. To find all solutions of the inequality

$$P(x) > 0, \qquad (5)$$

it is sufficient to know all real roots of the polynomial $P(x)$, their multiplicities, and the sign of the polynomial $P(x)$ at an arbitrarily chosen point which does not coincide with the root of the polynomial.

Example 1.1. Solve the inequality

$$x^2 (x + 2) (x - 1)^3 (x^2 + 1) > 0. \qquad (*)$$

Solution. Let us arrange the real roots of the polynomial appearing on the left-hand side of the inequality on the number axis Ox (Fig. 3.1). For $x > 1$ the polynomial is positive since all factors appear ing on the left-hand side of the inequality are positive. We shall move along the Ox axis from right to left. When passing through the

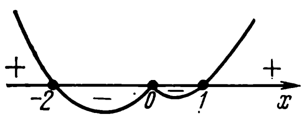

Fig. 3.1

point $x = 1$ the polynomial changes sign and becomes negative since $x = 1$ is a root of multiplicity 3; when passing through the point $x = 0$ the polynomial does not change sign since $x = 0$ is a root of multiplicity 2; when passing through the point $x = -2$ the polynomial again changes sign and becomes positive. The intervals of the constancy of sign of the given polynomial are shown schematically in Fig. 3.1. Using this figure, it is easy to write the set of solutions of inequality (*): $x \in (-\infty; -2) \cup (1, +\infty)$.

The solution of a *rational* inequality, i.e. inequality of the form

$$\frac{P(x)}{Q(x)} > 0, \qquad (6)$$

where $P(x)$ and $Q(x)$ are polynomials, can be reduced to that of equivalent inequality (5) as follows: multiplying both sides of inequality (6) by the polynomial $[Q(x)]^2$, which is positive for all permissible values of the unknown x (i.e. for those values of x for which $Q(x) \neq 0$), we obtain a rational inequality

$$P(x)\,Q(x) > 0,$$

which is equivalent to inequality (6), it can be solved by the method of intervals.

Example 1.2. Solve the rational inequality

$$\frac{x^2(x-1)^3(x+2)}{x-3} < 0. \qquad (*)$$

Solution. Multiplying both sides of the inequality by $(x-3)^2$, we obtain an inequality which is equivalent to inequality (*):

$$x^2(x-1)^3(x+2)(x-3) < 0.$$

The set of solutions of the last inequality can be found by the method of intervals: $x \in (-\infty;\ -2) \cup (1;\ 3)$.

Answer. $x \in (-\infty;\ -2) \cup (1;\ 3)$.

Solve the following inequalities.

1.1. $\dfrac{1}{2-x}+\dfrac{5}{2+x}<1.$ **1.2.** $\dfrac{1}{x+2}<\dfrac{3}{x-3}.$

1.3. $(x+1)(3-x)(x-2)^2 \geqslant 0.$ **1.4.** $\dfrac{x^2-3x+2}{x^2+3x+2} \geqslant 1.$

1.5. $\dfrac{x^3-x^2+x-1}{x+8} \leqslant 0.$ **1.6.** $\dfrac{2x-5}{x^2-6x-7}<\dfrac{1}{x-3}.$

1.7. $\dfrac{3}{x^2-x+1} \geqslant 1.$ **1.8.** $\dfrac{x^3-2x^2-5x+6}{x-2}>0.$

1.9. $\dfrac{2-x^2}{1-x} \leqslant x.$ **1.10.** $\dfrac{x-4}{4x^2-4x-3}<0.$

1.11. $\dfrac{x^2-2x+3}{x^2-4x+3}>-3.$

Irrational inequalities. An irrational inequality is an inequality in which the unknown quantities (or certain functions of the unknowns) are under the radical sign. To find the set of solutions of an irrational inequality, it is necessary, as a rule, to raise both sides of the inequality into a natural power. In that case, since it is impossible in principle to verify by substitution the solutions obtained, it is necessary to make sure that in transforming the inequalities we each time get an inequality which is equivalent to the initial inequality.

When solving irrational inequalities, one should remember that when raising both sides of an inequality into an odd power, we always obtain an inequality which is equivalent to the initial inequality. Now if we raise both sides of an inequality into an even power, we get an inequality which is equivalent to the initial inequality and has the same sign only in the case when both sides of the initial inequality are nonnegative.

Example 1.3. Solve the inequality

$$\sqrt{x-5}-\sqrt{9-x}>1. \qquad (*)$$

Solution. The set of permissible values is $x \in [5, 9]$. Inequality (*) is equivalent to the inequality

$$\sqrt{x-5}>\sqrt{9-x}+1, \qquad (**)$$

whose both sides are nonnegative. Squaring both sides of inequality (**), we get an equivalent inequality

$$2x-15>2\sqrt{9-x}. \qquad (***)$$

(1) If $2x-15 \leqslant 0$, i.e. $x \leqslant 15/2$, then the left-hand side of the inequality is negative or equal to zero and the right-hand side is positive. Therefore, for any value of x on the interval $[5, 15/2]$ inequality (***) is not satisfied*.

(2) If $2x-15>0$, i.e. $x>15/2$, then both sides of the inequality are nonnegative, and after the squaring we get an inequality which is equivalent to inequality (***):

$$(2x-15)^2>4(9-x).$$

Thus the set of solutions of inequality (*) is obtained as the set of solutions of the system of inequalities

$$5 \leqslant x \leqslant 9,$$
$$2x-15>0,$$
$$(2x-15)^2>4(9-x),$$

whence we get $x \in \left(\frac{14+\sqrt{7}}{2}, 9\right]$.

Answer. $x \in \left(\frac{14+\sqrt{7}}{2}, 9\right]$.

Solve the following inequalities.

1.12. $\sqrt{1-3x}-\sqrt{5+x}>1$. **1.13.** $\sqrt{4-\sqrt{1-x}}-\sqrt{2-x}>0$.

1.14. $\sqrt{x^2+4x-5}-2x+3>0$. **1.15.** $x+4<\sqrt{x+46}$.

1.16. $\sqrt{2-\sqrt{3+x}}<\sqrt{x+4}$. **1.17.** $\frac{\sqrt{24-2x-x^2}}{x}<1$.

1.18. $\frac{\sqrt{x+5}}{1-x}<1$. **1.19.** $\frac{4-\sqrt{x+1}}{1-\sqrt{x+3}}\leqslant 3$.

* If the initial inequality (*) had an opposite sign, then all the values of x satisfying the inequality $x \leqslant 15/2$ and belonging to the set of permissible values of the initial inequality would be solutions of the given inequality.

1.20. $\sqrt{8-x^2}-\sqrt{25-x^2}>x$. 1.21. $\frac{1}{\sqrt{1+x}}>\frac{1}{2-x}$.

1.22. $\sqrt{x^2-x-2}\geqslant 2x+3$. 1.23. $\sqrt{x^2+3x+4}>-2$.

1.24. $\frac{\sqrt{x^2-16}}{\sqrt{x-3}}+\sqrt{x-3}>\frac{5}{\sqrt{x-3}}$.

1.25. $\sqrt{3x^2+5x+7}-\sqrt{3x^2+5x+2}>1$.

Inequalities containing an unknown under the sign of an absolute value. The signs of an absolute value in inequalities can be removed in the same way as in similar equations.

Example 1.4. Solve the inequality

$$|x^2-1|-2x<0.$$

Solution. The expression under the sign of an absolute value can assume a positive as well as a negative value.

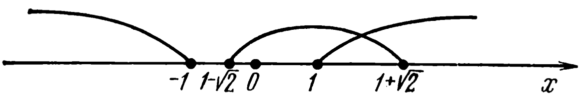

Fig. 3.2

(1) Let us assume that $x^2-1\geqslant 0$. Then, according to the definition of an absolute value $|x^2-1|=x^2-1$, the initial inequality assumes the form

$$x^2-2x-1<0.$$

The solution of this inequality is $1-\sqrt{2}<x<1+\sqrt{2}$. The intersection of the sets of solutions of the inequalities $x^2-1\geqslant 0$ and

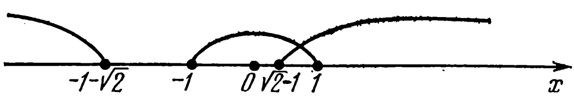

Fig. 3.3

$x^2-2x-1<0$ yields the first set of solutions of the given inequality (Fig. 3.2): $x\in[1,1+\sqrt{2})$.

(2) Suppose now that $x^2-1<0$. Then, according to the definition of an absolute value $|x^2-1|=-(x^2-1)$, the given inequality assumes the form $-x^2+1-2x<0$, or, what is the same,

$$x^2+2x-1>0.$$

The solution of the last inequality is $x<-1-\sqrt{2}$ and $x>\sqrt{2}-1$. The intersection of the sets of solutions of the inequalities $x^2-1<0$

and $x^2 + 2x - 1 > 0$ yields the second set of solutions of the given inequality (Fig. 3.3): $x \in (\sqrt{2} - 1, 1)$.

Thus, the solutions of the given inequality are the values $x \in (\sqrt{2} - 1, \sqrt{2} + 1)$.

Answer. $x \in (\sqrt{2} - 1, \sqrt{2} + 1)$.

Solve the following inequalities.

1.26. $|x - 3| > -1$. **1.27.** $|4 - 3x| \leqslant 1/2$.
1.28. $x^2 + 2|x + 3| - 10 \leqslant 0$. **1.29.** $|x^2 - 1| - 2x > 0$.
1.30. $x^2 + x - 10 < 2|x - 2|$. **1.31.** $x^2 - |3x + 2| + x \geqslant 0$.
1.32. $|x^2 - 3| + 2x + 1 \geqslant 0$. **1.33.** $\frac{9}{|x-5|-3} \geqslant |x-2|$.

2. Exponential Inequalities

The elementary *exponential inequalities* are inequalities of the form

$$a^x > b, \quad a^x < b, \tag{1}$$

where a and b are certain numbers ($a > 0$, $a \neq 1$).

Depending on the values of the parameters a and b, the set of solutions of the inequality $a^x > b$ can be in the following forms:

(1) $x \in (\log_a b, +\infty)$ for $a > 1$, $b > 0$;
(2) $x \in (-\infty, \log_a b)$ for $0 < a < 1$, $b > 0$);
(3) $x \in \mathbf{R}$ for $a > 0$, $b < 0$.

Depending on the values of a and b, the set of solutions of the inequality $a^x < b$ can be in the following forms:

(1) $x \in (-\infty, \log_a b)$, for $a > 1$, $b > 0$;
(2) $x \in (\log_a b, +\infty)$, for $0 < a < 1$, $b > 0$;
(3) $x = \varnothing$ for $a > 0$, $b < 0$ (i.e. the inequality has no solutions).

The set of solutions of the nonstrict inequalities $a^x \geqslant b$ and $a^x \leqslant b$ can be found as the union of the sets of solutions of the corresponding strict inequalities and the equation $a^x = b$.

Inequalities of form (1) can be generalized to the case when the exponent contains a certain function of x. Thus the set of solutions of the inequality

$$2^{f(x)} > 3 \tag{2}$$

can be found as the set of solutions of the inequality

$$f(x) > \log_2 3,$$

which is equivalent to inequality (2).

The methods of reducing complicated exponential inequalities to inequalities of form (1) or (2) are similar to the methods used in solving exponential equations. Thus, for instance, the solution of an exponential inequality of the form

$$P(a^x) > 0,$$

where $P(a^x)$ is a polynomial of the indicated argument, can be reduced, by the substitution $a^x = y$, to successive solutions of the inequality $P(y) > 0$ and of the elementary exponential inequalities of form (1) or systems of elementary exponential inequalities.

Example 2.1. Solve the inequality

$$9^x - 10 \cdot 3^x + 9 \leqslant 0.$$

Solution. We introduce the designation $3^x = y$. Since $9^x = (3^2)^x = (3^x)^2$, we have $9^x = y^2$ and for the variable y the given inequality assumes the form

$$y^2 - 10y + 9 \leqslant 0.$$

The solution of this quadratic inequality is $1 \leqslant y \leqslant 9$. This double inequality is equivalent to a system of two inequalities

$$y \geqslant 1, \ y \leqslant 9,$$

which assumes, for the unknown x, the form

$$\begin{matrix} 3^x \geqslant 1, \\ 3^x \leqslant 9, \end{matrix} \Leftrightarrow \begin{matrix} x \geqslant 0, \\ x \leqslant 2, \end{matrix} \Leftrightarrow 0 \leqslant x \leqslant 2.$$

Answer. $x \in [0, 2]$.

Solve the following inequalities.

2.1. $4^x + 2^{x+1} - 6 \leqslant 0$. **2.2.** $4^{-x+1/2} - 7 \cdot 2^{-x} - 4 < 0$.

2.3. $25^{-x} + 5^{-x+1} \geqslant 50$. **2.4.** $4^{x^2} - 3 \cdot 2^{x^2} + 1 \geqslant 0$.

2.5. $2 \cdot 3^{2x^2} + 4 \leqslant 3^{x^2+2}$. **2.6.** $\left(\frac{1}{3}\right)^{\sqrt{x+4}} > \left(\frac{1}{3}\right)^{\sqrt{x^2+3x+4}}$.

2.7. $98 - 7^{x^2+5x-48} \geqslant 49^{x^2+5x-49}$. **2.8.** $5 \cdot 4^x + 2 \cdot 25^x \leqslant 7 \cdot 10^x$.

2.9. $\sqrt{13^x - 5} \leqslant \sqrt{2 \cdot (13^x + 12)} - \sqrt{13^x + 5}$.

2.10. $9^{\sqrt{x^2-3}} + 3 < 3^{\sqrt{x^2-3}-1} \cdot 28$.

2.11. $5^{2x-10-3\sqrt{x-2}} - 4 \cdot 5^{x-5} < 5^{1+3\sqrt{x-2}}$.

2.12. $\left(\frac{1}{4}\right)^x < 2^{3-x} - 25^{1/\log_3 5}$.

2.13. $25^x - 2^{2\log_4 6 - 1} < 10 \cdot 5^{x-1}$.

2.14. $5^{2x+1} + 6^{x+1} > 30 + 5^x \cdot 30^x$.

2.15. $\sqrt{8 + 2^{\sqrt{3-x}+1} - 4^{\sqrt{3-x}}} + 2^{\sqrt{3-x}+1} > 5$.

2.16. $\sqrt{4^{x+1} + 17} - 5 > 2^x$. **2.17.** $3^{x+1} < \frac{9^{4x^2}}{\sqrt{27}}$.

2.18. $4^x \leqslant 3 \cdot 2^{\sqrt{x}+x} + 4^{1+\sqrt{x}}$.

2.19. $\begin{matrix} 3^{2x+1} - 3^{x+2} + 6 > 0, \\ 3^{2x+2} - 2 \cdot 3^{x+2} - 27 < 0. \end{matrix}$

2.20. $|3^{\tan \pi x} - 3^{1 - \tan \pi x}| \geqslant 2$.

3. Logarithmic Inequalities

The elementary *logarithmic inequalities* are inequalities of the form

$$\log_a x > b, \quad (1)$$

$$\log_a x < b, \quad (2)$$

where a and b are certain real numbers ($a > 0$, $a \neq 1$). Depending on the values of a, the sets of solutions of inequality (1) can be
(1) $x \in (a^b, +\infty)$ for $a > 1$;
(2) $x \in (0, a^b)$ for $a < 1$,
and those of inequality (2) can be
(1) $x \in (0, a^b)$ for $a > 1$;
(2) $x \in (a^b, +\infty)$ for $0 < a < 1$.

The set of solutions of the nonstrict inequalities $\log_a x \geqslant b$ and $\log_a x \leqslant b$ can be found as the union of the sets of solutions of the corresponding strict inequality and the equation

$$\log_a x = b.$$

An inequality of the form

$$\log_a f(x) > b \quad (3)$$

is equivalent to the following systems of inequalities*:
(1) $f(x) > 0$, $f(x) > a^b$ for $a > 1$;
(2) $f(x) > 0$, $f(x) < a^b$ for $a < 1$,
and an inequality of the form

$$\log_a f(x) < b \quad (4)$$

is equivalent to the following systems of inequalities:
(1) $f(x) > 0$, $f(x) < a^b$ for $a > 1$;
(2) $f(x) > 0$, $f(x) > a^b$ for $0 < a < 1$.

More complicated logarithmic inequalities can be reduced to inequalities of forms (1)—(4) by the methods similar to those used in solving logarithmic equations. Thus, for instance, the set of solutions of inequalities of the form

$$P(\log_a x) > 0, \quad (5)$$

as well as of the inequalities $P < 0$, $P \geqslant 0$, $P \leqslant 0$, where P is a polynomial of the indicated argument, can be found as follows. A new unknown $y = \log_a x$ is introduced and inequality (5) is solved as an algebraic inequality with respect to the unknown y. Then the solution of the initial inequality reduces to that of the corresponding elementary inequalities (1), (2) or systems of those inequalities.

Example 3.1. Solve the logarithmic inequality

$$\log_4^2 x - \log_2 x - 15 > 0. \quad (*)$$

Solution. The set of permissible values of the unknown is $x > 0$. Reducing the logarithms to the same base (say, to the base 2), we get

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* If inequality (3) is nonstrict, the second inequalities of these systems are also nonstrict.

an inequality which is equivalent to inequality (*):

$$\frac{1}{4}\log_2^2 x-\log_2 x-15>0. \qquad (**)$$

We solve this inequality as a quadratic inequality with respect to the new unknown $y=\log_2 x$. As a result we find that logarithmic inequality (**) is equivalent to two elementary inequalities

$$\log_2 x>10,\ \log_2 x<-6.$$

The union of their sets of solutions yields the set of solutions of inequality (*): $x\in(0,\ 2^{-6})\cup(2^{10},\ +\infty)$.

Answer. $x\in(0,\ 2^{-6})\cup(2^{10},\ +\infty)$.

A logarithmic inequality of the form

$$\log_{g(x)} f(x)>c \qquad (6)$$

is equivalent to two systems of inequalities:

$$\begin{array}{ll} f(x)>0, & f(x)>0,\\ g(x)>1, & 0<g(x)<1,\\ f(x)>[g(x)]^c; & f(x)<[g(x)]^c. \end{array} \qquad (7)$$

Example 3.2. Solve the inequality

$$\log_x\left(2x-\frac{3}{4}\right)>2. \qquad (*)$$

Solution. The given logarithmic inequality is equivalent to two systems of inequalities:

$$\begin{array}{ll} 2x-3/4>0, & 2x-3/4>0,\\ x>1, & 0<x<1,\\ 2x-3/4>x^2, & 2x-3/4<x^2. \end{array}$$

The set of solutions of logarithmic inequality (*) can be obtained as the union of the sets of solutions of these two systems.

Answer. $x\in\left(\frac{3}{8},\ \frac{1}{2}\right)\cup\left(1,\ \frac{3}{2}\right)$.

Solve the following inequalities.

3.1. $\log_{1/2}(2x+3)>0$. **3.2.** $\log_2\frac{x-3}{x+2}<0$.

3.3. $\log_{5/8}\left(2x^2-x-\frac{3}{8}\right)\geqslant 1$. **3.4.** $\log_{1/4}\frac{2x-1}{x+1}<\cos\frac{2\pi}{3}$.

3.5. $\log_2^2 x+\log_2 x-2\leqslant 0$. **3.6.** $2\log_4(2x^2+3)<\log_2(x^2+6)$.

3.7. $\log_2\sqrt{x}-2\log_{1/4}^2 x+1>0$. **3.8.** $\log_{1/4}(2x+3)>\log_9 27$.

3.9. $\log(x-4)+\log x<\log 21$. **3.10.** $\log_7 x-\log_x\left(\frac{1}{7}\right)\geqslant 2$.

3.11. $\log_2[(x-3)(x+2)]+\log_{1/2}(x-3)<-\log_{1/\sqrt{2}} 3$.

3.12. $\log_{100}(x^2)+\log_{10}^2 x<2$.

3.13. $\log_3 (7-x) \leqslant \frac{9}{16} \log^2_{\sqrt{2}} \frac{1}{4} + \log_{7-x} 9.$

3.14. $\log_3 x - \log_3^2 x \leqslant \frac{3}{2} \log_{1/(2\sqrt{2})} 4.$

3.15. $\log_{1/2} (4-x) \geqslant \log_{1/2} 2 - \log_{1/2} (x-1).$

3.16. $\log_2 (3-x) - \log_2 \frac{\sin \frac{3\pi}{4}}{5-x} > \frac{1}{2} + \log_2 (x+7).$

3.17. $2 \log_{1/4} (x+5) > \frac{9}{4} \log_{1/(3\sqrt{3})} 9 + \log_{\sqrt{x+5}} 2.$

3.18. $\log_4 (3^x - 1) \cdot \log_{1/4} \left(\frac{3^x-1}{16}\right) \leqslant \frac{3}{4}.$

3.19. $\log_{1/2} x \leqslant \log_{1/4} x.$ 3.20. $\log_3 (3^{4x} - 3^{2x+1} + 3) < 2 \log_9 7.$

3.21. $\log_{10} |2x+3|^3 + 2 \log_{(2x+3)^3} 10 < 3.$

3.22. $\log_{x/2} 8 + \log_{x/4} 8 < \frac{\log_2 x^4}{\log_2 x^2 - 4}.$

3.23. $8^{\log_2 x} - 2x^2 > x - 2.$

3.24. $2 \log_4 x - \frac{1}{2} \log_2 (x^2 - 3x + 2) \leqslant \cos \frac{4\pi}{3}.$

3.25. $\log_{4/3} \cos x \geqslant \log_{4/9} \frac{3}{2}$ for $x \in (-2;\ 3).$

3.26. $\log_{1/2} |x-3| > -1.$ 3.27. $\log_2 (\sqrt{x+3} - x - 1) \leqslant 0.$

3.28. $\frac{\sqrt{\log^2_{1/2} x - 81} + 2}{\log_{1/2} x - 1} < 1.$ 3.29. $\frac{\log_{1/2} \sqrt{x+4}}{\log_{1/2} (x+2)} \leqslant 1.$

3.30. $\log_3 \sqrt{5-2x} \cdot \log_x 3 < 1.$ 3.31. $\log_2 x \sqrt{\log_x \left(\frac{\sqrt{x}}{2}\right)} \leqslant 1.$

3.32. $\sqrt{\log_4 \left(\frac{2x^2-3x+3}{2}\right)} + 1 > \log_2 \left(\frac{2x^2-3x+3}{2}\right).$

3.33. $\frac{1 - \sqrt{1 - 8 \log_2^2 x}}{2 \log_2 x} < 1.$ 3.34. $\frac{\log_8 x}{\log_2 (1+2x)} \leqslant \frac{\log_2 \sqrt[3]{1+2x}}{\log_2 x}.$

3.35. $\log_x \left(x^2 - \frac{3}{16}\right) > 4.$ 3.36. $\log_{2x-x^2} \left(x - \frac{3}{2}\right)^4 > 0.$

3.37. $\log_{x+4} (5x+20) \leqslant \log_{x+4} [(x+4)^2].$

3.38. $\log_{1/x} \frac{2(x-2)}{(x+1)(x-5)} \geqslant 1.$

3.39. $\log_{(x+1/x)} \left(x^2 + \frac{1}{x^2} - 4\right) \geqslant 1.$

3.40. $\log_{\sqrt{x+1} - \sqrt{x-1}} (x^2 - 3x + 1) \geqslant 0.$

3.41. $(\log_{|x+6|} 2) \log_2 (x^2 - x - 2) \geqslant 1.$

3.42. $\log_{1/\sqrt{5}}(6^{x+1}-36^x) \geqslant -2$.

3.43. $\log_3 \log_{9/16}(x^2-4x+3) \leqslant 0$.

3.44. $x^{\log x} > 10 \cdot x^{-\log x}+3$. **3.45.** $\log_x [\log_2(4^x-6)] \leqslant 1$.

3.46. $\left(\frac{1}{2}\right)^{\log_3 \log_{1/5}(x^2-4/5)} < 1$.

3.47. $12x+\sqrt{3x^4+4x^5-4x^6} \cdot \log_2 x^2 > 3\sqrt{3+4x-4x^2} + 4x^3 \log_4 x^4$.

The expressions on the left-hand and right-hand sides of inequalities 3.48-3.52 are positive and these inequalities can be solved by taking logarithms of both sides of the inequalities.

3.48. $x^{\log^2 x-3\log x+1} > 1000$.

3.49. $x^3 > 2^{15\log_{2\sqrt{2}}\sqrt{3}} \cdot 3^{\log_{\sqrt{x}} 3}$.

3.50. $(x^2-x-1)^{x^2-1} < 1$.

3.51. $|x|^{x^2-x-2} < 1$.

3.52. $x^{1/\log x} \log x < 1$.

Inequalities 3.53-3.58 are equivalent to systems of trigonometric inequalities or systems of algebraic and trigonometric inequalities.

3.53. $\log_{|\sin x|}(x^2-8x+23) > \frac{3}{\log_2|\sin x|}$.

3.54. $(\log_{\sin x} 2)^2 < \log_{\sin x}(4\sin^3 x)$.

3.55. $\log_{\cos(x^2)}\left(\frac{3}{2}-2x\right) < \log_{\cos(x^2)}(2x-1)$.

3.56. $\sqrt{\tan x-1}\,[\log_{\tan x}(2+4\cos^2 x)-2] \geqslant 0$.

3.57. $\log_5 \sin x > \log_{125}(3\sin x-2)$.

3.58. $\log_{\sin x+\sqrt{3}\cos x}\left(\frac{x^2}{2}-\frac{5x}{2}+3\right) \geqslant 0$.

Find all integers satisfying the following inequalities.

3.59*. $3^{\frac{5}{2}\log_3(12-3x)}-3^{\log_2 x} > 83$.

3.60. $x-\frac{1}{2} < 2\log_5(x+2)$.

3.61. $\log_{\left(2\cos\frac{2\pi}{7}-x+8\right)}\left(\frac{\sqrt{x+5}-1}{\sqrt{10-x}}\right) \geqslant 0$.

4. Equations and Inequalities with Parameters

Equations and inequalities with parameters are traditionally the most difficult problems from the course of elementary mathematics. The solution of these problems reduces, in essence, to the investigation of the behaviour of the functions entering into the equation, with the subsequent solution of equations or inequalities with numerical coefficients. When solving equations (inequalities) with parameters, it is necessary to find out at what values of the parameters the equation (inequality) possesses a solution and then find all the solutions.

Example 4.1. Solve the inequality

$$ax > 1/x$$

for all values of a.

Solution. Writing the inequality in the form

$$\frac{ax^2-1}{x} > 0,$$

we find that the initial inequality is equivalent to two systems of inequalities:

$$\begin{aligned} ax^2 - 1 &> 0, \\ x &> 0; \end{aligned} \qquad \begin{aligned} ax^2 - 1 &< 0, \\ x &< 0. \end{aligned}$$

Let us solve the first inequality of the first system writing it in the form

$$ax^2 > 1.$$

For $a > 0$ it is equivalent to the inequality $x^2 > 1/a$ whose set of solutions is $x < -1/\sqrt{a}$ and $x > 1/\sqrt{a}$. In that case the solutions of the first system are $x \in (1/\sqrt{a}, +\infty)$. For $a \leqslant 0$ the left-hand side of the inequality $ax^2 - 1 > 0$ is negative for any x and the inequality has no solutions, and, consequently, the first system on the whole has no solutions.

Let us consider the second system. For $a > 0$ the solutions of the inequality $ax^2 - 1 < 0$ are $x \in (-1/\sqrt{a}, 1/\sqrt{a})$, and the solutions of the system are $x \in (-1/\sqrt{a}, 0)$. For $a \leqslant 0$ the left-hand side of the inequality $ax^2 - 1 < 0$ is negative for any values of x, i.e. the inequality holds for all $x \in \mathbf{R}$, and, consequently, $x \in (-\infty, 0)$ are solutions of the system.

Answer. If $a < 0$, then $x \in (-\infty, 0)$; if $a > 0$, then $x \in (-1/\sqrt{a}, 0) \cup (1/\sqrt{a}, +\infty)$.

Algebraic Equations and Inequalities with Parameters.

4.1. Solve the equation $x^2 + |x| + a = 0$ for every real number a.

4.2. Find all values of the parameter a for which both roots of the quadratic equation

$$x^2 - 6ax + (2 - 2a + 9a^2) = 0$$

are real and exceed 3.

4.3. Find all values of the parameter a for which both roots of the quadratic equation

$$x^2 - ax + 2 = 0$$

are real and belong to the interval (0, 3).

4.4. For what real values of m is the inequality

$$x^2 + mx + m^2 + 6m < 0$$

satisfied for all $x \in (1; 2)$?

4.5. Find all values of m such that

$$mx^2 - 4x + 3m + 1 > 0$$

for all $x > 0$.

4.6. For what real m does the inequality

$$x^2 - (3m + 1)x + m > 0$$

yield an inequality $x > 1$?

4.7. Find all values of the parameter a for which the inequality $ax^2 - x + 1 - a < 0$ yields an inequality $0 < x < 1$.

4.8. Find all values of the parameter a for which the inequality $0 \leqslant x \leqslant 1$ yields an inequality

$$(a^2 + a - 2)x^2 - (a + 5)x - 2 \leqslant 0.$$

4.9. Find all values of a for which the inequality

$$2x^2 - 4a^2x - a^2 + 1 > 0$$

is valid for all x which do not exceed unity in the absolute value.

4.10. Find all values of the parameter p for which the equation

$$(3x)^2 + (3^{1/p+3} - 15)x + 4 = 0$$

has one solution.

4.11. For what values of the parameter α is the ratio of the roots of the equation

$$x^2 + \alpha x + \alpha + 2 = 0$$

equal to 2?

4.12. Find the values of the parameters p and q which are the only roots of the equation

$$x^2 + px + q = 0.$$

4.13. For every value of a determine the number of solutions of the following equations:

(a) $\sqrt{2|x| - x^2} = a$; (b) $|x^2 - 2x - 3| = a$.

4.14. Find all values of a for which the inequality

$$\frac{x - 2a - 1}{x - a} < 0$$

is satisfied for all x such that $1 \leqslant x \leqslant 2$.

4.15. Find all values of the parameter a for which the roots of the equation

$$x^2 + x + a = 0$$

are real and exceed a.

4.16. Solve the inequality

$$2 \mid x - a \mid < 2ax - x^2 - 2$$

for every value of the parameter a.

4.17. Solve the inequality

$$\sqrt{a+x} + \sqrt{a-x} > a$$

for every value of the parameter a.

4.18. Find all a for each of which the inequality

$$25y^2 + \frac{1}{100} \geqslant x - axy + y - 25x^2$$

is satisfied for all pairs of numbers $(x; y)$ such that $\mid x \mid = \mid y \mid$.

4.19 Find all values of k for each of which there is at least one common solution of the inequalities

$$x^2 + 4kx + 3k^2 > 1 + 2k, \quad x^2 + 2kx \leqslant 3k^2 - 8k + 4.$$

4.20. Find all values of the parameter a for each of which there is at least one value of x satisfying the conditions

$$x^2 + \left(1 - \frac{3}{2} a\right) x + \frac{a^2}{2} - \frac{a}{2} < 0, \quad x = a^2 - \frac{1}{2}.$$

4.21.* Find all values of α for each of which the inequality

$$3 - \mid x - a \mid > x^2$$

has at least one negative solution.

4.22. Find all values of α for which the solutions of the system of inequalities

$$\begin{aligned} x^2 + 6x + 7 + \alpha &\leqslant 0, \\ x^2 + 4x + 7 &\leqslant 4\alpha \end{aligned}$$

form an interval of length unity on the number axis.

4.23. Find all values of α for which the system of inequalities

$$\begin{aligned} x^2 + 4x + 3 &\leqslant \alpha, \\ x^2 - 2x &\leqslant 3 - 6\alpha \end{aligned}$$

has a unique solution.

4.24. Find all values of α for each of which there are four integers $(x; y; u; v)$ satisfying the system of equations

$$\begin{aligned} x^2 + y^2 &= (111 - \alpha)(\alpha - 89), \\ 50(u^2 - v^2) &= \alpha(15u + 5v - \alpha). \end{aligned}$$

4.25. Find all real values of the quantity h for which the equation

$$x(x + 1)(x + h)(x + 1 + h) = h^2$$

has four real roots.

4.26. Find all real values of the quantity h for which the equation

$$x^4 + (h - 1)x^3 + x^2 + (h - 1)x + 1 = 0$$

possesses not less than two distinct negative roots.

4.27. For what values of the quantity h is the polynomial

$$x^4 - 2^{\tan h} \cdot x^2 + (\cos h + \cos 2h)x + 2^{\tan h - 2}$$

the square of the quadratic trinomial with respect to x?

4.28. For what values of the parameters m and n is the polynomial

$$5x^6 - mx^5 + 2nx^2 - 2x + 3$$

divisible by $x^2 - 1$?

Exponential and Logarithmic Equations and Inequalities with Parameters.

4.29. Solve the equation

$$\sqrt{a(2^x - 2) + 1} = 1 - 2^x$$

for every value of the parameter a.

4.30. Solve the equation

$$144^{|x|} - 2 \cdot 12^{|x|} + a = 0$$

for every value of the parameter a.

4.31. For what values of a does the equation

$$\log_3 (9^x + 9a^3) = x$$

possess two solutions?

4.32. Find all real values of the parameter a for which every solution of the inequality

$$\log_{1/2} x^2 \geqslant \log_{1/2} (x + 2)$$

is a solution of the inequality

$$49x^2 - 4a^4 \leqslant 0.$$

4.33. Find the solutions of the inequality

$$a^2 - 9^{x+1} - 8 \cdot 3^x \cdot a > 0.$$

4.34. Find all values of the parameter α for which the inequality

$$4^x - \alpha \cdot 2^x - \alpha + 3 \leqslant 0$$

has at least one solution.

4.35. Find all values of the parameter α for which the inequality

$$\alpha \cdot 9^x + 4(\alpha - 1)3^x + \alpha > 1$$

is valid for all x.

4.36. Find all values of the real parameter c for which the inequality

$$1 + \log_5 (x^2 + 1) \geqslant \log_5 (cx^2 + 4x + c)$$

is valid for all x.

4.37. Find all values of the parameter c for which the inequality

$$1 + \log_2 \left(2x^2 + 2x + \frac{7}{2}\right) \geqslant \log_2 (cx^2 + c)$$

possesses at least one solution.

4.38. Find all values of a for which the inequality

$$\log_{a(a+1)} (|x| + 4) > 1$$

is satisfied for any value of x.

4.39. Find all values of a for which the inequality

$$\log_{a/(a+1)} (x^2 + 2) > 1$$

is satisfied for any value of x.

4.40. Find all the values of x, smaller than 3 in the absolute value, which satisfy the inequality

$$\log_{2a-x^2} (x - 2ax) > 1$$

for all $a \geqslant 5$.

4.41. Find all values of $x > 1$ which are solutions of the inequality

$$\log_{(x^2+x)/b} (x + 2b - 1) < 1$$

for all b satisfying the condition $0 < b \leqslant 2$.

4.42. Find the set of all pairs of numbers $(a; b)$ for each of which the equality

$$a \cdot e^x + b = e^{ax+b}$$

is valid for all x.

Trigonometric Equations and Inequalities with Parameters.

4.43. Solve the equation

$$\sin x + \cos (a + x) + \cos (a - x) = 2$$

for every real number a.

4.44. Solve the equation

$$(\log \sin x)^2 - 2a \log \sin x - a^2 + 2 = 0$$

for every value of the parameter a.

4.45. Find all values of b for each of which the inequality

$$\cos^2 x + 2b \sin x - 2b < b^2 - 4$$

is satisfied for any x.

4.46. Determine all values of a for each of which the equation

$$\cos^4 x - (a + 2) \cos^2 x - (a + 3) = 0$$

possesses solutions and find the solutions.

4.47. For what values of the parameter a does the equation

$$\sin^2 4x + (a^2 - 3) \sin 4x + a^2 - 4 = 0$$

possess four roots on the interval $[3\pi/2; 2\pi]$?

4.48. For what values of b does the equation

$$\frac{b \cos x}{2 \cos 2x - 1} = \frac{b + \sin x}{(\cos^2 x - 3 \sin^2 x) \tan x}$$

possess solutions? Find the solutions.

4.49. Solve the equation

$$\log_{|\sin x|} 2 \cdot \log_{\sin^2 x} 3 = a$$

for every value of the parameter a.

4.50. Solve the inequality

$$x^{\sin x - a} > 1$$

for every value of the parameter $a > 0$ provided that $x \in (0, \pi/2)$.

4.51. Find the set of all pairs of numbers $(a; b)$ for each of which the equality

$$a(\cos x - 1) + b^2 = \cos(ax + b^2) - 1$$

holds true for all x.

4.52. Determine the integral values of k for which the system

$$(\arctan x)^2 + (\arccos y)^2 = \pi^2 k$$
$$\arctan x + \arccos y = \pi/2$$

possesses solutions and find all the solutions.

4.53. Find all values of a for each of which every root of the equation

$$a \cos 2x + |a| \cos 4x + \cos 6x = 1$$

is also a root of the equation

$$\sin x \cos 2x = \sin 2x \cos 3x - \frac{1}{2} \sin 5x,$$

and, conversely, every root of the second equation is also a root of the first equation.

Chapter 4

Trigonometry

The Basic Formulas of Trigonometry

$$\sin^2\alpha+\cos^2\alpha=1,$$

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha},\qquad \cot\alpha=\frac{\cos\alpha}{\sin\alpha},$$

$$\sec\alpha=\frac{1}{\cos\alpha},\qquad \operatorname{cosec}\alpha=\frac{1}{\sin\alpha},$$

$$\tan\alpha=\frac{1}{\cot\alpha},\qquad \cot\alpha=\frac{1}{\tan\alpha},$$

$$1+\tan^2\alpha=\sec^2\alpha,\quad 1+\cot^2\alpha=\operatorname{cosec}^2\alpha,$$

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta,$$
$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta,$$

$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta},\quad \cot(\alpha\pm\beta)=\frac{\cot\alpha\cot\beta\mp1}{\cot\beta\pm\cot\alpha},$$

$$\sin 2\alpha=2\sin\alpha\cos\alpha,\quad \cos 2\alpha=\cos^2\alpha-\sin^2\alpha,$$

$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha},$$

$$\sin\alpha=\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}},\qquad \cos\alpha=\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}},$$

$$1+\cos\alpha=2\cos^2\frac{\alpha}{2},\quad 1-\cos\alpha=2\sin^2\frac{\alpha}{2},$$

$$\tan^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha},\quad \tan\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha}=\frac{1-\cos\alpha}{\sin\alpha},$$

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2},$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

$$\cos\alpha-\cos\beta=2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2},$$

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha \pm \beta)}{\cos\alpha\cos\beta}, \quad \cot\alpha \pm \cot\beta = \frac{\sin(\beta \pm \alpha)}{\sin\alpha\sin\beta},$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)],$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)],$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)].$$

Recursion formulas

Name of function	Value of the argument						
	$-\alpha$	$\frac{\pi}{2}-\alpha$	$\frac{\pi}{2}+\alpha$	$\pi-\alpha$	$\pi+\alpha$	$\frac{3\pi}{2}-\alpha$	$\frac{3\pi}{2}+\alpha$
$\sin\alpha$	$-\sin\alpha$	$\cos\alpha$	$\cos\alpha$	$\sin\alpha$	$-\sin\alpha$	$-\cos\alpha$	$-\cos\alpha$
$\cos\alpha$	$\cos\alpha$	$\sin\alpha$	$-\sin\alpha$	$-\cos\alpha$	$-\cos\alpha$	$-\sin\alpha$	$\sin\alpha$
$\tan\alpha$	$-\tan\alpha$	$\cot\alpha$	$-\cot\alpha$	$-\tan\alpha$	$\tan\alpha$	$\cot\alpha$	$-\cot\alpha$
$\cot\alpha$	$-\cot\alpha$	$\tan\alpha$	$-\tan\alpha$	$-\cot\alpha$	$\cot\alpha$	$\tan\alpha$	$-\tan\alpha$

1. Identity Transformations of Trigonometric Expressions

To prove trigonometric identities, use can be made of the formulas for abbreviated multiplication as well as the formulas connecting the principal trigonometric functions.

Example 1.1. Prove the identity

$$2(\sin^6\alpha + \cos^6\alpha) - 3(\sin^4\alpha + \cos^4\alpha) + 1 = 0. \qquad (*)$$

Solution. Let us use a formula for abbreviated multiplication

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2),$$

setting $x = \sin^2\alpha$, $y = \cos^2\alpha$ in it. Then we get

$$\sin^6\alpha + \cos^6\alpha = (\sin^2\alpha + \cos^2\alpha)(\sin^4\alpha - \sin^2\alpha\cos^2\alpha + \cos^4\alpha).$$

By virtue of the identity

$$\sin^2\alpha + \cos^2\alpha = 1, \qquad (**)$$

the left-hand side of equation (*) reduces to the form

$$2\sin^4\alpha - 2\sin^2\alpha\cos^2\alpha + 2\cos^4\alpha - 3\sin^4\alpha - 3\cos^4\alpha + 1 = 0.$$

After collecting terms, we obtain

$$1 - 2\sin^2\alpha\cos^2\alpha - \sin^4\alpha - \cos^4\alpha = 0. \qquad (***)$$

To make sure of the identity of (***), we square both sides of formula (**). Then we have

$$\sin^4\alpha + 2\sin^2\alpha\cos^2\alpha + \cos^4\alpha = 1.$$

Further proof is obvious.
Prove the validity of the following identities.

1.1. $\sin^6\alpha+\cos^6\alpha=1-\frac{3}{4}\sin^2 2\alpha.$

1.2. $\frac{1+\sin 2\alpha+\cos 2\alpha}{1+\sin 2\alpha-\cos 2\alpha}=\cot\alpha.$

1.3. $(\sin\alpha+\sin\beta)^2+(\cos\alpha+\cos\beta)^2=4\cos^2\frac{\alpha-\beta}{2}.$

1.4. $\tan\alpha+\tan 2\alpha-\tan 3\alpha=-\tan\alpha\tan 2\alpha\tan 3\alpha.$

1.5. $\frac{2\sin\alpha-\sin 2\alpha}{2\sin\alpha+\sin 2\alpha}=\tan^2\frac{\alpha}{2}.$

1.6. $\frac{\sin\alpha+2\sin 3\alpha+\sin 5\alpha}{\sin 3\alpha+2\sin 5\alpha+\sin 7\alpha}=\frac{\sin 3\alpha}{\sin 5\alpha}.$

1.7. $\sin^2 3\alpha-\sin^2 2\alpha=\sin 5\alpha\sin\alpha.$

1.8. $\frac{\sin\alpha-\sin 3\alpha-\sin 5\alpha+\sin 7\alpha}{\cos\alpha-\cos 3\alpha+\cos 5\alpha-\cos 7\alpha}=-\tan 2\alpha.$

1.9. $\frac{1}{\tan 3\alpha-\tan\alpha}-\frac{1}{\cot 3\alpha-\cot\alpha}=\cot 2\alpha.$

1.10. $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$
$$=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\alpha+\gamma}{2}.$$

1.11. $\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha=4\cos\alpha\cos 2\alpha\sin 4\alpha.$

1.12. $\frac{\sin 3\alpha\cos^3\alpha+\cos 3\alpha\sin^3\alpha}{3}=\frac{\sin 4\alpha}{4}.$

1.13. $\frac{\sin 2\alpha-\sin 3\alpha+\sin 4\alpha}{\cos 2\alpha-\cos 3\alpha+\cos 4\alpha}=\tan 3\alpha.$

1.14. $\sin 2\alpha(1+\tan 2\alpha\tan\alpha)+\frac{1+\sin\alpha}{1-\sin\alpha}$
$$=\tan 2\alpha+\tan^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

1.15. $\sin^6\frac{\alpha}{2}-\cos^6\frac{\alpha}{2}=\frac{\sin^2\alpha-4}{4}\cos\alpha.$

1.16. $\cos\left(\frac{3\pi}{2}+4\alpha\right)+\sin(3\pi-8\alpha)-\sin(4\pi-12\alpha)$
$$=4\cos 2\alpha\cos 4\alpha\sin 6\alpha.$$

1.17. $\cot^2\alpha-\cot^2\beta=\frac{\cos^2\alpha-\cos^2\beta}{\sin^2\alpha\sin^2\beta}.$

1.18. $\frac{\sin^2 x}{\sin x-\cos x}-\frac{\sin x+\cos x}{\tan^2 x-1}=\sin x+\cos x.$

1.19. Prove that if $\alpha+\beta+\gamma=\pi$, then $\cos\alpha+\cos\beta+\cos\gamma$

$$=1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

1.20. Prove that if α, β, γ are angles of a triangle, then

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1.$$

1.21. Prove that if $\cos(\alpha+\beta)=0$, then

$$\sin(\alpha+2\beta)=\sin\alpha.$$

1.22. Prove that if $\sin^2\beta=\sin\alpha\cos\alpha$, then

$$\cos 2\beta=2\cos^2(\pi/4+\alpha).$$

1.23. Prove that if $\tan\alpha$ and $\tan\beta$ are roots of the equation $x^2+px+q=0$, then there holds an identity

$$\sin^2(\alpha+\beta)+p\sin(\alpha+\beta)\cos(\alpha+\beta)+q\cos^2(\alpha+\beta)=q.$$

1.24. Show that if the angles α and β are related as

$$\frac{\sin\beta}{\sin(2\alpha+\beta)}=\frac{n}{m},\quad |n|<|m|,$$

then the equality

$$\frac{1+\tan\beta/\tan\alpha}{m+n}=\frac{1-\tan\alpha\tan\beta}{m-n}$$

holds true.

1.25. It is known that α, β, γ form an arithmetic progression. Prove that

$$\frac{\sin\alpha-\sin\gamma}{\cos\gamma-\cos\alpha}=\cot\beta.$$

1.26. Prove that if $\alpha+\beta+\gamma=\pi$, then

$$\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1.$$

1.27. Prove the identity

$$\sin\left(\frac{\pi}{3}-\alpha\right)\cos\left(\frac{\pi}{6}+\alpha\right)+\cos\left(\frac{\pi}{3}-\alpha\right)\sin\left(\frac{\pi}{6}+\alpha\right)=1.$$

1.28. Prove that if $\sin^2\beta=\sin\alpha\cos\alpha$, then

$$\cos 2\beta=2\sin^2\left(\frac{\pi}{4}-\alpha\right).$$

1.29. Prove the identity

$$\frac{1+\sin 2\alpha}{\cos(2\alpha-2\pi)\tan\left(\alpha-\frac{3\pi}{4}\right)}-\frac{1}{4}\sin 2\alpha\left[\cot\frac{\alpha}{2}+\cot\left(\frac{3\pi}{2}+\frac{\alpha}{2}\right)\right]=-\sin^2\alpha.$$

1.30. Prove the identity

$$4\cos\left(\frac{\pi}{6}-\frac{\alpha}{2}\right)\sin\left(\frac{\pi}{3}-\frac{\alpha}{2}\right)=\frac{\sin\frac{3\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

1.31. Prove the identity

$$\frac{1+\sin 2\alpha}{\sin\alpha+\cos\alpha}-\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\sin\alpha.$$

1.32. Prove that if $\alpha+\beta+\gamma=\pi$, then

$$\sin^2\alpha-\cos^2\beta-\cos^2\gamma=2\cos\alpha\cos\beta\cos\gamma.$$

1.33. Simplify the following expression for $\alpha \in [0, 2\pi:]$

$$\frac{\sqrt{1+\cos\alpha}+\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}-\sqrt{1-\cos\alpha}}.$$

1.34. Simplify the following expression:

$$\frac{2\sin\alpha+\sin 2\alpha}{2\cos\alpha+\sin 2\alpha}\cdot\frac{1-\cos\alpha}{1-\sin\alpha}$$

and find possible values of α.

2. Calculating the Values of Trigonometric Functions

Problems connected with calculations of the values of trigonometric expressions without resort to tables are usually solved by means of identity transformations which reduce the required expression to a form containing only the tabular values of the trigonometric functions.

Example 2.1. Calculate without using tables

$$\tan 20^\circ \tan 40^\circ \tan 80^\circ.$$

Solution.

$$\frac{\sin 20^\circ\sin 40^\circ\sin 80^\circ}{\cos 20^\circ\cos 40^\circ\cos 80^\circ}=\frac{\sin 20^\circ\cdot 2\sin 20^\circ\cos 20^\circ\cdot 2\sin 40^\circ\cos 40^\circ}{\cos 20^\circ\cos 40^\circ\cos 80^\circ}$$

$$=\frac{2\sin 20^\circ(\cos 20^\circ-\cos 60^\circ)}{\cos 80^\circ}=\frac{\sin 40^\circ-\sin 20^\circ}{\cos 80^\circ}$$

$$=\frac{2\cos 30^\circ\sin 10^\circ}{\cos 80^\circ}=\frac{2\cos 30^\circ\cos(90^\circ-10^\circ)}{\cos 80^\circ}=\sqrt{3}.$$

Answer. $\sqrt{3}$.

Calculate the following expressions without using tables:

2.1. $\dfrac{\sin 24^\circ \cos 6^\circ - \sin 6^\circ \sin 66^\circ}{\sin 21^\circ \cos 39^\circ - \cos 51^\circ \sin 69^\circ}$.

2.2. $\sin^2 70^\circ \sin^2 50^\circ \sin^2 10^\circ$. 2.3. $\sin 15^\circ$.

2.4. $\sin \frac{3\pi}{10} \sin \frac{\pi}{10}$. 2.5. $8 \cos \frac{4\pi}{9} \cos \frac{2\pi}{9} \cos \frac{\pi}{9}$.

2.6. $\dfrac{1}{\sin \frac{\pi}{18}} - \dfrac{\sqrt{3}}{\cos \frac{\pi}{18}}$.

2.7. $\sin^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \sin^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8}$.

2.8.* $\sin 18^\circ$. 2.9.* $\sin 42^\circ$.

Problems on calculations of a trigonometric function from the known value of another function.

Example 2.2. Calculate

$$\frac{2 \sin 2\alpha - 3 \cos 2\alpha}{4 \sin 2\alpha + 5 \cos 2\alpha},$$

if $\tan \alpha = 3$.

Solution. Express $\sin 2\alpha$ and $\cos 2\alpha$ in terms of $\tan \alpha$ to obtain

$$\frac{2 \sin 2\alpha - 3 \cos 2\alpha}{4 \sin 2\alpha + 5 \cos 2\alpha} = \frac{4 \tan \alpha - 3 + 3 \tan^2 \alpha}{8 \tan \alpha + 5 - 5 \tan^2 \alpha}.$$

Replacing $\tan \alpha$ by its value 3 on the right-hand side, we obtain

$$\frac{4 \cdot 3 - 3 + 3 \cdot 9}{8 \cdot 3 + 5 - 5 \cdot 9} = -\frac{9}{4}.$$

Answer. $-9/4$.

2.10. Calculate $\sin \alpha$ if $\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} = 1.4$.

2.11. Calculate $1 + 5 \sin 2\alpha - 3 \cos^{-1} 2\alpha$ if $\tan \alpha = -2$.

2.12. Find the value of $\tan^4 \alpha + \cot^4 \alpha$ if $\tan \alpha + \cot \alpha = a$.

2.13. Calculate the value of $\sin^3 \alpha - \cos^3 \alpha$ if $\sin \alpha - \cos \alpha = n$.

2.14. Knowing that $\tan \frac{\alpha}{2} = m$, find $\dfrac{1 - 2 \sin^2 \frac{\alpha}{2}}{1 + \sin \alpha}$.

2.15. Calculate $\cos (\theta - \varphi)$ if $\cos \theta + \cos \varphi = a$, $\sin \theta - \sin \varphi = b$, $a^2 + b^2 \neq 0$.

2.16. The sum of three positive numbers α, β, γ is equal to $\pi/2$. Calculate the product $\cot \alpha \cot \gamma$ if it is known that $\cot \alpha$, $\cot \beta$, $\cot \gamma$ form an arithmetic progression.

2.17. Calculate $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}$ if

$$\sin \alpha + \sin \beta = a, \quad \cos \alpha + \cos \beta = b.$$

2.18*. Find $\tan (\alpha + 2\beta)$ if

$\sin (\alpha + \beta) = 1$, $\sin (\alpha - \beta) = 1/2$, where $\alpha, \beta \in [0, \pi/2]$.

2.19. Find the ratio $\cot\beta/\cot\alpha$ if it is known that

$$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{p}{q}.$$

2.20. Find $\tan\frac{\alpha}{2}$ if it is known that $\sin\alpha+\cos\alpha=\frac{1}{5}$.

2.21. Calculate $\tan\frac{\alpha}{2}$ if $\frac{\sin 3\alpha}{\sin\alpha}=\frac{11}{25}$.

2.22. Form an equation to find $\cos\frac{\alpha}{3}$ if $\cos\alpha=m$.

2.23. Find $\tan\frac{\alpha}{2}$ if it is known that

$$\frac{\cos\alpha}{1+\sin\alpha}=\frac{1-m}{1+m}.$$

2.24. Calculate $\sin 2\alpha$ if $\tan\alpha$ satisfies the relation

$$\tan^2\alpha - a\tan\alpha + 1 = 0$$

and it is known that $a > 0$ and $0 < \alpha < \pi/4$.

Calculating trigonometric functions from their inverses.

Example 2.3. Calculate the value of $\tan\left(\frac{1}{2}\operatorname{arccot} 3\right)$.

Solution. If we designate $\alpha = \operatorname{arccot} 3$, we have $\cot\alpha = 3$, $0 < \alpha < \pi/2$. Let us calculate the values of $\sin\alpha$ and $\cos\alpha$. We get

$$\sin\alpha=\frac{1}{\sqrt{1+\cot^2\alpha}}=\frac{1}{\sqrt{1+3^2}}=\frac{1}{\sqrt{10}},$$

$$\cos\alpha=\frac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}=\frac{3}{\sqrt{10}}.$$

Using the formula $\tan\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha}$, we obtain

$$\tan\left(\frac{\alpha}{2}\right)=\frac{1}{\sqrt{10}}\Big/\left(1+\frac{3}{\sqrt{10}}\right)=\frac{1}{\sqrt{10}+3}.$$

Answer. $\frac{1}{\sqrt{10}+3}$.

Calculate the following expressions.

2.25. $\sin\left(2\arccos\frac{1}{4}\right)$. **2.26.** $\cos\left[\arcsin\left(-\frac{1}{2}\right)\right]$.

2.27. $\sin\left(\arcsin\frac{3}{5}+\arcsin\frac{8}{17}\right)$. **2.28.** $\tan\left(2\arcsin\frac{2}{3}\right)$.

2.29. $\arcsin(\sin 2)$. **2.30.** $\tan\left(\arcsin\frac{1}{3}+\arccos\frac{1}{4}\right)$.

2.31. $\sin(\arctan 2+\arctan 3)$. **2.32.** $\cos\left(\arcsin\frac{1}{3}-\arccos\frac{2}{3}\right)$.

2.33. $\sin\left(2\arctan\frac{1}{3}\right)+\cos(\arctan 2\sqrt{3})$.

Verifying the validity of equalities containing inverse trigonometric functions. When solving these problems, bear in mind that the sum of inverse trigonometric functions calculated from positive quantities is contained in the interval $[0, \pi]$, and the difference is contained in the interval $[-\pi/2, \pi/2]$.

Example 2.4. Verify the equality

$$\arcsin\frac{4}{5}+\arccos\frac{2}{\sqrt{5}}=\operatorname{arccot}\frac{2}{11}.$$

Solution. Let us calculate the cotangent of the left-hand and right-hand sides of the equation:

$$\cot\left(\arcsin\frac{4}{5}+\arccos\frac{2}{\sqrt{5}}\right)$$

$$=\frac{\cot\left(\arcsin\frac{4}{5}\right)\cot\left(\arccos\frac{2}{\sqrt{5}}\right)-1}{\cot\left(\arcsin\frac{4}{5}\right)+\cot\left(\arccos\frac{2}{\sqrt{5}}\right)}=\frac{2}{11},$$

$$\cot\left(\operatorname{arccot}\frac{2}{11}\right)=\frac{2}{11}.$$

We obtain

$$\cot\left(\arcsin\frac{4}{5}+\arccos\frac{2}{\sqrt{5}}\right)=\cot\left(\operatorname{arccot}\frac{2}{11}\right).$$

Since $\arcsin\frac{4}{5}+\arccos\frac{2}{\sqrt{5}}$ belongs to the interval $(0, \pi)$, which is the interval of monotonicity of the cotangent function, the equality of the values of the cotangent functions yields the equality of the values of the arguments, and this is what we wished to prove.

Verify the following equalities:

2.34. $\arcsin\frac{\sqrt{3}}{2}+\arccos\frac{\sqrt{3}}{2}=\frac{\pi}{2}$.

2.35. $\arctan 1+\arctan 2=\pi-\arctan 3$.

2.36. $\arccos\sqrt{\frac{2}{3}}-\arccos\frac{\sqrt{6}+1}{2\sqrt{3}}=\frac{\pi}{6}$.

2.37. Prove that if

$$\arctan\alpha+\arctan\beta+\arctan\gamma=\pi,$$

then $\alpha+\beta+\gamma=\alpha\cdot\beta\cdot\gamma$.

2.38. Prove that

$$\arctan \frac{\sqrt{2}+1}{\sqrt{2}-1} - \arctan \frac{\sqrt{2}}{2} = \frac{\pi}{4}.$$

2.39. Prove that

$$\arctan 3 - \arcsin \frac{\sqrt{5}}{5} = \frac{\pi}{4}.$$

2.40. Prove that

$$\arcsin \frac{5}{13} + \arcsin \frac{12}{13} = \frac{\pi}{2}.$$

2.41. Prove that

$$\frac{\pi}{2} + \arcsin \frac{77}{85} = \arcsin \frac{8}{17} + \arccos \left(-\frac{3}{5}\right).$$

2.42.** Verify whether the equality

$$\arccos x + \arccos \left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right) = \frac{\pi}{3}$$

holds for $x \in \left[\frac{1}{2},\ 1\right]$.

2.43.* Verify whether the following equality holds true:

$$\arcsin \left(\frac{\sqrt{2}}{2} x + \frac{\sqrt{2-2x^2}}{2}\right) - \arcsin x = \frac{\pi}{4}.$$

The summation of the finite series of trigonometric functions

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n \qquad (*)$$

can often be carried out by selecting the so-called *generating function*, i.e. a function possessing the property

$$f(k+1) - f(k) = u_k.$$

If the function $f(k)$ is found, the sum (*) can be represented in the form

$$S_n = f(n+1) - f(1). \qquad (**)$$

Example 2.5. Sum up

$$S_n = \sin \alpha + \sin (\alpha + h) + \sin (\alpha + 2h) + \ldots + \sin (\alpha + nh).$$

Solution. We use the fact that

$$\cos\left(\alpha + \frac{2k+1}{2} h\right) - \cos\left(\alpha + \frac{2k-1}{2} h\right) = -2 \sin (\alpha + kh) \sin \frac{h}{2}.$$

Then we can take

$$f(k) = -\frac{1}{2 \sin \frac{h}{2}} \cos \left(\alpha + \frac{2k-1}{2} h\right)$$

as a generating function. According to (**) we get

$$S_n = -\frac{1}{2\sin\frac{h}{2}}\left[\cos\left(\alpha+\frac{2n+1}{2}h\right)-\cos\left(\alpha-\frac{h}{2}\right)\right].$$

Transforming the expression in brackets into a product, we get

$$S_n = \frac{\sin\left(\alpha+\frac{n}{2}h\right)\sin\left(\frac{n+1}{2}h\right)}{\sin\frac{h}{2}}.$$

Find the following sums.

2.44. $\sin\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\sin 3\alpha\sin 4\alpha+\ldots$
$\ldots+\sin n\alpha\sin(n+1)\alpha$.

2.45. $\cos 3\alpha+\cos 5\alpha+\cos 7\alpha+\ldots+\cos(2n+1)\alpha$.

2.46. $\tan\alpha+\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{4}\tan\frac{\alpha}{4}+\cdots+\frac{1}{2^n}\tan\frac{\alpha}{2^n}$.

2.47. $\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}+\cos\frac{9\pi}{13}+\cos\frac{11\pi}{13}$.

2.48. $\cos^2\alpha+\cos^2\left(\alpha+\frac{\pi}{n}\right)$

$$+\cos^2\left(\alpha+\frac{2\pi}{n}\right)+\ldots+\cos^2\left(\alpha+\frac{(n-1)\pi}{n}\right).$$

2.49. $\cos\frac{\pi}{19}+\cos\frac{3\pi}{19}+\cos\frac{5\pi}{19}+\ldots+\cos\frac{17\pi}{19}$.

2.50. $\cos\frac{\pi m}{n}+\cos\frac{3\pi m}{n}+\cos\frac{5\pi m}{n}+\ldots+\cos\frac{(2n-1)\pi m}{n}$.

2.51. $\dfrac{\sin\alpha+\sin 2\alpha+\ldots+\sin n\alpha}{\cos\alpha+\cos 2\alpha+\ldots+\cos n\alpha}$.

3. Trigonometric Equations

Solutions of Elementary Trigonometric Equations.

$\sin x = a$, $x = (-1)^k \arcsin a + \pi k$ $(|a| \leqslant 1)$, $k \in Z$,

$\cos x = a$, $x = \pm\arccos a + 2\pi k$ $(|a| \leqslant 1)$, $k \in Z$,

$\tan x = a$, $x = \arctan a + \pi k$, $k \in Z$,

$\cot x = a$, $x = \operatorname{arccot} a + \pi k$, $k \in Z$.

Equations of the form

$$P(\sin x) = 0,\ P(\cos x) = 0,\ P(\tan x) = 0,\ P(\cot x) = 0,$$

where P is a polynomial of the indicated arguments, can be solved as algebraic equations with respect to those arguments with a subsequent solution of elementary trigonometric equations.

Solve the following equations, first reducing them to algebraic equations with respect to one trigonometric function.

3.1. $2\sin^2 x + \sin x - 1 = 0.$
3.2. $\tan^3 x + 2\tan^2 x + 3\tan x = 0.$
3.3. $4\sin^4 x + \cos 4x = 1 + 12\cos^4 x.$
3.4. $6\cos^2 x + \cos 3x = \cos x.$

3.5*. $$\sqrt{\frac{1}{16} + \cos^4 x - \frac{1}{2}\cos^2 x} + \sqrt{\frac{9}{16} + \cos^4 x - \frac{3}{2}\cos^2 x} = \frac{1}{2}.$$

Equations of the form

$$a_0 \sin^n x + a_1 \sin^{n-1} x \cos x + a_2 \sin^{n-2} x \cos^2 x + \ldots + a_n \cos^n x = 0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers and the sum of the exponents in $\sin x$ and $\cos x$ in each summand is equal to n, are said to be *homogeneous with respect to* $\sin x$ and $\cos x$. For $\cos x \neq 0$ equations of this form are equivalent to the equations

$$a_0 \tan^n x + a_1 \tan^{n-1} x + \ldots + a_n = 0.$$

Example 3.1. Solve the equation

$$3\sin^2 x - 5\sin x \cos x + 8\cos^2 x = 2.$$

Solution. To reduce this equation to a homogeneous equation, we use the fundamental trigonometric identity

$$\sin^2 x + \cos^2 x = 1,$$

writing the equation in the form

$$3\sin^2 x - 5\sin x \cos x + 8\cos^2 x = 2(\sin^2 x + \cos^2 x).$$

Collecting terms, we obtain

$$\sin^2 x - 5\sin x \cos x + 6\cos^2 x = 0.$$

Dividing both sides of the equation by $\cos^2 x$, we pass to an equivalent equation

$$\tan^2 x - 5\tan x + 6 = 0,$$

whose solution reduces to that of elementary equations.

Solve the following equations reducing them to homogeneous equations.

3.6. $2\sin x \cos x + 5\cos^2 x = 4.$
3.7. $8\sin 2x - 3\cos^2 x = 4.$
3.8. $4\cos^2 \frac{x}{2} + \frac{1}{2}\sin x + 3\sin^2 \frac{x}{2} = 3.$
3.9. $\sin^4 x - \cos^4 x = 1/2.$
3.10. $2\sin^3 x + 2\cos x \sin^2 x - \sin x \cos^2 x - \cos^3 x = 0.$
3.11. $3 - 7\cos^2 x \sin x - 3\sin^3 x = 0.$
3.12. $2\sin^3 x - \sin^2 x \cos x + 2\sin x \cos^2 x - \cos^3 x = 0.$
3.13. $\sin^4 x + \cos^4 x = \sin 2x - 0.5.$

3.14. $\sin^6 2x+\cos^6 2x=\frac{3}{2}(\sin^4 2x+\cos^4 2x)+\frac{1}{2}(\sin x+\cos x)$.
3.15. $\cos^6 x+\sin^6 x-\cos^2 2x=1/16$.
3.16. $\sin^8 x+\cos^8 x=\cos^2 2x$.
3.17.* Solve the equation

$$\sin^6 x+\cos^6 x=a(\sin^4 x+\cos^4 x)$$

for all real values of a.

The method of an auxiliary angle. Equations of the form

$$a\cos x+b\sin x=c$$

are equivalent to the elementary trigonometric equation

$$\sin(x+\varphi)=\frac{c}{\sqrt{a^2+b^2}}, \qquad (*)$$

where φ can be found from the system

$$\sin\varphi=\frac{b}{\sqrt{a^2+b^2}},\quad \cos\varphi=\frac{a}{\sqrt{a^2+b^2}}.$$

Example 3.2. Solve the equation

$$3\sin x+4\cos x=5.$$

Solution. Since $\sqrt{3^2+4^2}=5$, the given equation is equivalent to the equation

$$\sin(x+\varphi)=1,$$

where φ can be found from the system of equations

$$\sin\varphi=4/5,\quad \cos\varphi=3/5.$$

Since $\sin\varphi$ and $\cos\varphi$ are greater than zero, we can take $\varphi=\arcsin\frac{4}{5}$ as φ and then the solution of the given equation has the form

$$x=-\arcsin\frac{4}{5}+\frac{\pi}{2}+2\pi n.$$

Answer. $x=-\arcsin\frac{4}{5}+\frac{\pi}{2}+2\pi n \quad (n\in Z)$.

Solve the following equations by introducing an auxiliary angle.

3.18. $\sin 8x-\cos 6x=\sqrt{3}(\sin 6x+\cos 8x)$.

3.19. $\sin 11x+\frac{\sqrt{3}}{2}\sin 7x+\frac{1}{2}\cos 7x=0$.

3.20. $\sin 10x+\cos 10x=\sqrt{2}\sin 15x$.
3.21. Find all solutions of the equation

$$\sqrt{1+\sin 2x}-\sqrt{2}\cos 3x=0$$

contained between π and $3\pi/2$.

3.22. $4\cos^2 x=2+\frac{1}{2}\cos 2x\left(\frac{\sqrt{3}}{\cos 2x}+\frac{1}{\sin 2x}\right)$.

3.23. $4 \sin 3x + 3 \cos 3x = 5.2$.

A trigonometric equation of the form

$$R (\sin kx, \cos nx, \tan mx, \cot lx) = 0, \qquad (*)$$

where R is a rational function of the indicated arguments and (k, n, m, and l are natural numbers) can be reduced to a rational equation with respect to the arguments $\sin x$, $\cos x$, $\tan x$, and $\cot x$ by means of the formulas for trigonometric functions of the sum of angles (in particular, the formulas for double and triple angles) and then reduce equation (*) to a rational equation with respect to the unknown $t = \tan \frac{x}{2}$ by means of the formulas of *universal trigonometric substitution*

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \quad \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}},$$

$$\tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}, \quad \cot x = \frac{1 - \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}}.$$

Example 3.3. Solve the equation

$$(\cos x - \sin x)\left(2 \tan x + \frac{1}{\cos x}\right) + 2 = 0.$$

Solution. We designate $t = \tan \frac{x}{2}$ and, using the formulas of the universal trigonometric substitution, write the equation in the form

$$\frac{3t^4 + 6t^3 + 8t^2 - 2t - 3}{(t^2 + 1)(1 - t^2)} = 0,$$

its roots are $t_1 = 1/\sqrt{3}$, $t_2 = -1/\sqrt{3}$. Thus the solution of the equation reduces to that of two elementary equations

$$\tan \frac{x}{2} = \frac{1}{\sqrt{3}}, \quad \tan \frac{x}{2} = -\frac{1}{\sqrt{3}}. \qquad (*)$$

Verification shows that the numbers πn which are roots of the equation $\cos \frac{x}{2} = 0$, are not the roots of the given equation, and, consequently, all solutions of the initial equation can be found as solutions of equation (*).

Answer. $x = \pm \frac{\pi}{3} + 2\pi k \quad (k \in Z)$.

Solve the following equations by the method of universal substitution.

3.24. $\sin x + \cot \frac{x}{2} = 2$. **3.25.** $\cot \left(\frac{\pi}{4} - x\right) = 5 \tan 2x + 7$.

3.26. $3 \sin 4x = (\cos 2x - 1) \tan x$.

3.27. $(1+\cos x)\sqrt{\tan \frac{x}{2}-2+\sin x=2\cos x}$.

Equations of the form

$$R(\sin x + \cos x,\ \sin x \cos x) = 0, \qquad (*)$$

where R is a rational function of the arguments in brackets, can be reduced to the equation with respect to the unknown $l = \sin x + \cos x$, if use is made of the trigonometric identity

$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x$$
$$= 1 + 2 \sin x \cos x,$$

which yields an equation

$$\sin x \cos x = \frac{t^2-1}{2}. \qquad (**)$$

Taking this equation into account, we can reduce equation (*) to the form

$$R\left(t,\ \frac{t^2-1}{2}\right)=0.$$

In just the same way, by the substitution $\sin x - \cos x = t$, we can reduce the equation of the form

$$R(\sin x - \cos x,\ \sin x \cos x) = 0$$

to an equation

$$R=\left(t,\ \frac{1-t^2}{2}\right)=0.$$

Example 3.4. Solve the equation

$$\sin x + \cos x - 2\sqrt{2} \sin x \cos x = 0.$$

Solution. Designating $\sin x + \cos x = t$ and using the equation $\sin x \cos x = (t^2 - 1)/2$, we reduce the equation to a new equation with respect to t:

$$\sqrt{2}\, t^2 - t - \sqrt{2} = 0.$$

The numbers $t_1 = \sqrt{2}$, $t_2 = -1/\sqrt{2}$ are roots of this quadratic equation.

Thus the solution of the initial equation reduces to the solution of two trigonometric equations:

$$\sin x + \cos x = \sqrt{2},\quad \sin x + \cos x = -1/\sqrt{2}.$$

Multiplying both sides of these equations by the number $1/\sqrt{2}$, we reduce them to two simpler equations:

$$\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x = 1 \Leftrightarrow \sin x \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos x = 1$$
$$\Leftrightarrow \sin\left(x+\frac{\pi}{4}\right) = 1,$$

$$\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x = -\frac{1}{2} \Leftrightarrow \sin\left(x+\frac{\pi}{4}\right) = -\frac{1}{2}.$$

The solutions of the equations $\sin\left(x+\frac{\pi}{4}\right)=1$ and $\sin\left(x+\frac{\pi}{4}\right)=-\frac{1}{2}$ are

$$x=\frac{\pi}{4}+2\pi k, \qquad k\in Z,$$

$$x=(-1)^{n+1}\frac{\pi}{6}-\frac{\pi}{4}+\pi n, \qquad n\in Z.$$

Solve the following equations.

3.28. $5(\sin x+\cos x)+\sin 3x-\cos 3x=2\sqrt{2}\,(2+\sin 2x)$.
3.29. $\sin x+\cos x+\sin x\cos x=1$.
3.30. $\sin x+\cos x-2\sin x\cos x=1$.
3.31.* Find the solution of the equation

$$\frac{1}{\cos x}+\frac{1}{\sin x}+\frac{1}{\sin x\cos x}=a$$

for all real values of a.

Some trigonometric equations can sometimes be simplified by lowering their degrees. If the exponents of the sines and cosines entering into an equation are even, the lowering of the degree can be done by half-argument formulas.

Example 3.5. Solve the equation

$$\sin^{10}x+\cos^{10}x=\frac{29}{16}\cos^4 2x.$$

Solution. Using half-angle formulas we can represent the given equation in the form

$$\left(\frac{1-\cos 2x}{2}\right)^5+\left(\frac{1+\cos 2x}{2}\right)^5=\frac{29}{16}\cos^4 2x.$$

Designating $\cos 2x=t$, we represent the given equation in the form

$$\left(\frac{1-t}{2}\right)^5+\left(\frac{1+t}{2}\right)^5=\frac{29}{16}t^4.$$

Removing the brackets and collecting terms, we arrive at a biquadratic equation

$$24t^4-10t^2-1=0,$$

whose only real root is $t^2=1/2$. Returning to the initial unknown, we obtain

$$\cos^2 2x=\frac{1}{2}\Leftrightarrow 1+\cos 4x=1\Leftrightarrow \cos 4x=0\Leftrightarrow$$

$$x=\frac{\pi}{8}+\frac{\pi k}{4}, \qquad k\in Z.$$

Answer. $x=\frac{\pi}{8}+\frac{\pi k}{4}\quad(k\in Z)$.

Solve the following equations.

3.32. $\sin^2 6x + 8\sin^2 3x = 0$.

3.33. $\sin^2 x + a\sin^2 2x = \sin\frac{\pi}{6}$. Investigate the solution.

3.34. $\sin^8 x + \cos^8 x = \frac{17}{32}$. **3.35.** $\cos 2x + 4\sin^4 x = 8\cos^6 x$.

3.36. $\cos 4x - 2\cos^2 x - 22\sin^2 x + 1 = 0$.

3.37. $\cos^2 3x + \cos^2 4x + \cos^2 5x = \frac{3}{2}$.

3.38. $\sin^2 3x + \sin^2 4x = \sin^2 5x + \sin^2 6x$.

3.39. $\cos^2\frac{x}{2} + \cos^2\frac{3}{2}x - \sin^2 2x - \sin^2 4x = 0$.

3.40. $\sin^4 x + \cos^4 x = \cos^2 2x + 0.25$.

3.41. $2 + \cos 4x = 5\cos 2x + 8\sin^6 x$.

3.42. $\sin^4 x + \cos^4 x = \frac{5}{8}$.

3.43. $8\sin^2 x + 6\cos^2 x = 13\sin 2x$.

3.44. $\sin^3 x(1 + \cot x) + \cos^3 x(1 + \tan x) = 2\sqrt{\sin x\cos x}$.

Solve the following equations applying the methods presented above.

3.45. $2\cos 2x = \sqrt{6}(\cos x - \sin x)$.

3.46. $\sin^3 x + \cos^3 x = 1 - \frac{1}{2}\sin 2x$.

3.47. $\sin 3x + \sin x + 2\cos x = \sin 2x + 2\cos^2 x$.

3.48. $\sin 5x \sin 4x = -\cos 6x \cos 3x$.

3.49. $\tan x + \sin 2x = \frac{1}{\cos x}$.

3.50. $2\tan x + \tan 2x = \tan 4x$.

3.51. $\cos 3x + \sin 5x = 0$.

3.52. $\sin x \cos 5x = \sin 9x \cos 3x$.

3.53. $1 + \sin x + \cos x + \sin 2x + \cos 2x = 0$.

3.54. $1 + \sin x + \cos 3x = \cos x + \cos 2x + \sin 2x$.

3.55. $\sin^2 x(\tan x + 1) = 3\sin x(\cos x - \sin x) + 3$.

3.56. $\sin 2x \sin 6x = \cos x \cos 3x$.

3.57. $\cos(x + 1)\sin 2(x + 1) = \cos 3(x + 1)\sin 4(x + 1)$.

3.58. $\sin x \sin 7x = \sin 3x \sin 5x$.

3.59. $\cos x \sin 7x = \cos 3x \sin 5x$.

3.60. $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$.

3.61. $\cos 2x - \cos 8x + \cos 6x = 1$.

3.62. $\sin x \sin 2x \sin 3x = \frac{1}{4}\sin 4x$.

3.63. $\sin^3 x \cos 3x + \sin 3x \cos^3 x = 0$.

3.64. $\tan x + \tan 2x = \tan 3x$.

3.65. $(1 - \tan x)(1 + \sin 2x) = 1 + \tan x$.

3.66. $(1 + \sin 2x)(\cos x - \sin x) = 1 - 2\sin^2 x$.

3.67. $\tan(x + \alpha) + \tan(x - \alpha) = 2\cot x$.

3.68. $\sin^2 2z + \sin^2 3z + \sin^2 4z + \sin^2 5z = 2.$

3.69. $\sin x \cos x \cos 2x \cos 8x = \frac{1}{4} \sin 12x.$

3.70. $\sin 2x \sin 6x - \cos 2x \cos 6x = \sqrt{2} \sin 3x \cos 8x.$

3.71. $\tan x + \cot 2x = 2 \cot 4x.$

3.72. $\cos 3x - \cos 2x = \sin 3x.$

3.73. $2 \sin 3x - \frac{1}{\sin x} = 2 \cos 3x + \frac{1}{\cos x}.$

3.74. $\cot^2 x - \tan^2 x = 32 \cos^3 2x.$

3.75. $\tan 2x + \cot x = 8 \cos^2 x.$

3.76. $\sin^2 2x - \tan^2 x = \frac{9}{2} \cos 2x.$

3.77. $\sin 2x - \tan x = 2 \sin 4x.$

3.78. $\frac{\sin 4x}{\sin (x - \pi/4)} = \sqrt{2} (\sin x + \cos x).$

3.79. $\cos 3x \tan 5x = \sin 7x.$

3.80. $\frac{\cos^2 2x}{\cos x + \cos (\pi/4)} = \cos x - \cos \frac{\pi}{4}.$

3.81. $\sin x \cot 3x = \cos 5x.$

3.82. $\frac{\cos x}{\cos 3x} - \frac{\cos 5x}{\cos x} = 8 \sin x \sin 3x.$

3.83. $\frac{\cos x}{\cos 3x} - \frac{\cos 3x}{\cos x} = -2 \cos 2x.$

3.84. $\frac{1}{\cos x \cos 2x} + \frac{1}{\cos 2x \cos 3x} + \frac{1}{\cos 3x \cos 4x} = 0.$

3.85. $\frac{\sin 3x}{\cos 2x} + \frac{\cos 3x}{\sin 2x} = \frac{2}{\sin 3x}.$

3.86.* $\cos \left(x - \frac{\pi}{4}\right) + \cos \left(x + \frac{\pi}{4}\right) = \frac{2}{3} \cos 2x.$

3.87. $4 \sin 3x + \sin 5x - 2 \sin x \cos 2x = 0.$

3.88. $\sin \left(x + \frac{\pi}{3}\right) \sin \left(x - \frac{\pi}{3}\right) = \sin x.$

3.89. $3 \cos x + 2 \cos 5x + 4 \cos 3x \cos 4x = 0.$

3.90. $3 \sin 5x = \cos 2x - \cos 8x - \sin 15x.$

3.91. $\cos 2x - \sin 3x - \cos 8x = \sin 10x - \cos 5x.$

3.92. $\sin 2x - \cos 2x = \tan x.$

3.93. $\cos 3x - \sin 5x - \cos 7x = \sin 4x - \cos 2x.$

3.94. $\sin 2x + \cos 2x = 2 \tan x + 1.$

3.95. $4 \sin^2 x + 3 \tan^2 x = 1.$

3.96. $4 \sin x \sin 2x \sin 3x = \sin 4x.$

3.97. $\frac{\sin 4x + \sin 2x - 4 \sin 3x + 2 \cos x - 4}{\sin x - 1} = 0.$

3.98.* $\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = y^2 - 4y + 5.$

3.99. $\cos 2x = \frac{1+\sqrt{3}}{2}(\cos x+\sin x)$.

3.100. $\cot x-\tan x = \frac{\cos x-\sin x}{\frac{1}{2}\sin 2x}$.

3.101. $\sin 7x + \sin 3x + 2\sin^2 x = 1$.
3.102. $\cos x - \cos 17x = 1 + 2\sin 8x \sin x - \cos 16x$.
3.103. $\sin x - \cos x = 4\sin x \cos^2 x$.
3.104. $2\cos 2x(\cot x - 1) = 1 + \cot x$.

3.105. $\tan x+2\cot 2x=\sin x\left(1+\tan x\tan\frac{x}{2}\right)$.

3.106. $2\cot 2x - \cot x = \sin 2x + 3\sin x$.

3.107. $\sin^4 x-\cos^4 x=\cos\left(\frac{3\pi}{2}-x\right)$.

3.108. $\sin 2x+\sin^4\frac{x}{2}=\cos^4\frac{x}{2}$.

3.109. $\cos x=\sqrt{3}\sin x+2\cos 3x$.

3.110. $\frac{1+\tan x}{1-\tan x}=(\sin x+\cos x)^2$.

3.111. $\sin 3x + \sin x = 4\sin^3 x$.
3.112. Find all values of x and y which satisfy the equation

$$12\sin x + 5\cos x = 2y^2 - 8y + 21.$$

The solution of some trigonometric equations sometimes presupposes a subsequent verification of the conditions which must be satisfied by the roots obtained. If the conditions are such that the roots of the equation must belong to the given interval, then the solution of the problem of isolating those roots reduces to the solution of some inequalities in integers.

Example 3.6. Find all solutions of the equation

$$(\tan^2 x - 1)^{-1} = 1 + \cos 2x, \qquad (*)$$

which satisfy the inequality $2^{x+1} - 8 > 0$.

Solution. Let us reduce the initial trigonometric equation to the form

$$(1+\cos 2x)\left(1+\frac{1}{2\cos 2x}\right)=0.$$

The following values of x are solutions of this equation

$$x=-\frac{\pi}{2}+\pi n, \quad x=\pm\frac{\pi}{3}+\pi k, \quad n,\ k\in Z.$$

By the hypothesis, we must choose those values of x which satisfy the inequalities

$$2^{x+1} - 8 > 0, \quad \cos x \neq 0.$$

The values we need are

$$x = \pm \frac{\pi}{3} + \pi n, \quad n \in \mathbf{N}.$$

Answer. $x = \pm \frac{\pi}{3} + \pi n, \ n \in \mathrm{N}$

3.113. Find all solutions of the equation

$$\sqrt{\sin (1-x)} = \sqrt{\cos x},$$

which satisfy the condition $x \in [0, 2\pi]$.

3.114. Find all solutions of the equation

$$\cos^4 x - 3 \cos 3x = 3 \cos x - \cos^3 x \cos 3x,$$

lying on the interval $[-\pi, 3\pi/2]$.

3.115. Find all solutions of the equation

$$\sin \frac{x}{2} - \cos \frac{x}{2} = 1 - \sin x,$$

which satisfy the condition

$$\left| \frac{x}{2} - \frac{\pi}{2} \right| \leqslant \frac{3\pi}{4}.$$

3.116. Find all solutions of the equation

$$\frac{1}{2} (\cos 5x + \cos 7x) - \cos^2 2x + \sin^2 3x = 0,$$

which satisfy the condition $|x| < 2$.

Solve the following equations.

3.117*. $\tan x^2 = \cot 5x$.

3.118*. $\sin \frac{5}{x} = \cos 3x$.

3.119*. $\sin x = \cos \sqrt{x}$.

3.120.** Prove that the equation

$$\sin (\cos x) = \cos (\sin x)$$

does not possess real roots.

3.121*. $\sin \left(\frac{2\pi}{5} \cos x \right) = \cos \left(\frac{2\pi}{5} \sin x \right)$.

3.122*. $\sin (\pi \cot x) = \cos (\pi \tan x)$.

3.123. Find the roots of the equation $\sin (x - 2) = \sin (3x - 4)$ belonging to the interval $(-\pi; \pi)$.

When additional conditions are represented by an inequality containing trigonometric functions, the required roots are isolated on the interval which is equal to the least common multiple of the periods of the trigonometric functions entering into the equations and inequalities.

Example 3.7. Find all solutions of the equation

$$1+(\sin x-\cos x)\sin\frac{\pi}{4}=2\cos^2\frac{5}{2}x, \qquad (*)$$

which satisfy the condition

$$\sin 6x<0. \qquad (**)$$

Solution. Let us simplify the initial equation:

$$1+(\sin x-\cos x)\sin\frac{\pi}{4}$$
$$=2\cos^2\frac{5x}{2}\Longleftrightarrow 1+(\sin x-\cos x)\frac{\sqrt{2}}{2}$$
$$=1+\cos 5x\Longleftrightarrow \cos 5x+\cos\left(x+\frac{\pi}{4}\right)=0,$$
$$2\cos\left(3x+\frac{\pi}{8}\right)\cos\left(2x-\frac{\pi}{8}\right)=0.$$

Thus initial equation $(*)$ is equivalent to the equations

$$\cos\left(3x+\frac{\pi}{8}\right)=0,\ \cos\left(2x-\frac{\pi}{8}\right)=0, \qquad (***)$$

whose roots are equal, respectively, to

$$x=\frac{\pi}{8}+\frac{\pi n}{3},\qquad n\in Z,$$
$$x=\frac{5\pi}{16}+\frac{\pi n}{2},\qquad n\in Z.$$

The least common multiple of the periods of the trigonometric functions entering into equation $(*)$ and inequality $(**)$ is equal to 2π. From the obtained solutions of the equation belonging to the interval $[0, 2\pi)$ the numbers $5\pi/16$ and $5\pi/16+\pi$ satisfy inequality $(**)$. All the solutions of the problem can be obtained by adding number, which are multiples of 2π, to each root obtained.

Answer. $x=\frac{5\pi}{16}+\pi k\quad (k\in Z)$.

3.124. Find all solutions of the equation

$$5-8\cos\left(x-\frac{3}{2}\pi\right)=2\sin\left(2x-\frac{7}{2}\pi\right),$$

which satisfy the inequality $\cos x>0$.

3.125. Find all solutions of the equation

$$\sqrt{\tan x+\sin x}+\sqrt{\tan x-\sin x}=\sqrt{3\tan x}$$

(a) on the interval $[0, \pi]$, (b) throughout the real axis.

3.126. Solve the equation

$$\sqrt{2+\tan x-\cos^2 x}-\sqrt{\frac{16}{9}+\tan x}=\sqrt{\frac{2}{9}-\cos^2 x}.$$

3.127.** Find all solutions of the equation

$$\sin\left(x-\frac{\pi}{4}\right)-\cos\left(x+\frac{3\pi}{4}\right)=1,$$

which satisfy the inequality $\frac{2\cos 7x}{\cos 3+\sin 3}>2^{\cos 2x}$.

3.128*. Find all solutions of the equation

$$\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}\cos x},$$

which satisfy the inequality $\log_{\sin^2 3}(1+\cos(2x+4))<\cos 4x$.

3.129*. Find all solutions of the equation

$$\sin\left(4x+\frac{\pi}{4}\right)+\cos\left(4x+\frac{5\pi}{4}\right)=\sqrt{2},$$

which satisfy the inequality $\frac{\cos 2x}{\cos 2-\sin 2}>2^{-\sin 4x}$.

3.130*. Find all solutions of the equation

$$\sin\left(2x-\frac{\pi}{4}\right)=\sqrt{2}\sin^2 x,$$

which satisfy the inequality $\log_{\cos^2 3}(1+\sin(7x+5))<\sin 8x$.

Some trigonometric equations can be solved by using the estimate of the left-hand and right-hand sides of the equation.

Example 3.8. Solve the equation

$$\tan\left(\frac{\pi}{4}+x\right)+\tan\left(\frac{\pi}{4}-x\right)=2. \quad (*)$$

Solution. Using the recursion formula, we obtain

$$\tan\left(\frac{\pi}{4}+x\right)=\cot\left(\frac{\pi}{2}-\frac{\pi}{4}-x\right)=\cot\left(\frac{\pi}{4}-x\right).$$

Since

$$\tan\left(\frac{\pi}{4}-x\right)=\frac{1}{\cot(\pi/4-x)},$$

the left-hand side of the given equation is the sum of two mutually inverse quantities. It is known that

$$a+1/a\geqslant 2$$

for $a>0$. Thus equality (*) can be obtained only for

$$\tan\left(\frac{\pi}{4}+x\right)=1, \quad (**)$$

and, consequently, equation (**) is equivalent to (*).

Answer. $x=\pi n$ $(n\in Z)$.

Solve the following equations.

3.131. $\sin x+\sin 5x=2$.

3.132. $\sin x \sin y = 1$.

3.133. $3^{\log \tan x} + 3^{\log \cot x} = 2$.

3.134.** $\cos x + \cos y - \cos(x+y) = \frac{3}{2}$.

3.135. $\sin x + \sin y = 2$.

3.136. $\sin x + \sin y + \sin z = -3$.

3.137. $\log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$.

3.138. $\cos^2 x + \cos x \cos y + \cos^2 y = 0$.

3.139. Prove that the equation

$$(\sin x + \sqrt{3}\cos x)\sin 4x = 2$$

has no solutions.

4. Systems of Trigonometric Equations

Systems of equations in which the unknowns are arguments of trigonometric functions are called *systems of trigonometric equations*. Systems of trigonometric equations, can be solved by both the methods of solving systems of algebraic equations and those for solving trigonometric equations.

Example 4.1. Solve the system of equations

$$\sin x \sin y = \sqrt{3}/4,$$
$$\cos x \cos y = \sqrt{3}/4.$$

Solution. Adding up the equations of the system, we arrive at an equation

$$\sin x \sin y + \cos x \cos y = \frac{\sqrt{3}}{2} \Leftrightarrow \cos(x-y) = \frac{\sqrt{3}}{2}.$$

Subtracting the first equation of the system from the second, we arrive at an equation

$$\cos x \cos y - \sin x \sin y = 0 \Leftrightarrow \cos(x+y) = 0.$$

Thus the initial system is equivalent to the system

$$\left\{\begin{array}{l} \cos(x-y) \frac{\sqrt{3}}{2}, \\ \cos(x+y) = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x-y = \pm\frac{\pi}{6} + 2\pi n, \\ x+y = \frac{\pi}{2} + \pi k, \end{array}\right. \quad n, k \in Z,$$

whence

$$x = \frac{\pi}{3} + \frac{\pi}{2}(2n+k), \qquad x = \frac{\pi}{6} + \frac{\pi}{2}(2n+k),$$
$$y = \frac{\pi}{6} + \frac{\pi}{2}(k-2n), \qquad y = \frac{\pi}{3} + \frac{\pi}{2}(k-2n).$$

Answer. $\frac{\pi}{3}+\frac{\pi}{2}(2n-k)$, $\frac{\pi}{6}+\frac{\pi}{2}(k-2n)$;

$\frac{\pi}{6}+\frac{\pi}{2}(2n+k)$, $\frac{\pi}{3}+\frac{\pi}{2}(k-2n)$ $(k,\ n \in Z)$.

Solve the following systems of equations,

4.1. $\sin x \cos y = -1/2,$
$\cos x \sin y = 1/2.$

4.2. $\sin x \cos y = 0.36,$
$\cos x \sin y = 0.175.$

4.3. $\sin x \sin y = 3/4,$
$\tan x \tan y = 3.$

4.4. $\cos x \cos y = \frac{1+\sqrt{2}}{4},$
$\cot x \cot y = 3+2\sqrt{2}.$

4.5. $\sin x - \sin y = 1/2,$
$\cos x + \cos y = \sqrt{3}/2.$

4.6. $\sin 2x + \sin 2y = 3(\sin x + \sin y),$
$\cos 2x + \cos 2y = \cos x + \cos y.$

4.7. $\sin x \cot y = \sqrt{6}/2,$
$\tan x \cos y = \sqrt{3}/2.$

4.8. $\tan x = \sin y,$
$\sin x = 2 \cot y.$

4.9. $\sin y = 5 \sin x,$
$3 \cos x + \cos y = 2.$

4.10. $3 \tan (y/2) + 6 \sin x = 2 \sin (y - x),$
$\tan (y/2) - 2 \sin x = 6 \sin (y + x).$

4.11. $\sin^2 x + \cos x \sin y = \cos 2y,$
$\cos 2x + \sin 2y = \sin^2 y + 3 \cos y \sin x.$

4.12. $2 \sin^2 y + \sin 2y = \cos (x + y),$
$\cos^2 x + 2 \sin 2y + \sin^2 y = \cos (x - y).$

4.13. $\sin^2(-2x) - (3-\sqrt{2}) \tan 5y = \frac{3\sqrt{2}-1}{2},$

$\tan^2 5y + (3-\sqrt{2}) \sin(-2x) = \frac{3\sqrt{2}-1}{2}.$

4.14. $\sin^2 3x + (4-\sqrt{3}) \cot(-7y) = 2\sqrt{3} - 3/4,$
$\cot^2(-7y) + (4-\sqrt{3}) \sin 3x = 2\sqrt{3} - 3/4.$

4.15. $4 \sin y - 6\sqrt{2} \cos x = 5 + 4 \cos^2 y,$
$\cos 2x = 0.$

4.16. $1 + 2 \cos 2x = 0,$
$\sqrt{6} \cos y - 4 \sin x = 2\sqrt{3}(1 + \sin^2 y).$

4.17. $2\sqrt{3} \cos x + 6 \sin y = 3 + 12 \sin^2 x,$
$4\sqrt{3} \cos x + 2 \sin y = 7.$

4.18. $\sqrt{2}\, y + \sqrt{12} \cot x = 4,$
$2\sqrt{2}\, y - \sqrt{27} \cot x = 1.$

4.19. $3 \tan 3y + 2 \cos x = 2 \tan 60°,$
$2 \tan 3y - 3 \cos x = -\frac{5}{3} \cos 30°.$

4.20. $\sin(y-3x) = 2\sin^3 x,$
$\cos(y-3x) = 2\cos^3 x.$

4.21. $\sin(x-y) = 3\sin x\cos y - 1,$
$\sin(x+y) = -2\cos x\sin y.$

4.22. Find the solutions of the system

$$|\sin x|\sin y = -1/4,$$
$$\cos(x+y) + \cos(x-y) = 3/2,$$

which satisfy the conditions $x \in (0, 2\pi)$, $y \in (\pi, 2\pi)$.

4.23. $\tan^2 x + \cot^2 x = 2\sin^2 y,$
$\sin^2 y + \cos^2 z = 1.$

4.24. $x + y = \pi/3,$
$\tan x/\tan y = 3/4.$

4.25. $4^{\sin x} + 3\cdot 9^{\cos y} = 3,$
$4^{-\sin x} + 5\cdot 81^{\cos y+1/2} = 11/2.$

4.26. $x + y + z = \pi,$
$\tan x\tan y = 2,$
$\tan x + \tan y + \tan z = 6.$

4.27. $\tan x/\tan y = 2,$
$\tan y/\tan z = 3,$
$x + y + z = \pi.$

If one of the equations of the system is rational with respect to the arguments of trigonometric functions, then the solution of the system usually reduces to the solution of a trigonometric equation for one of the unknowns.

Example 4.2. Solve the system of equations

$$x+y=2\pi/3,$$
$$\frac{\sin x}{\sin y}=2.$$

Solution. Let us reduce the second equation of the system to the form

$$\sin x = 2\sin y. \qquad (*)$$

Using the first equation of the system, we exclude the unknown y from equation $(*)$:

$$\sin x=2\sin\left(\frac{2\pi}{3}-x\right) \Longleftrightarrow \sin x=\sqrt{3}\cos x+\sin x.$$

The resulting equation is equivalent to the elementary trigonometric equation

$$\cos x = 0. \qquad (**)$$

Substituting the roots of equation $(**)$ into the first equation of the system, we obtain the values of the unknown y.

Answer. $x=\frac{\pi}{2}+\pi k,\quad y=\frac{\pi}{6}-\pi k\ (k\in Z).$

Solve the following systems.

4.28. $$x-y=\pi/18,$$
$$\sin\left(x+\frac{\pi}{18}\right)\sin\left(y+\frac{\pi}{9}\right)=\frac{1}{2}.$$

4.29. $\frac{1-\tan x}{1+\tan x}=\tan y,$

$x-y=\pi/6.$

4.30. $\tan x+\cot y=3,$

$|x-y|=\pi/3.$

4.31. $\sin x+\sin y=\sin(x+y),$

$|x|+|y|=1.$

4.32. For what values of a do the solutions of the system

$$8\cos x\cos y\cos(x-y)+1=0$$
$$x+y=a$$

exist? Find the solutions.

5. Equations Containing Inverse Trigonometric Functions

Solutions of elementary equations.

Equation		Solution
$\arcsin x=a$	$(\lvert a\rvert\leqslant\pi/2)$	$x=\sin a$
$\arccos x=a$	$(0\leqslant a\leqslant\pi)$	$x=\cos a$
$\arctan x=a$	$(\lvert a\rvert<\pi/2)$	$x=\tan a$
$\operatorname{arccot} x=a$	$(0<a<\pi)$	$x=\cot a$

Equations of the form

$$P(y(x))=0,$$

where P is a certain rational function and $y(x)$ is one of the arc functions, reduce to the elementary equations

$$y(x)=y_i,$$

where y_i are the roots of the equation $P(t)=0$.

Example 5.1. Solve the equation

$$2\arcsin^2 x-\arcsin x-6=0.$$

Solution. Introducing a new unknown $y=\arcsin x$, we get an equation

$$2y^2-y-6=0,$$

whose solution is $y_1=2$, $y_2=-1.5$. Consequently, the initial equation decomposes into two elementary equations

$$\arcsin x=2,\ \arcsin x=-1/5.$$

Since $2 > \pi/2$, and $|-1.5| < \pi/2$, the only solution is $x = -\sin 1.5$.

Answer. $x = -\sin 1.5$.

Solve the following equations.

5.1. $\arctan^2 \frac{x}{3} - 4 \arctan \frac{x}{3} - 5 = 0$.

5.2. $\arctan^2 (3x + 2) + 2 \arctan (3x + 2) = 0$.

5.3. $2 \arcsin x = \frac{\pi}{3} + \frac{\pi^2/9}{\arcsin x}$.

5.4. $3 \arctan^2 x - 4\pi \arctan x + \pi^2 = 0$.

5.5. Find the solutions of the equation $2 \arccos x = a + \frac{a^2}{\arccos x}$ for real values of a.

If an equation includes expressions containing various arc functions, or those arc functions depend on different arguments, then the equation is usually reduced to its algebraic consequence by calculating a certain trigonometric function of both sides of the equation. The resulting extraneous roots can be isolated by verification. If a tangent or a cotagent is chosen as a direct function, then the solutions that do not belong to the domain of definition of those functions can be lost. Therefore, before calculating the tangent or cotangent of both sides of the equation, we must ascertain that there are no roots of the initial equation among the points that do not belong to the domain of definition of those functions.

Example 5.2. Solve the equation

$$\arcsin 6x + \arcsin 6\sqrt{3}x = -\pi/2. \qquad (*)$$

Solution. Let us transfer $\arcsin 6x$ into the right-hand side of the equation and calculate the sine of both sides of the resulting equation:

$$\sin (\arcsin 6x) = \sin (-\arcsin 6\sqrt{3}x - \pi/2).$$

Transforming the right-hand side of this equation by the recursion formulas, we obtain an algebraic consequence of the initial equation:

$$6x = -\sqrt{1 - 108x^2}.$$

We square both sides of the equation. After collecting terms, we get an equation

$$144x^2 = 1,$$

whose roots are the numbers $1/12$ and $-1/12$.

Let us perform a verification. Substituting the value $x = -1/12$ into equation $(*)$, we obtain

$$\arcsin\left(-\frac{1}{2}\right) + \arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{2}.$$

Thus $x = -1/12$ is a root of the initial equation.

Substituting $x = 1/12$ into (*) we note that the left-hand side of the resulting relation is positive and the right-hand side is negative. Thus the value $x = 1/12$ is an extraneous root of equation (*).

Answer. $x = -1/12$.

Example 5.3. Solve the equation

$$2 \arctan (2x + 1) = \arccos x. \qquad (*)$$

Solution. Calculating the cosine of both sides of the equation, we obtain

$$\cos [2 \arctan (2x + 1)] = x.$$

We can reduce the left-hand side of this equation to the form

$$\cos [2 \arctan (2x+1)] = \frac{1-(2x+1)^2}{1+(2x+1)^2} = \frac{2x^2+2x}{1+2x+2x^2}.$$

Thus the algebraic consequence of equation (*) is an equation

$$\frac{2x^2+2x}{1+2x+2x^2} = x \Longleftrightarrow 2x^3 - x = 0,$$

whose roots are equal to $0, \sqrt{2}/2, -\sqrt{2}/2$. To find out which of these numbers satisfy the initial equation, we make a verification. For $x = 0$ both sides of the equation are equal to $\pi/2$. For $x = \sqrt{2}/2$ the right-hand and left-hand sides of the equation are equal to $\pi/4$ and $2 \arctan (\sqrt{2} + 1)$, respectively. But $\sqrt{2} + 1 > 1$, and, consequently, $\arctan (\sqrt{2} + 1) > \pi/4$; hence $x = \sqrt{2}/2$ is not a root of the initial equation. For $x = -\sqrt{2}/2$ the left-hand side of the equation is negative and the right-hand side is positive. Consequently, $x = -\sqrt{2}/2$ is not a root of equation (*) either.

Answer. $x = 0$.

Solve the following equations.

5.6. $\arccos \frac{x}{2} = 2 \arctan (x-1)$. **5.7.** $\arccos x - \pi = \arcsin \frac{4x}{3}$.

5.8. $\arctan \left(x+\frac{1}{2}\right) + \arctan \left(x-\frac{1}{2}\right) = \frac{\pi}{4}$.

5.9. $\arctan 2x + \arctan 3x = \frac{3\pi}{4}$.

5.10. $\arcsin x + \arccos (x-1) = \pi$.

5.11. $2 \arccos x + \arcsin x = \frac{11\pi}{6}$.

5.12. $2 \arccos \left(-\frac{x}{2}\right) = \arccos (x+3)$.

5.13. $2 \arcsin x = \arcsin \sqrt{2}\, x$.

5.14. $\arctan \frac{x}{3} + \arctan \frac{x}{2} = \arctan x$.

5.15. $\arccos x - \arcsin x = \frac{\pi}{6}$.

5.16. $\arcsin x + \arcsin \frac{x}{\sqrt{3}} = \frac{\pi}{2}$.

5.17. $\arcsin 2x = 3 \arcsin x$.

5.18. $\arccos x - \arcsin x = \arccos \sqrt{3}x$.

5.19. $\arcsin x - \arccos x = \arcsin (3x - 2)$.

Some equations containing the unknown under the sign of an arc function are identities on the common domain of definition of the left-hand and right-hand sides of the equation. The process of solution of such an equation consists in seeking that domain.

Example 5.4. Solve the equation

$$2 \arccos x = \arcsin (2x \sqrt{1-x^2}). \qquad (*)$$

Solution. According to the definition of the function $y = \arccos x$, we have

$$x = \cos y, \text{ where } 0 \leqslant y \leqslant \pi, \quad |x| \leqslant 1.$$

Substituting x thus expressed into the right-hand side of the equation, we obtain

$$\arcsin (2 \cos y \sin y) = \arcsin (\sin 2y).$$

According to the definition of the function $y = \arcsin x$, we infer that

$$\arcsin (\sin 2y) = 2y \quad \text{for} \quad -\frac{\pi}{4} \leqslant y \leqslant \frac{\pi}{4}.$$

Thus the left-hand side of equation (*) is equal to the right-hand side for all $y \in [0, \pi/4]$. Returning to the initial unknown, we find that $x \in [\sqrt{2}/2, 1]$.

Answer. $x \in [\sqrt{2}/2, 1]$.

Solve the following equations.

5.20. $\arcsin x = \arccos \sqrt{1-x^2}$.

5.21. $\arccos x = \pi - \arcsin \sqrt{1-x^2}$.

5.22. $\arccos x = \arctan \frac{\sqrt{1-x^2}}{x}$.

5.23. $\arcsin (2x \sqrt{1-x^2}) = \arccos (2x^2 - 1)$.

5.24. $2 \arccos x = \arccos (2x^2 - 1)$.

5.25. $2 \arctan x = \arcsin \left(\frac{2x}{1+x^2}\right)$.

5.26. $2 \arctan x = \arccos \left(\frac{1-x^2}{1+x^2}\right)$.

5.27. $\arccos x = \arctan \frac{x}{\sqrt{1-x^2}}$.

5.28. $2 \arccos x = \operatorname{arccot} \dfrac{2x^2-1}{2x\sqrt{1-x^2}}$.

5.29. Solve the equation $\arcsin x = 2 \arcsin a$ for all real values of a.

5.30. $\arccos x = \arcsin 2a$.

6. Trigonometric Inequalities

Solutions of elementary trigonometric inequalities.

Inequality	Set of solutions of inequality ($n \in Z$)
$\sin x > a$ ($\|a\| < 1$)	$x \in (\arcsin a + 2\pi n,\ \pi - \arcsin a + 2\pi n)$
$\sin x < a$ ($\|a\| < 1$)	$x \in (-\pi - \arcsin a + 2\pi n,\ \arcsin a + 2\pi n)$
$\cos x > a$ ($\|a\| < 1$)	$x \in (-\arccos a + 2\pi n,\ \arccos a + 2\pi n)$
$\cos x < a$ ($\|a\| < 1$)	$x \in (\arccos a + 2\pi n,\ 2\pi - \arccos a + 2\pi n)$
$\tan x > a$	$x \in (\arctan a + \pi n,\ \pi/2 + \pi n)$
$\tan x < a$	$x \in (-\pi/2 + \pi n,\ \arctan a + \pi n)$
$\cot x > a$	$x \in (\pi n,\ \operatorname{arccot} a + \pi n)$
$\cot x < a$	$x \in (\operatorname{arccot} a + \pi n,\ \pi + \pi n)$

Solve the following inequalities.

6.1. $\sin x > -1/2$. 6.2. $\tan x > 2$.

6.3. $\cot x > -3$. 6.4. $\sin(x-1) \leqslant -\sqrt{3}/2$.

6.5. $\sin x^2 \leqslant 1/2$. 6.6. $\sin x + \cos x > -\sqrt{2}$.

6.7. $\cos(\sin x) < 0$. 6.8. $\sin(\cos x) \geqslant 0$.

Inequalities of the form $R(y) > 0$, $R(y) < 0$, where R is a certain rational function and y is a trigonometric function (sine, cosine, tangent, or cotangent), are usually solved in two stages: first the rational inequality is solved for the unknown y and then follows the solution of an elementary trigonometric inequality.

Example 6.1. Solve the inequality

$$2 \sin^2 x - 7 \sin x + 3 > 0.$$

Solution. Designating $\sin x = y$, we get an inequality

$$2y^2 - 7y + 3 > 0,$$

whose set of solutions is $y < 1/2$, $y > 3$. Returning to the initial unknown, we find that the given inequality is equivalent to two inequalities:

$$\sin x < 1/2 \text{ and } \sin x > 3.$$

The second inequality has no solutions and the solution of the first is

$$x \in \left(-\frac{7\pi}{6}+2\pi n,\ \frac{\pi}{6}+2\pi n\right), \quad n \in Z.$$

Answer. $x \in \left(-\frac{7}{6}\pi+2\pi n,\ \frac{\pi}{6}+2\pi n\right)$ $(n \in Z)$.

Solve the following inequalities.

6.9. $\cot^3 x+\cot^2 x-\cot x-1<0$. 6.10. $2\cos 2x+\sin 2x>\tan x$.

6.11. $\tan x+\cot x<-3$. 6.12. $\sin 2x>\cos x$.

6.13. $\sin x+\sqrt{3}\cos x<0$. 6.14. $\sqrt{3-4\cos^2 x}>2\sin x+1$.

6.15. $\sqrt{3\sin x+1}>4\sin x+1$. 6.16. $\frac{2\sin^2 x+\sin x-1}{\sin x-1}<0$.

6.17. $5+2\cos 2x \leqslant 3\,|2\sin x-1|$.

Solution of inequalities by factorization.

Example 6.2. Solve the inequality

$$\cos x+\cos 2x+\cos 3x>0.$$

Solution. Transforming the sum of the end terms into a product, we get an inequality

$$\cos 2x+2\cos 2x\cos x>0.$$

Putting $\cos 2x$ before the brackets, we obtain

$$\cos 2x\,(2\cos x+1)>0.$$

This inequality is equivalent to two systems of elementary inequalities:

$$\cos 2x<0, \qquad \cos 2x>0,$$
$$\cos 2x<-1/2, \qquad \cos x>-1/2.$$

Uniting the solutions of these systems, we get a solution of the initial inequality.

Answer. $\left(\frac{2\pi}{3}+2\pi n,\ \frac{3\pi}{4}+2\pi n\right) \cup \left(\frac{5\pi}{4}+2\pi n,\ \frac{4\pi}{3}+2\pi n\right) \cup \left(-\frac{\pi}{4}+2\pi n,\ \frac{\pi}{4}+2\pi n\right)$ $(n \in Z)$.

Solve the following inequalities.

6.18. $\sin x\sin 2x-\cos x\cos 2x>\sin 6x$.

6.19. $2\sin x\sin 2x\sin 3x<\sin 4x$.

6.20. $\sin x\sin 3x>\sin 5x\sin 7x$.

6.21. $\cos^3 x\sin 3x+\cos 3x\sin^3 x<3/8$. 6.22. $\sin x \geqslant \cos 2x$.

6.23. $2\tan 2x \leqslant 3\tan x$. 6.24. $\sin x<|\cos x|$.

7. Inequalities Containing Inverse Trigonometric Functions

Solutions of elementary inequalities.

Inequality	Solution
$\arcsin x > a$ $(\lvert a\rvert < \pi/2)$	$x \in (\sin a,\ 1]$
$\arcsin x < a$ $(\lvert a\rvert \leqslant \pi/2)$	$x \in [-1,\ \sin a)$
$\arccos x > a$ $(0 < a < \pi)$	$x \in [-1,\ \cos a)$
$\arccos x < a$ $(0 < a \leqslant \pi)$	$x \in (\cos a,\ 1]$
$\arctan x > a$ $(\lvert a\rvert < \pi/2)$	$x \in (\tan a,\ +\infty)$
$\arctan x < a$ $(\lvert a\rvert < \pi/2)$	$x \in (-\infty,\ \tan a)$
$\operatorname{arccot} x > a$ $(0 < a < \pi)$	$x \in (-\infty,\ \cot a)$
$\operatorname{arccot} x < a$ $(0 < a < \pi)$	$x \in (\cot a,\ +\infty)$

Solve the following inequalities.
7.1. $\arcsin x \leqslant 5$. **7.2.** $\arcsin x \geqslant -2$.
7.3. $\arccos x \leqslant \arccos \frac{1}{4}$. **7.4.** $\arccos x > \pi/6$.
7.5. $\arctan x > -\pi/3$. **7.6.** $\operatorname{arccot} x > 2$.

Inequalities of the form $R(y) > 0$, $R(y) < 0$, where R is a certain rational function and y is an inverse trigonometric function (arc sine, arc cosine, arc tangent, arc cotangent), are usually solved in two stages: first the inequality is solved for the unknown y and then follows the solution of an elementary inequality containing the inverse trigonometric function.

Example 7.1. Solve the inequality

$$\operatorname{arccot}^2 x - 5 \operatorname{arccot} x + 6 > 0.$$

Solution. Designating $\operatorname{arccot} x = y$, we write the initial inequality in the form

$$y^2 - 5y + 6 > 0,$$

whose solutions are $y < 2$ and $y > 3$. Returning to the initial unknown, we find that the initial inequality reduces to two elementary inequalities:

$$\operatorname{arccot} x < 2 \text{ and } \operatorname{arccot} x > 3$$

whose solutions are $x \in (\cot 2,\ \infty)$ and $x \in (-\infty,\ \cot 3)$ respectively. Uniting these solutions, we get the solution of the initial inequality.

Answer. $(\cot 2,\ \infty) \cup (-\infty,\ \cot 3)$.

Solve the following inequalities.
7.7. $\arctan^2 x - 4 \arctan x + 3 > 0$, 7.8. $\log_2 (\arctan x) > 1$.
7.9. $2^{\arctan x} + 2^{-\arctan x} \geqslant 2$. 7.10. $4 (\arccos x)^2 - 1 \geqslant 0$.

To solve the inequalities connecting the values of various inverse trigonometric functions, or the values of one trigonometric function calculated from different arguments, it is convenient to calculate the value of a certain trigonometric function of both sides of the inequality. It should be born in mind that the resulting inequality is equivalent to the initial one only in the case when the sets of values of the right-hand and left-hand sides of the initial inequality belong to the same interval of monotonicity of that trigonometric function.

Example 7.2. Solve the inequality

$$\arcsin x > \arccos x. \qquad (*)$$

Solution. The set of permissible values of x entering into the inequality is $x \in [-1, 1]$. For $x < 0$ we have $\arcsin x < 0$, $\arccos x > 0$. Consequently, the values $x < 0$ are not solutions of the inequality. For $x \geqslant 0$ both the right-hand side and the left-hand side of the inequality have values belonging to the interval $[0, \pi/2]$. Since the sine function is monotone increasing on the interval $[0, \pi/2]$, it follows that for $x \in [0, 1]$ inequality $(*)$ is equivalent to the inequality

$$\sin (\arcsin x) > \sin (\arccos x) \Leftrightarrow x > \sqrt{1 - x^2}.$$

For the values of the unknown we consider, the last inequality is equivalent to the inequality

$$2x^2 > 1. \qquad (**)$$

Thus the solutions of inequality $(**)$ belonging to the interval $[0, 1]$ are solutions of the initial inequality.

Answer. $x \in (\sqrt{2}/2, 1]$.

Solve the following inequalities.
7.11. $\arcsin x < \arccos x$. 7.12. $\arccos x > \arccos x^2$.
7.13. $\arctan x > \operatorname{arccot} x$. 7.14. $\arcsin x < \arcsin (1 - x)$.
7.15. $\tan^2 (\arcsin x) > 1$.

8. Proving the Validity of Trigonometric Inequalities

The proof of the validity of inequalities relating the values of trigonometric functions on the entire number axis or on its interval is usually based on the investigation of the properties of functions: monotonicity, boundedness, etc.

Example 8.1. Prove that

$$\cos \frac{\alpha+\beta}{2} \geqslant \frac{\cos \alpha + \cos \beta}{2},$$

if $\alpha, \beta \in (-\pi/2, \pi/2)$.

Solution. To prove the given inequality, it is sufficient to represent its right-hand side in the form

$$\frac{\cos \alpha + \cos \beta}{2} = \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$$

and take into account that $\frac{\alpha-\beta}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and, consequently $0 < \cos\frac{\alpha-\beta}{2} < 1$.

Prove that the following inequalities hold true for $x \in [0, \pi/2]$

8.1. $\sin x \cos x \leqslant 1/2$. 8.2. $\sin x + \cos x \leqslant \sqrt{2}$.

8.3. $\tan x + \cot x \geqslant 2$. 8.4. $\tan x \geqslant \sin x$.

8.5. $\sin 2x \leqslant 2 \sin x$. 8.6. $\sqrt{\cos x} \leqslant \sqrt{2}\cos\frac{x}{2}$.

8.7. Prove that

$$\sin\frac{\alpha+\beta}{2} \geqslant \frac{\sin\alpha+\sin\beta}{2},$$

if $\alpha, \beta \in [0, \pi]$.

Prove that the following relations hold true for any real x.

8.8*. $|a\cos x + b\sin x| \leqslant \sqrt{a^2+b^2}$.

8.9**. $$\frac{a+c-\sqrt{a^2+b^2+c^2-2ac}}{2} \leqslant a\sin^2 x + b\sin x\cos x + c\cos^2 x \leqslant \frac{a+c+\sqrt{a^2+b^2+c^2-2ac}}{2}.$$

To prove the validity of trigonometric inequalities, use is often made of algebraic inequalities which establish a relationship between the geometric mean and the arithmetic mean of two or several positive numbers.

Example 8.2. Prove that

$$\frac{1}{\sin\frac{\alpha}{2}} + \frac{1}{\sin\frac{\beta}{2}} + \frac{1}{\sin\frac{\gamma}{2}} \geqslant 6, \qquad (*)$$

if $\alpha+\beta+\gamma = \pi$ and $\alpha, \beta, \gamma > 0$.

Solution. Since $\sin\frac{\alpha}{2}$, $\sin\frac{\beta}{2}$, $\sin\frac{\gamma}{2}$ are nonnegative, we can use the inequality relating the arithmetic mean of the three numbers and their geometric mean and obtain

$$\frac{1}{\sin\frac{\alpha}{2}} + \frac{1}{\sin\frac{\beta}{2}} + \frac{1}{\sin\frac{\gamma}{2}} \geqslant \frac{3}{\sqrt[3]{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}}. \qquad (**)$$

Transforming now the radicand, we get

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{1}{2}\sin\frac{\alpha}{2}\left[\cos\frac{\beta-\gamma}{2} - \cos\frac{\beta+\gamma}{2}\right].$$

Since $\alpha+\beta+\gamma=\pi$, we have

$$\cos\frac{\beta+\gamma}{2}=\cos\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)=\sin\frac{\alpha}{2},$$

$$\cos\frac{\beta-\gamma}{2}<1,$$

and, consequently,

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\leqslant\frac{1}{2}\sin\frac{\alpha}{2}\left(1-\sin\frac{\alpha}{2}\right).$$

The greatest value of the function appearing on the right-hand side of the inequality $f(y)=\frac{1}{2}y(1-y)$ $\left(y=\sin\frac{\alpha}{2}\right)$, on the interval [0, 1] is equal to 1/4. Consequently,

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\leqslant\frac{1}{8}.$$

Snce the value of a fraction decreases with an increase in the denominator, we can substitute the value of the denominator obtained into the right-hand side of (**), extract a cubic root of 1/8 and make sure of the validity of the initial inequality.

Prove the following inequalities.

8.10*. Prove that

$$(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)\leqslant 1/8$$

if $\alpha+\beta+\gamma=\pi$ and $\alpha, \beta, \gamma>0$.

8.11*. Prove that

$$\frac{1}{\cos\alpha}+\frac{1}{\cos\beta}+\frac{1}{\cos\gamma}\geqslant 6$$

if $\alpha+\beta+\gamma=\pi$ and $\alpha, \beta, \gamma\in(0, \pi/2)$.

8.12*. Prove that

$$\tan^2\alpha+\tan^2\beta+\tan^2\gamma\geqslant 9$$

if $\alpha+\beta+\gamma=\pi$ and $\alpha, \beta, \gamma\in(0, \pi/2)$.

8.13. Prove that

$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\leqslant\frac{3}{8}\sqrt{3}$$

if $\alpha+\beta+\gamma=\pi$.

8.14**. Prove that

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}\geqslant\frac{3}{4}$$

if $\alpha+\beta+\gamma=\pi$.

8.15. Prove that

$$\cos\alpha + \cos\beta + \cos\gamma \leqslant 3/2$$

if $\alpha + \beta + \gamma = \pi$.

In Example 8.2 we had to find the greatest value of the function $\frac{1}{2}\sin\frac{\alpha}{2}\left(1-\sin\frac{\alpha}{2}\right)$. Making a substitution, we have found that the required value coincides with the greatest value of the function $f(y)=\frac{1}{2}y(1-y)$ on the interval $[0, 1]$. A similar technique is often used when it is required to find the sets of values of some complex trigonometric expressions.

8.16. Prove that

$$-4 \leqslant \cos 2x + 3\sin x \leqslant 2\frac{1}{8}.$$

8.17. Prove that

$$\cos^3 x + \cos^6 x \leqslant 1/4.$$

8.18. Prove that

$$\sin^2 x + p\sin x + q \geqslant \frac{-p^2-4q}{4}$$

if $|p| < 2$.

8.19. Prove that $\sin^2 x \cos^2 x \leqslant 1/4$.

Inequalities relating trigonometric functions and certain polynomials considered on certain intervals of variation of the arguments can be proved by using combinations of the techniques discussed above. In that case, the proof is often based on the inequality

$$\sin x \leqslant x \leqslant \tan x, \tag{1}$$

which is valid for $x \in [0, \pi/2]$.

Example 8.3. Prove that

$$x - \frac{x^3}{4} < \sin x$$

on the interval $(0, \pi)$.

Solution. Let us represent the function $\sin x$ as

$$\sin x = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = 2\tan\frac{x}{2}\left(1-\sin^2\frac{x}{2}\right).$$

Using (1), we obtain

$$\tan\frac{x}{2} \geqslant \frac{x}{2}, \quad 1-\sin^2\frac{x}{2} \geqslant 1-\frac{x^2}{4}.$$

Substituting the estimates obtained into the right-hand side of the initial inequality, we make sure of its validity.

8.20. Prove that the inequality

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{16}$$

is valid on the interval $(0, \pi)$.

8.21. Prove that the inequality

$$x - \frac{x^3}{2} < \tan x$$

is valid on the interval $(0, \pi/2)$.

8.22. Prove that

$$1 - \cos x \leqslant \frac{x^2}{2}.$$

Chapter 5

Complex Numbers

The notation for the complex number z in the form $a + bi$, where a and b are real numbers, is known as an *algebraic form* of a complex number. The number a is called a *real part* of the complex number and is designated as Re z, and the number b is called an *imaginary part* of a complex number and is designated as Im z. The symbol i is called an *imaginary unit.*

Two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are *equal* if $a_1 + a_2$ and $b_1 = b_2$.

The complex number $-a - bi$ is said to be *opposite* to the complex number $a + bi$.

Rules for operating with complex numbers. Assume that $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are two complex numbers. The *sum* $z_1 + z_2$, the *difference* $z_1 - z_2$, the *product* $z_1 \cdot z_2$, and the *quotient* z_1/z_2 ($z_2 \neq 0$) of the complex numbers z_1 and z_2 can be calculated by the formulas

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + (b_1 + b_2)\, i, \\ z_1 - z_2 &= (a_1 - a_2) + (b_1 - b_2)\, i, \\ z_1 \cdot z_2 &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)\, i, \\ \frac{z_1}{z_2} &= \frac{a_1 b_2 + a_2 b_1}{a_2^2 + b_2^2} + \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\, i. \end{aligned} \tag{1}$$

Addition and multiplication of complex numbers are commutative and associative and multiplication is distributive with respect to addition.

The complex number $a - bi$ is a *complex conjugate* of $z = a + bi$. It is designated as $\bar{z}$. The property of complex conjugate numbers is $z \cdot \bar{z} = a^2 + b^2$.

Assume that $z = a + bi$ is a nonzero complex number ($a^2 + b^2 \neq 0$). The *modulus (absolute value) of the complex number*, designated as $|z|$, is a quantity $r = \sqrt{a^2 + b^2}$; the *argument of the complex number* z is an angle φ found from the conditions

$$\sin \varphi = \frac{b}{\sqrt{a^2 + b^2}}, \quad \cos \varphi = \frac{a}{\sqrt{a^2 + b^2}}.$$

The argument of the complex number z is designated as $\arg z$. The notation for the complex number $z = a + bi$ in the form

$$z = r(\cos \varphi + i \sin \varphi)$$

is known as a *trigonometric form* of notation of a complex number.

The *principal value* of the argument of the complex number z ($\arg z$) is the value of φ which belongs to the interval $[0, 2\pi]$. In terms of *geometry*, the *modulus* of a complex number can be represented as a line segment (radius vector) with the points (0, 0) and (a, b) as its ends, the argument of a complex number is an angle which the radius vector forms with the positive direction of the x-axis (Fig. 5.1).

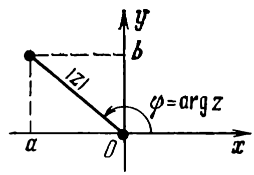

Fig. 5.1

Two complex numbers written in trigonometric form are equal if and only if their moduli are equal and the arguments differ by $2\pi k$ ($k \in \mathbb{Z}$).

The *product* and the *quotient* of two nonzero complex numbers $z_1 = r_1(\cos \varphi_1 + i \sin \varphi_1)$ and $z_2 = r_2(\cos \varphi_2 + i \sin \varphi_2)$, written in trigonometric form, are the numbers

$$z_1 \cdot z_2 = r_1 \cdot r_2 [\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)],$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)].$$

The *nth degree* of the complex number $z = r(\cos \varphi + i \sin \varphi)$ can be calculated by *de Moivre's formula*

$$z^n = r^n (\cos n\varphi + i \sin n\varphi).$$

The *root of degree n* of the complex number z is a complex number w satisfying the equation

$$w^n = z.$$

All solutions of this equation are designated as $\sqrt[n]{z}$ and for the number z written in the trigonometric form $z = r(\cos \varphi + i \sin \varphi)$ are calculated by the formula

$$\sqrt[n]{z} = \sqrt[n]{r} \left(\cos \frac{\varphi + 2\pi k}{n} + i \sin \frac{\varphi + 2\pi k}{n} \right),$$

where $k = 0, 1, 2, \ldots, n - 1$.

1. Operations with Complex Numbers

When calculating a product and a quotient of complex numbers, it is often convenient to represent complex numbers in trigonometric form.

Example 1.1. Represent the complex number $z = -3 + i$ in trigonometric form.

Solution. By the definition of the modulus of a complex number, we have

$$|z| = \sqrt{(-3)^2+1} = \sqrt{10}.$$

Designating the argument of the complex number as φ, we get

$$\sin\varphi = \frac{1}{\sqrt{10}}, \quad \cos\varphi = -\frac{3}{\sqrt{10}},$$

whence it follows that the angle φ belongs to the second quarter and is equal to $\arccos(-3/\sqrt{10})$. Consequently, the complex number $z = -3 + i$ has a trigonometric form:

$$z = \sqrt{10}\left[\cos\left(\arccos\left(-\frac{3}{\sqrt{10}}\right)\right) + i\sin\left(\arccos - \frac{3}{\sqrt{10}}\right)\right].$$

Represent the following complex numbers in trigonometric form.
1.1. 1. **1.2.** -3. **1.3.** i.
1.4. $1 + i$. **1.5.** $-1 + i$. **1.6.** $1 + i\sqrt{3}$.
1.7. $3 - 4i$. **1.8.** $-3 - 4i$.
1.9. $-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. **1.10.** $\sin\alpha - i\cos\alpha$. **1.11.** $\left(\frac{1}{i-1}\right)^{100}$.

The roots of degree k of complex numbers are usually calculated by representing the complex numbers in trigonometric form.

Example 1.2. Calculate $\sqrt[3]{i}$.

Solution. We write the complex number $i = 0 + 1 \cdot i$ in trigonometric form. Since $|i| = 1$, $\arg i = \pi/2$, the number i is in the trigonometric form

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}.$$

According to the definition of a root of degree n of the complex number $\sqrt{i}$, these are complex numbers z satisfying the equation

$$z^3 = i. \qquad (*)$$

Writing the required number z in the trigonometric form $z = r(\cos\varphi + i\sin\varphi)$ and using de Moivre's formula, we write equation $(*)$ in the form

$$r^3(\cos 3\varphi + i\sin 3\varphi) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}.$$

By the definition of the equality of two complex numbers written in trigonometric form, we obtain

$$r^3 = 1, \; 3\varphi = \frac{\pi}{2} + 2\pi k, \; k \in Z,$$

whence it follows that

$$r = \sqrt[3]{1} = 1, \quad \varphi = \frac{\pi}{6} + \frac{2\pi k}{3}.$$

Among the infinite set of numbers z with modulus 1 and argument $\varphi = \frac{\pi}{6} + \frac{2\pi k}{3}$, those numbers obtained for $k=0$, $k=1$, $k=2$ are distinct:

$$\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\cdot\frac{1}{2},$$

$$\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2},$$

$$\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i.$$

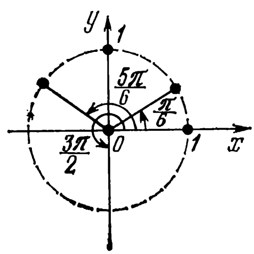

Fig. 5.2

They are precisely the values of $\sqrt[3]{i}$. In geometrical interpretation the complex numbers $\sqrt[3]{i}$ can be represented as points lying on a unit circle (since $r = 1$), whose radius vectors form the angles $\pi/6$, $5\pi/6$, and $3\pi/2$ with the positive direction of the x-axis (Fig. 5.2).

Calculate the following expressions using the trigonometric notation for complex numbers.

1.12. $\sqrt{2i}$. **1.13.** $\sqrt{-8i}$. **1.14.** $\sqrt{3-4i}$.

1.15. $\sqrt[4]{-1}$. **1.16.** $\sqrt[4]{2-2i\sqrt{3}}$. **1.17.** $\sqrt[4]{i}$.

1.18. $\sqrt[7]{3+4i}$. **1.19.** $\sqrt[3]{1}$.

2. Geometrical Representation of Complex Numbers

An algebraic notation for complex numbers which satisfy certain relations is usually used to represent them geometrically in the coordinate system Oxy.

Example 2.1. Find the set of points of the coordinate plane Oxy which represent the complex numbers z for which $|z + i - 2| \leqslant 2$.

Solution. Let us write the complex number z in algebraic form $z = x + iy$. Then we have

$$z + i - 2 = (x - 2) + i(y + 1).$$

By the definition of the modulus of a complex number, we have

$$|z+i-2| = \sqrt{(x-2)^2+(y+1)^2},$$

and hence the inequality $|z + i - 2| \leqslant 2$ assumes the form

$$\sqrt{(x-2)^2+(y+1)^2} \leqslant 2 \Longleftrightarrow (x-2)^2+(y+1)^2 \leqslant 2^2.$$

The set of points of the coordinate plane Oxy, which satisfy the last inequality, is the set of all points lying in the interior of the circle and on the circle with centre at the point $(2, -1)$ and radius 2.

Example 2.2. Find the set of points, belonging to the coordinate plane Oxy, for which the real part of the complex number $(1+i)z^2$ is positive.

Solution. Let us represent the number z in the algebraic form: $z = x + iy$. Then

$$z^2 = x^2 - y^2 + i(2xy),$$

$$(1+i)z^2 = (1+i)[x^2 - y^2 + i(2xy)]$$

$$= (x^2 - 2xy - y^2) + i(x^2 + 2xy - y^2).$$

By the hypothesis, the real part of the complex number $(1+i)z^2$ is positive:

$$x^2 - 2xy - y^2 > 0. \quad (*)$$

Assuming that $y \neq 0$, and dividing both sides of the inequality by y^2, we get an inequality

$$\left(\frac{x}{y}\right)^2 - 2\left(\frac{x}{y}\right) - 1 > 0.$$

Solving this quadratic inequality, we obtain

$$x/y > 1 + \sqrt{2}, \quad x/y < 1 - \sqrt{2}. \quad (**)$$

For $y > 0$, we can write inequalities $(**)$ in the form

$$y < \frac{1}{1+\sqrt{2}} x, \quad y < \frac{1}{1-\sqrt{2}} x.$$

The set of points of the plane Oxy which satisfy these conditions is hatched in Fig. 5.3.

For $y < 0$, inequalities $(**)$ can be written in the form

$$y > \frac{1}{1+\sqrt{2}} x, \quad y > \frac{1}{1-\sqrt{2}} x$$

and the set of points of the coordinate plane Oxy which satisfy these inequalities is hatched in Fig. 5.4.

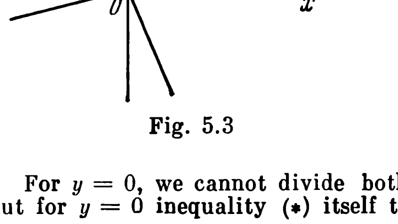

Fig. 5.3

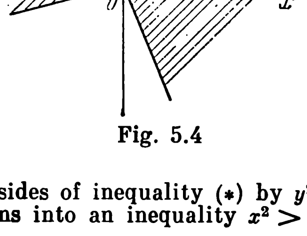

Fig. 5.4

For $y = 0$, we cannot divide both sides of inequality $(*)$ by y^2, but for $y = 0$ inequality $(*)$ itself turns into an inequality $x^2 > 0$

whose solution is any real number x different from zero, i.e. any point on the x-axis except for zero is a solution of inequality (*).

Uniting these three cases, we definitely establish that the required set is constituted by the angles, without their boundaries, whose sides

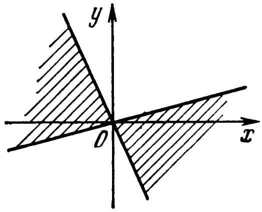

Fig. 5.5

are the straight lines $y = \frac{1}{1+\sqrt{2}}x$ and $y = \frac{1}{1-\sqrt{2}}x$ containing the x-axis (Fig. 5.5).

Find the set of points of the coordinate plane Oxy which represent the complex numbers $z = x + iy$, for which

2.1. $|z| = 1$. **2.2.** $z = |z|$. **2.3.** $\arg z = \pi/3$.

2.4. $1 < |z| < 4$. **2.5.** $|2z - 1| > 2$. **2.6.** $||z| + i| < 10$.

2.7. $|z + 1| = |z - 1|$. **2.8.** $|z + i| = |z + 2|$.

2.9. $|z + i| > |z|$. **2.10.** $1 \leqslant |z + i| \leqslant 4$.

2.11. $(1 - i)\bar{z} = (1 + i)z$.

2.12. On the coordinate plane Opq represent the set of points (p, q) such that the roots which may be complex of the equation

$$x^2 + px + q = 0$$

do not exceed unity in absolute value.

2.13. Indicate all points of a complex plane such that (1) $z\alpha$, (2) $z + \alpha$ are real numbers ($\alpha = a + bi$ is a given complex number).

2.14. Find the set of points of the coordinate plane Oxy which satisfy the inequality

$$\log_{\frac{1}{2}} \frac{|z-1|+4}{3|z-1|-2} > 1.$$

2.15. On the plane Oxy find the set of all points whose coordinates satisfy the following conditions: the number $z^2 + z + 1$ is real and positive.

3. Solution of Equations in the Set of Complex Numbers

The solution of equations in the set of complex numbers reduces to the solution of systems of equations in the set of real numbers resulting from the comparison of real and imaginary parts of the left-hand and right-hand sides of the initial equation.

Example 3.1. Solve the equation

$$2z = |z| + 2i$$

in complex numbers.

Solution. We write the complex number z in the algebraic form $z = x + iy$, where x and y are real numbers. Then $|z| = \sqrt{x^2 + y^2}$ and the given equation assumes the form

$$2x + 2iy = \sqrt{x^2 + y^2} + 2i.$$

From the definition of the equality of two complex numbers we get a system of equations for determining x and y:

$$2x - \sqrt{x^2 + y^2} = 0,$$
$$2y - 2 = 0.$$

From the second equation we find $y = 1$. Substituting $y = 1$ into the first equation of the system, we get an equation $2x = \sqrt{x^2 + 1}$ whose solution is $x = \frac{1}{\sqrt{3}}$. Thus the complex number $z = \frac{1}{\sqrt{3}} + i$ is a solution of the given equation.

Answer. $z = \frac{1}{\sqrt{3}} + i.$

Ex mple :.2. For every real number $a > 0$ find all complex numbers z satisfying the equation

$$z|z| + az + i = 0.$$

Solution. We write the complex number z in the algebraic form $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2}$ and the equation assumes the form

$$(x + iy)\sqrt{x^2 + y^2} + a(x + iy) + i = 0$$
$$\Leftrightarrow (x\sqrt{x^2 + y^2} + ax) + i(y\sqrt{x^2 + y^2} + ay + 1) = 0 + 0 \cdot i.$$

It follows from the definition of the equality of two complex numbers that the last equation is equivalent to a system of two equations:

$$x\sqrt{x^2 + y^2} + ax = 0,$$
$$y\sqrt{x^2 + y^2} + ay + 1 = 0, \qquad (*)$$

whose solutions are sought in the set of real numbers.

It can be easily seen that the set of solutions of the first equation of system (*) can be found as a union of the sets of solutions of two equations:

$$x = 0, \qquad \sqrt{x^2 + y^2} + a = 0.$$

The second equation has no solutions by virtue of the condition $a > 0$. Substituting $x = 0$ into the second equation of system (*), we get an equation for the real number y:

$$y|y| + ay + 1 = 0,$$

whose set of solutions is obtained as a union of the sets of solutions of two systems:

$$y \geqslant 0, \qquad\qquad y < 0,$$
$$y^2 + ay + 1 = 0, \quad -y^2 + ay + 1 = 0.$$

Bearing in mind the condition $a > 0$, it is easy to verify that the first system has no solutions and the second system has a unique solution $y = \dfrac{a - \sqrt{a^2 + 4}}{2}$. Thus, the purely imaginary number

$$z = i\,\frac{a - \sqrt{a^2 + 4}}{2}$$

in a solution of the given equation.

Solve the following equations.

3.1. $(2 + i)\,z^2 - (5 - i)\,z + 2 - 2i = 0$. **3.2.** $z^2 + \bar{z} = 0$.

3.3. $|\,z\,| - iz = 1 - 2i$. **3.4.** $z^2 = (\bar{z})^3$.

3.5. $(x + y)^2 + 6 + ix = 5\,(x + y) + i\,(y + 1)$ (x, y are real numbers).

3.6. For what real values of x and y does the equality

$$\frac{x - 2 + (y - 3)\,i}{1 + i} = 1 - 3i$$

hold true?

3.7. Prove that the equation $z^3 + iz - 1 = 0$ does not have real solutions.

3.8. Calculate $z^{14} + 1/z^{14}$ if z is a root of the equation

$$z + 1/z = 1.$$

3.9. Solve the equation

$$z^3 - z^2 + z - 1 = 0$$

in complex numbers.

3.10. Solve the following systems in complex numbers:

(a) $z^5 w^7 = 1,\quad z^2 - \bar{w}^3 = 0,$

(b) $z^{13} w^{19} = 1,\quad z^5 w^7 = 1,\quad z^2 + w^2 = -2.$

3.11. What condition should be satisfied by the complex number $a + bi$ for it to be representable in the form

$$a + bi = \frac{1 - ix}{1 + ix},$$

where x is a real number.

3.12. Among the complex numbers z find all those for which

$$\log_{14}(13 + |\,z^2 - 4i\,|) + \log_{196}\frac{1}{(13 + |\,z^2 + 4i\,|)^2} = 0.$$

3.13. For every real number $a \geqslant 1$ find all complex numbers z which satisfy the condition

$$z + a | z + 1 | + i = 0.$$

3.14*. Find the real values of the parameter a for which at least one complex number $z = x + iy$ satisfies both the equality

$$| z + \sqrt{2} | = a^2 - 3a + 2$$

and the inequality

$$| z + i\sqrt{2} | < a^2.$$

3.15. Find the real values of the parameter a for which at least one complex number $z = x + iy$ satisfies both the equality

$$| z - ai | = a + 4,$$

and the inequality

$$| z - 2 | < 1.$$

3.16. Find the complex number z, the least in the absolute value, which satisfies the condition

$$| z - 2 + 2i | = 1.$$

Chapter 6

Sequences

1. The Definition and the Properties of Sequences

A set of quantities enumerated in a definite order is called a *sequence of numbers*. The elements of a number sequence are *terms of the sequence*; the first term is designated as x_1, the second term as x_2, the nth term as x_n and so on. The whole number sequence is designated as $x_1, x_2, \ldots, x_n, \ldots$ or (x_n), $n \in \mathrm{N}$*. The concept of a sequence can also be introduced by means of the concept of a function: an *infinite number sequence* (x_n), $n \in \mathrm{N}$, is a numerical function $f(n)$ defined on the set of all natural numbers. The formula which makes it possible to calculate any term of a sequence from its number n is known as the *formula for the general term* of the sequence.

The sequence (x_n), $n \in \mathrm{N}$, is said to be *bounded* if there are two numbers m and M such that the double** inequality

$$m \leqslant x_n \leqslant M \tag{1}$$

holds true for all $n \in \mathrm{N}$. The sequence (x_n), $n \in \mathrm{N}$, is said to be *monotone increasing* if the inequality

$$x_{n+1} > x_n \tag{2}$$

holds true for any natural n, and *monotone decreasing* if the inequality

$$x_{n+1} < x_n \tag{3}$$

holds true for any natural n. The sequence (x_n), $n \in \mathrm{N}$, is said to be *nondecreasing* (*nonincreasing*) if inequality (2) ((3) respectively) is nonstrict.

Increasing, decreasing, nonincreasing, and nondecreasing sequences are known as *monotone* sequences. The definition of monotonicity can be generalized to the sequences which possess this property only beginning with a certain term. In that case, the corresponding inequality must be satisfied for all $n > n_0$, where n_0 is the number of the term beginning with which the sequence becomes monotonic.

Example 1.1. Prove that the sequence defined by the formula for the general term

$$x_n = \frac{3n-1}{5n+2}$$

is increasing.

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* We shall also designate number sequences as (y_n), (z_n), $n \in \mathrm{N}$.

** We shall omit the word "double" everywhere for brevity.

Solution. Let us consider the difference

$$x_{n+1}-x_n=\frac{3(n+1)-1}{5(n+1)+2}-\frac{3n-1}{5n+2}$$

and verify the validity of the inequality $x_{n+1} - x_n > 0$ for all $n \in$ N:

$$\frac{3(n+1)-1}{5(n+1)+2}-\frac{3n-1}{5n+2}>0, \quad \text{or} \quad \frac{11}{(5n+7)(5n+2)}>0.$$

Since the last inequality holds for all $n \in$ N, it follows, according to (2), that the sequence is increasing.

1.1. Prove that the sequence

$$y_n=\frac{6-n}{5n-1}, \quad n\in \mathbf{N},$$

is decreasing.

1.2. Is the sequence

$$y_n=\frac{2n-3}{n}, \quad n\in \mathbf{N},$$

monotonic?

1.3*. Is the sequence

$$y_n=\frac{2^n}{n!}, \quad n\in \mathbf{N},$$

monotonic?

1.4. Find the ratios between a, b, c, d for which the sequence

$$y_n=\frac{an+b}{cn+d}, \quad n\in \mathbf{N}.$$

is increasing.

Example 1.2. Find the greatest term of the sequence

$$y_n = -n^2 + 5n - 6,\ n \in \mathbf{N}.$$

Solution. Let us consider the function

$$y(x) = -x^2 + 5x - 6.$$

It assumes the greatest value at the point $x = 2.5$, the function $y(x)$ being increasing on the interval $(-\infty, 2.5)$ and decreasing on the interval $(2.5, +\infty)$. Consequently, returning to the sequence, we can write

$$y_1 < y_2 \quad \text{and} \quad y_2 > y_n, \quad n \geqslant 4.$$

Thus either y_2 or y_3 is the greatest value of the function, but $y_2 = y_3 = 0$.

Answer. The second and third terms of the sequence are the greatest.

Find the greatest and the least terms of the following sequences.

1.5. $y_n = n^2 - 1,\ n \in$ N.

1.6. $y_n = 6n - n^2 - 5$, $n \in \mathbf{N}$.

1.7*. $x_n = 2n + \frac{512}{n^2}$, $n \in \mathbf{N}$.

1.8. The sequence (x_n), $n \in \mathbf{N}$, is defined by the formula for the general term $x_n = \frac{2n-3}{n}$. Findthe natural values of n which satisfy the following conditions:

$$\text{(a) } |x_n - 2| < 0.1; \text{ (b) } |x_n - 2| < 0.01.$$

1.9*. How many terms of the sequence

$$y_n = |n^2 - 5n + 6|, \quad n \in \mathbf{N},$$

satisfy the inequality $2 < y_n < 6$?

1.10.* Beginning with what number do the terms of the sequence

$$y_n = n^2 - 5n + 6, \quad n \in \mathbf{N},$$

satisfy the inequality $x_{n+1} > x_n$?

1.11*. Beginning with what number n is the sequence, defined by the formula for the general term $y_n = nq^n$, monotone if $0 < q < 1$?

If a sequence is defined by the formula for the general term $x_n = f(n)$, then the boundedness of the sequence from above and from below can be derived from the boundedness of the function $f(x)$ for $x \in [1, \infty)$.

Example 1.3. Is the sequence

$$x_n = \frac{3n+8}{2n}, \quad n \in \mathbf{N},$$

bounded?

Solution. Let us consider the function

$$f(x) = \frac{3x+8}{2x},$$

which defines the terms of the given sequence for $x = n$, $n \in \mathbf{N}$. We find the interval of variation of the function on the interval $[1, \infty)$.

Writing the function $f(x)$ in the form

$$f(x) = 3/2 + 4/x, \qquad (*)$$

we ascertain that the function decreases monotonically for $x \geqslant 1$. Consequently, the function has the greatest value for $x = 1$, and it is equal to $11/2$. It can be seen from notation $(*)$ that

$$f(x) > 3/2$$

for all $x \in [1, +\infty)$. Consequently, $f(n) \in (3/2, 11/2]$.

Answer. The sequence is bounded: all its terms are contained in the interval $(3/2, 11/2]$.

Find out whether the following sequences are bounded.

1.12. $x_n = 2^{(-1)^n}$, $n \in \mathrm{N}$.

1.13. $x_n = \dfrac{3+n^2}{2+n^2}$, $n \in \mathrm{N}$.

1.14*. $x_n = \dfrac{1}{n^2-2n+3}$, $n \in \mathrm{N}$.

1.15*. $x_n = \left(\dfrac{1}{(n+1)!} - \dfrac{1}{(n+2)!}\right)(n+2)$, $n \in \mathrm{N}$.

2. The Limit of a Sequence

We say that the number a is the *limit of the sequence* (x_n), $n \in \mathrm{N}$, and write $\lim x_n = a$, if, for any $\varepsilon > 0$, there is a number $n_0(\varepsilon)$ such that the inequality $|x_n - a| < \varepsilon$ holds for all $n > n_0(\varepsilon)$. If the sequence (x_n), $n \in \mathrm{N}$, has a limit, it is said to be *convergent*.

Here is the *necessary condition for the convergence* of a number sequence: for a sequence to be convergent it is necessary that it should be bounded.

Example 2.1. Prove that

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

Solution. To prove that the limit of the sequence $x_n = \dfrac{n+1}{n}$, $n \in \mathrm{N}$, is equal to unity, it is sufficient to indicate the way of constructing the number $n_0(\varepsilon)$, appearing in the definition of a limit, for any $\varepsilon > 0$. Let us specify $\varepsilon > 0$ and set up an inequality

$$\left|\frac{n+1}{n} - 1\right| < \varepsilon, \qquad (*)$$

which is equivalent to the inequality $1/n < \varepsilon$. Consequently, if we choose the number $[1/\varepsilon] + 1$* as the number $n_0(\varepsilon)$, then inequality (*) is satisfied for all $n > n_0(\varepsilon)$. Thus we have proved the assertion that unity is a limit of the sequence $x_n = \dfrac{n+1}{n}$, $n \in \mathrm{N}$.

Prove the following:

2.1. $\lim\limits_{n\to\infty} \dfrac{3n-2}{2n} = 1.5$. **2.2.** $\lim\limits_{n\to\infty} \dfrac{1}{n^2+1} = 0$.

2.3. $\lim\limits_{n\to\infty} \dfrac{5n-3}{6n+2} = \dfrac{5}{6}$. **2.4.** $\lim\limits_{n\to\infty} \dfrac{6n-1}{1/2-n} = -6$.

2.5. $\lim\limits_{n\to\infty} \dfrac{n+1}{n^2+1} = 0$. **2.6.** $\lim\limits_{n\to\infty} \dfrac{n^2}{n^2+n} = 1$.

It is convenient to use the following geometric interpretation of the concept of a limit of a sequence in order to solve some problems which require proving the convergence of sequences.

* The symbol $[z]$ designates the integral part of z.

The number a is a *limit of the sequence* (x_n), $n \in \mathbf{N}$, if, for any positive number ε there is a number $n = n_0(\varepsilon)$ such that all terms of the sequence, beginning with x_{n_0+1}, belong to the ε-neighbourhood of the number a, i.e. to the interval $(a - \varepsilon, a + \varepsilon)$.

Using the geometric interpretation indicated above, let us ascertain the validity of the following statements:

2.7*. If a sequence converges to a certain number, it is bounded. (*The necessary criterion of convergence*).

2.8*. It is known that $\lim\limits_{n\to\infty} x_n = a$, $a < q$. Prove that almost all terms of the sequence (x_n), $n \in \mathbf{N}$ (with the exception, perhaps, of a finite number of terms), are smaller than q.

2.9*. It is known that $\lim\limits_{n\to\infty} a_n = p$, $\lim\limits_{n\to\infty} b_n = q$, $p \neq q$. Is there a limit of the sequence $a_1, b_1, a_2, b_2, \ldots, a_n b_n, \ldots$?

2.10. Using the result of the preceding problem, prove that the sequence

$$x_n = 1 + (-1)^n$$

has no limit.

2.11*. Find whether the sequence

$$x_n = \sin n \frac{\pi}{2}$$

has a limit.

2.12*. Find whether the sequence

$$x_n = \frac{1}{n} \sin n \frac{\pi}{2}$$

has a limit.

2.13. Find whether the following sequences have limits:

$$(a)\ x_n = 1 + \frac{(-1)^n}{n}, \quad (b)\ x_n = \left[1 + \frac{(-1)^n}{n}\right].$$

3. Calculating the Limits of Sequences

Limits of sequences are usually calculated with the use of the following *properties of convergent sequences*:

If two sequences (x_n), $n \in \mathbf{N}$, and (y_n), $n \in \mathbf{N}$, converge to $\lim\limits_{n\to\infty} x_n$ and $\lim\limits_{n\to\infty} y_n$, then

the sequence $(x_n \pm y_n)$, $n \in \mathbf{N}$, converges to

$$\lim_{n\to\infty} (x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n, \quad (1)$$

the sequence $(x_n y_n)$, $n \in \mathbf{N}$, converges to

$$\lim_{n\to\infty} x_n y_n = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n), \quad (2)$$

if, in addition, $\lim_{n\to\infty} y_n \neq 0$, then the sequence $\left(\frac{x_n}{y_n}\right)$, $n \in \mathbf{N}$, converges and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}. \qquad (3)$$

When calculating limits, the direct application of formulas (1)-(3) is usually preceded by some identity transformations. When limits of the form

$$\lim_{n\to\infty} \frac{x_n}{y_n},$$

where x_n and y_n are infinitely increasing sequences, are calculated, the division of the numerator and the denominator of the fraction by the same expression is such a transformation.

Example 3.1. Calculate the limit

$$\lim_{n\to\infty} \frac{5n+1}{7-9n}.$$

Solution. Since the numerator and the denominator are infinite sequences, it is impossible to make a direct use of formula (3). Let us divide the numerator and the denominator by n. We can now apply formula (3) to the resulting fraction:

$$\lim_{n\to\infty} \frac{5+1/n}{7/n-9} = \frac{\lim\limits_{n\to\infty} (5+1/n)}{\lim\limits_{n\to\infty} (7/n-9)}.$$

Applying now formula (1), we obtain

$$\frac{\lim\limits_{n\to\infty} (5+1/n)}{\lim\limits_{n\to\infty} (7/n-9)} = \frac{\lim\limits_{n\to\infty} 5 + \lim\limits_{n\to\infty} (1/n)}{\lim\limits_{n\to\infty} (7/n) - \lim\limits_{n\to\infty} 9}.$$

Taking into account that the limit of a constant is equal to that constant, and

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

we obtain

$$\lim_{n\to\infty} \frac{5n+1}{7-9n} = -\frac{5}{9}.$$

Answer. $-5/9$.

Calculate the following limits by means of a division of the numerator and the denominator of the fraction by the leading power of n:

3.1. $\lim\limits_{n\to\infty} \frac{3n^2-7n+1}{2-5n-6n^2}$. 3.2. $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n^3+2n-1}}{n+2}$.

3.3. $\lim\limits_{n\to\infty} \frac{(n+1)^4-(n-1)^4}{(n+1)^4+(n-1)^4}$. 3.4. $\lim\limits_{n\to\infty} \frac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}+\sqrt{n^3+1}}$.

In a number of cases, calculation of limits of expressions containing exponential functions with natural argument can be based on the following equality:

$$\lim_{n\to\infty} q^n = 0 \quad \text{if} \quad |q| < 1. \tag{4}$$

Example 3.2. Calculate the limit

$$\lim_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{2^n+3^n}.$$

Solution. We divide the numerator and the denominator of the fraction by 3^{n+1}. Applying formula (4), we obtain

$$\lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^{n+1}+1}{\frac{1}{3}\left(\frac{2}{3}\right)^n+\frac{1}{3}} = \frac{\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^{n+1}+1}{\frac{1}{3}\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n+\frac{1}{3}} = \frac{0+1}{\frac{1}{3}\cdot 0+\frac{1}{3}} = 3.$$

Answer. 3.

Calculate the following limits.

3.5. $\lim\limits_{n\to\infty} \frac{2^n+3^n}{2^n-3^n}$. 3.6*. $\lim\limits_{n\to\infty} \frac{3\cdot 2^{n+1}-7\cdot 3^n+1}{2^{n+1}-5\cdot 3^{n+1}+6}$.

It is sometimes convenient to use the following *property of sequences*, when calculating limits: if the terms of two sequences, (a_n), $n \in \mathrm{N}$, and (b_n), $n \in \mathrm{N}$ are related as

$$|a_n| \leqslant |b_n| \quad \text{for all} \quad n \in \mathrm{N},$$

then the equality of the limit of the sequence (b_n) to zero yields the equality to zero of the limit of the sequence (a_n), $n \in \mathrm{N}$.

Example 3.3. Calculate the limit

$$\lim_{n\to\infty} \left(\frac{n}{2n+1}\right)^n.$$

Solution. Let us first make sure that the inequality

$$\frac{n}{2n+1} < \frac{1}{2}$$

is valid for all $n \in \mathrm{N}$. Dividing the numerator and the denominator of the fraction appearing on the left-hand side of the inequality by n, we get an obvious inequality

$$\frac{1}{2+\frac{1}{n}} < \frac{1}{2}.$$

Using the property of powers, we infer that the inequality

$$\left(\frac{1}{2+1/n}\right)^n < \left(\frac{1}{2}\right)^n$$

is valid for all $n \in \mathbf{N}$. Since $\lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$ according to (4), it follows from the property of sequences cited above that

$$\lim_{n\to\infty}\left(\frac{n}{2n+1}\right)^n = 0.$$

Answer. 0.

Calculate the following limits.

3.7*. $\lim\limits_{n\to\infty}\left(\frac{3n+1}{4n+5}\right)^n$.

3.8*. $\lim\limits_{n\to\infty}\left[\frac{n^2-1}{(2n+1)(n+2)}\right]^n$.

3.9*. $\lim\limits_{n\to\infty}\frac{n \sin n!}{n^2+1}$.

When limits containing irrational expressions are calculated, the irrational expression is often transferred from the denominator to the numerator and vice versa.

Example 3.4. Calculate the limit

$$\lim_{n\to\infty}(\sqrt{n^2+2n}-n).$$

Solution. Let us multiply and divide the expression under the limit sign by the conjugate expression. Then we get

$$\lim_{n\to\infty}\frac{(\sqrt{n^2+2n}-n)(\sqrt{n^2+2n}+n)}{\sqrt{n^2+2n}+n} = \lim_{n\to\infty}\frac{n^2+2n-n^2}{\sqrt{n^2+2n}+n}$$

$$= \lim_{n\to\infty}\frac{2n}{\sqrt{n^2+2n}+n}. \quad (**)$$

Dividing the numerator and the denominator by n, we obtain

$$\lim_{n\to\infty}\frac{2n}{\sqrt{n^2+2n}+n} = \lim_{n\to\infty}\frac{2}{\sqrt{1+2/n}+1} = 1.$$

Answer. 1.

Calculate the following limits.

3.10. $\lim\limits_{n\to\infty}(\sqrt{n+2}-\sqrt{n})$.

3.11. $\lim\limits_{n\to\infty}(\sqrt{n^2-5n+6}-n)$.

3.12. $\lim\limits_{n\to\infty} n(\sqrt{n^2+1}-n)$.

3.13*. $\lim\limits_{n\to\infty}(n+\sqrt[3]{1-n^3})$.

If the limit of the sequence (x_n), $n \in \mathbf{N}$, is known to exist, then it is convenient to use the following formula for its calculation:

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{n+1}. \tag{5}$$

Example 3.5. Find the limit of the sequence $x_n = q^n$, $n \in \mathbf{N}$, for $|q| < 1$ (i.e. prove formula (4)).

Solution. We designate the required limit as a. Note that the given sequence can be written in a recurrent form:

$$x_{n+1} = qx_n.$$

Passing to the limit in this equality and using (5), we get

$$\lim_{n \to \infty} x_n = q \lim_{n \to \infty} x_n,$$

or $a = aq$, or $a(1 - q) = 0$, and since $|q| < 1$, we have $a = 0$.

Answer. $\lim\limits_{n \to \infty} x_n = 0$.

3.14*. The sequence (x_n), $n \in \mathbf{N}$, whose first term is $x_1 = \sqrt{2}$ can be determined from the recursion formula

$$x_{n+1} = \sqrt{2 + x_n}.$$

Find the limit of (x_n), $n \in \mathbf{N}$, if it is known to exist.

3.15*. The sequence (x_n), $n \in \mathbf{N}$, whose first term is $x_1 = 1$, can be determined from the recursion relation

$$x_{n+1} = x_n^2 + (1 - 2a)\, x_n + a^2.$$

Find the limit of (x_n), $n \in \mathbf{N}$, if it is known to exist.

3.16. The sequence is defined by the recursion relation

$$x_{n+1} = \frac{1}{2}\left(\frac{a}{x_n} + x_n\right),$$

where $x_1 > 0$, $a > 0$. Find the limit of (x_n), $n \in \mathbf{N}$, if it is known to exist.

4. The Arithmetic Progression

A sequence whose first term a_1 is specified and each successive term, beginning with the second, is equal to the sum of the preceding term and a constant d, is known as an *arithmetic progression*:

$$a_{n+1} = a_n + d, \tag{1}$$

a_n is the nth term of the progression and d is the common difference, or simply difference, of the progression. The formula for the general term of the arithmetic progression is

$$a_n = a_1 + d\,(n - 1). \tag{2}$$

The sum S_n of n terms of the progression can be calculated by the formula

$$S_n = \frac{a_1 + a_n}{2}\, n = \frac{2a_1 + d\,(n-1)}{2}\, n. \tag{3}$$

The property of the terms of an arithmetic progression. Any term of an arithmetic progression (except for the first) is equal to half the sum of the terms which are equidistant from it:

$$a_n = \frac{a_{n-k} + a_{n+k}}{2}, \; k < n, \tag{4}$$

for $k = 1$ we get

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}. \tag{5}$$

An arithmetic progression is completely defined if a_1 and d are known.

Example 4.1. When we divide the ninth term of an arithmetic progression by its second term, we get 5 as a quotient, and when we divide the thirteenth term of that progression by the sixth term, we get 2 as a quotient and 5 as a remainder. Find the first term and the common difference.

Solution. The hypothesis can be written as the following system of equations:

$$a_9 = a_2 \cdot 5,$$
$$a_{13} = 2a_6 + 5.$$

Using the formula for the general term of an arithmetic progression, we obtain a system

$$a_1 + 8d = (a_1 + d) \cdot 5,$$
$$a_1 + 12d = 2(a_1 + 5d) + 5,$$

containing only two unknowns, a_1 and d. Collecting terms in the equations of the system, we get a system

$$4a_1 = 3d,$$
$$a_1 - 2d + 5 = 0,$$

whose solution is $a_1 = 3$, $d = 4$.

Answer. $a_1 = 3$, $d = 4$.

4.1. The sum of the first and the fifth term of an arithmetic progression is equal to 5/3, and the product of the third term by the fourth term is equal to 65/72. Find the sum of the first seventeen terms of the progression.

4.2*. Find the arithmetic progression if it is known that

$$a_1 + a_3 + a_5 = -12, \quad a_1 a_2 a_3 = 80.$$

4.3*. The sum of three numbers which are consecutive terms of an arithmetic progression is 2, and the sum of the squares of those numbers is 14/9. Find the numbers.

4.4. Given in an arithmetic progression: $a_p = q$ and $a_q = p$; find the formula for the general term of the progression a_n $(p \neq q)$.

4.5*. Show that if for the positive numbers a, b, and c the numbers a^2, b^2, c^2 are consecutive terms of an arithmetic progression, then the numbers $\frac{1}{b+c}$, $\frac{1}{a+c}$, $\frac{1}{a+b}$ are also consecutive terms of an arithmetic progression.

4.6*. If $a_1, \ldots, a_n$ are terms of an arithmetic progression, none of which is equal to zero, then the following identity holds true:

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_na_{n+1}}=\frac{n}{a_1a_{n+1}}.$$

When solving problems in which use is made of the concept of the sum of terms of an arithmetic progression, it is convenient to employ the following formula relating the nth term and the sum of n terms:

$$a_{n+1} = S_{n+1} - S_n. \tag{6}$$

Example 4.2. It is known that for any n the sum S_n of the terms of a certain progression is expressed by the formula

$$S_n = 4n^2 - 3n.$$

Find the general term of the progression.

Solution. Using (6), we obtain

$$a_{n+1} = S_{n+1} - S_n = 4(n+1)^2 - 3(n+1) - (4n^2 - 3n) = 8n + 1,$$
$$a_n = 8(n-1) + 1 = 8n - 7.$$

Answer. $a_n = 8n - 7$.

Example 4.3. Find the sum of all even two-digit numbers.

Solution. The first even two-digit number is 10, and the last is 98. Using the formula for the general term of the progression for $d = 2$, $a_1 = 10$, $a_n = 98$, we get

$$n=1+\frac{98-10}{2}=45.$$

Substituting the value of n we have obtained into the formula $S_n = \frac{a_1+a_n}{2}n$, we find

$$S_n=\frac{98+10}{2}\cdot 45=54\cdot 45=2430.$$

Answer. 2430.

4.7. Solve the equation $2 + 5 + 8 + 11 + \ldots + x = 155$.

4.8. Ten roubles were paid for the manufacture and installation of the first reinforced concrete ring, and then 2 roubles more were paid for each successive ring than for the preceding one. In addition, 40 roubles were paid when the job was completed. The average cost of the manufacture and installation of one ring turned out to be $22\frac{4}{9}$ roubles. How many rings were installed?

4.9. Solve the equation $\frac{x-1}{x}+\frac{x-2}{x}+\ldots+\frac{1}{x}=3$.

4.10*. The sum of the first m terms of an arithmetic progression is equal to that of its n first terms ($m \neq n$). Prove that in this case the sum of the first $m + n$ terms is equal to zero.

4.11*. Find the sum of all even three-digit numbers which are divisible by 3.

4.12. Find the arithmetic progression in which the ratio of the sum of the first n terms to the sum of n terms succeeding them does not depend on n.

4.13*. Find the sum $50^2 - 49^2 + 48^2 - 47^2 + \ldots + 2^2 - 1$.

4.14*. Find the sum of the first nineteen terms of the arithmetic progression $a_1, a_2, \ldots$ if it is known that

$$a_1 + a_8 + a_{12} + a_{16} = 224.$$

4.15. Find $a_1 + a_6 + a_{11} + a_{16}$ if it is known that $a_1, a_2, \ldots$ is an arithmetic progression and

$$a_1 + a_4 + a_7 + \ldots + a_{16} = 147.$$

4.16*. Find the sequence in which the sum of any number of terms, beginning with the first, is four times as large as the square of the number of terms.

4.17*. Prove that if S_n, S_{2n}, S_{3n} are the sums of n, $2n$, $3n$ terms of an arithmetic progression, then

$$S_{3n} = 3(S_{2n} - S_n).$$

4.18*. It is known that the equality $\frac{S_m}{S_n} = \frac{m^2}{n^2}$ holds for a certain arithmetic progression and for a certain pair of natural numbers m and n. Prove that

$$\frac{a_m}{a_n} = \frac{2m-1}{2n-1}.$$

4.19*. For what values of the parameter a are there values of x such that

$$5^{1+x} + 5^{1-x}, \quad \frac{a}{2}, \quad 25^x + 25^{-x}$$

are three successive terms of an arithmetic progression?

4.20. Find the values of x for which three numbers, $\log 2$, $\log(2^x - 1)$ and $\log(2^x + 3)$ are three successive terms of an arithmetic progression?

4.21. Prove that if u_1, u_2, u_3 $(u_1 \neq u_2)$ are terms (not necessarily successive) of an arithmetic progression, then there is a rational number λ such that

$$\frac{u_3 - u_2}{u_2 - u_1} = \lambda.$$

4.22*. Prove that the numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cannot be terms (not necessarily adjacent) of an arithmetic progression.

4.23*. Can the numbers 2, $\sqrt{6}$, 4.5 be terms of an arithmetic progression?

4.24*. The lengths of the sides of a quadrilateral form an arithmetic progression. Can a circle be inscribed into the quadrilateral?

5. The Geometric Progression

A sequence whose first term b_1 is specified and each successive term, beginning with the second, results from the multiplication of its predecessor by the same number q is called a *geometric progression*:

$$b_n = b_{n-1}q; \tag{7}$$

b_n is the nth term of the progression, and q is its common ratio, or simply ratio. The formula for the general term of a geometric progression is

$$b_n = b_1q^{n-1}. \tag{8}$$

The sum of n terms of a geometric progression can be calculated by the formula

$$S_n = \frac{b_1(1-q^n)}{1-q}. \tag{9}$$

If $|q| < 1$, then the progression is *infinitely decreasing*. The limit of the sum of its terms, $S = \lim_{n\to\infty} S_n$, is called the *sum of an infinitely decreasing geometric progression*. It is calculated by the formula

$$S = \frac{b_1}{1-q}. \tag{10}$$

The *property of the terms of a geometric progression*. The square of any (except for the initial) term of a geometric progression is equal to the product of the terms which are equidistant from it:

$$b_n^2 = b_{n-k}b_{n+k}. \tag{11}$$

A geometric progression is completely defined if b_1 and q are known.

Example 5.1. Find four successive terms of a geometric progression of which the second term is smaller than the first by 35, and the third term is larger than the fourth by 560.

Solution. Assume that b_1, b_2, b_3, b_4 are four successive terms of the geometric progression. The hypothesis can be written as the following system of equations:

$$b_1 - 35 = b_2,$$
$$b_3 - 560 = b_4.$$

Using the formula for the general term, we write the system as

$$b_1 - b_1q = 35,$$
$$b_1q^2 - b_1q^3 = 560.$$

Substituting $b_1(1-q)$ into the second equation of the system, we get for q an equation $q^2 = 16$, whose roots are equal to 4 and -4. Then, using the values $q = 4$ and $q = -4$ we get from the first equation of the system the respective values $b_1 = -35/3$, $b_1 = 7$.

Answer. $\left(-\frac{35}{3}, -\frac{35\cdot 4}{3}, -\frac{35\cdot 16}{3}, -\frac{35\cdot 64}{3}\right)$;

$(7, -28, 112, -448)$.

5.1. Prove that for any even number of terms of a geometric progression S_{odd} (the sum of the odd terms) and S_{even} (the sum of the even terms) are related as $qS_{odd} = S_{even}$.

5.2. Find the first and the fifth term of a geometric progression if it is known that the common ratio is equal to 3 and the sum of the first six terms is equal to 1820.

5.3. Find four successive terms of a geometric progression if it is known that the sum of the extreme terms is equal to -49 and the sum of the middle terms is 14.

5.4. In a geometric progression with positive terms $S_2 = 4$ and $S_3 = 13$. Find S_4.

5.5. The sum of the first three terms of a geometric progression is 13 and their product is 27. Find the numbers.

5.6. The sum of the first three terms of a geometric progression is 13 and the sum of the squares of those numbers is 91. Find the numbers.

5.7. Find three numbers which are three successive terms of a geometric progression if their sum is 21 and the sum of the inverse values is 7/12.

5.8. The sum of the first four terms of a geometric progression is 30 and the sum of their squares is 340. Find the given numbers.

5.9. The product of the first three terms of a geometric progression is 64 and the sum of their cubes is 584. Find the progression.

5.10. The sum of the first three terms of a geometric progression is 31 and the sum of the first and the third term is 26. Find the progression.

5.11. The number of the terms of a geometric progression is even. The sum of all its terms is thrice as large as the sum of the odd terms. Find the common ratio of the progression.

5.12. Given a geometric progression with positive terms. Express the product of its first n terms in terms of their sum S_n and in terms of the sum S'_n of the inverse values of those terms.

5.13. The sum of any five successive terms of an increasing geometric progression is 19 times as large as the third of them. Find the progression if it is known that its mth term is unity.

5.14. Calculate the sum of the squares of n terms of a geometric progression whose first term is equal to u_1 and the common ratio $q \neq 1$.

5.15. Prove that the ratio of the sum of the squares of the odd number of the terms of a geometric progression to the sum of the first powers of those terms is a certain polynomial with respect to q (q is the common ratio of the progression).

5.16. Prove that if S_n, S_{2n}, S_{3n} are the sums of the first n, $2n$, $3n$ terms of a geometric progression, then

$$S_n (S_{3n} - S_{2n}) = (S_{2n} - S_n)^2.$$

5.17*. Find the sum

$$\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\cdots+\left(x^n+\frac{1}{x^n}\right)^2, \quad x\neq \pm 1.$$

5.18*. Find the sum

$$S_n = \frac{1}{2^0}+\frac{2}{2^1}+\frac{3}{2^2}+\frac{4}{2^3}+\cdots+\frac{n}{2^{n-1}}.$$

5.19*. Find the sum

$$S_n = x + 2x^2 + 3x^3 + \ldots + nx^n, \quad x \neq 1.$$

Example 5.2. Find the nonzero common ratio of an infinitely decreasing geometric progression whose every term is 4 times as large as the sum of all its successive terms. (It is assumed that $b_1 \neq 0$.)

Solution. Let us derive an equation which by the hypothesis relates the nth term of the progression with the sum of the terms beginning with the $(n+1)$th term. We have

$$b_n = 4\,\frac{b_{n+1}}{1-q}.$$

Expressing b_n, b_{n+1} in terms of b_1 and q, we get an equation

$$b_1 q^{n-1} = 4\,\frac{b_1 q^n}{1-q},$$

which, upon the division of the right-hand and left-hand sides by $b_1 q^{n-1}$, assumes the form $1 = \frac{4q}{1-q}$. Its root is $q = 1/5$.

Answer. $q = 1/5$.

5.20. The sum of an infinitely decreasing geometric progression is 16 and the sum of the squares of its terms is $153\frac{3}{5}$. Find the fourth term and the common ratio of the progression.

5.21*. Find the common ratio of an infinitely decreasing geometric progression whose every term is related to the sum of all successive terms as 2 to 3.

5.22. In an infinitely decreasing geometric progression with positive terms the sum of the first three terms is 10.5 and the sum of the progression is 12. Find the progression.

5.23. The sum of an infinitely decreasing geometric progression is 4 and the sum of the cubes of its terms is 192. Find the first term and the common ratio of the progression.

5.24. The first term of an infinitely decreasing geometric progression is unity and its sum is S. Find the sum of the squares of the terms of the progression.

5.25*. At what value of x is the progression

$$\frac{a+x}{a-x},\ \frac{a-x}{a+x},\ \left(\frac{a-x}{a+x}\right)^3,\ \ldots,\ \text{where } a > 0,$$

infinitely decreasing? Find the sum of the terms of the progression.

5.26. The side of the square is equal to a. The midpoints of its sides have been connected by line segments, and a new square resulted. The sides of the resulting square were also connected by segments so that a new square was obtained, and so on. Find the limit P of the sum of the perimeters and the limit S of the sum of the areas of the squares.

5.27. Find the condition under which three numbers a, b, and c are, respectively, the kth, the pth, and the mth term of a geometric progression.

5.28. Can the numbers 11, 12, and 13 be terms (not necessarily adjacent) of the same geometric progression?

6. Mixed Problems on Progressions

Mixed problems on progressions are problems whose solution requires the use of the properties of both arithmetic and geometric progressions.

Example 6.1. Three numbers are successive terms of a geometric progression. If we subtract 4 from the third number, these numbers will become successive terms of an arithmetic progression. Now if we subtract unity from the second and from the third term of the resulting arithmetic progression, the numbers obtained will again become successive terms of a geometric progression. Find the numbers.

Solution. Let us designate the required numbers as a, b, c. To derive the first equation relating a, b, and c we use the property of the terms of a geometric progression:

$$b^2 = ac.$$

The second equation can be obtained from the hypothesis and from the property of the terms of an arithmetic progression:

$$2b = a + c - 4.$$

And, finally, the last condition of the problem can be written as an equation

$$(b - 1)^2 = a(c - 5).$$

To solve the system

$$b^2 = ac,$$
$$2b = a + c - 4,$$
$$(b - 1)^2 = a(c - 5),$$

we subtract the third equation from the first. We obtain a linear equation $2b - 1 = 5a$ relating b and a. Expressing now the unknowns a and c from the system of linear equations

$$2b - 1 = 5a,$$
$$2b = a + c - 4$$

in terms of b, we get

$$a = \frac{2b-1}{5}, \quad c = \frac{8b+21}{5}.$$

We exclude the unknowns a and c from the system substituting their expressions in terms of b into the first equation of the system. Then we get a quadratic equation for b:

$$9b^2 - 34b + 21 = 0,$$

whose roots are equal to 3 and 7/9. Substituting these values of b into the expressions for a and c, we get the required numbers.

Answers. (1, 3, 9), (1/9, 7/9, 49/9).

6.1. Find three numbers which are successive terms of a geometric progression if it is known that an increase of the second number by 2 makes the three numbers terms of an arithmetic progression, and if then the last number is increased by 9, the resulting numbers become again terms of a geometric progression.

6.2. Three numbers the third of which is 12 are three successive terms of a geometric progression. If we take 9 instead of 12, then the three numbers become three successive terms of an arithmetic progression. Find the numbers.

6.3. Given a three-digit number whose digits are three successive terms of a geometric progression. If we subtract 792 from it, we get a number written by the same digits in the reverse order. Now if we subtract 4 from the hundred's digit of the initial number and leave the other digits unchanged, we get a number whose digits are successive terms of an arithmetic progression. Find the number.

6.4. Given four numbers, the first three of which are three successive terms of a geometric progression and the last three are successive terms of an arithmetic progression. The sum of the extreme numbers is 32 and that of the middle numbers is 24. Find the numbers.

6.5. The first terms of an arithmetic and a geometric progression are the same and equal to 2, the third terms are also equal, and the second terms differ by 4. Find the progressions if all their terms are positive.

6.6. The first term of an arithmetic progression is 1 and the sum of the first nine terms is equal to 369. The first and the ninth term of the geometric progression coincide with the first and the ninth term of the arithmetic progression. Find the seventh term of the geometric progression.

6.7. Among the 11 terms of an arithmetic progression, the first, the fifth, and the eleventh term are three successive terms of a certain geometric progression. Find the formula for the general term of the arithmetic progression if its first term is equal to 24.

6.8. The second term of a certain arithmetic progression is the mean proportional of the first and the fourth term. Show that the fourth, the sixth, and the ninth term of the progression are successive terms of a geometric progression. Find the common ratio of that progression.

6.9. Prove that if a, b, and c are simultaneously the 5th, the 17th, and the 37th term of an arithmetic and of a geometric progression, then

$$a^{b-c}b^{c-a}c^{a-b}=1.$$

6.10. Prove that if a, b, and c are three successive terms of a geometric progression, then

$$\frac{1}{\log_a N},\quad \frac{1}{\log_b N},\quad \frac{1}{\log_c N}$$

are successive terms of an arithmetic progression. (The numbers a, b, and c are assumed to be positive and not equal to unity.)

7. Miscellaneous Problems

Verify whether the following sequences are bounded.

7.1. $x_n=1+(-1)^n\frac{1}{n}$. 7.2. $x_n=n(1-(-1)^n)$.

7.3*. $x_n=\frac{3n+5}{2n-3}$.

7.4*. The general term of a sequence is represented as

$$x_n = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n}.$$

How many terms of the sequence are smaller than $\frac{1023}{1024}$?

7.5. Prove that the sequence

$$u_1 = 3, \ u_{n+1} = \frac{2 + u_n^2}{2u_n}, \quad n \in \mathrm{N}.$$

is decreasing.

7.6. Prove that the sequence

$$u_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$$

is increasing.

7.7*. Let a_n be a side of a regular 2^{n+1}-gon inscribed into a circle of radius 1. Prove that the sequence (a_n), $n \in \mathrm{N}$, is decreasing and the sequence of the perimeters (P_n), $n \in N$, is increasing.

7.8. The leg of an isosceles right-angled triangle is divided into n equal parts and inscribed rectangles are constructed on the resulting line segments. Find the limit of the sequence (S_n), $n \in \mathrm{N}$, of the areas formed by the stepped figures.

7.9. Find the "area" of the figure, bounded by the parabola $y = x^2$, the interval [0, 1] of the abscissa axis, and the straight line $x = 1$, as the limit of the sequence of the areas of the stepped figures consisting of rectangles constructed in the same way as in the preceding problem.

7.10. Find $\lim\limits_{n \to \infty} \sqrt[3]{(n^3+1} - n)$.

7.11*. Find a three-digit number which is divisible by 45 and whose digits are terms of an arithmetic progression.

7.12.** Prove that if $S_n = n^2 p$, $S_k = k^2 p$, $k \neq n$, in an arithmetic progression, then $S_p = p^3$.

7.13*. Prove that if $a_1, \dots, a_n$ are terms of an arithmetic progression with the common difference d, then

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \dots + \frac{1}{\sqrt{a_n} + \sqrt{a_{n+1}}} = \frac{\sqrt{a_{n+1}} - \sqrt{a_1}}{d}.$$

7.14. Four numbers a, b, c, d are terms of a geometric progression. Prove that

$$(a - c)^2 + (b - c)^2 + (b - d)^2 = (a - d)^2.$$

7.15*. Solve the following system of equations:

$$\frac{x}{y}=\frac{y}{z}=\frac{z}{u}=\frac{u}{s}=\frac{s}{t},$$
$$x=8u,$$
$$x+y+z+u+s+t=15\frac{3}{4}.$$

7.16*. Prove the equality

$$(\underbrace{66\ldots6}_{n \text{ digits}})^2+\underbrace{88\ldots8}_{n \text{ digits}}=\underbrace{44\ldots4}_{2n \text{ digits}}.$$

7.17*. Assume that x_1 and x_2 are roots of the equation $x^2-3x+A=0$, and x_3 and x_4 are roots of the equation $x^2-12x+B=0$. The sequence x_1, x_2, x_3, x_4 is known to be an increasing geometric progression. Find A and B.

When calculating limits of sequences whose terms are the results of summation, use is often made of the following formulas:

$$1+2+\ldots+n=\frac{n(n+1)}{2}. \qquad (1)$$

$$1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}. \qquad (2)$$

$$1^3+2^3+\ldots+n^3=\frac{n^2(n+1)^2}{4}. \qquad (3)$$

Using (1) and (2) and the formulas for the sums of n terms of an arithmetic and a geometric progression, calculate the following limits:

7.18. $\lim\limits_{n\to\infty}\dfrac{1+2+\ldots+2^n}{1+5+\ldots+5^n}$.

7.19. $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2}+\dfrac{2}{n^2}+\ldots+\dfrac{n-1}{n^2}\right)$.

7.20*. $\lim\limits_{n\to\infty}\left(\sqrt{3}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{3\sqrt{3}}+\ldots+\dfrac{\sqrt{3}}{3^{n-1}}\right)$.

7.21. $\lim\limits_{n\to\infty}\left(\dfrac{1+3+5\ldots+(2n+1)}{n+1}-\dfrac{2n-1}{2}\right)$.

7.22. $\lim\limits_{n\to\infty}\left(\dfrac{1^2}{n^3}+\dfrac{2^2}{n^3}+\ldots+\dfrac{(n-1)^2}{n^3}+\dfrac{1}{n}\right)$.

7.23. $\lim\limits_{n\to\infty}\left(\dfrac{7}{10}+\dfrac{29}{10^2}+\ldots+\dfrac{5^n+2^n}{10^n}\right)$.

7.24*. $\lim\limits_{n \to \infty} \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} \right)$.

7.25*. $\lim\limits_{n \to \infty} \left(\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \ldots + \frac{1}{a_n a_{n+1}} \right)$,

where (a_n), $n \in \mathbf{N}$, is an arithmetic progression with the common difference d whose terms are nonzero.

7.26. $\lim\limits_{n \to \infty} \left(\frac{1}{n^4} + \frac{8}{n^4} + \ldots + \frac{n^3}{n^4} \right)$.

Chapter 7

Limit of a Function. Continuity of a Function

1. Limit of a Function

Let (a, b) be an interval of the number axis and $x_0 \in (a, b)$. We assume that the function $y = f(x)$ is defined at all the points of the interval (a, b), except, maybe, the point x_0. We say that the number A is a *limit of the function $y = f(x)$ at the point x_0* and write $\lim\limits_{x\to x_0} f(x) = A$ if for any $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ such that the inequality $|f(x) - A| < \varepsilon$ holds for all $x \in (a, b)$ satisfying the inequality $0 < |x - x_0| < \delta(\varepsilon)$.

We say that the number A is a *limit of the function $y = f(x)$ for x tending to infinity* and write $\lim\limits_{x\to\infty} f(x) = A$ if for any $\varepsilon > 0$ there is a number $n_0(\varepsilon)$ such that the inequality $|f(x) - A| < \varepsilon$ is satisfied for all $x > n_0(\varepsilon)$.

The function $f(x)$ is said to be *bounded on the interval* $[a, b]$ if there are numbers m and M such that the inequality $m \leqslant f(x) \leqslant M$ is satisfied for all $x \in [a, b]$.

The function $f(x)$ is said to be *infinitely small as $x \to x_0$* if $\lim\limits_{x\to x_0} f(x) = 0$.

The function $f(x)$ is said to be *infinitely great as $x \to x_0$* if for any number $E > 0$ there is a number $\delta(\varepsilon)$ such that the inequality $|f(x)| > E$ is valid for all $x \in (a, b)$ satisfying the inequality $0 < |x - x_0| < \delta(E)$. In that case we write $\lim\limits_{x\to x_0} f(x) = \infty$.

To prove that the number A is a limit of the function $f(x)$ as $x \to x_0$, it is sufficient to find, for any ε, a number $\delta(\varepsilon)$ entering into the definition of a limit.

Example 1.1. Prove that

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$$

Solution. To find the required number $\delta(\varepsilon)$ for the given ε, we derive an inequality

$$0 < \left|\frac{x^2-4}{x-2} - 4\right| < \varepsilon. \qquad (*)$$

For $x \neq 2$ it is equivalent to the inequality

$$0 < |x - 2| < \varepsilon, \qquad (**)$$

implies that $\delta(\varepsilon) = \varepsilon$ can be taken as $\delta(\varepsilon)$, and by virtue of the equivalence of inequalities (*) and (**) for all values of x, satisfying inequality (**), inequality (*) is satisfied.

Example 1.2. Prove that

$$\lim_{x \to 0} 2^{1/x^2} = \infty.$$

Solution. To find the required $\delta(E)$ for the given E, we derive an inequality

$$2^{1/x^2} > E. \qquad (*)$$

Taking logarithms of both its sides to the base 2, we get an equivalent inequality

$$\frac{1}{x^2} > \log_2 E, \qquad (**)$$

solving which with respect to x, we obtain

$$|x| < \left(\frac{1}{\log_2 E}\right)^{1/2}.$$

Thus we can take

$$\delta(E) = \left(\frac{1}{\log_2 E}\right)^{1/2},$$

as $\delta(E)$.

Prove that

1.1. $\lim_{x \to 3} (x+5) = 8$. **1.2.** $\lim_{x \to 2} (x^2 - 4) = 0$.

1.3. $\lim_{x \to a} \sqrt{x} = \sqrt{a}$, $a > 0$. **1.4.** $\lim_{x \to 1} (6 - 2x) = 4$.

1.5. $\lim_{x \to \infty} \frac{1}{x} = 0$. **1.6.** $\lim_{x \to a} 2^{\frac{x}{|x-a|}} = \infty$.

1.7. $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

When solving certain problems, it is convenient to use the following definition of a limit of a function. Suppose the function $f(x)$ is defined at all the points of the interval (a, b), except, maybe, for the point $x_0 \in (a, b)$. We say that the number A is a *limit of the function* $f(x)$ as x tends to x_0 if for any sequence of the values of the argument (x_n) tending to x_0 $(x_n \neq x_0)$,

$$\lim_{n \to \infty} f(x_n) = A$$

for the corresponding sequence of the values of the function.

Example 1.3. Prove that the function $f(x) = \frac{|x|}{x}$ has no limit as $x \to 0$.

Solution. Let us take two sequences of the values of the arguments which converge to zero: $x_n^{(1)} = 1/n$, $x_n^{(2)} = -1/n$. Then

$$\lim_{n\to\infty} f(x_n^{(1)}) = \lim_{n\to\infty} \frac{1/n}{1/n} = \lim_{n\to\infty} 1 = 1,$$

$$\lim_{n\to\infty} f(x_n^{(2)}) = \lim_{n\to\infty} \frac{1/n}{-1/n} = \lim_{n\to\infty} (-1) = -1.$$

$$\lim_{n\to\infty} f(x_n^{(1)}) \neq \lim_{n\to\infty} f(x_n^{(2)}).$$

We have thus constructed two sequences of the values of the argument, different from zero, whose limit is zero, and which are such that the respective sequences of the values of the function converge to different numbers (one to 1 and the other to -1). But since it is stated in the definition of a limit that the limit of the sequence of the values of the function should be the same number for each of the sequences of the values of the argument, we have thus proved that the given function has no limit as $x \to 0$.

Prove that the following functions have no limits.

1.8*. $f(x) = \sin(1/x)$ as $x \to 0$.

1.9. $f(x) = e^{-1/x}$ as $x \to 0$.

2. Methods of Calculating Limits of Functions

If there exist $\lim_{x\to a} f_1(x)$ and $\lim_{x\to a} f_2(x)$, then there are limits

$$\lim_{x\to a} cf_1(x) = c \lim_{x\to a} f_1(x), \qquad (1)$$

$$\lim_{x\to a} [f_1(x) \pm f_2(x)] = \lim_{x\to a} f_1(x) \pm \lim_{x\to a} f_2(x), \qquad (2)$$

$$\lim_{x\to a} [f_1(x) f_2(x)] = \lim_{x\to a} f_1(x) \lim_{x\to a} f_2(x), \qquad (3)$$

$$\lim_{x\to a} [f_1(x)/f_2(x)] = \lim_{x\to a} f_1(x) / \lim_{x\to a} f_2(x) \ (\lim_{x\to a} f_2(x) \neq 0). \qquad (4)$$

When seeking the limit of the ratio of two polynomials dependent on x, as $x \to \infty$, it is first necessary to divide both terms of the ratio by x^n, where n is the highest degree of the polynomials.

Example 2.1. Find $\lim_{x\to\infty} \frac{(x-3)(x-2)}{2x^2-5x+3}$.

Solution. Let us divide the numerator and the denominator of the fraction by x^2. Then we have

$$\lim_{x\to\infty} \frac{(x-3)(x-2)}{2x^2-5x+3} = \lim_{x\to\infty} \frac{\left(\frac{x-3}{x}\right)\left(\frac{x-2}{x}\right)}{2-5/x+3/x^2}.$$

Using now formulas (4), (3) and also (2) and (1) (for $c = -1$), we obtain

$$\frac{\lim\limits_{x\to\infty}(1-3/x)\lim\limits_{x\to\infty}(1-2/x)}{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}\frac{5}{x}+\lim\limits_{x\to\infty}\frac{3}{x^2}}. \qquad (*)$$

Using the equation $\lim\limits_{x\to\infty}\frac{a}{x}=0$, we get

$$\frac{\left(1-\lim\limits_{x\to\infty}\frac{3}{x}\right)\left(1-\lim\limits_{x\to\infty}\frac{2}{x}\right)}{2-\lim\limits_{x\to\infty}\frac{5}{x}+\lim\limits_{x\to\infty}\frac{3}{x^2}}=\frac{1}{2}.$$

Answer. 1/2.

Calculate the following limits.

2.1. $\lim\limits_{x\to\infty}\frac{(x+1)^2}{(x-3)(x+2)}$. 2.2. $\lim\limits_{x\to\infty}\frac{-x^2+5x-6}{(x-1)(x-2)(x-3)}$.

2.3. $\lim\limits_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$. 2.4. $\lim\limits_{x\to\infty}\frac{2x+5}{x+\sqrt{x}}$.

If $P(x)$ and $Q(x)$ are polynomials and $Q(a) \neq 0$, then the limit of the ratio

$$\lim_{x\to a}\frac{P(x)}{Q(x)}$$

can be found directly by means of formulas (1)-(4). Now if $P(a) = 0$ and $Q(a) = 0$, then, writing the polynomials $P(x)$ and $Q(x)$ as

$$P(x) = (x-a)^k P_1(x), \quad Q(x) = (x-a)^n Q_1(x)$$

(k and n are the multiplicities of the root $x = a$ of the polynomials $P(x)$ and $Q(x)$), it is necessary to cancel the numerator and the denominator of the fraction $P(x)/Q(x)$ by the common factor before passing to the limit.

Example 2.2. Calculate

$$\lim_{x\to 3}\frac{x^2-5x+6}{x^2-9}.$$

Solution. We write the expression under the limit sign as

$$\frac{x^2-5x+6}{x^2-9}=\frac{(x-3)(x-2)}{(x-3)(x+3)}=\frac{x-2}{x+3}.$$

The limit of the obtained fraction can be calculated by means of formulas (1)-(4):

$$\lim_{x\to 3}\frac{x-2}{x+3}=\frac{1}{6}.$$

Answer. 1/6.

Calculate the following limits.

2.5. $\lim\limits_{x\to -1}\frac{x^3+1}{x^2+1}$. 2.6. $\lim\limits_{x\to -1}\frac{(x+1)^3}{x^3+1}$.

2.7. $\lim\limits_{x\to 1/2}\frac{8x^3-1}{6x^2-5x+1}$. 2.8. $\lim\limits_{h\to 0}\frac{(x+h)^3-x^3}{h}$.

2.9. $\lim\limits_{x\to 2}\left(\frac{1}{2-x}-\frac{3}{8-x^3}\right)$.

2.10. $\lim\limits_{x\to a}\left[\frac{(x^3-2ax^2-a^2x+2a^3)(x-2a)^{-1}+2a(x+a)}{x^3-a^2x}-\frac{2}{x-a}\right]$.

2.11. (a) $\lim\limits_{x\to 1/2} f(x)$, (b) $\lim\limits_{x\to 3/2} f(x)$,

where

$$f(x)=\frac{\frac{|x-1|}{2}+x\,|x-1|+2-\frac{2}{x}}{\sqrt{x-2+1/x}}.$$

2.12. (a) $\lim\limits_{x\to 1} f(x)$, (b) $\lim\limits_{x\to -1} f(x)$,

where $f(x)=\frac{x\,|x-3|}{(x^2-x-6)\,|x|}$.

2.13*. $\lim\limits_{x\to 1}\frac{x^4-x^3-x+1}{x^3-5x^2+7x-3}$.

Calculation of the limits of irrational expressions can sometimes be simplified by an introduction of new variables.

Example 2.3. Calculate

$$\lim_{x\to 1}\frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}.$$

Solution. We introduce the designation $\sqrt[3]{x}=t$. Then, for the variable t, the expression under the limit sign can be written in the form

$$\frac{t^2-2t+1}{(t^3-1)^2}.$$

The number to which the new variable t tends, as $x \to 1$, can be found as the limit of the function $t(x) = \sqrt[3]{x}$ as $x \to 1$, i.e.

$$\lim_{x \to 1} t(x) = \lim_{x \to 1} \sqrt[3]{x} = 1.$$

Thus we have

$$\lim_{t \to 1} \frac{t^2-2t+1}{(t^3-1)^2} = \lim_{t \to 1} \frac{(t-1)^2}{(t-1)^2(t^2+t+1)^2} = \lim_{t \to 1} \frac{1}{(t^2+t+1)^2} = \frac{1}{9}.$$

Answer. 1/9.

Calculate the following limits.

2.14. $\lim\limits_{x \to 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$. **2.15.** $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$.

2.16. $\lim\limits_{x \to 1} \frac{\sqrt{x}+\sqrt{x-1}-1}{\sqrt{x^2-1}}$.

The limit of a fraction containing an irrational expression can sometimes be calculated by transferring the irrational expression from the numerator to the denominator and vice versa.

Example 2.4. Calculate

$$\lim_{x \to 0} \frac{\sqrt{x^2+1}-1}{x}.$$

Solution. Multiplying the numerator and the denominator of the fraction under the limit sign by the expression which is a conjugate of the numerator, we obtain

$$\lim_{x \to 0} \frac{\sqrt{x^2+1}-1}{x} = \lim_{x \to 0} \frac{x^2+1-1}{x(\sqrt{x^2+1}+1)} = \lim_{x \to 0} \frac{x^2}{x(\sqrt{x^2+1}+1)} = 0.$$

Answer. 0.

Calculate the following limits.

2.17. $\lim\limits_{x \to \infty} (\sqrt{x^2+1}-x)$. **2.18.** $\lim\limits_{x \to 0} \frac{\sqrt{x^2+4}-2}{\sqrt{x^2+9}-3}$.

2.19. $\lim\limits_{x \to 3} \frac{x-3}{\sqrt{x+1}-2}$. **2.20.** $\lim\limits_{x \to 9} \frac{3-\sqrt{x}}{\sqrt{x-5}-2}$.

2.21. $\lim\limits_{x \to 4} \frac{\sqrt{x}-2}{\sqrt{x+5}-3}$. **2.22.** $\lim\limits_{x \to 2} \frac{\sqrt{x+2}-2}{\sqrt{x+7}-3}$.

2.23. $\lim\limits_{x\to-\infty} x\left(\sqrt{4x^2+7}+2x\right)$. 2.24. $\lim\limits_{x\to 3} \frac{x-3}{\sqrt[3]{x^2-1}-2}$.

2.25. $\lim\limits_{x\to\infty} \frac{9x-\sqrt{x^2-4}}{x}$. 2.26. $\lim\limits_{x\to 3} \frac{\sqrt{x^2+7}-4}{x^2-5x+6}$.

When calculating the limits of expressions containing trigonometric functions, use is often made of the limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Example 2.5. Find the following limit:

$$\lim_{x\to 0} \frac{1-\cos 2x}{x}.$$

Solution. We transform the numerator of the fraction by the formula

$$1-\cos 2x = 2\sin^2 x.$$

Then we obtain

$$\lim_{x\to 0} \frac{1-\cos 2x}{x} = \lim_{x\to 0} \frac{2\sin^2 x}{x} = 2\lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \sin x = 0.$$

Answer. 0.

Example 2.6. Find the following limit:

$$\lim_{x\to 0} \frac{\arcsin x}{x}.$$

Solution. We designate $y = \arcsin x$; then $x = \sin y$. Since $\arcsin x$ tends to zero as $x \to 0$, we have

$$\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{y\to 0} \frac{y}{\sin y} = 1.$$

Answer. 1.

Calculate the following limits.

2.27. $\lim\limits_{x\to 0} \frac{\sin nx}{\sin mx}$. 2.28*. $\lim\limits_{x\to a} \frac{\sin x-\sin a}{x-a}$.

2.29*. $\lim\limits_{n\to\infty} \left(n \sin \frac{\pi}{n}\right)$. 2.30*. $\lim\limits_{x\to -2} \frac{\tan \pi x}{x+2}$.

2.31. $\lim\limits_{x\to\pi/4} \dfrac{\sin x - \cos x}{1-\tan x}$. 2.32*. $\lim\limits_{x\to\pi/3} \dfrac{\tan^3 x - 3\tan x}{\cos(x+\pi/6)}$.

2.33*. $\lim\limits_{x\to\pi/3} \dfrac{\sin(x-\pi/3)}{1-2\cos x}$. 2.34. $\lim\limits_{x\to\pi} \dfrac{1-\sin(x/2)}{\pi - x}$.

2.35*. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{x}$.

3. Continuity of a Function at a Point

The function $f(x)$, defined on the interval (a, b), is said to be *continuous at the point* $x_0 \in (a, b)$ if

(1) there is a limit $\lim\limits_{x\to x_0} f(x)$;

(2) this limit is equal to the value of the function at the point x_0, i.e.

$$\lim_{x\to x_0} f(x) = f(x_0).$$

The proof of the continuity of the function $f(x)$ at the point x_0 consists in verifying the validity of the equality

$$\lim_{x\to x_0} f(x) = f(x_0). \tag{1}$$

Example 3.1. Prove that the function

$$f(x) = 3x^2 + 5$$

is continuous at the point $x = 2$.

Solution. Applying the theorems on limits, we have

$$\lim_{x\to 2} (3x^2 + 5) = 3 \lim_{x\to 2} x^2 + 5 = 17.$$

On the other hand, the value of the function at the point 2 is also equal to 17. Consequently, equality (1) is satisfied and the given function is continuous at the point $x = 2$.

Prove the continuity of the following functions at the indicated points.

3.1. $f(x) = x^2 - 2x + 1$ at the point 1.

3.2. $f(x) = \dfrac{1+\cos 2x}{\cos x}$ at the point $x = \dfrac{\pi}{4}$.

3.3. $f(x) = \begin{cases} x^2, & x \geqslant 1 \\ 1, & x < 1 \end{cases}$ at the point $x = 1$.

3.4. $f(x)=\begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x=0 \end{cases}$ at the point $x=0$.

3.5. $f(x)=\begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x=0 \end{cases}$ at the point $x=0$.

3.6. $f(x)=\begin{cases} (1+x)^{1/x}, & x \neq 0, \\ e, & x=0 \end{cases}$ at the point $x=0$.

3.7. $f(x)=\begin{cases} \dfrac{\sqrt{1+x}-\sqrt[3]{1+x}}{x}, & x \neq 0, \\ 1/6, & x=0 \end{cases}$ at the point $x=0$.

When proving the continuity of the function $f(x)$, defined on the interval (a, b) at the point $x_0 \in (a, b)$, it is sometimes more convenient to verify the validity not of equality (1) but of the equality

$$\lim_{\Delta x \to 0} [f(x_0+\Delta x)-f(x_0)]=0, \tag{2}$$

whose validity ensures the continuity of the function at the point x_0.

Example 3.2. Prove that the function

$$f(x) = \sin x$$

is continuous for any value of the argument x.

Solution. Let us form a difference $f(x+\Delta x) - f(x)$ for the given function:

$$\sin(x+\Delta x)-\sin x=2\sin\frac{\Delta x}{2}\cos\left(\frac{2x+\Delta x}{2}\right).$$

Taking advantage of the fact that

$$\lim_{\Delta x \to 0} \frac{\sin(\Delta x/2)}{\Delta x/2}=1, \quad \left|\cos\left(x+\frac{\Delta x}{2}\right)\right| \leqslant 1,$$

we obtain, by means of formulas (2) and (5) from Sec. 2,

$$\lim_{\Delta x \to 0} [\sin(x+\Delta x)-\sin x]=0.$$

Prove the continuity of the following functions on the whole domain of their definition.

3.8. $f(x) = x^2$. 3.9*. $f(x) = \cos x$.

3.10*. $f(x) = \ln x$. 3.11*. $f(x) = e^x$.

The following theorems are often used to prove the continuity.

If the functions $f(x)$ and $g(x)$ are continuous at the point x_0, then their sum, difference, product, and quotient (provided that $g(x_0) \neq 0$) are continuous at the point x_0.

Example 3.3. Prove the continuity of the function

$$f(x) = \frac{2x^2 - 2}{x^2 + 1}$$

throughout the number axis.

Solution. Since $f(x)$ is a ratio of two polynomials, the denominator being positive everywhere, the continuity of $f(x)$ at any point $x \in R$ follows from the continuity of the numerator and the denominator at that point.

3.12. Prove that the fractional rational function $w = \dfrac{az + b}{cz + d}$ $(ad - bc \neq 0)$ is continuous in its domain of definition.

3.13. Is the function $y = \tan x$ continuous throughout the number axis?

If $\lim\limits_{x \to x_0} f(x)$ exists, but the function is not defined at the point x_0, then we say that x_0 is the *point of removable discontinuity*. In that case we can extend the definition of the function $f(x)$ "by continuity", setting

$$\tilde{f}(x_0) = \lim_{x \to x_0} f(x). \qquad (3)$$

Example 3.4. Define the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

at the point $x = 2$ by continuity.

Solution. The point $x = 2$ does not belong to the domain of definition of the given function, but

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

Let us complete the definition of the function $f(x)$ at the point $x = 2$ by the value equal to 4. Then we get a function

$$\tilde{f}(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{for } x \neq 2, \\ 4 & \text{for } x = 2, \end{cases}$$

which coincides with the initial function throughout the domain of definition of the initial function and is continuous on the entire number axis.

Answer. $\tilde{f}(2) = 4$.

Define the following functions by continuity at the indicated points.

3.14. $f(x) = \dfrac{\sin x}{x}$ at the point $x=0$.

3.15. $f(x) = \dfrac{e^x - e^{-x}}{x}$ at the point $x=0$.

3.16. $f(x) = \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$ at the point $x=0$.

3.17. $f(x) = \dfrac{3-\sqrt[4]{x}}{9-\sqrt{x}}$ at the point $x=81$.

Choose the parameters such that $f(x)$ become continuous at an indicated point (if the point is not specified, then on the entire number line):

3.18. $f(x) = \begin{cases} \dfrac{x^2-5x+6}{x-3}, & x \neq 3, \\ A, & x=3. \end{cases}$

3.19. $f(x) = \begin{cases} 2^{-1/x^2}, & x \neq 0, \\ A, & x=0. \end{cases}$

3.20. $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin 2x}, & x \neq 0, \\ A, & x=0 \end{cases}$ at the point $x=0$.

3.21. $f(x) = \begin{cases} \dfrac{1-\cos x}{\sin^2 x}, & x \neq 0, \\ A, & x=0 \end{cases}$ at the point $x=0$.

3.22. $f(x) = \begin{cases} \dfrac{x^2}{1-\cos mx}, & x \neq 0, \\ A, & x=0 \end{cases}$ at the point $x=0$.

3.23*. $f(x) = \begin{cases} (1-x)\tan \dfrac{\pi x}{2}, & x \neq 1, \\ A, & x=1 \end{cases}$ at the point $x=1$.

Suppose the function $f(x)$ is defined on the interval (a, x_0). We call the number A a *left-hand limit* (or *limit on the left*) of the function $f(x)$ *at the point* x_0 and write

$$\lim_{x \to x_0 - 0} f(x) = A$$

if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that the inequality

$$|f(x) - A| < \varepsilon$$

holds for any $x \in (a, x_0)$ satisfying the inequality $x_0 - \delta(\varepsilon) < x$.

The function $f(x)$ is said to be *continuous at the point* x_0 *from the left* if the point x_0 belongs to the domain of definition of the function and

$$\lim_{x \to x_0 - 0} f(x) = f(x_0).$$

The right-hand limit of a function and the continuity of a function on the right are similarly defined.

For the function $f(x)$ to be continuous at the point x_0, it is necessary and sufficient that it be continuous on the left and on the right of the point x_0.

Example 3.5. What conditions should be satisfied by the parameters a and b for the function

$$f(x) = \begin{cases} x-1 & \text{for } x \leqslant 1, \\ ax^2+bx & \text{for } x > 1 \end{cases}$$

to be continuous?

Solution. Let us calculate the left-hand and right-hand limits of the given function at the point $x = 1$:

$$\lim_{x \to 1-0} (x-1) = 0, \quad \lim_{x \to 1+0} (ax^2+bx) = a+b.$$

Since at the point $x = 1$ the given function is continuous on the left and $f(1) = 0$, it follows that for this function to be continuous it is necessary and sufficient that the equality $a + b = 0$ be satisfied.

Answer. $a + b = 0$.

Choose parameters, entering into the definition of the function, such that the function $f(x)$ become continuous:

3.24. $f(x) = \begin{cases} ax+1, & x \leqslant \pi/2; \\ \sin x + b, & x > \pi/2. \end{cases}$

3.25. $f(x) = \begin{cases} x^2, & x \leqslant 1; \\ ax, & x > 1. \end{cases}$

3.26. $f(x) = \begin{cases} |x^2-5x+6|, & x > 2; \\ ax-b, & x \leqslant 2. \end{cases}$

3.27. $f(x) = \begin{cases} |x^2-5x+6|, & x < 3; \\ ax-b, & x \geqslant 3. \end{cases}$

3.28. $f(x) = \begin{cases} 2^{1/(x-1)}, & x < 1; \\ ax^2+bx+1, & x \geqslant 1. \end{cases}$

3.29. $f(x)=\begin{cases} -\dfrac{3}{x^2+1}+1, & x>0; \\ x^2+b, & x\leqslant 0. \end{cases}$

3.30. $f(x)=\begin{cases} x^2+x+1, & x\geqslant -1; \\ \sin(\pi(x+a)), & x<1. \end{cases}$

3.31. $f(x)=\begin{cases} x+3, & x\leqslant 3; \\ a\cdot 2^x, & x>3. \end{cases}$

4. Miscellaneous Problems

Calculate the following limits.

4.1. $\lim\limits_{x\to 0}\dfrac{\cos x\sin x-\tan x}{x^2\sin x}$. 4.2. $\lim\limits_{x\to\pi/4}\dfrac{1-\cot^3 x}{2-\cot x-\cot^3 x}$.

Verify the validity of the following inequalities.

4.3. $\lim\limits_{x\to\infty}\dfrac{2x+3}{x+\sqrt[3]{x}}>\lim\limits_{x\to -1/2}\dfrac{2x^2-5x-3}{4x^2-18x-10}+\lim\limits_{x\to 0}\dfrac{\sin^2\dfrac{x}{2}}{x^2}$.

4.4. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-1}{x}+\lim\limits_{x\to\pi/4}\dfrac{\sqrt{2}-2\cos x}{\sin(x-\pi/4)}>\log 0.005$.

Calculate the following limits.

4.5. $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^2+5x-6}$. 4.6. $\lim\limits_{x\to 0}\dfrac{\sin(2x-\pi)}{\cos 3(x+\pi/2)}$.

4.7. $\lim\limits_{x\to\pi}\dfrac{\sin 2x}{1+\cos^3 x}:\lim\limits_{x\to 0}\dfrac{\sqrt{2x+1}-1}{\sqrt{3x+4}-2}$.

4.8. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\sin^3 x}$. 4.9. $\lim\limits_{x\to a}\dfrac{\tan x-\tan a}{x-a}$.

4.10. $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{\cos x-\cos a}$. 4.11. $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{\tan x-\tan a}$.

4.12. $\lim\limits_{x\to\pi/4}\dfrac{\sin x-\cos x}{\tan x-\cot x}$. 4.13. $\lim\limits_{x\to\pi/4}\dfrac{\sin x-\cos x}{\tan x-1}$.

4.14. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$. 4.15. $\lim\limits_{x\to\pi/4}\dfrac{\cos(\pi/4+x)}{1-\tan x}$.

4.16. $\lim\limits_{x\to\pi/4}\dfrac{\cos 2x}{\cos x-\sqrt{2}/2}$. 4.17. $\lim\limits_{x\to 3\pi/2}\dfrac{\sin 2x}{1+\sin x}$.

4.18. $\lim\limits_{x\to\pi/2}\dfrac{\cos x}{\pi-2x}$. 4.19. $\lim\limits_{x\to 0}\dfrac{\tan x}{1-\cos x}$.

4.20. $\lim\limits_{x\to\pi/3} \frac{\sin(\pi/3-x)}{2\cos x-1}$. 4.21. $\lim\limits_{x\to 0} \frac{x\sin x}{1-\cos x}$.

4.22. $\lim\limits_{x\to\pi/2} \frac{\sec 2x+1}{\cos x}$. 4.23. $\lim\limits_{x\to\pi} \frac{1-\cos 2x}{\tan^2 x}$.

4.24. $\lim\limits_{x\to\pi/2} \frac{1+\cos 2x}{\cot^2 x}$. 4.25. $\lim\limits_{x\to a} \frac{\sin^2 x-\sin^2 a}{\sin(x-a)}$.

4.26. $\lim\limits_{x\to\pi/2} \frac{1-\sin x}{\cos^2 x}$. 4.27. $\lim\limits_{x\to\pi/4} \frac{\sin(\pi/4-x)}{\cos^2 x-1/2}$.

4.28. $\lim\limits_{x\to 0} \frac{\tan x-\sin x}{\sin^3 x}$. 4.29. $\lim\limits_{x\to a} \frac{\tan^2 x-\tan^2 a}{\tan(x-a)}$.

4.30. $\lim\limits_{x\to a} \frac{\cos^2 x-\cos^2 a}{\sin\frac{x-a}{2}}$. 4.31. $\lim\limits_{x\to 0} \frac{\tan 3x-\tan^3 x}{\tan x}$.

4.32. $\lim\limits_{x\to\pi/4} \frac{\tan^2 x+\tan x-2}{\sin x-\cos x}$.

4.33. $\lim\limits_{x\to\pi/4} \left[(\sin x-\cos x)\tan\left(\frac{\pi}{4}+x\right)\right]$.

4.34. $\lim\limits_{x\to\pi/2} [(1-\sin x)\tan^2 x]$. 4.35. $\lim\limits_{x\to a} \left[(a-x)\sec\frac{\pi x}{2a}\right]$.

4.36. $\lim\limits_{x\to a} \left(\sin\frac{a-x}{2}\tan\frac{\pi x}{2a}\right)$.

4.37. $\lim\limits_{x\to\arctan 3} \frac{\tan^2 x-2\tan x-3}{\tan^2 x-4\tan x+3}$.

4.38. $\lim\limits_{x\to 0} \left(\frac{1}{x}\tan\frac{x}{2}\right)$. 4.39. $\lim\limits_{x\to\infty} \left(2^x\sin\frac{a}{2^x}\right)$.

4.40. $\lim\limits_{x\to 0} \frac{1-\cos^3 x}{x\sin x\cos x}$.

Extend the definition of the following functions by continuity.

4.41. $f(x)=\frac{x-3}{\sqrt[3]{x^3-1}-2}$ at the point $x=3$.

4.42. $f(x)=\frac{2-\sqrt{x+4}}{\sin 2x}$ at the point $x=0$.

4.43. $f(x)=\frac{\cos^2 x-\sin^2 x-1}{\sqrt{x^2+1}-1}$ at the point $x=0$.

Find the parameters for which the function $f(x)$ is continuous.

4.44. $f(x)=\begin{cases} \dfrac{\sqrt{x^2+7}-4}{x^2-5x+6}, & x \neq 3, \text{ at the point } x=3. \\ A & x=3. \end{cases}$

4.45. $f(x)=\begin{cases} \dfrac{2^{x+2}-16}{4^x-2^4}, & x \neq 2, \\ A, & x=2. \end{cases}$

Chapter 8
The Derivative and Its Applications

1. Calculating the Derivatives

If the function $f(x)$ is defined on the interval (a, b), then the *derivative of the function* $f(x)$ *at the point* $x_0 \in (a, b)$ is the limit of the ratio of the increment of the function

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) \tag{1}$$

to the increment of the independent variable Δx ($\Delta x = x - x_0$) as Δx tends to zero:

$$\lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x}. \tag{2}$$

If the limit exists, then we say that the function $f(x)$ *possesses a derivative at the point* x_0 or that $f(x)$ is *differentiable at* x_0. The derivative of the function $f(x)$ at x_0 is designated as $f'(x_0)$. If the limit does not exist, then we say that the function $f(x)$ is *not differentiable at* x_0.

The problem connected with calculations of the derivative, proceeding from its definition, consists in direct calculations of limit (2).

Example 1.1. Calculate the derivative of the function

$$f(x) = \sin x.$$

Solution. Let us derive the increment of the function $\Delta f(x_0)$:

$$\Delta f(x_0) = \sin(x_0 + \Delta x) - \sin x_0.$$

To find the limit

$$\lim_{\Delta x \to 0} \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x},$$

we use the formula

$$\sin(x_0 + \Delta x) - \sin x_0 = 2 \sin \frac{\Delta x}{2} \cos\left(x_0 + \frac{\Delta x}{2}\right).$$

Taking into account the continuity of the function $\cos x$, we get

$$\lim_{\Delta x \to 0} \frac{2 \sin \frac{\Delta x}{2} \cos\left(x_0 + \frac{\Delta x}{2}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\Delta x / 2} \lim_{\Delta x \to 0} \cos\left(x_0 + \frac{\Delta x}{2}\right) = \cos x_0.$$

Since the point x_0 is arbitrary, we can infer that

$$(\sin x)' = \cos x.$$

Answer. $(\sin x)' = \cos x$.

Proceeding from the definition of a derivative, calculate the derivatives of the following functions.
1.1. $f(x) = 1/x$. 1.2. $f(x) = \cos x$.
1.3*. $f(x) = e^x$. 1.4*. $f(x) = \ln x$.
1.5*. $f(x) = x^n$. 1.6. $f(x) \equiv c$.

The one-sided limits

$$\lim_{\Delta x \to -0} \frac{\Delta f(x_0)}{\Delta x}, \tag{3}$$

$$\lim_{\Delta x \to +0} \frac{\Delta f(x_0)}{\Delta x} \tag{4}$$

are called, respectively, the *left* and the *right derivative of the function $f(x)$ at the point* x_0 and are designated as $f'_-(x_0)$ and $f'_+(x_0)$. For the derivative $f'(x_0)$ to exist, it is necessary and sufficient that both derivatives (left and right) exist at x_0 and be equal:

$$f'_+(x_0) = f'_-(x_0). \tag{5}$$

Example 1.2. Prove that the function

$$f(x) = \begin{cases} x, & x \geqslant 1, \\ x^2, & x < 1, \end{cases}$$

is not differentiable at the point $x = 1$.

Solution. The increment of the function at $x = 1$ is

$$\Delta f(1) = f(1+\Delta x) - f(1) = \begin{cases} \Delta x, & \Delta x \geqslant 0, \\ (1+\Delta x)^2 - 1, & \Delta x < 0, \end{cases}$$

or, after transformations,

$$\Delta f(1) = \begin{cases} \Delta x, & \Delta x \geqslant 0, \\ 2\Delta x + (\Delta x)^2, & \Delta x < 0. \end{cases}$$

Consequently, by definition (3), (4), we have

$$f'_-(1) = \lim_{\Delta x \to -0} \frac{2\Delta x + (\Delta x)^2}{\Delta x} = 2,$$

$$f'_+(1) = \lim_{\Delta x \to +0} \frac{\Delta x}{\Delta x} = 1.$$

Since $f'_+(1) \neq f'_-(1)$, the derivative $f'(x)$ does not exist at $x = 1$.

Prove that the following functions are not differentiable at the indicated points.
1.7. $f(x) = |x|$ for $x = 0$.

1.8. $f(x) = |x^2 - 5x + 6|$ for $x = 2$ and $x = 3$.

1.9. $f(x) = \begin{cases} x & \text{for } x \leqslant 1, \\ 2 - x & \text{for } x > 1. \end{cases}$

1.10*. Show that the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

does not have either a right or a left derivative at the point $x = 0$.

1.11. Prove that the function $f(x) = x|x|$ is differentiable at the point $x = 0$.

Table of derivatives of the principal elementary functions.

$$(x^a)' = ax^{a-1}, \tag{6}$$

$$(a^x)' = a^x \ln a, \quad a > 0; \quad (e^x)' = e^x, \tag{7}$$

$$(\log_a x)' = \frac{1}{x \ln a}, \quad a > 0, \ a \neq 1; \ (\ln x)' = \frac{1}{x}, \tag{8}$$

$$(\sin x)' = \cos x, \tag{9}$$

$$(\cos x)' = -\sin x, \tag{10}$$

$$(\tan x)' = \frac{1}{\cos^2 x}, \tag{11}$$

$$(\cot x)' = -\frac{1}{\sin^2 x}, \tag{12}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \tag{13}$$

$$(\arctan x)' = \frac{1}{1+x^2}. \tag{14}$$

Rules of differentiation. Assume that c is a constant and $f(x)$ and $g(x)$ are differentiable functions. Then

$$c' = 0, \tag{15}$$

$$[f(x) + g(x)]' = f'(x) + g'(x), \tag{16}$$

$$[f(x)\, g(x)]' = f'(x)\, g(x) + g'(x)\, f(x), \tag{17}$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\, g(x) - g'(x)\, f(x)}{g^2(x)}. \tag{18}$$

Theorem on the differentiation of a composite function. Assume that $y = f(x)$ possesses a derivative at the point x_0, and $g(y)$ possesses a derivative at the point $y_0 = f(x_0)$; then the composite function $F(x) = g[f(x)]$ also possesses a derivative at x_0, which is

$$F'(x_0) = g'(y_0)\, f'(x_0). \tag{19}$$

Example 1.3. Calculate the derivative of the function

$$F(x) = (x^2 + x + 1)^{100}.$$

Solution. Setting $y = f(x) = x^2 + x + 1$, $g(y) = y^{100}$, we get

$$g'(y) = 100y^{99}, \quad f'(x) = 2x + 1.$$

Then, in accordance with (19), we get

$$F'(x) = 100\,(x^2 + x + 1)^{99}(2x + 1).$$

Calculate the derivatives of the following composite functions.

1.12. $y = \dfrac{2+\sqrt{x}}{2-\sqrt{x}}$.

1.13. $y = \sqrt{\dfrac{1-x^2}{1+x^2}}$.

1.14. $y = \sqrt{\sin\sqrt{x}}$.

1.15. $y = e^{\sqrt{\ln(ax^2+bx+c)}}$.

1.16. $y = \arctan\dfrac{1+x}{1-x}$.

1.17. $y = \dfrac{(a+bx^n)^m}{(a-bx^n)^m}$.

1.18. $y = \dfrac{1}{15}\cos^3 x\,(3\cos^2 x - 5)$.

1.19. $y = \dfrac{(\tan^2 x - 1)(\tan^4 x + 10\tan^2 x + 1)}{3\tan^3 x}$.

1.20. $y = \ln\cos\dfrac{x-1}{x}$.

Beginning with simplification of the expressions, calculate the derivatives of the following functions

1.21. $f(x) = \dfrac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}$.

1.22. $f(x) = \dfrac{\sqrt[3]{x+\sqrt{2-x^2}}\;\sqrt[6]{1-x\sqrt{2-x^2}}}{\sqrt[3]{1-x^2}}$.

1.23. $f(x) = \dfrac{\left[(1-x^2)^{-1/2} + 1 + \dfrac{1}{(1-x^2)^{-1/2}-1}\right]^{-2}}{2-x^2-2\sqrt{1-x^2}}$.

1.24. $f(x) = \dfrac{(x^{2/m} - 9x^{2/n})\left(\sqrt[n]{x^{1-m}} - 3\sqrt[n]{x^{1-n}}\right)}{(x^{1/m}+3x^{1/n})^2 - 12x^{(m+n)/(mn)}}$.

1.25. $f(x) = (\sqrt{1-x^4}+1) : \left(\dfrac{1}{\sqrt{1+x^2}} + \sqrt{1-x^2}\right)$.

1.26. $f(x)=\dfrac{\left(\sqrt[3]{\left(\frac{1}{2}\right)^{-3}-t^3}+\sqrt[3]{\frac{t^5+2t^4+4t^3}{4-4t+t^2}}\right)}{\frac{1}{\sqrt{2}-\sqrt{t}}+\frac{1}{\sqrt{t}+\sqrt{2}}}$.

1.27. $f(x)=\dfrac{(\sqrt{x}+2)\left(\frac{2}{\sqrt{x}}-1\right)-(\sqrt{x}-2)\left(\frac{2}{\sqrt{x}}+1\right)}{(2-\sqrt{x+2}):\left(\sqrt{\frac{2}{x}+1}-\frac{2}{\sqrt{x}}\right)}$.

1.28. $f(x)=\dfrac{\sqrt{2x+2\sqrt{x^2-1}}}{\left(\frac{\sqrt{x-1}}{\sqrt{x+1}}+\frac{\sqrt{x+1}}{\sqrt{x-1}}+2\right)^{1/3}}$.

1.29. $f(x)=\dfrac{\sqrt{\sqrt{\frac{x-1}{x+1}}+\sqrt{\frac{x+1}{x-1}}-2}\,(2x+\sqrt{x^2-1})}{\sqrt{(x+1)^3}-\sqrt{(x-1)^3}}$.

1.30. $f(x)=\dfrac{\sqrt{1+\sqrt{1-x^2}}\,(\sqrt{(1+x)^3}-\sqrt{(1-x)^3})}{2+\sqrt{1-x^2}}$.

1.31.

$$f(x)=\left(\frac{\sqrt[4]{ax^3}-\sqrt[4]{a^3x}}{\sqrt{a}-\sqrt{x}}+\frac{1+\sqrt{ax}}{\sqrt[4]{ax}}\right)^{-2}\sqrt{1+2\sqrt{\frac{a}{x}}+\frac{a}{x}}.$$

If the function $f(x)$ is defined on an interval $[a, b]$, then the value of its left derivative on the right end and that of the right derivative on the left end are taken, respectively, as the values of its derivatives at the end-points of that interval.

Example 1.4. Calculate the derivative of the function

$$f(x)=\sqrt{x^2-2x+1}$$

on the interval $[0, 2]$.

Solution. Since the expression under the radical sign is a perfect square, we can, according to the definition of the modulus, represent the given function in the following form:

$$f(x)=|x-1|=\begin{cases} x-1, & x\in(1, 2], \\ 1-x, & x\in[0, 1). \end{cases} \qquad (*)$$

Differentiating $f(x)$ separately on the intervals $[0, 1)$ and $(1, 2]$, we obtain

$$f'(x)=\begin{cases} -1, & x\in[0, 1), \\ 1, & x\in(1, 2]. \end{cases}$$

Since the left and right derivatives of $f(x)$ at the point $x = 1$ do not coincide, the derivative does not exist at $x = 1$; we take the values of the left derivative of function (*) at the point 2 and of its right derivative at the point 0 as the values of $f'(x)$ at the end-points of the interval [0, 2].

Answer. $f'(x)=\begin{cases} -1, & x\in[0, 1), \\ 1, & x\in(1, 2]. \end{cases}$

Calculate the derivatives of the following functions.

1.32*. $f(x)=x\,\dfrac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}-1}$.

1.33. $$f(x)=\frac{2x\sqrt{\frac{1}{4}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)^2-1}}{2\sqrt{\frac{1}{4}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)^2-1}-\frac{1}{2}\left(\sqrt{\frac{1}{x}}-\sqrt{x}\right)}.$$

1.34. $$f(x)=\frac{\sqrt{1+\left(\frac{x^2-1}{2x}\right)^2}}{(x^2+1)\frac{1}{x^2}}.$$

1.35*. $f(x)=\sqrt{x+2\sqrt{2x-4}}+\sqrt{x-2\sqrt{2x-4}}$.

2. Intervals of Monotonicity and Extrema of a Function

We say that the function $y=f(x)$ *increases on the interval* (a, b) if for any x_1 and x_2, belonging to (a, b), the inequality $x_1<x_2$ yields an inequality $f(x_1)<f(x_2)$. We say that the function $y=f(x)$ *decreases on the interval* (a, b) if for any x_1 and x_2, belonging to (a, b), the inequality $x_1<x_2$ yields an inequality $f(x_1)>f(x_2)$.

Sufficient conditions for a function to be monotonic. Assume that the function $y=f(x)$ is defined and differentiable on the interval (a, b). For the function to be increasing on the interval (a, b), it is sufficient that the following condition be satisfied:

$$f'(x)>0 \quad \text{for any} \quad x\in(a, b).$$

For the function to be decreasing on the interval (a, b), it is sufficient that the following condition be satisfied:

$$f'(x)<0 \quad \text{for any} \quad x\in(a, b).$$

The points, belonging to the interval (a, b), at which the derivative is zero or does not exist are called *critical* or *stationary points* of the function $y=f(x)$. It follows from the definition of a critical point that if the derivative of a function changes sign, it can occur only when it passes through a critical point. Thus the intervals of decrease and increase (the intervals of monotonicity) of the function $f(x)$ are bounded by critical points. Therefore, to determine the intervals of monotonicity of a function, it is necessary

(1) to find the critical points of $f(x)$;
(2) to determine the sign of the derivative $f'(x)$ inside the intervals bounded by critical points.

Example 2.1. Find whether the function

$$f(x) = xe^{-3x},$$

increases or decreases.

Solution. We find the derivative

$$f'(x) = e^{-3x} - 3xe^{-3x} = e^{-3x}(1 - 3x).$$

The derivative $f'(x)$ exists everywhere and vanishes at the point $1/3$. The point $x = 1/3$ divides the number axis into two intervals, $(-\infty, 1/3)$ and $(1/3, +\infty)$. Since the function e^{-3x} is always positive, the sign of the derivative is defined by the second factor. Consequently, $f'(x) > 0$ on the interval $(-\infty; 1/3)$ and $f'(x) < 0$ on the interval $(1/3; +\infty)$.

Answer. The function $f(x)$ increases on the interval $(-\infty, 1/3)$ and decreases on the interval $(1/3, +\infty)$.

Find the intervals of increase and decrease of the following functions.

2.1. $f(x) = \dfrac{x^2 - 2}{2x + 3}$. **2.2.** $f(x) = \dfrac{x}{\ln x}$.

2.3. $f(x) = \dfrac{3}{2}x - \sin^2 x$. **2.4.** $f(x) = 2\ln(x - 2) - x^2 + 4x + 1$.

2.5. $f(x) = \dfrac{2x - 1}{(x - 1)^2}$. **2.6.** $f(x) = \dfrac{3 - x^2}{x}$.

2.7*. Find the set of all values of the parameter a for each of which the function

$$f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$$

increases for all $x \in \mathbf{R}$ and has no critical points.

2.8*. Find all values of the parameter a for each of which the function

$$y(x) = 8ax - a\sin 6x - 7x - \sin 5x$$

increases and has no critical points for all $x \in \mathbf{R}$.

We say that the function $y = f(x)$ *possesses a maximum* (or *minimum*) *at the point* x_0 if there is a δ-neighbourhood of the point x_0, belonging to the domain of definition of the function, such that the inequality

$$f(x) < f(x_0) \quad (f(x) > f(x_0) \text{ respectively})$$

holds for all $x \neq x_0$ belonging to the interval $(x_0 - \delta, x_0 + \delta)$. The points of maximum and minimum are known as *points of extremum* and the values of the function at those points are called *extremal values*.

The necessary condition for the existence of an extremum of a function. Assume that the function $f(x)$ is differentiable on the interval (a, b). If the function $f(x)$ attains its extremum at a certain point $x_0 \in (a, b)$, then $f'(x_0) = 0$.

The sufficient condition for the existence of an extremum of a function. Assume that the function is defined and continuous on the interval (a, b) and is differentiable throughout the interval (except, maybe, for a finite number of points). If the derivative changes sign when it passes through a critical point, then the critical point is a point of extremum of the function. It is a point of maximum if the sign changes from plus to minus and a point of minimum if the sign changes from minus to plus.

Example 2.2. Find the extremum of the function

$$f(x) = \sqrt{2x^2 - x + 2}.$$

Solution. We find the derivative:

$$f'(x) = \frac{1}{2} \frac{4x - 1}{\sqrt{2x^2 - x + 2}}. \qquad (*)$$

Then we equate the derivative $f'(x)$ to zero:

$$\frac{1}{2} \frac{4x - 1}{\sqrt{2x^2 - x + 2}} = 0.$$

From this we find a critical point: $x_0 = 1/4$. It can be seen from expression (*) that $f'(x) > 0$ for $x > 1/4$ and $f'(x) < 0$ for $x < 1/4$, i.e. when the derivative passes through the point $x_0 = 1/4$ it changes sign from minus to plus. Consequently, $x_0 = 1/4$ is a point of minimum, with $f(x_0) = \sqrt{15/8}$. The denominator of expression (*) is positive for all $x \in \mathbf{R}$. Consequently, the function has no other critical points except for $x = 1/4$.

Answer. $x_{\min} = 1/4$, $y_{\min} = \sqrt{15/8}$.

Find the extrema of the following functions.

2.9. $f(x) = \frac{(x-2)^2 (x+4)}{4}$.

2.10. $f(x) = x + \sin 2x$.

2.11. $f(x) = xe^{x-x^2}$.

2.12. $f(x) = \frac{2x}{x^2+9}$.

2.13. $f(x) = 2x^3 + 3x^2 - 12x + 5$.

2.14. $f(x) = \frac{x}{\ln x}$.

2.15. $f(x) = 2x^3 - 6x^2 - 18x + 7$.

2.16. $f(x) = \frac{x^2 - 2x + 2}{x - 1}$.

The methods of finding an extremum of a function make it possible to ascertain the validity of certain transcendental inequalities.

Example 2.3. Prove the validity of the inequality

$$e^x - x > 1 \quad \text{for} \quad x \neq 0. \qquad (*)$$

Solution. Let us consider the function

$$f(x) = e^x - 1 - x$$

and find its extremum. Solving the equation $f'(x) = 0$, i.e. the equation $e^x - 1 = 0$, we get $x = 0$.

For $x = 0$ the function $f(x)$ attains its only minimum since when passing through the point $x = 0$ the derivative $f'(x)$ changes sign from minus to plus. Since $f(0) = 0$, the inequality $f(x) > 0$ holds true for all $x \neq 0$, i.e. $e^x - 1 - x > 0$, or $e^x - x > 1$. And this is what we wished to prove.

Prove the following inequalities.

2.17. $x - x^3/6 < \sin x < x$ for $x > 0$.

2.18. $\cos x > 1 - x^2/2$ for $x \neq 0$.

2.19. $\ln(1 + x) < x$ for $x > 0$.

3. The Greatest and the Least Value of a Function

Assume that the function $f(x)$ is defined and continuous on a finite interval $[a, b]$. To find the greatest (the least) value of the function it is necessary to find all maxima (minima) of the function on the interval (a, b), choose the greatest (the least) of them, and compare it with the values of the function at the points a and b. The greatest (the least) of these numbers is precisely the *greatest* (the *least*) *value of the function* $f(x)$ *on the interval* $[a, b]$; it is designated as $\max\limits_{x \in [a,b]} f(x)$ ($\min\limits_{x \in [a,b]} f(x)$). When we seek the greatest or the least value of a function, it may turn out that inside the interval $[a, b]$ the derivative exists at all points of the interval and does not vanish at any point of the interval (i.e. the critical points of the function are absent). This means that in the interval being considered the function increases or decreases and, consequently, it attains its greatest and least values at the end-points of the interval.

Example 3.1. Find the least and the greatest value of the function $f(x) = \frac{x}{8} + \frac{2}{x}$ on the interval $[1, 6]$.

Solution. Since

$$f'(x) = \frac{1}{8} - \frac{2}{x^2},$$

the only critical point getting into the given interval is the point $x = 4$. Comparing the values of the function at the point $x = 4$, with its values at the end-points of the interval

$$f(4) = 1, \quad f(1) = 2\frac{1}{8}, \quad f(6) = 1\frac{1}{12},$$

we infer that the function $f(x)$ attains its least value at the point $x = 4$ and its greatest value at the left end-point of the interval for $x = 1$.

Answer. $\max\limits_{x \in [1, 6]} f(x) = f(1) = 2\frac{1}{8}$; $\min\limits_{x \in [1, 6]} f(x) = f(4) = 1$.

Find the greatest and the least value of the following functions on the indicated intervals.

3.1. $f(x) = x^5 - x^3 + x + 2$, $x \in [-1, 1]$.

3.2. $f(x) = 3x^4 + 4x^3 + 1$, $x \in [-2, 1]$.

3.3. $f(x) = \cos^2 \frac{x}{2} \sin x$, $x \in [0, \pi]$.

3.4. $f(x) = \frac{1}{2} \cos 2x + \sin x$, $x \in \left[0, \frac{\pi}{2}\right]$.

3.5. $f(x) = \frac{x}{2} - \frac{1}{4} \sin 2x + \frac{1}{3} \cos^3 x - \cos x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Take into account the following remark when seeking the greatest (the least) values of certain functions.

If the continuous function $F(x)$ on the interval $[a, b]$ can be represented as $F(x) = f[g(x)]$, where $g(x)$ and $f(y)$ are continuous functions on the intervals $x \in [a, b]$ and $y \in [c, d]$ respectively, $c = \min\limits_{x \in [a,b]} g(x)$, $d = \max\limits_{x \in [c,d]} g(x)$, then

$$\max_{x \in [a,b]} F(x) = \max_{y \in [c,d]} f(y) \quad \text{and} \quad \min_{x \in [a,b]} F(x) = \min_{y \in [c,d]} f(y).$$

Example 3.2. Find the greatest and the least value of the function

$$F(x) = \frac{\sin 2x}{\sin(x + \pi/4)} \text{ on the interval } \left[0, \frac{\pi}{2}\right].$$

Solution. Using the formulas

$$\sin\left(\frac{\pi}{4} + x\right) = \frac{\sqrt{2}}{2} (\sin x + \cos x),$$

$$\sin 2x = (\sin x + \cos x)^2 - 1,$$

we represent the given function as a composite function

$$F(x) = f[g(x)],$$

where

$$f(y) = \frac{y^2 - 1}{y} \sqrt{2}, \quad g(x) = \sin x + \cos x.$$

Let us find the greatest and the least value of $g(x)$. The critical points of $g(x)$ are the roots of the equation

$$\cos x - \sin x = 0,$$

from which only $x = \pi/4$ gets into the interval $[0, \pi/2]$. Comparing $g(0)$, $g(\pi/4)$, and $g(\pi/2)$, we infer that the interval $[1, \sqrt{2}]$ is the domain of variability of $g(x)$. It is easy to see that

$$f'(y) = \sqrt{2}\left(1 + \frac{1}{y^2}\right) > 0$$

on the whole domain of definition of $f(y)$, for $y \in [1, \sqrt{2}]$ inclusive. Consequently, the function $f(y)$ increases on the interval $[1, \sqrt{2}]$

and attains its greatest and least values at the right and left end-points of the interval, respectively:

$$\max_{y\in[1,\sqrt{2}]} f(y) = f(\sqrt{2}) = 1,$$
$$\min_{g\in[1,\sqrt{2}]} f(y) = f(1) = 0.$$

The same values are the greatest and least values of the initial function $F(x)$ as well.

Answer. $\max_{x\in[0,\pi/2]} F(x) = 1, \quad \min_{x\in[0,\pi/2]} F(x) = 0.$

Find the greatest and the least value of the following functions.

3.6. $f(x) = \dfrac{\sin 2x}{\sin(\pi/4+x)}, \quad x \in \left[\pi;\ \dfrac{3\pi}{2}\right].$

3.7. $f(x) = \dfrac{1}{\sin x+4} - \dfrac{1}{\cos x-4}, \quad x \in \mathbf{R}.$

3.8. $f(x) = \tan x + \cot x,\ x \in [\pi/6,\ \pi/3].$

3.9*. Find the least value of the function

$$f(x) = \left(\frac{2+\cos x}{\sin x}\right)^2$$

on the interval $[0, \pi]$.

Seeking the greatest and the least value of functions containing the sign of an absolute value.

Example 3.3. Find the greatest and the least value of the function

$$f(x) = |x^2 - 5x + 6| \qquad (*)$$

on the interval $[0, 2.4]$.

Solution. To express the modulus in expressio $(*)$, we find the roots of the equation $f(x) = 0$. Solving the equation $x^2 - 5x + 6 = 0$, we get $x = 2$, $x = 3$. Thus,

$$f(x) = \begin{cases} x^2-5x+6 & \text{for } x \in (-\infty,\ 2) \cup (3,\ \infty); \\ -(x^2-5x+6) & \text{for } x \in [2,\ 3]. \end{cases} \qquad (**)$$

It can be seen from $(**)$ that on the interval in question $[0, 2.4]$ the function $f(x)$ admits of two representations depending on the values of the argument:

$$f(x) = \begin{cases} -(x^2-5x+6), & x \in (2,\ 2.4], \\ x^2-5x+6, & x \in [0,\ 2]. \end{cases}$$

We calculate the derivative of the function $f(x)$:

$$f'(x) = \begin{cases} -(2x-5), & x \in (2,\ 2.4], \\ 2x-5, & x \in [0,\ 2). \end{cases}$$

For $x \in (2, 2.4]$ we have $f'(x) > 0$ and, consequently, $f(x)$ increases, and for $x \in [0, 2)$ we have $f'(x) < 0$ and, consequently, $f(x)$ decreases; the point $x = 2$ is a critical point since the derivative $f'(x)$ does not exist at that point. Comparing the values of the function at the endpoints of the interval [0, 2.4] with its value at the critical point, we infer that

$$\max_{x\in[0,2.4]} f(x) = f(0) = 6, \qquad \min_{x\in[0,2.4]} f(x) = f(2) = 0.$$

Answer. $\max\limits_{x\in[0,2.4]} f(x) = f(0) = 6$, $\min\limits_{x\in[0,2.4]} f(x) = f(2) = 0$.

Find the greatest and the least value of the following functions on the indicated intervals.

3.10. $f(x) = \left|\frac{1+x}{1-x}\right|$, $x \in [-2, 0]$.

3.11. $f(x) = \sqrt{1-2x+x^2} + \sqrt{1+2x+x^2}$,
(a) $x \in [0, 2]$, (b) $x \in [-2, 0]$.

3.12. $f(x) = \sqrt{1-2x+x^2} - \sqrt{1+2x+x^2}$, $x \in (-\infty, +\infty)$.

3.13. $f(x) = |x^2+2x-3| + \frac{3}{2}\ln x$, $x \in \left[\frac{1}{2}, 4\right]$.

3.14. Find the points of minimum of the function

$$f(x) = 4x^3 - x\,|x-2|, \quad x \in [0, 3],$$

and its greatest value on that interval.

3.15. Find the greatest and the least value of the function

$$f(x) = \sqrt{x(10-x)}.$$

3.16*. Find the greatest and the least value of the function

$$f(x) = (x-1)^2\sqrt{x^2-2x+3}, \quad x \in [0, 3].$$

4. Some Problems Which Reduce to Those on Seeking the Greatest and the Least Value and the Extrema

The hypotheses of some problems do not state explicitly that it is required to find the greatest and the least valuee and the extrema. These are, for instance, problems on seeking the set of values of a function.

Example 4.1. Find the image of the interval [—1, 3] under the mapping specified by the function

$$f(x) = 4x^3 - 12x.$$

Solution. To find the image of the given interval, we must find the set of values of the function $f(x)$ for $x \in [-1, 3]$, which, by virtue of the continuity of the initial function, is an interval $[\min\limits_{x\in[-1,3]} f(x); \max\limits_{x\in[-1,3]} f(x)]$. Thus the initial problem reduces to that on seeking the greatest and the least value of the function $f(x)$ on the interval [—1, 3].

The critical points of $f(x)$ can be found from the equation

$$12x^2 - 12 = 0,$$

whose roots are $x_1 = 1$ and $x_2 = -1$. Comparing the values of $f(x)$ at the critical points and at the end-points of the interval, we obtain

$$\max_{x\in[-1,\,3]} f(x) = f(3) = 72, \quad \min_{x\in[-1,\,3]} f(x) = f(1) = -8.$$

Consequently, the image of the interval $[-1, 3]$ under the mapping specified by the initial function is the interval $[-8, 72]$.

Answer. $[-8, 72]$.

4.1. Find the set onto which the derivative of the function $f(x) = x(\ln x - 1)$ maps the ray $[1, \infty)$.

4.2. Find the image of the interval $[0, 0.5]$ under the mapping given by the derivative of the function $f(x) = \tan 3x$.

4.3. Find the intersection of the sets onto which the derivatives of the functions $y = \dfrac{x+3}{x-5}$, $y = \sqrt{6x+5}$ map the interval $[0, 1]$.

4.4*. Into what interval does the function $y = \dfrac{x-1}{x^2-3x+3}$ transform the entire real line?

4.5. Find the ranges of the following functions:

(a) $y = \dfrac{x^2}{x^4+1}$, (b) $y = \dfrac{x}{x^2+1}$.

4.6*. Prove that the following inequality holds true:

$$\frac{x}{ax^2+b} \leqslant \frac{1}{2\sqrt{ab}}.$$

4.7*. Prove that the inequality

$$\min_{x\in[-\pi,\,\pi]} f(x) > -7/9$$

holds true for the function $f(x) = \cos x \sin 2x$.

4.8*. Prove that the inequality

$$\max_{x\in[-\pi,\,\pi]} f(x) < 0.77$$

is satisfied for the function $f(x) = \sin x \sin 2x$.

4.9*. Prove that the inequality

$$\cos x \sqrt{\sin x} \leqslant 2^{1/2} \cdot 3^{-3/4}$$

holds true for $x \in [0, \pi/2]$.

4.10. Prove that the inequality

$$1 \leqslant \sqrt[3]{\frac{x^2}{2x-1}} \leqslant \sqrt[3]{\frac{4}{3}}$$

holds true for $x \in [3/4, 2]$.

4.11*. Prove that the function

$$y=\frac{x^2+x+1}{x^2+1}$$

cannot have values greater than 3/2 and values smaller than 1/2 for any real values of x.

4.12*. Find all a for each of which there is at least one pair of numbers (x, y) satisfying the conditions

$$x^2+(y+3)^2<4, \quad y=2ax^2.$$

4.13*. The sum of the third and the ninth term of an arithmetic progression is equal to the least value of the quadratic trinomial $2x^2-4x+10$. Find the sum of the first eleven terms of the progression.

4.14*. For what value of the parameter a do the values of the function

$$y=x^3-6x^2+9x+a$$

at the point $x=2$ and at the points of extremum, taken in a certain order, form a geometric progression?

4.15. The sum of the terms of an infinitely decreasing geometric progression is equal to the greatest value of the function

$$f(x)=x^3+3x-9$$

on the interval $[-2, 3]$; the difference between the first and the second term of the progression is equal to $f'(0)$. Find the common ratio of the progression.

4.16. The sum of an infinitely decreasing geometric progression is equal to the least value of the function

$$f(x)=3x^2-x+\frac{25}{12},$$

and the first term of the progression is equal to the square of its common ratio. Find the common ratio of the progression.

4.17*. Find the least value of a for which the equation

$$\frac{4}{\sin x}+\frac{1}{1-\sin x}=a$$

has at least one solution on the interval $(0, \pi/2)$.

4.18*. Prove that the function

$$z=(x+1/x)^2+(y+1/y)^2$$

is not smaller than 12.5 if $x>0$, $y>0$, $x+y=1$.

4.19*. Show that the function

$$z=2x^2+2xy+y^2-2x+2y+2$$

is not smaller than -3.

4.20*. For what value of a does the sum of the squares of the roots of the equation

$$x^2-(a-2)x-a-1=0$$

assume the least value?

4.21*. Prove that the inequality

$$-\frac{1}{2} \leqslant \frac{x}{1+x^2} \leqslant \frac{1}{2}$$

holds true for all values of $x \in \mathbf{R}$.

4.22. Prove that the inequality $0 \leqslant 2x + 3\sqrt[3]{x^2} \leqslant 1$ is valid on the interval

$$\left[-\frac{3}{2};\ 10^{-3}\sqrt{\log_6 \sqrt[5]{25} + \log_8 \sqrt[7]{49}}\right].$$

4.23. Prove that the inequality

$$\frac{1}{\sin(\pi/3+\alpha)} + \frac{1}{\sin(\pi/3-\alpha)} \geqslant \frac{4\sqrt{3}}{3}$$

is valid for $\alpha \in [0, \pi/3)$.

4.24. Prove that the inequality

$$x^4 + y^4 + \frac{2}{x^2y^2} \geqslant 4$$

is valid for all x and y.

4.25*. Prove the validity of the inequalities

$$\frac{9-\sqrt{85}}{2} < \frac{-2x+3}{x^2+6x+10} < \frac{9+\sqrt{85}}{2}.$$

4.26*. Prove that $1/4 < \sin^6 x + \cos^6 x \leqslant 1$.

4.27. Prove that the inequality

$$5e^{1/3} < (3x^2 - 7x + 7)\, e^x < \frac{11}{3}\sqrt[3]{e^2}$$

is valid for $x \in [0, 2/3]$.

5. Textual Problems on Finding the Greatest and the Least Value and the Extrema

To solve a problem on an extremum and on the greatest (least) value, it is first necessary, using the hypothesis, to form a function $f(x)$ and determine the interval of the variability of its argument, and then find the extremum or the greatest (least) value of the function on the interval obtained.

Example 5.1. Represent the number 26 as the sum of three positive terms the sum of whose squares is the least, if it is known that the second term is thrice as large as the first.

Solution. We designate the unknown terms as x, y, z. By the hypothesis, the terms we have introduced satisfy the following system of equations:

$$x + y + z = 26, \qquad (*)$$
$$y = 3x.$$

Using (*), we express the unknowns y and z in terms of x:

$$y = 3x, \quad z = 26 - 4x. \qquad (**)$$

Let us now form a function whose minimum we have to find:

$$S(x) = x^2 + 9x^2 + (26 - 4x)^2.$$

In this case, the interval of the variability of the argument is determined from the condition that all the terms be positive. Solving the system of inequalities

$$x > 0,$$
$$26 - 4x > 0,$$

we find that the required interval is $\left(0, \frac{13}{2}\right)$. We have thus reduced the problem to seeking the minimum of the function $S(x)$ on the interval $\left(0, \frac{13}{2}\right)$. The only critical point of the function $S(x)$ on the interval $\left(0, \frac{13}{2}\right)$ is the point $x = 4$. When passing through that point, the derivative of the function $S(x)$ changes sign from minus to plus, and consequently, $S(x)$ decreases on the interval $(0, 4)$ and increases on the interval $\left(4, \frac{13}{2}\right)$. Thus, for $x = 4$ the function $S(x)$ attains its minimum. Substituting $x = 4$ into equation (**), we get the values of the other unknowns.

Answer. $26 = 4 + 12 + 10$.

5.1. Represent the number 18 as a sum of two positive terms so that the sum of their squares be the least.

5.2. Represent the number 36 as a product of two factors so that the sum of their squares be the least.

5.3. Represent the number 180 as a sum of three positive terms so that the ratio of two of them be 1 : 2 and the product of the three terms be the greatest.

5.4. Represent the given positive number a as a sum of two positive terms so that their product be the greatest.

5.5*. The parabola $y = x^2 + px + q$ cuts the straight line $y = 2x - 3$ at a point with abscissa 1. For what p and q is the distance between the vertex of the parabola and the Ox axis the least? Find that distance.

5.6. Find the least distance from the point M with coordinates $(0, -2)$ to points (x, y) such that

$$y = \frac{16}{\sqrt{3x^3}} - 2, \quad x > 0.$$

5.7. Inscribe a rectangle of the greatest area into the segment of the parabola $y^2 = 2px$ cut off by the straight line $x = 2a$.

Geometric Problems.

Example 5.2. Find the height of a cone of the greatest volume if its generatrix is equal to l.

Solution. The volume of the cone the area of whose base is S and the height is H, can be calculated by the formula

$$V = \frac{1}{3} SH,$$

where $S = \pi R^2$, and R is the radius of the circle lying at the base of the cone. By the Pythagorean theorem, R and H are related as

$$R^2 + H^2 = l^2.$$

Using this equality, we express V as a function of only one variable H:

$$V = \frac{1}{3} \pi (l^2 - H^2) H.$$

Solving the equation

$$V'(H) = \frac{\pi}{3} (l^2 - 3H^2) = 0,$$

we find two critical points of the function $V(H)$: $H_1 = +l/\sqrt{3}$ and $H_2 = -l/\sqrt{3}$, from which only the point H_1 belongs to the interval $(0, l)$. When passing through the point H_1, the function $V'(H) = \frac{1}{3}(l^2 - 3H^2)$ changes sign from plus to minus and, consequently, on the interval $(0, l/\sqrt{3})$ the function $V(H)$ increases and on the interval $(l/\sqrt{3}, l)$ it decreases. Thus $H = l/\sqrt{3}$ is the height of the cone of the maximum volume for the given length of the generatrix l.

Example 5.3. Inscribe a rectangle of the greatest area into the trapezoid $ABCD$, one of whose nonparallel sides AB (8 cm long) is perpendicular to the base, so that one of its sides lies on the larger base of the trapezoid. The bases of the trapezoid are 6 and 10 cm in length, respectively. Calculate the area of the rectangle.

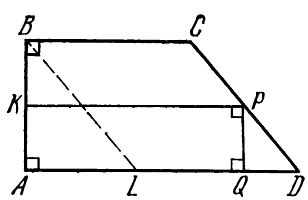

Fig. 8.1

Solution. Let us consider two cases separately. The first case: the vertex P of the rectangle lies on one of the nonparallel sides CD of the trapezoid (see Fig. 8.1). The second case: the vertex P lies on the base BC of the trapezoid. In the first case we designate the sides of the rectangle $|AQ| = x$ and $|AK| = y$. We set up an equation relating the unknowns x and y. For that purpose we draw an auxiliary line segment BL parallel to the side CD and consider two right triangles ABL and QPD. The legs of those triangles are equal, respectively to $|AB| = 8$, $|AL| = 4$, $|QD| = 10 - x$, $|PQ| = y$. We obtain the required equation from the condition of the similarity of the triangles ABL and QPD:

$$\frac{y}{10 - x} = 2, \quad \text{or} \quad y = 20 - 2x.$$

The area of the rectangle $AKPQ$ is

$$S(x) = x(20 - 2x).$$

In the first case the interval of the variability of x can be found from the condition that the point Q is the projection of the point P lying on the side CD and, consequently, $x \geqslant 6$. Thus the problem has reduced to seeking the least value of the function $S(x)$ on the interval $[6, 10]$. The only critical point of the function $S(x)$, $x = 5$, does not belong to the interval obtained. Consequently, the derivative of the function $S(x)$ does not change sign on this interval. Calculating the derivative of $S(x)$ at an arbitrary point of the interval $[6, 10]$, we ascertain that it is negative. Thus the greatest value of $S(x)$ is attained, at the left end of the interval, i.e.

$$\max_{x \in [6, 10]} S(x) = S(6) = 48 \text{ cm}^2.$$

In the second case, the area of the rectangles does not exceed 48 cm² since with the same nonparallel side equal to 8 cm, the lengths of their bases cannot be greater than 6 cm.

Answer. 48 cm².

5.8. From all the cones inscribed into a ball of radius R find that the area of whose lateral surface is the greatest.

5.9. Find the dimensions of the cylinder having the greatest volume if the area of its total surface is 2π.

5.10*. Among all the right triangles of area S find that for which the area of the circumscribed circle is the least.

5.11*. A trapezoid $ABCD$ is inscribed into a semi-circle of radius l so that the base AD of the trapezoid is a diameter and the vertices B and C lie on the circumference. Find the base angle φ of the trapezoid $ABCD$ which has the greatest perimeter.

5.12*. From all the triangles with the same base and the same vertex angle α choose the triangle with the greatest perimeter.

5.13. Given an isosceles triangle ABC and a rectangle whose two vertices lie on the base AC and the other two, on the sides AB and BC, inscribed into it. Find the greatest value of the area of the rectangle if $|AC| = 12$, $|BD| = 10$, BD is the height of the triangle ABC.

5.14. Given various trapezoids whose two nonparallel sides and the smaller base are equal to a. Find the value of the greater base of the trapezoid which has the greatest area.

5.15. The length of the side of a square $ABCD$ is 10 cm. The segments AA_1, BB_1, CC_1, DD_1, each of length x, are laid off on its sides, with $A_1 \in AB$, $B_1 \in BC$, $C_1 \in CD$, $D_1 \in DA$. Prove that the quadrilateral $A_1B_1C_1D_1$ is a square and find the value of x for which the area of the square is the least.

5.16. An isosceles triangle is inscribed into a circle of radius R. For what magnitude of the vertex angle α of the triangle is the height H drawn to the lateral side, of the greatest length? Find that length.

5.17*. Find the magnitude of the vertex angle α of an isosceles triangle of the given area S such that the radius r of the circle inscribed into the triangle is the greatest.

With the velocity with two components the path of the body (or the projection of the path onto a certain direction) is a function of two or more variables whose relationship can be established from physical considerations.

5.18. A traveller has to cross a river. At what angle α should he sail to attain the least drift if the speed of the boat is V_b and that of the river is V_r?

5.19*. A stone is thrown at an angle α to the horizontal with a velocity V_0. At what angle α will the stone travel the furthest?

5.20*. Find the lowest height $h = |OB|$ of the entrance $ABCD$ to a vertical tower that will admit a rigid bar of length l. The end of the bar will slide along a horizontal line which passes across the base AB of the tower. The width of the tower is $|AB| = d < l$.

5.21. A section of the text must occupy 384 cm^2 on the page. The upper and the lower margin of the page must each be 3 cm wide and the right and the left margins must each be 2 cm wide. If the only consideration is the economy of paper, what the optimal size of the page is?

5.22. A beam of rectangular cross section must be sawn from a round log of diameter d. What should the width x and the height y of the cross section be for the beam to offer the greatest resistance: (a) to compression, (b) to bending?

Remark. The compressive strength of a beam is proportional to the area of the cross section and the bending strength is proportional to the product of the width of the section by the square of its height.

5.23. A lamp is hung above the centre of a round table of radius r. At what height h of the lamp above the table is an object lying on the edge of the table illuminated best? (The illumination of an object is directly proportional to the cosine of the angle of incidence of the light and inversely proportional to the square of the distance from its source.)

5.24. A rectangular area must be designed so that it is surrounded by a guard net on three sides and a long brick wall on the fourth side. What are the most advantageous dimensions of the area that will maximize the surface area, if l running metres of net are available?

5.25. A straight line segment AB of length a connects two light sources A (with intensity p) and B (with intensity q). Find the point M which is illuminated the worst. (The illumination of a point is inversely proportional to the square of the distance from the light source.)

5.26*. A boat is 3 km away from the nearest point A of the river bank. A passenger on the boat wants to reach a point B which is on the bank and 5 km away from A. The boat can travel at 4 km/h, while the passenger, having left the boat, can walk at 5 km/h. At what point along the bank should the passenger alight from the boat in order to reach B in the shortest time?

5.27*. A rain drop with an initial mass m_0 falls under gravity, evaporating uniformly as it does, the decrease in the mass being proportional to time (the proportionality factor is k). How many seconds after the drop starts falling is the kinetic energy of the drop the greatest and what is it equal to? (The air resistance is neglected.)

5.28*. The cost of fuelling a ship is proportional to the cube of its speed. It is known that at 10 km/h the cost of the fuel is 30 roubles per hour; the other expenses, which are independent of the speed, are 480 roubles per hour. At what speed is the total cost per kilometer travelled the least? What then is the total cost per hour?

5.29. To transport a factory's produce from point N to a town A a highway NP is being built to connect the factory with a railway AB which passes through town A. Carriage by road is twice as expensive as that by rail. To what point P should the highway lead for the total transportation cost of the produce from point N to town A by road and by rail to be the least? The distance between N and the railway is 100 km, while A is a km away from the railway station, located on the circle that passes through both A and N, which are the end-points of a diameter of that circle.

When solving problems concerning the time of attaining the least distance between two objects moving at an angle to each other, use the fact that the distance between the objects after a time moment t is one side of a triangle the other two sides of which are functions of the distances covered by the objects up until that moment.

5.30. Two cars are moving towards each other at constant speeds of 40 and 50 km/h respectively along two intersecting streets. Assuming that the streets intersect at right angles and knowing that at a certain moment in time the cars are respectively 2 and 3 km away from the intersection, when will the two cars be closest?

5.31*. Three points, A, B, and C, are located so that $\angle ABC = 60°$. A car starts from point A the moment a train leaves point B. The car travels towards B at 80 km/h and the train travels towards point C at 50 km/h. How long after they started will the car and the train be closest if $|AB| = 200$ km?

5.32*. Two airplanes are flying horizontally at the same altitude along paths that are 120° apart and at the same speed of v km/h. At a certain moment one of the airplanes reaches the point where the two paths intersect. At that moment the other airplane is a km away (not having reached the intersection). How soon after will the planes be closest?

5.33. Determine the diameter of a round hole in a dam for which the flow rate Q of water per second will be greatest given that $Q = cy\sqrt{h - y}$, where h is the depth of the lowest point of the hole, and assuming that h and the coefficient c are constant.

5.34. The cost of a diamond is proportional to the square of its mass. In the process of grinding the diamond was broken in two. How big were the parts if it is known that the accident entailed the maximum loss in value?

5.35. An electric circuit contains two resistances in parallel. For what ratio of resistances is the resistance of the circuit the least if the total resistance of the two resistances connected in series is R?

5.36*. A courier has to reach a point B on one bank of a river from a point A on the other bank. Knowing that the courier can walk along the bank k times faster than he can row along the river, find the angle α at which the courier must cross the river in order to reach point B in the shortest time. The width of the river is h and the distance between A and B (along the bank) is d.

5.37*. Points A and B are in different optical media divided by a straight line. The speed of light in the first medium is v_1 and in the second v_2. Using *Fermat's principle*, which states that a light ray propagates along the path AMB which will be traversed in the least time, derive the law for the refraction of a light ray.

5.38*. Using Fermat's principle, derive the law for the reflection of a light ray from a plane surface in a homogeneous medium.

5.39. If a current I flows in an electric circuit with a resistance R, then the heat liberated per unit time is proportional to I^2R. Determine how the current I must be divided into two branches I_1 and I_2 with the aid of two wires, whose resistances are R_1 and R_2, respectively, for the liberation of heat to be the least.

5.40. A rectangular area of 9000 m^2 must be surrounded by a fence, with two opposite sides being made of brick and the other two of wood. One metre of wooden fencing costs 10 roubles while one metre of brick walling costs 25 roubles. What is the least amount of money that must be allotted to the construction of such a fence?

5.41*. A number of identical buildings must be constructed to yield a total living area of 40 000 m^2. The cost of a single building depends on the cost of the foundation, which is proportional to the square root of the living area in the building, and on the cost of the surface area of the structure, which is proportional to the cube of the square root of the value of the living area. The construction of a building that has 1600 m^2 of living area costs 184.8 thousand roubles, the cost of the surface area being 32% of the cost of the foundation. How many buildings must be constructed for the cost to be the least? Find how much the whole project will cost.

When solving certain problems on finding the greatest (or least) value of the quantity in the hypothesis, it is more convenient to seek the greatest (or least) value of another quantity which is a monotonic function of the first quantity.

5.42*. A statue 4 metres high sits on a column 5.6 metres high. How far from the column must a man, whose eye level is 1.6 metres from the ground, stand in order to see the statue at the greatest angle?

5.43*. A tourist bus is travelling along a straight highway. An ancient palace is located to one side of the highway and a road has been built from the main entrance at right angles to the highway. How far from the point where the roads intersect should the bus stop for the tourists to have the best view of the façade of the palace if the length of the palace is $2a$, the façade is at 60° to the highway, and the distance from the main entrance, which is the centre of symmetry of the palace, to the highway is b?

5.44*. A certain force must be applied to move a body of weight P at rest on a horizontal plane. The friction force is proportional to the force pressing the body onto the plane and opposes the force acting to move the body; the proportionality factor (the coefficient of friction) is k. At what angle α to the horizontal must the moving force be applied for it to be the least? Find the least value of the force.

5.45*. A body of weight P is lying on an inclined plane. A thread runs from the body over a pulley at the top of the inclined plane to a weight p ($p < P$). At what angle α will the body be kept on the inclined plane by the least weight if the coefficient of friction is k and $\alpha \in [\arctan k, \pi/2)$?

5.46. A lever of the second order has its fulcrum at A and a load P is suspended from point B ($AB = a$). The weight per unit length of the lever is p. What should the length of the lever be for the load P to be counterbalanced by the least force? (The moment of the balancing force must be equal to the sum of the moments of the load P and the lever.)

Sometimes problems having the greatest or the least value formulations can be more simply stated by using geometrical considerations.

Example 5.4. The beds of two rivers (within a certain region) are a parabola $y = x^2$ and a straight line $x - y - 2 = 0$. The rivers must be connected by a straight canal. Through what points must it pass for the canal to be the shortest?

Solution. The loci of all the points which are at a distance d from the straight line are two straight lines, parallel to the given line and one on either side of it. The points in the interior of the resulting strip are obviously closer than d from the line while those outside the strip are farther than d from the line. If the given straight line does not cut the parabola, then we can increase the width of the strip until we touch the parabola. The resulting point of tangency is the point at which the parabola is closest to the initial line. Consequently, to find this point, it is sufficient to find the coordinates of the point of tangency which is parallel to the given line. From the condition of parallelism (see Sec. 6), we have

$$2x = 1 \Longrightarrow x = 1/2 \quad \text{and} \quad y = 1/4.$$

To find the point on the straight line (the other end of the canal), we write the equation of the straight line which is perpendicular to the line $x - y - 2 = 0$ and which passes through the point $(1/2, 1/4)$.

$$y - 1/4 = -(x - 1/2), \quad \text{or} \quad y = -x + 3/4.$$

Solving the system of equations

$$y = -x + 3/4,$$
$$y = x - 2,$$

we get $x = 11/8$, $y = -5/8$.

Answer. The coordinates of the ends of the canal are $(1/2, 1/4)$ and $(11/8, -5/8)$.

5.47. A straight line l passes through the points $(3, 0)$ and $(0, 4)$. The point A lies on the parabola $y = 2x - x^2$. Find the distance p from point A to the straight line when A coincides with the origin, and indicate the coordinates of the point A (x_0, y_0) on the parabola for which the distance from the parabola to the straight line is the least.

5.48*. Four points A, B, C, D lie in that order on the parabola $y = ax^2 + bx + c$. The coordinates of A, B, and D are known:

$A(-2, 3)$, $B(-1, 1)$, $D(2, 7)$. Find the coordinates of C for which the area of the quadrilateral $ABCD$ is the greatest.

5.49*. Given two points $A(-2, 0)$ and $B(0, 4)$ and a straight line l on a coordinate plane: $y = x$. Find the perimeter of the triangle AMB, where M is a point with an abscissa of 3 which lies on the line l. At which position of the point M on the line l is the perimeter of the triangle AMB the least?

5.50*. Given an angle $\angle AOB$ and a point M in its interior. How should a straight line be drawn through the point M for that line to cut off a triangle of the least area from the angle?

5.51. Given an angle $\angle AOB$ and a point M in its interior. How should a triangle of the least perimeter be constructed such that one of its vertices is at point M, the second is on the side AO and the third is on the side BO of the given angle?

5.52. Consider various trapezoids inscribed in a circle of radius R such that the centre of the circle is in the interior of the trapezoid and one of the bases is $R\sqrt{3}$ long. Find the length of the nonparallel side of the trapezoid that has the greatest area.

5.53. Given a regular triangular pyramid $DABC$ (D is the vertex and ABC is the base with $|AB| = a$, $|AD| = b$). A plane α, which is parallel to the edges AD and BC, cuts the pyramid. How far from the edge AD should the plane be drawn for the area of the cross section to be the greatest?

5.54. Consider various right parallelepipeds whose bases are squares and whose each lateral face has a perimeter of 6 cm. Find the parallelepiped with the greatest volume and calculate the volume.

5.55. A rectangle is inscribed in a sector of a circle of radius R, the central angle of the sector being a right angle. One of the vertices of the rectangle coincides with the centre of the circle and the opposite vertex lies on the circumference. Find the lengths of the sides of the rectangle that has the greatest area.

5.56. The chord AB is as long as the radius of a circle. The chord CD, which is parallel to AB, is drawn so that the area of the quadrilateral $ABCD$ is at a maximum. Find the smaller of the central angles, that which rests on the chord CD.

5.57*. A rectangle with the greatest area is inscribed in a sector of a circle of radius R (the central angle of the sector being α). Calculate the area.

6. Geometrical Applications of a Derivative

Assume that the function $y = f(x)$ is differentiable at a point x_0 and $y_0 = f(x_0)$. The straight line defined by the equation

$$y = y_0 + f'(x_0)(x - x_0) \tag{1}$$

is said to be *tangent to the graph of the function* $y = f(x)$ *at the point* $M(x_0, y_0)$. Writing equation (1) in the form

$$y - y_0 = f'(x_0)(x - x_0), \tag{2}$$

we can infer that of all the straight lines passing through the point $M(x_0, y_0)$, that straight line is tangent to the graph of the function $f(x)$ whose angular coefficient is equal to $f'(x_0)$ (an angular coefficient is a tangent of the angle of inclination of a straight line to the positive direction of the Ox axis).

The straight line which is perpendicular to the tangent line at the point of tangency is called a *normal* to the graph of the function $y = f(x)$ at that point. The equation of the normal has the form

$$(y - y_0) f'(x_0) + (x - x_0) = 0. \tag{3}$$

The *angle between the graphs* of the functions

$$y = f_1(x) \quad \text{and} \quad y = f_2(x)$$

at their common point $M(x_0, y_0)$ is an angle α between the tangents to those graphs at the point $M(x_0, y_0)$. The tangent of an angle is calculated by the formula

$$\tan\alpha = \left|\frac{f_2'(x_0) - f_1'(x_0)}{1 + f_1'(x_0) f_2'(x_0)}\right|. \tag{4}$$

If the expression $1 + f_1'(x_0) f_2'(x_0)$ in the denominator vanishes, then the curves intersect at right angles.

To derive an equation of a tangent (normal) to the graph of the function $y = f(x)$ at the point whose abscissa is known and is equal to x_0, it is sufficient to find the values of $f'(x_0)$ and $y_0 = f(x_0)$ and substitute them into equation (1) (equation (3) respectively). The coordinates of a point on the graph of a function at which a tangent must be drawn are determined from the conditions of the problem.

Conditions of parallelism and perpendicularity of two straight lines. Assume that the straight lines are defined by the equations $y = k_1x + b_1$ and $y = k_2x + b_2$. For those straight lines to be parallel, it is necessary and sufficient that $k_1 = k_2$. For those lines to be perpendicular, it is necessary and sufficient that $k_1k_2 = -1$.

Example 6.1. On the curve $y = x^2 - 7x + 3$ find a point at which the tangent is parallel to the straight line $y = -5x + 3$.

Solution. From the condition of parallelism of two straight lines it follows that the angular coefficient of the tangent at the required point must be equal to -5. Then we can find the abscissa of the point of tangency using the equation $y'(x) = 2x - 7$:

$$2x - 7 = -5 \Rightarrow x = 1,$$

and the ordinate can be found by the substitution of $x = 1$ into the equation of the curve:

$$y(1) = -3.$$

Answer. The required point has the coordinates $(1, -3)$.

6.1. Find the points on the curve $y = x^3 - 3x + 2$ at which the tangent line is parallel to the straight line $y = 3x$.

6.2. Write the equation of a horizontal tangent to the graph of the function $y = e^x + e^{-x}$.

6.3. Write the equation of the tangent line to the graph of the function $y = \cos(2x - \pi/3) + 2$ at the point with abscissa $x_0 = \pi/2$.

6.4. What angle is formed by the abscissa axis and the tangent to the parabola $y = x^2 + 4x - 17$ drawn at the point $M(5/2, -3/4)$? Write the equation of the tangent.

6.5*. The straight line $y = -\frac{3}{4}x - \frac{3}{32}$ is known to be a tangent to the graph of the function $f(x) = \frac{1}{2}x^4 - x$. Find the coordinates of the point of tangency.

6.6. Show that the coordinates of the point of intersection of the tangents to the curve $y = 1 - x^2/a^2$ drawn through the points with the ordinates $y = 0$ do not depend on the parameter a. Find the coordinates of the intersection point.

6.7. Calculate the area of the triangle bounded by the coordinate axes and the tangent to the graph of the function $y = x/(2x - 1)$ at the point with abscissa $x_0 = 1$.

6.8. Find the equation of the common tangent to the curves

$$y = x^2 + 4x + 8 \quad \text{and} \quad y = x^2 + 8x + 4.$$

6.9. At what value of $x_0 \in [0, \pi/2]$ are the tangents to the graph of the function

$$f(x) = \sin x + \sin 2x$$

at the points with abscissas x_0 and $x_0 + \pi/2$ parallel?

6.10. Find all values of x_0 for each of which the tangents to the graphs of the functions

$$y(x) = 3\cos 5x, \; y(x) = 5\cos 3x + 2$$

at the points with abscissa x_0 are parallel.

6.11. Find the coordinates of the points of intersection of the Ox axis and the tangents to the graph of the function

$$y(x) = \frac{x+1}{x-3}$$

which form an angle $3\pi/4$ with the Ox axis.

6.12*. On the graph of the function $y(x) = x^3 - 3x^2 - 7x + 6$ find all points at each of which the tangents to that graph cut off on the positive semi-axis Ox a line segment half that on the negative semi-axis Oy. Find the lengths of the cut-off segments.

6.13*. The cord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y = 1/(1 - x)$ at the point $x = 2$ and is bisected by that point. Find a.

6.14. Write the equation of the tangent to the graph of the function $f(x) = |x^2 - |x||$ at the point with abscissa $x = -2$.

6.15. Two tangents to the graph of the function $y = \sqrt{17(x^2 + 1)}$ intersect at right angles to a certain point of the Oy axis. Write the equations of the tangents.

Example 6.2. Determine the angle at which the sine line

$$y = \frac{1}{\sqrt{3}} \sin 3x$$

cuts the abscissa axis at the origin.

Solution. By definition, the required angle is equal to the angle of inclination to the abscissa axis of a tangent drawn to the sine line through the origin. Thus the tangent of the required angle coincides with the angular coefficient of that tangent and is equal to the value of the derivative of the function $y = \frac{1}{\sqrt{3}} \sin 3x$ calculated for $x = 0$. Since

$$y' = \frac{3}{\sqrt{3}} \cos 3x,$$

it follows that $\tan \alpha = 3/\sqrt{3}$, and, consequently, $\alpha = \pi/3$.

Answer. $\alpha = \pi/3$.

6.16. Show that the tangents drawn to the graph of the function

$$y = \frac{x-4}{x-2}$$

at the points of its intersection with the coordinate axes are parallel.

6.17. At what points does the tangent to the graph of the function

$$f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$$

form an angle of 45° with the Ox axis?

6.18. At what angle is the tangent inclined to the Ox axis if it is drawn to the curve $y = 2x^3 - x$ at the point of intersection of that curve with the Oy axis?

6.19*. Show that the curves defined by the equations

$$xy = a^2, \quad x^2 - y^2 = b^2$$

intersect at right angles.

6.20*. Show that the families of curves defined by the equations

$$y = ax, \quad y^2 + x^2 = c^2$$

are perpendicular for any a and c.

When it is required to find the equation of a tangent to the graph of the function $y = f(x)$ passing through the given point $M(x_1, y_1)$ which does not belong to the graph of the function, the abscissa x_0 and the ordinate y_0 of the point of tangency can be found from the system of equations

$$y_1 - y_0 = f'(x_0)(x_1 - x_0), \qquad (5)$$
$$f(x_0) = y_0.$$

Example 6.3. At what point of the curve

$$y = x^2 - 5x + 6 \qquad (*)$$

should a tangent to that curve be drawn for the tangent to pass through the point $M_1(1, 1)$?

Solution. We form system (5):

$$1 - y_0 = (2x_0 - 5)(1 - x_0),$$
$$y_0 = x_0^2 - 5x_0 + 6.$$

Substituting y_0 from the second equation into the first, we get a quadratic equation

$$x_0^2 - 2x_0 = 0.$$

Hence the required points have the coordinates (2, 0) and (0, 6).

Answer. (2, 0), (0, 6).

6.21. At what point of the curve

$$y = ax^2 + bx + c$$

should a tangent line to that curve be drawn for the tangent to pass through the origin? Find the values of a, b, and c for which the problem has a solution.

6.22. At what point of the curve

$$y = x^2 - 5x + 6$$

should a tangent line be drawn for it to pass through the point $M(a, b)$? Find the values of a and b for which the problem has a solution.

6.23. Write the equation of a tangent line to the curve

$$y = (x + 1)/x,$$

if it is known that the tangent passes through the point $M(a, b)$. How many solutions does the problem have depending on the choice of the point? Find the solutions.

6.24*. Write the equation of the straight line passing through the point with the coordinates (1/2, 2), and touching the graph

$$y(x) = -\frac{x^2}{2} + 2$$

and cutting at two points the graph of the function

$$y(x) = \sqrt{4 - x^2}.$$

6.25. At what point M_0 of the curve $y = \sqrt{2}x^{3/2}$ is the tangent perpendicular to the straight line $4x + 3y + 2 = 0$?

It is known that the equality of the discriminant of a quadratic equation to zero signifies that the corresponding parabola touches the straight line $y = 0$, i.e. the abscissa axis. Similar considerations can sometimes be used to find the equations of tangent lines.

Example 6.4. Find the tangents to the circle

$$x^2 + y^2 = 25$$

which are parallel to the straight line

$$2x - y + 1 = 0.$$

Solution. All straight lines which are parallel to the straight line

$$2x - y + 1 = 0$$

are described by equations of the form

$$y = 2x + c.$$

For the circle and this straight line to intersect, the following system must be consistent:

$$2x + c = y,$$
$$x^2 + y^2 = 25.$$

Substituting y from the second equation into the first, we get

$$x^2 + (2x + c)^2 = 25.$$

For a unique solution to exist, the discriminant of the last equation must be equal to zero. From this condition we get for c the following possible values: $c = 5\sqrt{5}$ and $c = -5\sqrt{5}$.

Answer. $y = 2x + 5\sqrt{5}$, $y = 2x - 5\sqrt{5}$.

6.26*. At what angle can the circle

$$x^2 + y^2 = 16$$

be seen from the point $(+8, 0)$?

6.27. Point M moved along the circle

$$(x - 4)^2 + (y - 8)^2 = 20,$$

then it broke away from it and, moving along a tangent to the circle, cut the Ox axis at the point $(-2, 0)$. Find the point of the circle at which the moving point broke away.

6.28. Find the condition under which the straight line $y = kx + b$ touches the parabola $y^2 = 2px$.

6.29*. Find the locus of points from which the parabola $y = x^2$ can be seen at right angles.

6.30. Find the angle between the tangents to the graph of the function $y = x^2$ passing through the point with the coordinates $(0, -1)$.

6.31*. A right angle moves so that its sides touch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ all the time. Find the locus of the vertices of the angle.

7. Mechanical Applications of a Derivative

If the path traversed by a body by the time moment t is defined by the function

$$y = f(t), \tag{1}$$

then the speed of the movement v at the time moment t is equal to the derivative of the function $f(t)$:

$$v = f'(t), \tag{2}$$

and the acceleration is equal to the derivative of the speed:

$$a = [f'(t)]'. \tag{3}$$

Example 7.1. With the speed of b m/s a man approaches the foot of a tower h m high. What is the speed of his approach to the vertex of the tower when he is at a distance of l m from the foot?

Solution. We designate the distance from the man to the foot of the tower at the time moment t as $x(t)$. Then the distance $y(t)$ from the man to the vertex of the tower at time t has the form

$$y(t) = \sqrt{h^2 + x^2(t)}.$$

Differentiating $y(t)$ with respect to t, we obtain

$$y'(t)=\frac{x(t)\,x'(t)}{\sqrt{h^2+x^2(t)}}$$

and, taking into account that $x'(t) = b$ and the distance between the man and the foot of the tower is l, we get

$$v=\frac{bl}{\sqrt{h^2+l^2}}.$$

Answer. $v=\dfrac{bl}{\sqrt{h^2+l^2}}$.

Having found the law of motion, calculate the speed in the following problems.

7.1*. The lower end of the ladder which is 5 m high is slipping along the floor in the direction from the wall against which it is put. What is the speed of the upper end of the ladder at the moment when the lower end is at a distance of 3 m from the wall if the speed of the lower end is constant and equal to 2m/s?

7.2. A man approaching a vertical wall is illuminated by a lantern from behind, the lantern being at a distance of l m from the wall. The speed of the man is v m/s. With what speed does his shadow change if the man is h m tall?

7.3. A point moves along the hyperbola $y = 10/x$ so that its abscissa grows uniformly at a speed of a unity per second. At what speed does its ordinate change when the point passes the position (5, 2)?

7.4*. Two points having the laws of motion

$$x_1=100+5t,\quad x_2=\frac{1}{2}t^2,\quad t\geqslant 0$$

move along the Ox axis. What is the relative velocity of those points at the moment of their meeting (x is in cm and t is in seconds)?

7.5.** A wheel of radius R is rolling without slipping along a straight line. The centre of the circle moves with the velocity v. A nail is in the rim of the wheel. Find the velocity of the nail at the time moment t.

7.6*. A point moves with the angular velocity w along a circle of radius R with centre at the origin. What is the rate of the variation of the abscissa of the point when it passes the Ox axis?

7.7*. A body is thrown at the angle α to the horizontal with the velocity v. What is the maximum height that the body can reach?

7.8. The angle α (in radians) through which a wheel rotates in t s is equal to $\alpha = 3t^2 - 12t + 36$. Find the angular velocity of the wheel at the moment $t = 4$ s and at the moment when the wheel stops.

7.9*. Two bodies move at an angle of 60° towards each other; the equation of motion of the first body is

$$S_1(t) = t^2 - 2t,$$

and the equation of motion of the other body is

$$S_2(t) = 2t.$$

At the time moment $t = 0$ the bodies were at the same point. With what speed does the distance between them increase?

7.10. A horse runs along a circle with a speed of 20 km/h. A lantern is at the centre of the circle. A fence is along the tangent to the circle at the point at which the horse starts. With what speed does the shadow of the horse move along the fence at the moment when it covers 1/8 of the circle?

7.11*. A rocket moves rectilinearly according to the law $S(t) = v_0 t + at^2/2$. In the time t_1 after the beginning of the movement, a certain object separates from it and continues moving by inertia. At what time moment t and what new velocity v must be imparted to the object that, continuing a uniform motion, it would overtake the rocket at the moment t_2 having the same velocity as the rocket? What is the law of motion of the object?

7.12*. A rocket is launched along a straight line from a certain point. The law of motion of the rocket is $S = t^2/2$, $t \geqslant 0$. At what time moment t_0, reckoning from the beginning of motion, must the engines be switched off for the rocket, which continues moving by inertia with the velocity obtained, to be at the distance S_1 from the initial point at the time moment t_1?

Chapter 9

The Antiderivative and the Integral

1. Integration

The differentiable function $F(x)$ is said to be an *antiderivative* (or *primitive*) *of the function* $f(x)$ on a given interval if the equality

$$F'(x) = f(x) \tag{1}$$

is valid for all values of x belonging to that interval. If $F(x)$ is an antiderivative of $f(x)$ on a certain interval, then the expression

$$\int f(x)\,dx = F(x) + C, \tag{2}$$

where C is an arbitrary constant (the constant of integration), is called an *indefinite integral of the function* $f(x)$.

Principal Rules of Integration.

$$\int af(x)\,dx = a\int f(x)\,dx, \tag{3}$$

where a is a constant quantity;

$$\int [f_1(x) \pm f_2(x)]\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx; \tag{4}$$

if $\int f(x)\,dx = F(x) + C$, then

$$\int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + C \tag{5}$$

($a \neq 0$ and b are constants).

Table of Indefinite Integrals.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1. \tag{6}$$

$$\int \frac{dx}{x} = \ln|x| + C. \tag{7}$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C, \quad a > 0. \tag{8}$$

$$\int e^x\,dx = e^x + C.$$

$$\int \sin x\,dx = -\cos x + C. \tag{9}$$

$$\int \cos x\,dx = \sin x + C. \tag{10}$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C. \tag{11}$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C. \tag{12}$$

$$\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right| + C. \tag{13}$$

$$\int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C. \tag{14}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C, \quad a \neq 0. \tag{15}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C, \quad a \neq 0. \tag{16}$$

An infinite integral (an antiderivative) of a given function is calculated by using the integration rules to reduce the indefinite integral of the given function to the tabular one.

Example 1.1. Find all antiderivatives of the function

$$f(x) = \frac{(x^m - x^n)^2}{\sqrt{x}},$$

where m and n are integers.

Solution. Let us reduce $f(x)$ to the form

$$f(x) = x^{2m-1/2} - 2x^{m+n-1/2} + x^{2n-1/2}.$$

Using now integration rules (3), (4) and formula (6), we obtain

$$\int (x^{2m-1/2} - 2x^{m+n-1/2} + x^{2n-1/2})\,dx$$

$$= \frac{1}{2m+1/2}x^{2m+1/2} - \frac{2}{m+n+1/2}x^{m+n+1/2} + \frac{1}{2n+1/2}x^{2n+1/2} + C$$

$$= \frac{2x^{2m}\sqrt{x}}{4m+1} - \frac{4x^{m+n}\sqrt{x}}{2m+2n+1} + \frac{2x^{2n}\sqrt{x}}{4n+1} + C.$$

Answer. $\dfrac{2x^{2m}\sqrt{x}}{4m+1} - \dfrac{4x^{m+n}\sqrt{x}}{2m+2n+1} + \dfrac{2x^{2n}\sqrt{x}}{4n+1} + C.$

Using the rules of integration and the table of indefinite integrals, find the antiderivatives of the following functions.

1.1*. $f(x) = \dfrac{\sqrt{2+x^2} - \sqrt{2-x^2}}{\sqrt{4-x^4}}$. 1.2. $f(x) = \dfrac{x - 2\sqrt{x}}{\sqrt[3]{x}}$.

1.3*. $f(x)=x\sqrt{1-x}$. **1.4*.** $f(x)=\dfrac{x}{\sqrt{1-x/2}}$.

1.5*. $f(x)=\dfrac{x-1}{(2x-1)^3}$. **1.6*.** $f(x)=\dfrac{x}{2}\sqrt{1+x}$.

Having simplified the integrand, find the following indefinite integrals.

1.7. $\displaystyle\int \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}\,dx.$

1.8. $\displaystyle\int \frac{2\sqrt{1+\frac{1}{4}\left(\sqrt{\frac{1}{t}}-\sqrt{t}\right)^2}}{\sqrt{1+\frac{1}{4}\left(\sqrt{\frac{1}{t}}-\sqrt{t}\right)}-\frac{1}{2}\left(\sqrt{\frac{1}{t}}-\sqrt{t}\right)}\,dt.$

1.9. $\displaystyle\int \sqrt{\frac{2x}{(1+x)\sqrt[3]{1+x}}}\;\sqrt[3]{\frac{4+8/x+4/x^2}{\sqrt{2}}}\,dx.$

1.10. $\displaystyle\int \left(\frac{1-x^{-2}}{x^{1/2}-x^{-1/2}}-\frac{2}{x^{3/2}}+\frac{x^{-2}-x}{x^{1/2}-x^{-1/2}}\right)dx.$

1.11. $\displaystyle\int \left(\frac{\sqrt{x}}{2}-\frac{1}{2\sqrt{x}}\right)^2\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}-\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)dx.$

1.12. $\displaystyle\int \frac{\sqrt{1-x^2}-1}{x}\left(\frac{1-x}{\sqrt{1-x^2}+x-1}+\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}\right)dx.$

1.13. $\displaystyle\int \frac{\left((1-x^2)^{-1/2}+1+\frac{1}{(1-x^2)^{-1/2}-1}\right)^{-2}}{(2-x^2-2\sqrt{1-x^2})}\,dx.$

1.14. $\displaystyle\int \left(\frac{x^{-6}-64}{4+2x^{-1}+x^{-2}}\,\frac{x^2}{4-4/x+1/x^2}-\frac{4x^2(2x+1)}{1-2x}\right)dx.$

1.15. $\displaystyle\int \frac{(x^{2/m}-9x^{2/n})(x^{(1-m)/m}-3x^{(1-n)/n})}{(x^{1/m}+3x^{1/n})^2-12x^{(m+n)/(mn)}}\,dx.$

1.16. $\displaystyle\int \frac{\left(2-x+4x^2+\frac{5x^2-6x+3}{x-1}\right)}{2x+1+2x/(x-1)}\,dx.$

1.17. $\displaystyle\int \frac{\sqrt{1-x^2}+1}{\sqrt{1-x}+1/\sqrt{1+x}}\,dx.$

1.18. $\displaystyle\int \frac{(\sqrt{x}+2)\left(\frac{2}{\sqrt{x}}-1\right)-(\sqrt{x}-2)\left(\frac{2}{\sqrt{x}}+1\right)}{(2-\sqrt{x-2}):\left(\sqrt{\frac{2}{x}+1}-\frac{2}{\sqrt{x}}\right)}\,dx.$

1.19. $\int \frac{\frac{x^4+5x^3+15x-9}{x^6+3x^4}+\frac{9}{x^4}}{(x^3-4x+3x^2-12)/x^5}\,dx.$

1.20. $\int \frac{\sqrt{x+\sqrt{2-x^2}}\,\sqrt[6]{1-x\sqrt{2-x^2}}}{\sqrt[3]{1-x^2}}\,dx.$

1.21*. $\int 4\cos\frac{x}{2}\cos x\sin\frac{21}{2}x\,dx.$

1.22. $\int -4\sin\frac{x}{2}\sin x\sin\frac{11}{2}x\,dx.$

1.23. $\int 2\sqrt{2}\cos\alpha\sin\left(\frac{\pi}{4}+2\alpha\right)d\alpha.$

1.24. $\int 2\sin^2(3\pi-2x)\cos^2(5\pi+2x)\,dx.$

1.25. $\int \cot\left(\frac{3}{4}\pi-2x\right)\cos 4x\,dx.$

1.26. $\int \left[\sin^2\left(\frac{9\pi}{8}+\frac{x}{4}\right)-\sin^2\left(\frac{7\pi}{8}+\frac{x}{4}\right)\right]dx.$

1.27. $\int [\cos^2(45°-x)\cos^2(60°+x)-\cos 75°\sin(75°-2x)]\,dx.$

1.28. $\int \frac{\sin 2x+\sin 5x-\sin 3x}{\cos x+1-2\sin^2 2x}\,dx.$

1.29. $\int \left[\frac{\cot^2 2x-1}{2\cot 2x}-\cos 8x\cot 4x\right]dx.$

1.30. $\int \frac{\cos 4x+1}{\cot x-\tan x}\,dx.$

1.31. $\int \left[\sin\alpha\sin(x-\alpha)+\sin^2\left(\frac{x}{2}-\alpha\right)\right]dx.$

1.32. $\int \left[\frac{1+\sin 2\alpha}{\cos(2\alpha-2\pi)\cot\left(\alpha-\frac{5}{4}\pi\right)}+\cos^2\alpha\right]d\alpha.$

1.33. $\int \tan^2 x\,dx.$ 1.34. $\int \cot^2 x\,dx.$

2. Problems on the Properties of Antiderivatives

The function

$$G(x) = F(x) + C, \quad (1)$$

where $F(x)$ is an arbitrary antiderivative of $f(x)$ and the constant C satisfies the equation

$$F(x_0) + C = y_0, \quad (2)$$

is an antiderivative of the function $f(x)$, the graph of the antiderivative passing through the point $M(x_0, y_0)$.

Example 2.1. For the function $f(x) = \cos^2 x$ find the antiderivative whose graph passes through the point $M(\pi/2, \pi/4)$.

Solution. Let us calculate the indefinite integral of the function $f(x) = \cos^2 x$:

$$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx = \frac{1}{2}x + \frac{\sin 2x}{4} + C.$$

To choose the required antiderivative from all the antiderivatives we have found, we derive, in accordance with (2), an equation

$$\frac{1}{2}\cdot\frac{\pi}{2} + \frac{\sin \pi}{4} + C = \frac{\pi}{4},$$

whose root is $C = 0$.

Answer. $F(x) = \frac{1}{2}x + \frac{\sin 2x}{4}$.

2.1. Find the equation of a curve, passing through the point $A(1, 2)$, the slope of whose tangent line at each point is thrice as large as the square of the abscissa of that point.

2.2. Find the equation of a curve, passing through the point $A(1, 1)$ whose slope at each point is equal to double the abscissa of that point.

2.3. Find the equation of a curve passing through the point $A(0, -1)$ if all its tangent lines are parallel to the straight line $y = 5x-3$.

If the graphs of the differentiable functions $y = f_1(x)$ and $y = f_2(x)$ touch each other at the point $M(x_0, y_0)$, then the following relations are satisfied:

$$f_1(x_0) = f_2(x_0), \tag{3}$$

$$f_1'(x_0) = f_2'(x_0). \tag{4}$$

Example 2.2. Find all the antiderivatives of the function $y = x + 2$ which touch the curve $y = x^2$.

Solution. Since the function $y = x + 2$ is a derivative of any of its antiderivatives, it follows, according to (4), that the equation for finding the abscissa of the point of tangency has the form

$$2x = x + 2.$$

The root of this equation is $x = 2$. The value of the function $y = x^2$ at the point $x = 2$ is equal to 4. Consequently, among all the antiderivatives of the functions $y = x + 2$, i.e. the functions $f(x) = \frac{1}{2}x^2 + \quad + C$, we must find that whose graph passes through the point $M(2, 4)$. The constant C can be found from the condition $f(2) = 4$: $\frac{1}{2}\cdot 4 + 2\cdot 2 + C = 4 \Rightarrow C = -2$.

Answer. $F(x) = \frac{1}{2}x^2 + 2x - 2$.

2.4. Find the antiderivative of the function $f(x) = x$ whose graph touches the straight line $y = x - 1$.

2.5. Find all the antiderivatives of the function $f_1(x) = x^2$ whose graphs touch the parabola $f_2(x) = x^2 + 1$.

2.6. Find all the antiderivatives of the function $f(x) = 3/x$ whose graphs touch the curve $y = x^3$.

If a body moves with a speed changing according to the law

$$v = f(t), \tag{5}$$

then the relationship between the path covered by the body and the time t can be represented as

$$S(t) = F(t) + C, \tag{6}$$

where $F(t)$ is a certain antiderivative of the function $f(t)$, and the constant C can be found from auxiliary conditions.

Example 2.3. A body moves rectilinearly with a speed changing according to the law

$$v = 2t \text{ m/s}.$$

Find the law of motion of the body if it is known that it covered 15 m in the first two seconds.

Solution. The set of all antiderivatives of the function $v(t) = 2t$ is $S(t) = t^2 + C$. According to an auxiliary condition we have

$$S(2) = 2^2 + C = 15,$$

whence we obtain $C = 11$. Thus the law of motion of the body is

$$S(t) = t^2 + 11.$$

2.7. A particle moves rectilinearly with a speed of $v(t) = \sin t \cos t$ m/s. Find the equation of motion of the particle if for $t = \pi/3$ s the distance covered is 17/8 m.

2.8. A first traveller left point A with the speed varying according to the law $v(t) = 2t$, and at the same time a second traveller started from point B, which is 4 km away from A, in pursuit of the first traveller with a constant speed of $2p$ km/h. For what values of p will the second traveller overtake the first one? Find the value of p for which the travellers will meet only once.

3. The Definite Integral

The *definite integral on the interval* $[a, b]$ of the continuous function $f(x)$ is an increment $F(b) - F(a)$ of any antiderivative F of that function on the interval $[a, b]$ and is designated as

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{(\textit{the Newton-Leibniz formula}).} \tag{1}$$

Here a and b are the lower and the upper limit of integration, respectively; $f(x)$ is an integrand. The difference $F(b) - F(a)$ appearing on the right-hand side of formula (1) is sometimes designated as $F(x)\,|_a^b$.

To calculate the definite integral of the function f on the interval $[a, b]$, it is necessary to find any antiderivative of the function and calculate the difference of its values at the right-hand and left-hand ends of the interval $[a, b]$. The calculation of the definite integral of the function $f(x)$ on $[a, b]$ is known as the *integration* of that function.

Simple Rules of Integration.

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \tag{2}$$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx. \tag{3}$$

$$\int_a^b f(kx+p)\,dx = \frac{1}{k}\int_{ka+p}^{kb+p} f(t)\,dt, \quad k \neq 0. \tag{4}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad c \in [a, b]. \tag{5}$$

Example 3.1. Calculate the definite integral

$$\int_0^{\pi/2} \cos^4 x\,dx.$$

Solution. Let us represent the integrand as

$$\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4}(1+2\cos 2x+\cos^2 2x)$$

$$= \frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right) = \frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x.$$

The function

$$F(x) = \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x$$

is an antiderivative of the function $\cos^4 x$. Let us now use the Newton-Leibniz formula to calculate the definite integral:

$$\int_0^{\pi/2} \cos^4 x\,dx = \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x\,\Big|_0^{\pi/2} = \frac{3}{16}\pi.$$

Answer. $3\pi/16$.

Calculate the following integrals.

3.1. $\int_1^4 \frac{4x-2\sqrt{x}}{x}\,dx$.

3.2. $\int_0^{\pi/2} \cos x \sin x\,dx$.

3.3. $\int_0^{\pi} \cos x \sin 3x\,dx$.

Example 3.2. Calculate the integral

$$\int_3^{-18} \sqrt[3]{2-\frac{x}{3}}\,dx.$$

Solution. We rewrite the integral in the form

$$\int_3^{-18} \left(2-\frac{x}{3}\right)^{1/3} dx.$$

Using formula (4) for $k = -1/3$, $p = 2$, we find the lower and upper limits of integration of the right-hand side of formula (4). We have

$$-\frac{1}{3}\cdot 3+2=1$$

and

$$(-18)\left(-\frac{1}{3}\right)+2=8,$$

respectively. Thus we obtain

$$\int_3^{-18} \left(2-\frac{x}{3}\right)^{1/3} dx = -3\int_1^8 t^{1/3}\,dt = \frac{3t^{4/3}}{4}(-3)\Big|_1^8$$

$$= -36+\frac{9}{4} = -33.75.$$

Answer. -33.75.

Calculate the following integrals.

3.4. $\int_{-8}^{8} \frac{dx}{\sqrt{5+x/2}}$. **3.5*.** $\int_{-4}^{2} \frac{x\,dx}{\sqrt{2-x/2}}$.

3.6*. $\int_{1/3}^{5/3} (x-2)\sqrt{3x-1}\,dx.$ 3.7*. $\int_{1}^{2} \frac{dx}{\sqrt{x-1}+\sqrt{1+x}}.$

3.8. $\int_{2}^{9} \sqrt[3]{x-1}\,dx.$ 3.9. $\int_{0}^{7/3} \frac{x+1}{\sqrt[3]{3x+1}}\,dx.$

3.10. $\int_{0}^{\pi/4} (\sin 2t-\cos 2t)^2\,dt.$ 3.11. $\int_{0}^{1} \frac{x\,dx}{(x+1)^2}.$

3.12. $\int_{0}^{\pi/2} \sin x \cos 3x\,dx.$ 3.13. $\int_{\pi/6}^{\pi/4} (\tan x+\cot x)^{-1}\,dx.$

3.14. $\int_{0}^{\pi} \cos^2\left(\frac{3}{8}\pi-\frac{x}{4}\right)-\cos^2\left(\frac{11}{8}\pi+\frac{x}{4}\right)\,dx.$

3.15. $\int_{\pi/6}^{\pi/3} \left[1-\frac{1}{1-\sin^{-1}(2x+3\pi/2)}\right]dx.$

If the integrand contains a variable under the sign of absolute value, then the calculation of the definite integral with the given limits of integration can be reduced to the calculation of the sum of the definite integrals with integrands which no longer contain a variable under the sign of the absolute value.

Example 3.3. Calculate

$$\int_{1}^{5} (|x-3|+|1-x|)\,dx.$$

Solution. We can represent the integrand as

$$f(x)=\begin{cases} 4-2x, & x\leqslant 1,\\ 2, & 1<x<3,\\ 2x-4, & x\geqslant 3.\end{cases}$$

Using property (5) of the definite integral, we get

$$\int_{1}^{3} (|x-3|+|1-x|)\,dx+\int_{3}^{5} (|x-3|+|1-x|)\,dx$$

$$=\int_{1}^{3} 2\,dx+\int_{3}^{5} (2x-4)\,dx=2x\big|_1^3+(x^2-4x)\big|_3^5=4+8=12.$$

Answer. 12.

Calculate the following integrals.

3.16. $\int_{-1}^{1} \sqrt{x^2-2x+1}\, dx$. 3.17. $\int_{0}^{2} \sqrt{x^2-2x+1}\, dx$.

3.18. $\int_{-\pi/4}^{\pi/2} \sqrt{1-\cos^2 x}\, dx$. 3.19. $\int_{0}^{\pi} \sqrt{1-\sin 2x}\, dx$.

3.20. $\int_{3}^{5} \left(\sqrt{x+2\sqrt{2x-4}}+\sqrt{x-2\sqrt{2x-4}}\right) dx$.

3.21. $\int_{0}^{3} \left(\frac{1}{\sqrt{x^2+4x+4}}+\sqrt{x^2-4x+4}\right) dx$.

3.22. $\int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2\right]^{1/2} dx$.

3.23. $\int_{\pi/2}^{3\pi/2} \sqrt{1-\cos 2x}\, dx$. 3.24. $\int_{\pi/4}^{3\pi/2} \sqrt{1+\cos 2x}\, dx$.

4. Integrals with a Variable Upper Limit

An *integral with a variable upper limit*

$$F(x)=\int_{a}^{x} f(t)\, dt \qquad (1)$$

is the antiderivative of the function $f(x)$ ($F'(x) = f(x)$) whose value at the point a is zero.

Example 4.1. Find the greatest and the least value of the function

$$F(x)=\int_{0}^{x} (t+1)\, dt$$

on the interval [2, 3].

Solution. Let us find the critical points of the function $F(x)$. Since $F(x)$ is an antiderivative of the function $x+1$, we have $F'(x) = x+1$; the function $F'(x)$ does not vanish on the interval [2, 3] and is positive. Consequently, the function attains its greatest value on the right-hand end of the interval and the least value on the left-hand end:

$$\max_{x\in[2,\,3]} F(x)=F(3)=\int_0^3 (t+1)\,dt=\left(\frac{t^2}{2}+t\right)\Big|_0^3=7.5,$$

$$\min_{x\in[2,\,3]} F(x)=F(2)=\int_0^2 (t+1)\,dt=\left(\frac{t^2}{2}+t\right)\Big|_0^2=4.$$

Find the greatest and the least value of the following functions on the indicated intervals.

4.1. $F(x)=\int_0^x \sin t\,dt,\ x\in\left[0,\ \frac{\pi}{2}\right].$

4.2. $F(x)=\int_0^x (2t-5)\,dt,\ x\in[-1,\ 3],$

4.3. $F(x)=\int_0^x (t^2-5t+6)\,dt,\ x\in[0,\ 4].$

4.4. Find the greatest and the least value of the function

$$F(x)=\int_1^x |t|\,dt$$

on the interval $\left[-\frac{1}{2},\ \frac{1}{2}\right]$.

4.5. Write the equations of the tangent lines to the graph of the function

$$F(x)=\int_2^x (2t-5)\,dt$$

at the points where the graph cuts the abscissa axis.

4.6. Find the abscissas of the points of intersection of the graph of the functions

$$F_1(x)=\int_2^x (2t-5)\,dt,\quad F_2(x)=\int_3^x (2t-5)\,dt.$$

4.7. Find the points of intersection of the graphs of the functions

$$F_1(x)=\int_2^x (2t-5)\,dt,\quad F_2(x)=\int_0^x 2t\,dt.$$

4.8. Find the antiderivative of the function

$$F(x)=\int_3^x (2t-5)\,dt,$$

whose graph passes through the origin.

4.9. For the graph of the function

$$F(x)=\int_0^x 2|t|\,dt$$

find the tangent lines which are parallel to the bisector of the first coordinate angle.

Assume that a particle moves rectilinearly with the speed $v(t)$; A is a certain point on the trajectory of movement of the particle. If at the time moment $t = t_0$ the distance between the moving particle and the point A is equal to S_0, then at any time moment $t > t_0$ the distance between the moving particle and the point A can be calculated by the formula

$$S(t)=\int_{t_0}^{t} v(x)\,dx+S_0. \tag{2}$$

Example 4.2. The speed of the particle moving rectilinearly varies according to the law $v(t) = \sqrt{t} + 2t$ (km/h). At the moment of time $t = 1$ h the particle was 5 km away from the point A lying on its trajectory. How far from A will the particle be at the moment $t = 3$ h?

Solution. In accordance with (1) and (2), we represent the coordinate of the particle as the function of time in the form

$$S(t)=\int_1^t (\sqrt{x}+2x)\,dx+5.$$

Let us calculate the value of $S(t)$ for $t = 3$:

$$S(3)=\int_1^3 (\sqrt{t}+2t)\,dt+5=\left(\frac{2t^{3/2}}{3}+t^2\right)\Big|_1^3+5=2\sqrt{3}$$

$$+9-\frac{2}{3}-1+5=12\frac{1}{3}+2\sqrt{3}.$$

Answer. At the moment $t = 3$ h the particle will be at a distance of $12\frac{1}{3}+2\sqrt{3}$ (km) from the point A.

4.10. The speed of the body is proportional to the square of time. Find the relationship between the distance covered and time elapsed if it is known that during the first three seconds the body covered 18 cm and the movement began at the time moment $t = 0$.

4.11. The force acting on a particle varies uniformly relative to the distance covered. At the beginning of motion it was equal to 100 N, and when the particle covered 10 m, the force increased to 600 N. Find the function defining the relationship between work and time.

4.12. The body moves with uniform acceleration, and it is known that by the moment of time $t = 2$ s its speed was 4 m/s and the distance covered was 3 m. Find the law of motion of the body.

4.13. With constant acceleration the body covered the distance of 4 m from the point A during the first second, and during the first three seconds the distance between the body and the point A increased to 16 m. Find the relationship between the distance covered by the body and the time elapsed if it is known that at $t = 0$ the body was at A.

5. Problems Requiring the Use of the Properties of Antiderivatives and Integrals

5.1. Solve the inequality

$$\left[\ln \frac{1}{(3-x)^3}\right]' - \frac{\frac{6}{\pi}\int\limits_0^{\pi} \sin^2 \frac{x}{2}\, dx}{x+2} > 0.$$

5.2. Solve the inequality

$$\sqrt{5x-6-x^2} + \frac{\pi}{2}\int\limits_0^{x} dz > x \int\limits_0^{\pi} \sin^2 x\, dx.$$

5.3. Solve the inequality

$$\sqrt{x^2-x-12} - \int\limits_0^{x} dz < x \int\limits_0^{\pi/2} \cos 2x\, dx.$$

5.4. Find the numbers A and B such that the function of the form

$$f(x) = A \sin \pi x + B$$

satisfies the conditions

$$f'(1) = 2, \quad \int\limits_0^{2} f(x)\, dx = 4.$$

5.5. Find all the numbers a ($a > 0$) for each of which

$$\int\limits_0^{a} (2-4x+3x^2)\, dx \leqslant a.$$

5.6. Find all solutions of the equation

$$\int_0^{\alpha} (\cos x + \alpha^2)\, dx = \sin \alpha$$

belonging to the interval [2.3].

5.7. Two points begin moving along a straight line at the same time moment from the same point and in the same direction. The speeds of the points are $v_1(t) = 3t^2 + 2t$ m/s and $v_2(t) = 2t$ m/s respectively. In how many seconds will the distance between them equal 216 m?

5.8. Ascertain that any antiderivative of an odd continuous function, defined on the interval $[-a, a]$, is an even function.

5.9. Ascertain that an even continuous function, defined on the interval $[-a, a]$, has at least one odd antiderivative on that interval.

5.10. Verify whether the following assertion is true: for any antiderivative of the continuous function $f(x)$ to be even on the interval $[-a, a]$, it is necessary and sufficient that the function $f(x)$ be odd on that interval.

5.11. Find the values of A, B, and C for which the function of the form

$$f(x) = Ax^2 + Bx + C$$

satisfies the conditions

$$f'(1) = 8, \quad f(2) + f''(2) = 33, \quad \int_0^1 f(x)\, dx = \frac{7}{3}.$$

5.12. Find all the values of α ($\alpha \in [0, 2\pi]$) which satisfy the equation

$$\int_{\pi/2}^{\alpha} \sin x\, dx = \sin 2\alpha.$$

5.13. Find the positive values of a which satisfy the equation

$$\int_0^a (3x^2 + 4x - 5)\, dx = a^3 - 2.$$

5.14. Find all the values of α from the interval $[-\pi, 0]$ which satisfy the equation

$$\sin \alpha + \int_{\alpha}^{2\alpha} \cos 2x\, dx = 0.$$

6. Calculating Areas

The figure bounded by the graph of the continuous function $f(x)$ ($f(x) \geqslant 0$), the straight lines $x = a$ and $x = b$, and the Ox axis is a *curvilinear trapezoid*. Its area can be calculated by the formula

$$S = \int_a^b f(x)\, dx. \tag{1}$$

If the condition $f_2(x) \geqslant f_1(x)$ ($f_2(x) - f_1(x) \geqslant 0$) is fulfilled for all x from the interval $[a, b]$, then the area of the figure bounded by the graphs of the continuous functions $y = f_1(x)$ and $y = f_2(x)$ and the straight lines $x = a$ and $x = b$, can be calculated by the formula

$$S = \int_a^b [f_2(x) - f_1(x)]\, dx. \tag{2}$$

If the condition $\psi_2(y) \geqslant \psi_1(y)$ ($\psi_2(y) - \psi_1(y) \geqslant 0$) is fulfilled for all y from the interval $[c, d]$, then the area of the figure contained between the straight lines $y = c$ and $y = d$ and the graphs of the continuous functions $x = \psi_1(y)$ and $x = \psi_2(y)$ can be calculated by the formula

$$S = \int_c^d [\psi_2(y) - \psi_1(y)]\, dy. \tag{3}$$

Example 6.1. Find the area of the figure bounded by the lines

$$x = 0,\ x = \pi/2,\ f_1(x) = \sin x,\ f_2(x) = \cos x.$$

Solution. Since the sign of the difference $f_2(x) - f_1(x)$ does not remain constant on the interval $[0, \pi/2]$, we divide the interval into domains where the difference retains sign. For that purpose we set up an equation

$$f_2(x) - f_1(x) = 0,$$

whose only root, belonging to the interval $[0, \pi/2]$, is a point $x = \pi/4$. Since

$$\sin x \geqslant \cos x \quad \text{for} \quad x \in [\pi/4, \pi/2],$$
$$\sin x < \cos x \quad \text{for} \quad x \in [0, \pi/4],$$

we obtain, in accordance with (2),

$$S = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx =$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= (\sqrt{2} - 1) + (-1 + \sqrt{2}) = 2(\sqrt{2} - 1).$$

Answer. $S = 2(\sqrt{2} - 1)$.

Note that using the symmetry of a figure about the axis $x = \pi/4$, we could have calculated the figure by the formula

$$S = 2 \int_0^{\pi/4} (\cos x - \sin x)\, dx.$$

Calculate the areas of the figures bounded by the indicated lines.
6.1. $y = x^2 + x,\ y = x + 1$.
6.2. $y = -2x^2 + 3x + 6,\ \ y = x + 2$.
6.3. $y = 0,\ \ y = 20 - 2x^2 - 6x$.
6.4. $y = x^2,\ \ y = 1/x,\ \ y = 0,\ \ x = 2$.
6.5. $y = (1/2)^x,\ \ x - 2y + 2 = 0,\ \ x = 2$.
6.6. $y = 4x - x^2,\ \ y - x = 0$.
6.7. $y = 5/x,\ \ y = 6 - x$.
6.8. $y = x^3,\ y = 1/x,\ x = 2$.
6.9. $y = x^2 + 1,\ y = -x^2 + 3$.
6.10. $y = 1/\cos^2 x,\ \ y = 0,\ \ x = 0,\ \ x = \pi/4$.
6.11. $y = 2^x,\ y = 2,\ x = -1$.
6.12. $xy = 7,\ y = 0,\ x = 4,\ x = 12$.
6.13. $y = (x - 1)^2,\ y = x + 1$.
6.14 $y = -x^2 + \frac{7}{2}x + 1,\ \ y = 2^{-x},\ \ x = 2\ (x \leqslant 2)$.
6.15. $x = 1,\ x = 2,\ y = 0,\ \log_2 x + \log_2 y = 0$.
6.16. $y = 2x^2 + 1,\ y = x + 2,\ y = 1.5$.
6.17. $y = 2^x,\ y = 4^x,\ x = 1$.
6.18. $y = x^2,\ \ y = 2\sqrt{2x}$.
6.19. $2yx = 16 + x^2,\ y = 5$.
6.20. $y = -1 + 8x^2 - x^4,\ \ y = 15,\ \ x = 1\ \ (x \geqslant 1)$.
6.21. $y = 1/(1 + x^2),\ \ y = x^2/2$.
6.22. $3y = -x^2 + 8x - 7,\ \ y + 1 = 4/(x - 3)$.
6.23*. $y = \sqrt{x},\ y = \sqrt{4 - 3x},\ y = 0$.
6.24*. Find the area of the figure whose set of points satisfies the system of inequalities

$$x^2 + y^2 \leqslant r^2,\quad r > 0,$$
$$x - y \leqslant 0,\quad y \geqslant 0.$$

6.25. Calculate the area of a plane figure bounded by parts of the lines $\max(x, y) = 1$ and $x^2 + y^2 = 1$ lying in the first quadrant:

$$\max(x, y) = \begin{cases} x, & \text{if } x \geqslant y, \\ y, & \text{if } x < y. \end{cases}$$

6.26. Find the area of the figure bounded by the graphs of the functions $y = x^2$, $y = 2x - x^2$.

If the function $y = f(x)$ is strictly monotonic on the interval $[a, b]$, then it is sometimes convenient to reduce the calculation of the area bounded by the graph of the function on that interval and the Ox axis to the calculation of the area bounded by the graph ot the inverse function $x = g(y)$ on the interval $[c, d]$ and the Oy axis, where

$$c = \min\{f(a); f(b)\},$$
$$d = \max\{f(a); f(b)\}.$$

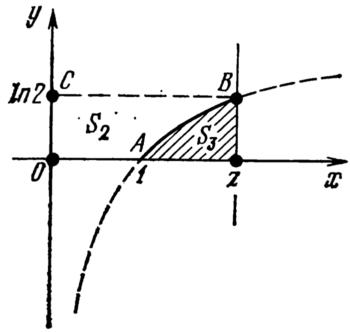

Fig. 9.1

Example 6.2. Calculate the area of the figure bounded by the graph of the function $y = \ln x$, the straight line $x = 2$, and Ox axis.

Solution. The inverse function of $y = \ln x$ is $x = e^y$. It can be seen from Fig. 9.1. that the area of the hatched figure S_3 is equal to the difference of the areas S_1 of the rectangle with sides 2 and ln 2 and S_2 of the curvilinear trapezoid $OABC$. According to (3) we have

$$S_2 = \int_0^{\ln 2} e^y\, dy = e^y \Big|_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1,$$

$$S_1 = 2 \ln 2.$$

Thus the required area is

$$S_3 = S_1 - S_2 = 2 \ln 2 - 1.$$

Find the areas of the figures bounded by the following lines.

6.27. $y = \arcsin x$, $x = 1$, $y = 0$.

6.28. $y = \arccos x$, $x = 0$, $y = 0$.

The areas of certain figures can be easily calculated using the known values of the areas of the parts of a circle of radius R.

Example 6.3. Calculate the area of the figure bounded by the lines

$$y = \sqrt{1 - x^2}, \quad y = 0.$$

Solution. Squaring both sides of the equation $y = \sqrt{1 - x^2}$, we obtain an equation of a circle of unit radius; $y^2 + x^2 = 1$. Thus the graph of the function $y = \sqrt{1 - x^2}$ is an upper semi-circle of radius 1. Consequently, the required area is equal to half the area of a circle of unit radius.

Find the areas of the figures bounded by the following lines.

6.29*. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

6.30*. $y^2 + x^2 + 2x = 0.$

6.31.** In the Cartesian system of coordinates Oxy the figure F is bounded by the Ox axis, the curve $y = 2x^2$, and the tangent to that curve; the abscissa of the point of tangency is equal to 2. Find the area of the figure F.

6.32. Calculate the area of the figure bounded by the parabola $y = x^2 - 2x + 2$, the tangent to it at the point M (3, 5), and the axis of ordinates. Make a drawing.

6.33. Calculate the area of the figure bounded by the lines $y = 1/x + 1$, $x = 1$ and the tangent drawn at the point (2, 3/2) to the curve $y = 1/x + 1$.

6.34*. Find the area of the figure bounded by the line $y = x^2 - 4x + 5$ and the straight lines touching it at the points with abscissas $x_1 = 1$ and $x_2 = 4$.

6.35. A tangent line drawn from the point (3/2, 0) to the parabola $y = 2x^2 - 6x + 9$ makes an acute angle with the positive direction of the Ox axis. Determine the area of the figure contained between the parabola, the Ox axis, the Oy axis, and the tangent line.

6.36**. What part of the area of a square is cut off by the parabola passing through two adjacent vertices of the square and touching the midpoint of one of its sides?

6.37*. What part of the area of a semi-circle is cut off by the parabola passing through the end points of the diameter of the semi-circle and touching the circumference at a point which is equidistant from the ends of the diameter?

6.38*. Find the area of the figure bounded by the straight line $y = -8x - 46$ and the parabola $y = 4x^2 + ax + 2$, if it is known that the tangent to the parabola at the point $x = -5$ makes an angle $\pi - \arctan 20$ with the Ox axis.

6.39*. For what value of a is the area of the figure bounded by the lines $y = 1/x$, $y = 1/(2x - 1)$, $x = 2$, $x = a$, equal to $\ln \frac{4}{\sqrt{5}}$?

6.40. For what value of a does the straight line $y = a$ bisect the area of the figure bounded by the lines $y = 0$, $y = 2 + x - x^2$?

6.41*. For what value of the parameter $a > 0$ is the area of the figure bounded by the curves $y = a\sqrt{x}$, $y = \sqrt{2 - x}$ and the Oy axis equal to the number b? For what values of b does the problem have a solution?

6.42*. Find the value of a for which the area of the figure bounded by the curve $y = \sin 2x$, the straight lines $x = \pi/6$, $x = a$, and the abscissa axis is equal to 1/2.

6.43*. Find all the values of the parameter b ($b > 0$) for each of which the area of the figure bounded by the curves $y = 1 - x^2$ and $y = bx^2$ is equal to a. For what values of a does the problem have a solution?

6.44*. Through the point (x_0, y_0) of the graph of the function $y = \sqrt{1 + \cos 2x}$ draw a normal to the graph, if it is known that the straight line $x = x_0$ divides the area bounded by the given curve, the Ox axis, and the straight lines $x = 0$ and $x = \frac{3}{4}\pi$ into equal parts.

7. Problems on Finding the Greatest (the Least) Areas

If it is required to find the position of curves depending on one or several parameters, for which the area of the figure bounded by those curves is maximum (minimum), then it is first necessary to form a function which expresses the dependence of the area of the parameters and then to solve the problem on finding the greatest (least) value of the function in the range of the parameters.

Example 7.1. Find all the values of the parameter a ($a \geqslant 1$) for each of which the area of the figure bounded by the straight lines $y = 1$ and $y = 2$ and the curves $y = ax^2$, $y = \frac{1}{2}ax^2$ is the greatest.

Solution. Let us calculate the value of the area for a fixed value of a. In the given case it is convenient to calculate the area assuming y to be an independent variable. Due to the symmetry of the parabolas $y = ax^2$ and $y = \frac{1}{2}ax^2$ about the Oy axis, the area of the figure lying in the half-plane $x > 0$ is equal to that of the figure lying in the half-plane $x < 0$, and, therefore, the required area is equal to double the area of the figure bounded by the lines $x = \sqrt{y/a}$, $x = \sqrt{2y/a}$, $y = 1$, $y = 2$:

$$S(a) = 2\int_1^2 \left(\sqrt{\frac{2y}{a}} - \sqrt{\frac{y}{a}}\right) dy = \frac{2}{\sqrt{a}}\int_1^2 (\sqrt{2y} - \sqrt{y})\, dy$$

$$= \frac{2}{\sqrt{a}}\left(\frac{2\sqrt{2}y^{3/2}}{3} - \frac{2}{3}y^{3/2}\right)\Big|_1^2$$

$$= \frac{2}{\sqrt{a}}\,\frac{2}{3}(\sqrt{2}-1)(2\sqrt{2}-1), \quad a \in [1, \infty)$$

The function $S(a)$ evidently decreases monotonically on the interval $[1, \infty)$ and assumes the greatest value at the left end of the interval $[1, \infty)$, i.e. for $a = 1$.

Answer. $a = 1$.

7.1. For what value of a is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight lines $y = 0$, $x = 0$, and $x = 1$ the least?

7.2. Find all the values of the parameter a ($a > 0$) for each of which the area of the figure bounded by the straight line $y = \frac{(a^2 - ax)}{1 + a^4}$ and the parabola $y = \frac{(x^2 + 2ax + 3a^2)}{1 + a^4}$ is the greatest.

7.3. For what positive a does the area S of a curvilinear trapezoid bounded by the lines

$$y = \frac{x}{6} + \frac{1}{x^2}, \quad y = 0, \quad x = a, \quad x = 2a$$

assume the least value?

7.4*. Let us designate as $S(k)$ the area contained between the parabola $y_1 = x^2 + 2x - 3$ and the straight line $y_2 = kx + 1$. Find $S(-1)$ and calculate the least value of $S(k)$.

Example 7.2. The tangent to the parabola $y = x^2$ has been drawn so that the abscissa x_0 of the point of tangency belongs to the interval [1, 2]. Find x_0 for which the triangle bounded by the tangent, the axis of ordinates, and the straight line $y = x_0^2$ has the greatest area.

Solution. The equation of the tangent at the point x_0 for the function $y = x^2$ has the form $y - x_0^2 = 2x_0 (x - x_0)$. The ordinate of the intersection point of the tangent and the Oy axis is

$$y_1 = x_0^2 - 2x_0^2 = -x_0^2,$$

and the area of the required right triangle can be calculated by the formula

$$S(x_0) = \frac{x_0 (x_0^2 + x_0^2)}{2} = x_0^3.$$

It is required to find the greatest value of $S(x_0)$, on the interval [1, 2]. The function $S(x_0)$ evidently increases on that interval and consequently,

$$\max_{x_0 \in [1, 2]} S(x_0) = S(2) = 8.$$

Answer. $x_0 = 2$.

7.5. The tangent to the graph of the function $y = \sqrt[3]{x^2}$ is such that the abscissa x_0 of the point of tangency belongs to the interval [1/2, 1]. For what value of x_0 is the area $S(x_0)$ of the triangle bounded by the tangent, the Ox axis, and the straight line $x = 2$ the least and what is it equal to?

7.6*. The curvilinear trapezoid is bounded by the curve $y = x^2 + 1$ and the straight lines $x = 1$ and $x = 2$. At what point of the given curve with abscissa $x \in [1, 2]$ should a tangent be drawn for it to cut off from the curvilinear trapezoid an ordinary trapezoid of the greatest area?

7.7*. Find the value of the parameter a for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$, and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the points of extremum of the function, is the least.

7.8*. For what values of a ($a \in [0, 1]$) does the area of the figure bounded by the graph of the function $y = f(x)$ and the straight lines $x = 0$, $x = 1$, $y = f(a)$, have the greatest value and for which values does it have the least value if $f(x) = x^\alpha + 3x^\beta$, $\alpha, \beta \in \mathbf{R}$ with $\alpha > 1$, $\beta > 1$?

7.9*. For what values of a does the area of the figure bounded by the graph of the function $\frac{x^3}{3} - x^2 + a$, the straight lines $x = 0$, $x = 2$, and the Ox axis, attain its minimum?

7.10*. For what values of a ($a \in [0, 1]$) does the area of the figure, bounded by the graph of the function $f(x)$ and the straight lines $x = 0$, $x = 1$, $y = f(a)$, is at a minimum, and for what values is it at a maximum, if $f(x) = \sqrt{1 - x^2}$?

7.11*. For what values of a does the area of the figure, bounded by the straight lines $x = x_1$, $x = x_2$, the graph of the function $y = |\sin x + \cos x - a|$, and the abscissa axis, where x_1 and x_2 are two successive extrema of the function $f(x) = \sqrt{2} \sin (x + \pi/4)$, have the least value?

8. Calculating Volumes

The volume V of a body resulting from the rotation of a curvilinear trapezoid, bounded by the lines $y = f(x)$ $(f(x) \geqslant 0)$, $x = a$, $x = b$ $(b > a)$, about the Ox axis, can be calculated by the formula

$$V = \pi \int_a^b f^2(x)\, dx. \qquad (1)$$

The volume V of a body resulting from the rotation of a curvilinear trapezoid, bounded by the graph of the function $x = \varphi(y)$ $(\varphi(y) \geqslant 0)$, the straight lines $y = c$, $y = d$ $(d > c)$, and the Oy axis, about the Oy axis, can be calculated by the formula

$$V = \pi \int_c^d \varphi^2(y)\, dy.$$

Example 8.1. Calculate the volume of the body resulting from the rotation of an arc of a sine curve (the graph of the function $y = \sin x$ on the interval $[0, \pi]$) about the Ox axis.

Solution. By formula (1) we find

$$V = \pi \int_0^\pi \sin^2 x\, dx = \pi \int_0^\pi \frac{1 - \cos 2x}{2}\, dx$$
$$= \pi \left(\frac{1}{2} x + \frac{\sin 2x}{4} \right) \Big|_0^\pi = \pi \cdot \frac{1}{2} \pi = \frac{1}{2} \pi^2.$$

Answer. $\pi^2/2$.

8.1. Calculate the volume of the body resulting from the rotation about the abscissa axis of a curvilinear trapezoid bounded by the hyperbola $xy = 2$, the straight lines $x = 1$ and $x = 2$, and the abscissa axis.

8.2*. Calculate the volume of the body resulting from the rotation of the figure bounded by the parabolas $y^2 = x$, $y = x^2$ about the abscissa axis.

8.3. A category line $y = (e^x + e^{-x})/2$ rotates about the abscissa axis. The surface that results is a *catenoid*. Calculate the volume of the body formed by the catenoid and two planes which are perpendicular to the abscissa axis and are at the distances a and b from the origin.

8.4*. Calculate the volume of the body resulting from the rotation of the figure, bounded by the parabola $y = 2x - x^2$ and the abscissa axis, about the axis of ordinates.

8.5*. Find the volume of the body resulting from the rotation of the curvilinear trapezoid, bounded by the lines $y = \arcsin x$, $y = \pi/2$ and $x = 0$, about the Oy axis.

8.6*. Find the volume of the body resulting from the rotation of the figure, bounded by the lines $y = \ln 2$, $y = \ln x$, $y = 0$ and $x = 0$, about the Oy axis.

9. Applications of the Definite Integral in the Fields of Mechanics and Physics

Calculating the path. The path S of a body, moving with the speed $v(t)$, covered during the time from the moment t_1 to the moment t_2, can be calculated by the formula

$$S = \int_{t_1}^{t_2} V(t)\, dt. \tag{1}$$

Example 9.1. A body moves rectilinearly with the speed

$$v(t) = 2t^2 - t + 1 \text{ (m/s)}.$$

Find the distance traversed in the first five seconds.

Solution. According to (1), we have

$$S(t) = \int_0^5 (2t^2 - t + 1)\, dt = \frac{2t^3}{3} - \frac{t^2}{2} + t \Big|_0^5$$

$$= \frac{250}{3} - \frac{25}{2} + 5 = 75\frac{5}{6}.$$

Answer. $75\frac{5}{6}$ m.

9.1. A body moves rectilinearly with the speed $v(t) = 2t + a$ (m/s). Find the value of a if it is known that during the time from $t_1 = 0$ to $t_2 = 2$ s the body covered the distance equal to 40 m.

9.2*. A body moves rectilinearly with the speed $v = 12t - t^2$ (m/s). Find the length of the path traversed by the body from the beginning of motion to its stop.

9.3. Two bodies began moving along a straight line at the same time moment from one point and in the same direction. One body moved with the speed $v_1(t) = 3t^2 + 2t$ (m/s) and the other with the speed $v_2(t) = 2t$ (m/s). What distance will separate the bodies in six seconds?

9.4. A particle moves rectilinearly under the action of a constant force with the acceleration of 2 m/s^2 and with the zero initial velocity. Three seconds after the beginning of motion, the action of the force ceases and the particle begins moving uniformly with the velocity attained. Find the law of motion $S(t)$ of the particle.

If a particle moves along the Ox axis under the action of the force $F(x)$, which depends on the coordinate x, then the work done by the force in transferring the particle from a to b $(b > a)$ can be calculated by the formula

$$A = \int_a^b F(x)\,dx. \tag{2}$$

Example 9.2. A force which acts on a particle depends linearly on the path traversed. At the beginning of motion it is 100 N, and after the particle has covered 10 m, the force increased to 600 N. Find the work done by the force on the path traversed.

Solution. It follows from the hypothesis that the force $F(x)$ acting on the particle varies according to the law $F(x) = ax + b$, where the parameters a and b can be found from the conditions

$$\begin{array}{ccc} F(0) = 100, & & b = 100, & & b = 100, \\ & \text{or} & & \text{or} & \\ F(10) = 600, & & 100a + 100 = 600, & & a = 50. \end{array}$$

Thus, $F(x) = 50x + 100$ and the work done by the force on the path traversed is equal, according to (2), to

$$A = \int_0^{10} (50x + 100)\,dx = 25x^2 + 100x \Big|_0^{10} = 25 \cdot 100 + 100 \cdot 10 = 3500.$$

Answer. 3500 J.

9.5*. A particle is acted upon by a force which varies in inverse proportion to the square of the distance to a certain object. It is known to equal 1 N when the distance to the object was 2 m. Calculate the work done by the force in transferring the particle from the point which is at the distance of 10 m from the object to the point which is at the distance of 3 m.

9.6*. Calculate the work done in compressing a spring by 15 cm, if it is known that the acting force is proportional to the compression of the spring and that a force of 30 N is needed to compress the spring by 1 cm.

Chapter 10

Problems on Deriving Equations

1. Motion Problems

A system of equations which must be derived proceeding from the hypotheses of motion problems usually contains the following quantities: a distance designated as S, velocities of moving bodies designated as $u, v, w, \ldots$, or as letters with indices $v_1, v_2, \ldots$; time designated as t, T. In cases when the motion is uniformly accelerated (or uniformly retarded), the acceleration is designated as a.

Uniform motion along a straight line. Accepted assumptions:

1. The motion on separate sections is assumed to be uniform; in this case the distance traversed is defined by the formula $S = vt$.

2. Rotations of moving bodies are assumed to be instantaneous, i.e. they do not take time; the velocity also changes instantaneously.

3. If a body moves down a river then its speed w (relative to the bank) consists of the speed of the body in dead water u (the inherent speed of the body) and the speed of the river flow a: $w = a + u$, and if the body flows up the river, then its speed (relative to the bank) is $w = u - a$. If the problem deals with the flow of a raft, then it is assumed that the raft moves with the speed of the river flow.

The hypothesis of problems on uniform motion sometimes includes a condition that either two bodies move towards each other or one body overtakes the other. In this case, if the initial distance between the bodies is S and the speeds of the bodies are v_1 and v_2, then

(1) if the bodies move towards each other, then the time at which they meet is $\dfrac{S}{v_1 + v_2}$;

(2) if the bodies move in the same direction ($v_1 > v_2$), then the time at which the first body overtakes the other is $\dfrac{S}{v_1 - v_2}$.

Example 1.1. A cyclist leaves town A for town B, and three hours later a motor-cyclist starts from town B in the direction towards the cyclist, the speed of the motor-cyclist being three times that of the cyclist. The cyclist and motor-cyclist meet half-way between A and B. If the motor-cyclist had left B two and not three hours after the cyclist left A, then they would have met 15 km closer to A. Find the distance between A and B.

Solution. We designate the required distance between A and B as S (km), the speeds of the cyclist and the motor-cyclist as v_c (km/h) and v_m (km/h) respectively. Then we tabulate the conditions of the problem and equations corresponding to those conditions:

Conditions of the problem	Equation
The speed of the motor-cyclist is three times that of the cyclist	$v_m = 3v_c$
The cyclist and motor-cyclist meet half-way between A and B, the motor-cyclist having left B 3 hours later than the cyclist left A	$\dfrac{\frac{S}{2}}{v_c} = \dfrac{\frac{S}{2}}{v_m} + 3$
If the motor-cyclist had left B 2 hours later than the cyclist left A, then they would have met 15 km closer to A	$\dfrac{\frac{S}{2} - 15}{v_c} = \dfrac{\frac{S}{2} + 15}{v_m} + 2$

Using the first equation, we can write the second and the third equation in the forms

$$\frac{S}{2v_c} = \frac{S}{6v_c} + 3,$$

$$\frac{S-30}{2v_c} = \frac{S+30}{6v_c} + 2.$$

From the first equation of the system we get $v_c = S/9$. Substituting $v_c = S/9$ into the second equation of the system, we get an equation enabling us to find S:

$$\frac{3S-180}{S} = 2 \Rightarrow S = 180.$$

Answer. The distance between A and B is 180 km.

Example 1.2. A raft starts from a pier and travels down the river. Five hours and twenty minutes later a motor-boat starts from the same pier and travels in the same direction. Having covered 20 km, it overtakes the raft. What is the speed of the raft if the inherent speed of the motor-boat is known to be 9 km/h higher than that of the raft?

Solution. Let us designate the inherent speed of the boat (i.e. its speed in dead water) as v_b (km/h) and the speed of the river flow as v_r (km/h). By the hypothesis, the inherent speed of the boat is 9 km/h higher than that of the raft:

$$v_b - v_r = 9.$$

Travelling down the river, the motor-boat covered 20 km in the time $20/(v_b + v_r)$; the raft covered the same 20 km in the time $20/v_r$. Since the raft covered 20 km during the time exceeding by 5 h 20 min = 16/3 h the time needed by the boat to cover the same distance, we have

$$\frac{20}{v_r} - \frac{20}{v_b + v_r} = \frac{16}{3}.$$

Thus the solution of the problem reduces to the solution of the system

$$v_b - v_r = 9,$$

$$\frac{20}{v_r} - \frac{20}{v_b + v_r} = \frac{16}{3}.$$

From the first equation we get $v_b = v_r + 9$. Substituting $v_b = v_r + 9$ into the second equation, we obtain an equation for v_r:

$$\frac{20}{v_r} - \frac{20}{2v_r + 9} = \frac{16}{3} \Rightarrow 8v_r^2 + 21v_r - 135 = 0.$$

Solving the last equation, we find that $v_r = 3$. (The second root of the equation $v_r = -45/8$ does not make sense in the context of this problem.)

Answer. The speed of the river flow (as well as the speed of the raft) is equal to 3 km/h.

1.1. A ship sails 4 km up the river and then another 33 km down the river having spent an hour on the whole trip. Find the speed of the ship in dead water if the speed of the river is 6.5 km/h.

1.2. A cutter sets out down the river at the same time as a raft in the same direction and travels down the river for $13\frac{1}{3}$ km, and then, without stopping, it travels $9\frac{1}{3}$ km in the reverse direction and meets the raft. How many times is the inherent speed of the cutter higher than the speed of the river flow?

1.3. Two cars leave the same point simultaneously and start in the same direction. The speed of the first car is 40 km/h, the speed of the other car is 125% that of the first. Thirty minutes later, a third car starts from the same point in the same direction. It overtakes the first car and 1.5 h later it overtakes the second car. What is the speed of the third car?

1.4. Three runners participate in a race for 120 m. The speed of the first runner exceeds that of the second by 1 m/s, and the speed of the second runner is equal to half the sum of the speeds of the first and the third runner. Find the speed of the third runner if the first runner is known to reach the finish 3 s sooner than the third.

1.5. An artificial reservoir is a rectangle with a 1 km difference between the sides. Two fishermen simultaneously leave one vertex of the rectangle for a point located at the opposite vertex. One fisherman crosses the river in a boat, the other walks along the bank. Find the size of the reservoir if each of them has the speed of 4 km/h and one of them arrives 30 min earlier than the other.

1.6. Two cyclists start simultaneously from two points 270 km apart and travel towards each other. The second cyclist covers 1.5 km less per hour than the first cyclist and meets him in as many hours as the first cyclist covers in an hour. Find the speed of each cyclist.

1.7. A tourist sailed 90 km down the river in a boat and then walked 10 km. He walked 4 h less than he sailed. If the tourist walked as long as he sailed and sailed as long as he walked, the distances would be equal. How much time did he walk and how much ti e did he sail?

1.8. The distance between two towns is S km. Two cars set out from these towns moving towards each other and will meet half-way if the first car starts t h earlier than the other. Now if they start simultaneously, they will meet in $2t$ h. Find the speed of each car assuming that the speeds are constant over the whole distance.

1.9. A messenger sets out on a moped from town A to town B located 120 km from A. An hour later, a second messenger sets out on a motor-cycle from A. He overtakes the first messenger, passes a commission to him and immediately starts back at the same speed. He arrives at A at the moment the first messenger arrives at B. What is the speed of the first messenger if the speed of the second is 50 km/h?

1.10. A ship starting from port A and going to port C must pass a lighthouse B on the way, the distance between A and B being 140 km and from B to C 100 km. Three hours later a high-speed cutter left port A in the direction of the ship and, having overtaken the ship, gave a command to increase the speed by 5 km/h. The command was immediately obeyed and, as a result, the ship passed the lighthouse B half an hour earlier and arrived at port C an hour and a half earlier. Find the initial speed of the ship and the speed of the cutter.

1.11. A first tourist, having cycled for 1.5 h at 16 km/h, makes a stop for 1.5 h and then continues travelling at the same speed. A second tourist starts after him 4 h later, driving a motocycle at a speed of 56 km/h. What distance will they cover before the second tourist overtakes the first one?

1.12. Two boats start simultaneously from pier A down the river to pier B. The first boat arrives at B two hours earlier than the second. If the boats started from the piers simultaneously travelling towards each other (the first from A and the second from B), they would meet 3 h later. The distance between the piers is 24 km. The speed of the second boat in dead water is thrice the speed of the river flow. Find the speed of the river flow.

1.13. First a motor boat went S km down the river and then twice that distance across the lake into which the river flows. The trip lasted an hour. Find the inherent speed of the motor boat if the speed of the river flow is v km/h.

1.14. At 9 a.m. self-propelled barge starts from point A up the river and some time later arrives at point B; two hours after it arrived at B, the barge starts back and arrives at A at 7.20 p.m. on the same day. Assuming that the speed of the river flow is 3 km/h and the inherent speed of the barge is constant, find the time the barge arrived at B if the distance AB is 60 km.

1.15. A car left town A for town B and two hours later stopped for 45 min. Then it continued travelling to town B, having increased the initial speed by 20 km/h, and later arrived at B. If the car had travelled at its initial speed without stopping, it would have spent the same time to travel from A to B. Find the initial speed of the car if the distance between A and B is 300 km.

1.16. A motor-cyclist left point A for point B, the distance between the points being 120 km. He started back at the same speed, but an hour later he had to stop for 10 min. Then he continued travelling to A, having increased the speed by 6 km/h. What was the initial speed of the motor-cyclist if it is known that it took him as much time to get back as to travel from A to B?

1.17*. A motor-cyclist covers 1 km 4 min faster than a cyclist. How many kilometres does each of them cover in 5 hours if it is known that during that time the motor-cyclist covers 100 km more than the cyclist?

1.18. According to the schedule, the train always covers a certain 120 km span at the same speed. Yesterday the train covered half the span with that speed and had to stop for 5 min. To reach the point of destination on time, the engine-driver had to increase the speed by 10 km/h on the second half of the span. Today, the train stopped at the middle of the same span, but the delay lasted for 9 min. What was the speed of the train on the second half of the span today if again the train reached the point of destination on time?

1.19. Two motor-cyclists began the race simultaneously from the same start and in the same direction, one at 80 km/h and the other at 60 km/h. A third motor-cyclist started half an hour later from the same point and in the same direction. Find the speed of the third motor-cyclist if it is known that he overtook the first motor-cyclist 1 h 15 min later than the second.

1.20. Two cyclists left point A at the same time and went in the same direction. The first cyclist travelled at a speed of 7 km/h and the second at a speed of 10 km/h. Thirty minutes later a third cyclist left point A and went in the same direction. He overtook the first cyclist, and 1.5 h later he overtook the second cyclist. Find the speed of the third cyclist.

1.21. A ship and a raft start simultaneously from pier A and travel down the river. Having reached pier B, located 324 km from A, the ship stopped there for 18 h and then started back to A. At the moment when it was 180 km from A, a second ship, which left A 40 h later than the first, overtook the raft which by that time had covered 144 km. Assuming the speed of the river flow to be constant, the speed of the raft to be equal to that of the river flow, and the speeds of the ships in dead water to be constant and equal, find the speeds of the ships and of the river flow.

1.22. Point A is located up the river relative to point B. A raft and a motor-boat left point A at the same time and went down the river, and a second motor-boat started from point B and travelled up the river. Some time later the boats met at point C and during that time the raft covered a third of the distance from A to C. If the first boat reached point B without stopping on the way, then during that time the raft would reach point C. If the second boat started from point A to point B and the first boat started from B to A, they would meet 40 km from A. What are the speeds of the boats in dead water and what is the distance between points A and B if the speed of the river flow is 3 km/h.

1.23. A car and a motor-cycle left town A for town B at the same time, and when the motor-cycle covered a sixth of the distance, a cyclist started from A in the same direction. By the moment the car arrived at town B, the cyclist covered a fourth of the distance. The speed of the motor-cyclist is 21 km/h lower than that of the car and as much higher than that of the cyclist. Find the speed of the car.

1.24. A cyclist and a pedestrian left simultaneously point A for point B which is 100 km from A. At the same time a car-driver started towards them from point B. An hour after the start, the car met the bicycle and then, having covered another 240/17 km, it met the pedestrian. The car-driver reversed the car, took the pedestrian with him and they drove on together after the cyclist and overtook him. Calculate the speeds of the cycle and the car if the speed of the pedestrian is known to be 5 km/h.

1.25. At noon sharp a pedestrian and a cyclist left point A for point B and a horseman left point B for point A. Two hours later the cyclist and the horseman met 3 km from the half-way point between A and B, and another 48 min later the pedestrian and the horseman met. Find the speed of each of them and the distance AB if the speed of the pedestrian is known to be half that of the cyclist.

1.26. Two cyclists started simultaneously towards each other from points A and B and met 12 km from point B. After the meeting, they arrived at points B and A respectively, started back immediately and met again 6 km from point A. Find the speeds of the cyclists and the distance AB if the second cyclist is known to arrive at point B an hour after the first cyclist arrived at A.

1.27. A passenger train covers the distance between towns A and B 4 hours quicker than a goods train. If it took each train the same time it had taken the other train to cover the distance from A to B, then the passenger train would cover 280 km more than the goods train. Now if each train increased its speed by 10 km/h, then the passenger train would cover the distance from A to B 2 h and 24 min quicker than the goods train. Find the distance between towns A and B.

1.28. At a ski race of 10 000 m the second skier took the start some time later than the first skier, the speed of the second skier being 1 m/s higher than that of the first. At the moment when the second skier overtook the first, the first skier increased his speed by 2 m/s and the speed of the second skier remained the same. As a result, the second skier finished 7 min and 8 s later than the first. If the race course were 500 m longer, the second skier would finish 7 min and 33 s later than the first. Find the time which passed between the starts of the first and the second skiers.

1.29. Two cyclists left point A for point B at the same time. The first of them stopped 42 minutes later, when he was within 1 km of B, and the second cyclist stopped in 52 minutes, when 2 km remained to B. If the first cyclist covered as many kilometres as the second and the second as many as the first, then the first of them would need 17 min less than the second. How many kilometres are there between points A and B?

1.30. The distance between a station and a settlement is 4 km. A boy and a car left the station for the settlement at the same time. Ten min later the boy met the car on its way back from the settlement; having walked another 1/14 km, he met the car once again, which arrived at the station and again set out for the settlement. Find the speeds of the boy and the car if it is known that they travelled uniformly and without stops.

1.31. The distance between a railway station and a beach is 4.5 km. A boy and a bus started simultaneously from the station to the beach. Fifteen minutes later the boy met the bus returning from the beach and walked another 9/28 km from the place of their first meeting before the bus overtook him again, having reached the station and set out again for the beach. Find the speeds of the boy and the bus assuming that the speeds are constant and neither the boy nor the bus stopped on the way but the bus stopped for 4 min at the station and at the beach.

1.32. A cyclist covers half the distance between points A and B two hours quicker than a pedestrian covers a third of that distance. During the time needed by the cyclist to cover the whole way from A to B, the pedestrian walks 24 km. If the cyclist increased his speed by 7 km/h, then during the time the pedestrian walks 18 km, the cyclist would cover the whole distance from A to B and 3 km more. Find the speed of the pedestrian.

1.33. A tug-boat has a minimum time to tow two pontoons down the river a distance of 1 km. It was decided that one pontoon would be sent down the river by itself and the boat would tow the other pontoon for some time, then would leave it and return to the first one and tow it to the point of destination. For how many kilometres must the second pontoon be transported by the boat for both pontoons to arrive at the point of destination at the same time, and how much time would the whole operation take if the inherent speed of the tug-boat is v km/h and the speed of the river flow is u km/h?

1.34*. A passenger knows that at the given part of the way the speed of the train is 40 km/h. As soon as a train going in the opposite direction began passing by the window, the passenger started his stop-watch and noted that the train passed the window during 3 s. Find the speed of the train going in the opposite direction if its length is known to be 75 m.

1.35. Two trains travel uniformly in opposite directions along straight parallel tracks which are 60 m apart. Each train is 100 m long. A switchman is at a distance of 40 m from the track which is closer to him. The first train obstructs from his view a part of the second train for 5 s. The speed of the first train is 16 m/s. Find the speed of the second train. (Neglect the width of the trains.)

1.36. Two identical ships start simultaneously from two piers: the first ship starts from pier A down the river and the other from pier B up the river. By the time they meet, the first ship covers thrice the distance covered by the second ship. Each ship arrives at the point of destination, stops there for some time, and then starts back. If, the first ship stops at B for 40 min more than the second ship stops at A then on their return trip they will meet 12 km from A. Now if the first ship stops at B for 40 min less than the second ship stops at A, then on their return trip they will meet 26 km from B. Find the distance between A and B and the speeds of the ships in dead water.

1.37. The piers A and B are on opposite banks of the lake. A ship sails from A to B, stops at B for 10 min, and returns to A, its speed in both directions being constant and equal to 18 km/h. At the moment the ship starts from A, a boat, sailing with a constant speed, starts from B to A towards the ship. The boat and the ship meet at 11 h 10 min. At 11 h 25 min the boat is 3 km from A. Starting from B to A after the stop, the ship overtakes the boat at 11 h 40 min. Find the time of arrival of the boat at A.

1.38. A column of motor-cycles travels with the speed of 15 km/h, the distance between every two motor-cycles being 50 m. A cyclist pedals in the opposite direction along the column (reckoning from the first motor-cycle). Having come up to the 45th motor-cycle, he increases his speed by 10 km/h and comes up to the last motor-cycle. Then he goes back and overtakes the first motor-cycle with the same (increased) speed. If the cyclist travelled with that (increased) speed from the very beginning, he would have returned to the head of the column 15/8 min earlier. Find the initial speed of the cyclist (neglect the lengths of the bicycle and the motor-cycle and the time it takes the cyclist to turn back).

When solving textual problems, it is first necessary to choose the unknowns which would enter a system of equations. The following principle may be at the basis of such a choice: the unknowns must be introduced so that the equations derived would make it easier to write the conditions of the problem. It is not obligatory for the required quantity to be among the chosen unknowns. As a rule, with such a choice of the unknowns the required quantity is a certain combination of the unknowns introduced, to find which it is not necessary to find all the unknowns entering into it separately.

In motion problems, it is usually convenient to choose as unknowns the distance (provided that it is not given) and the speeds of the moving objects appearing in the hypothesis.

Example 1.3. The towns A and B are on the river bank, the town B being located down the river. A raft starts from town A to town B at 9 a.m. At the same time, a boat starts from town B to town A and meets the raft 5 h later. Having reached town A, the boat goes back and arrives at town B at the same time as the raft. Will the boat and the raft have enough time to arrive at town B at 9 p.m. of the same day?

Solution. We isolate from the hypothesis the statements whose mathematical notation forms a system of equations of problem. There are two of them:

(1) the boat and the raft start simultaneously and meet 5 h later;

(2) the boat returns to town B at the same time as the raft.

We choose the following quantities as the unknowns: the distance between the towns, in km, which we designate as S; the velocity of the river flow and the speed of the boat in dead water, which we designate as v_r (km/h) and v_b (km/h) respectively. These unknowns enable us to write in a simple form the conditions of the problem as a system of equations.

The condition of the problem	The equation
The boat and the raft start simultaneously and meet 5 h later	$\dfrac{S}{v_r+(v_b-v_r)}=5$
The boat returns to B at the same time as the raft	$\dfrac{S}{v_b+v_r}+\dfrac{S}{v_b-v_r}=\dfrac{S}{v_r}$

The ratio S/v_r in the last equation is the time of travel of the raft, $S/(v_b - v_r)$ is the time of travel of the boat up the river, $S/(v_b + v_r)$

is the time of travel of the boat down the river. The conditions of the problem form a system of two equations in three unknowns:

$$\frac{S}{v_b}=5,$$

$$\frac{S}{v_b+v_r}+\frac{S}{v_b-v_r}=\frac{S}{v_r}.$$

We cannot find all three unknowns from the given system. It is, however, only required to find whether the boat and the raft have enough time to arrive at town B at 9 p.m., i.e. we have to find the time of travel of the boat or the raft. Since the time of travel of the raft is S/v_r, we have to find, from the system of equations, not the unknowns S, v_b, and v_r themselves, but only the ratio S/v_r. We divide both sides of the second equation by S/v_b, and the system assumes the form

$$\frac{S}{v_b}=5,$$

$$\frac{1}{v_r/v_b}=\frac{1}{1-v_r/v_b}+\frac{1}{1+v_r/v_b},$$

i.e. the system contains only two unknown quantities, S/v_b and v_r/v_b. From the second equation of the system we find that $v_r/v_b=-1\pm\sqrt{2}$. By the sense of the problem, only one value of the ratio v_r/v_b is suitable, namely, $v_r/v_b=\sqrt{2}-1$. It is now easy to find the ratio S/v_r:

$$\frac{S}{v_r}=\frac{S}{v_b}:\frac{v_r}{v_b}=\frac{5}{\sqrt{2}-1}=5(\sqrt{2}+1).$$

It is easy to prove that the value of the ratio S/v_r we have found exceeds 12 and, consequently, the boat and the raft will not manage to arrive at B at 9 p.m.

Answer. They will not have enough time.

1.39. A passenger train set out from town A for town B and at the same time a goods train started from B to A. The speed of each train is constant throughout the way. Two hours after they met, the distance between them was 280 km. The passenger train arrived at the point of destination 9 h and the goods train 16 h after their meeting. How much time did it take each train to cover the whole distance?

1.40. Two trains start towards each other at constant speeds, one from Moscow and the other from Leningrad. They can meet halfway if the train from Moscow starts 1.5 h earlier. If both trains started simultaneously, then 6 h later the distance between them would equal a tenth of the initial distance. How much time does it take each train to cover the distance between Moscow and Leningrad?

1.41. Two cyclists start simultaneously from points A and B towards each other and, travelling at constant speeds, meet $2\frac{2}{5}$ h later. If the first cyclist increased his speed by 50 per cent and the second by 20 per cent, then it would take the first cyclist 2/3 h more to cover the distance between A and B than the second. What time does it take each cyclist to cover the distance between A and B if they both travelled at their initial speeds?

1.42. Two cars started simultaneously from point A and drove along the highway and an hour later a third car started after them. An hour later the distance between the third and the first car decreased by a factor of 1.5 and that between the third and the second car decreased two times. The speed of which car, the first or the second, is greater and by how many times? (The third car is known not to overtake the first two.)

1.43. A local passenger train left point A for point B. Three hours later an express train started from A after them and overtook the local train half-way between points A and B. By the time the express train arrived at B, the local train covered 13/16 of the distance between A and B. How much time did it take the local train to cover the distance from A to B if the speeds of both trains are constant?

1.44. A cyclist left point A for point B. At the time he covered 1/4 of the distance from A to B a motor-cyclist left B for A and having reached point A, started back without delay and arrived at B at the same time as the cyclist. The time it took the motor-cyclist till his first meeting with the cyclist is equal to the time it took him to cover the distance from A to B. Assuming the speeds of the motor-cyclist to be different on his way from A to B and from B to A, find by how many times the speed of the motor-cyclist on his way from A to B is greater than the speed of the cyclist.

1.45. A bus starts from point A to point B. Having reached B, it travels in the same direction. At the moment the bus reaches point B a car starts from point A and runs in the same direction as the bus. It takes the car 3 h 20 min less to cover the distance from A to B than the bus. Find those times if their sum is half as large as the time it takes the car to overtake the bus.

1.46. Two cyclists and a pedestrian started from point A to point B at the same time. More than an hour after his start, the first cyclist had an accident with his bicycle and continued on foot, travelling 4.5 times slower than he travelled on the bicycle. The second cyclist overtook him 5/8 h after the accident and the pedestrian 10.8 h after the accident. By the time of the accident the second cyclist covered twice the distance the pedestrian covered by the moment 5/36 h later than the moment of the accident. How many hours after the start did the accident occur?

1.47. Two pedestrians started simultaneously, the first from A to B and the second from B to A. When the distance between them became one-sixth of the initial distance, a cyclist started from B to A. The first pedestrian met him the moment the second pedestrian covered 4/9 of the distance from B to A. The cyclist arrived at A at the same time as the first pedestrian arrived at B. Find the ratio of the speeds of the pedestrians to that of the cyclist.

1.48*. A motor-cyclist covered the distance of 200 km between towns A and B in 6 h. At first he travelled at the speed v_1, exceeding 15 km/h, and then at the speed v_2, the lengths of time of movement at each speed being proportional to the speeds. Four hours after he started, the motor-cyclist was 120 km from town A. Find the speeds v_1 and v_2.

1.49. A tractor moves from point A to point B. The radius of the front wheel of the tractor is smaller than that of the rear wheel. On the way from A to B the front wheel made 200 rotations more than the rear wheel. If the circumference of the front wheel was 5/4 times longer, then on the way from A to B it would make 80 rotations more than the rear wheel. Find the lengths of the circumferences of the front and the rear wheel of the tractor if the circumference of the rear wheel exceeds by 1 m that of the front wheel.

1.50. Three sports cars of different makes started simultaneously in a motor-rally. The crew of the first car lost three hours on repairs during the rally and, as a result, they finished an hour later than the second crew. Find the speeds of the cars if it is known that the ratio of the speed of the second car to that of the third car is 5 : 4, the speed of the third car is 30 km/h lower than that of the first car and that 3 hours elapsed between the finishes of the second and the third car.

1.51. A ship sails between two towns, its speed in good weather differs from that in bad weather. On Monday, during its voyage, good weather lasted an hour longer than bad weather. On Tuesday the ship ran in good weather as long as it ran in bad weather the previous day, and in bad weather it ran 1 h 4 min longer. On Wednesday, the ship ran in good weather 2 h 30 min longer than on the previous day, and it was 9 km from the point of destination when the weather became bad. On Thursday good weather lasted 0.5 h longer than on Tuesday, and then an accident occurred and the ship had to decrease its speed by 5 km/h. Find the distance which the ship covered a day and the speeds of the ship in good and in bad weather if it is known that after the accident the ship ran half an hour longer than before the accident.

1.52. Three pedestrians started simultaneously, each of them having his own route, t hours later it remained for the second pedestrian to walk half as much again as the first pedestrian had covered, and it remained for the first pedestrian to walk thrice as far as the third pedestrian had covered, $2t$ hours after he started, the first pedestrian had to walk half the distance the second had covered, and the third pedestrian had covered the distance that remained for the first and second pedestrians taken together. How long were the first and the second pedestrian en route?

Some problems contain conditions whose mathematical notation is an inequality.

Example 1.4. A fast (long-distance) train starts from point A to point B at 8 a.m. At the same time a local train and an express start from B to A, the speed of the local train being half that of the express. The fast train arrives at B at 1.50 p.m. and meets the express not earlier than 10.30 a.m. on the same day. Find the time of arrival of the local train at point A if it is known that the last train met the local train not less than on hour later it met the express.

Solution. We introduce the following unknowns to write the conditions of the problem: we desi nate the distance between the towns as S (km), the speed of the fast train as v_f (km/h), the speed of the local train as v_l (km/h). Then the speed of the express is $2v_l$ (km/h). We write the conditions of the problem as a system of equations and inequalities,

Condition of the problem	Equation, inequality
The fast train arrived at B at 1.50 p.m., i.e. in 5 h 50 min.	$\frac{S}{v_f}=\frac{35}{6}$
The fast train meets the express not earlier than 10.30 a.m., i.e. not earlier than in 2 h 30 min.	$\frac{S}{v_f+2v_f}\geqslant\frac{5}{2}$
The fast train met the local train not less than an hour later it had met the express	$\frac{S}{v_f+v_l}-\frac{S}{v_f+2v_l}\geqslant 1$

The last inequality means the following: the ratio $S/(v_f+2v_l)$ is the time which elapsed from the start to the meeting of the fast train and the express and the ratio $S/(v_f+v_l)$ is the time which elapsed to the meeting of the fast and the local train. The difference between these ratios is the time which passed from the meeting of the fast train and the express to the meeting of the fast train and the local train. By the hypothesis this difference is greater than or equal to 1.

The inequalities which express the conditions of the problem can also be written as follows:

$$\frac{S/v_c}{1+2v_l/v_f}\geqslant\frac{5}{2},$$

$$\frac{S/v_c}{1+v_l/v_f}-\frac{S/v_c}{1+2v_l/v_f}\geqslant 1.$$

Substituting the value of the ratio S/v_f into these inequalities and performing obvious transformations, we obtain the following system of inequalities:

$$\begin{array}{lcl} v_l/v_f \leqslant 2/3, & & v_l/v_f\leqslant 2/3, \\ & \text{or} & \\ 12\,(v_l/v_f)^2-17\,(v_l/v_f)+6\leqslant 0, & & 2/3\leqslant v_l/v_f\leqslant 3/4. \end{array}$$

The last system has a unique solution: $v_l/v_f=2/3$.

It is required to find the time of arrival of the local train at point A, i.e. the quantity S/v_l

$$\frac{S}{v_l}=\frac{S}{v_f}\frac{v_f}{v_l}=\frac{35}{6}\cdot\frac{3}{2}=\frac{35}{4}=8\frac{3}{4}.$$

Thus it takes the local train 8 h 45 min to accomplish the trip and it arrives at point A at 4.45 p.m.

1.53. A boat sails 10 km down the river and then 6 km up the river. The speed of the river flow is 1 km/h. What must be the range of the inherent speed of the boat for the whole trip to take from 3 to 4 hours?

1.54. A boat sails up the river, the speed of the river flow being v km/h. Having sailed l km, it "gets into a lake with dead water". What should the inherent speed of the boat be for the total distance of S km to be covered in not more than t h?

1.55. Two cyclists start simultaneously towards each other from points A and B which are 120 km apart and meet more than 5 hours later. The next day they start simultaneously in the same direction from points C and D which are 36 km apart. The cyclist, who is leading, pedals at a speed which is 6 km/h higher than it was on the previous day and the cyclist who is behind him travels at the same speed as the previous day. Will 2 h be enough for the second cyclist to overtake the first one?

1.56. A motor-boat, whose speed in dead water is 6 km/h, starts from pier A to pier B which is 12 km down the river. At the same time a motor-launch whose speed in dead water is 10 km/h starts from B to A. After their meeting, they turn back and return to their piers. Find all possible values of the speed of the river flow v for which the boat arrives at A not sooner than an hour after the motor-launch returns to B.

1.57. A raft starts from pier A down the river which flows at speed v km/h. An hour later a motor-launch whose speed in dead water is 10 km/h follows the raft. Having overtaken the raft, the motor-launch comes back. Find all the values of v for which the raft covers more than 15 km by the moment the motor-launch returns to A.

1.58. A village is located on a river bank and a school on a highway which crosses the river at right angles. In winter the schoolboy skies from the village to the school taking a straight road across the river and it takes him 40 min to get to the school. In spring, when the roads are bad, he walks along the river bank to the highway and then along the highway to the school and it takes him 1 h 10 min to accomplish the trip. In autumn he walks along the river bank half the distance between the village and the highway and then crosses the river, and in that case it takes him less than 57 min to get to the school. Find what is farther away: the village from the highway or the school from the river if it is known that the boy always walks at the same speed and skies at the speed 25% higher (assume that the river and the highway are straight lines).

Movement in a circle. If two bodies move along a circle of radius R with constant speeds v_1 and v_2 in opposite directions, then the time between their meetings can be calculated by the formula $2\pi R/(v_1 + v_2)$.

If two bodies move along a circle of radius R with constant speeds v_1 and v_2 ($v_1 > v_2$) in the same direction, then the time between their meetings can be calculated by the formula $2\pi R/(v_1 - v_2)$.

Example 1.5. Two bodies move in different directions along a circle 1 m in circumference with constant velocities and meet every 6 s. When they move in the same direction, the first body overtakes the second every 48 s. Find the linear velocities of the bodies.

Solution. We designate the velocities of the bodies as v_1 m/s and v_2 m/s respectively. Then, according to the hypothesis, we obtain the following systems of equations:

$$\begin{aligned} 1/(v_1 + v_2) &= 6, \\ 1/(v_1 - v_2) &= 48, \end{aligned} \Rightarrow \begin{aligned} v_1 + v_2 &= 1/6, \\ v_1 - v_2 &= 1/48. \end{aligned}$$

Solving the last system, we get $v_1 = 3/32$ and $v_2 = 7/96$.

Answer. The velocity of the first body is 3/32 m/s and that of the second body is 7/96 m/s.

Example 1.6. Three racers start simultaneously from the same point on the highway, which has the shape of a circle, and race in the same direction at constant speeds. The first racer overtook the second for the first time when he made his fifth circle and at the point which is diametrically opposite to the start, and half an hour later he overtook the third racer for the second time since he started. The second racer overtook the third one for the first time 3 h after the start. How many circles does the first racer make in an hour if the second racer makes a circle in not less than 20 min?

Solution. We designate the length of the circular track as S (km) and the speeds of the racers as v_1 (km/h), v_2 (km/h), and v_3 (km/h) respectively. By the hypothesis, $v_1 > v_2 > v_3$.

The first racer overtakes the second with the speed $v_1 - v_2$ and the third one with the speed $v_1 - v_3$, and the second racer overtakes the third with the speed $v_2 - v_3$. At the moment one racer overtakes another for the first time, he covers a distance which exceeds by S the distance covered by the other racer, when he overtakes him for the second time, he covers the distance $2S$ greater than that covered by the other racer, and so on.

We can write the hypothesis as the following equations.

The hypothesis	The equation
The first racer overtakes the second for the first time while making his fifth round, at a point which is diametrically opposite to the start (i.e. having made 4.5 of a circle).	$\frac{S}{v_1 - v_2} = \frac{\frac{9}{2}S}{v_1}$
Half an hour after his meeting with the second racer, the first racer overtakes the third racer for the second time (i.e. the first racer makes two rounds more than the third one).	$\frac{\frac{9}{2}S}{v_1} + \frac{1}{2} = \frac{2S}{v_1 - v_3}$
The second racer overtakes the third one for the first time 3 hours after the start.	$\frac{S}{v_2 - v_3} = 3$
The second racer makes a round in not less than 20 minutes.	$\frac{S}{v_2} \geqslant \frac{1}{3}$

To obtain a single-valued solution, it is necessary to take into consideration the inequality appearing among the conditions of the problem. By the hypothesis, it is required to find the ratio v_1/S rather than the unknowns S, v_1, v_2, and v_3. It is, therefore, convenient to introduce new unknowns x, y, z:

$$x = v_1/S, \quad y = v_2/S, \quad z = v_3/S,$$

defining the number of rounds made by each racer in an hour. For these unknowns the system of equations assumes the following form:

$$x - y = \frac{2}{9} x,$$

$$x - z = \frac{4x}{9+x},$$

$$y - z = \frac{1}{3}.$$

Adding the first and the last equation together and subtracting the second equation from the sum obtained, we get a quadratic equation for x:

$$2x^2 - 15x + 27 = 0.$$

From this we find two values of x: $x_1 = 9/2$, $x_2 = 3$. Now, directing our attention to the first equation of the system, we find two values of y: $y_1 = \frac{7}{9}x_1 = \frac{7}{2}$ and $y_2 = \frac{7}{9}x_2 = \frac{7}{3}$; taking the third equation of the system, we find the respective values of z_1 and z_2.

The value $y_1 = 7/2$ does not satisfy the inequality $y \leqslant 3$, and, consequently, the triple of numbers $x_1 = 9/2$, $y_1 = 7/2$, and $z_1 = 19/6$ is not a solution of the system. The value $y_2 = 7/3$ satisfies the last inequality of the system and, consequently, $x_2 = 3$ is the only solution of the problem.

Answer. The first racer makes three rounds in an hour.

1.59. Two bodies move in a circle uniformly in the same direction. The first body makes a circle 2 s quicker than the second and overtakes the second body every 12 s. How much time does it take each body to make a circle?

1.60. Two sportsmen run along the same closed track in a stadium. The speed of each runner is constant but the first sportsman makes a round 10 s quicker than the second. If they begin at the same start and run in the same direction, then they will meet again in 720 s. How much of the track does each runner cover per second?

1.61. Two points rotate uniformly along two concentric circles. One of them makes a full circle 5 s quicker than the other and, therefore, makes two rotations more per minute. How many rotations a minute does each point make?

1.62. Two points move along a circle of radius R uniformly in the same direction. One of them makes a full rotation t s quicker than the other, and the periods between successive meetings are equal to T. Find the speeds of the points.

1.63. Two skaters have to run a ring S km long. When the winner finished, the other skater had to run one more complete circle. Find the length of the race-course if the winner, who completed every full circle a s quicker than the other skater, ran the distance in t min.

1.64*. The hour and the minute hands of a clock coincide at midnight. At what time of the new day will the hour and the minute hands coincide again if the hands are assumed to move without jumps?

1.65. At a certain moment a watch is 2 min behind although it is running fast. If it was 3 min behind but was 1/2 min faster in 24 hours than it is now, it would show the correct time a day earlier than it would do now. By how many minutes a day is the watch fast?

1.66. By a signal of the trainer two ponies started simultaneously and ran uniformly along the outer circumference of the circus ring in opposite directions. The first pony ran faster than the second and by the moment of their meeting it covered 5 m more than the second. Continuing the run, the first pony ran to the trainer 9 s after meeting the second pony and the second pony ran to the trainer 16 s after the meeting. What is the diameter of the circus ring?

1.67. Points A and B are diametrically opposite points on a circular road 36 km long. A cyclist started from point A and made two circles. He made the first circle with a certain constant speed and then decreased his speed by 3 km/h. The time between his two passages through point B is known to be 5 h. Find the speed with which the cyclist made the first circle.

1.68. Three racers, first A, then B and then C, start with an interval of 1 min from the same point on a circular highway and race in the same direction at constant speeds. It takes each racer more than 2 min to make a circle. Having made three circles, racer A overtakes racer B for the first time at the start, and 3 min later he overtakes racer C for the second time. Racer B overtakes racer C for the first time also at the start having completed 4 circles. How many minutes does it take racer A to make a circle?

1.69. Three racers, A, B, and C, start simultaneously but from different points and race at constant speeds in the same direction along a circular highway. At the moment of the start racer B was ahead of racer A by 1/3 of the length of the highway, and racer C was ahead of racer B by the same distance. Racer A overtakes B for the first time when B completes his first circle and 10 min later he overtakes racer C for the first time. It takes racer B 2.5 min less to complete a circle than racer C. How many minutes does it take racer A to make a full circle?

1.70. A car and a motor-cycle start simultaneously from point A a circular highway and drive in the same direction at constant speeds. The car makes two circles in the same direction without a stop and at the moment when it overtakes the motor-cycle the latter reverses its direction, increases its speed by 16 km/h and 22.5 min after the reversal arrives at point A at the same time as the car. Find the length of the distance covered by the motor-cylce if that distance is 5.25 km shorter than the length of the highway.

Problems on uniformly accelerated motion. When solving problems of this kind, use is usually made of the following two formulas relating time t, the path traversed S, the initial velocity v_0, acceleration

a, and speed v:

$$S = v_0 t + at^2/2,$$
$$a = (v - v_0)/t,$$

where $a > 0$ if the motion is uniformly accelerated and $a < 0$ if the motion is uniformly decelerated.

Example 1.7. A car drives from point A to point B at a constant speed of 42 km/h. At point B it changes to a uniformly decelerated movement, decreasing its speed by a km/h until it comes to a complete stop. Then it immediately begins driving with a uniform acceleration of a km/h^2. What must be the value of a for the car to be as close to B as possible 3 h after the resumption of movement?

Solution. We designate the distance from point B to the place where the car stops as S_1 (km), the time it takes to travel that distance as t_1 (h), and the distance the car covered in 3 h travelling with uniform acceleration as S_2 (km). We tabulate the conditions of the problem with the aid of the unknowns introduced.

The hypothesis	Equation
At point B the car, which drives at 42 km/h, changes to a uniformly decelerated movement, its speed decreasing by a km/h, to a complete stop.	$\frac{42}{t_1} = a$ $S_1 = 42t_1 - \frac{at_1^2}{2}$
The car drives with a uniform acceleration of a km/h^2 for 3 h.	$S_2 = \frac{a \cdot 3^2}{2}$

We have thus obtained a system of three equations

$$a = 42/t_1,$$
$$S_1 = 42t_1 - at_1^2/2,$$
$$S_2 = 9a/2$$

to find four unknowns a, t_1, S_1, and S_2. This system does not yield a unique value for the required acceleration a. The hypothesis, however, contains one more condition making it possible to find the quantity a, namely, it is necessary to find the value of a such that the distance $S_1 + S_2$ be minimal. We write the distance $S_1 + S_2$ as a function of the acceleration a. The first equation of the system yields $t_1 = 42/a$, Substituting the $42/a$ for t_1 into the second equation of the system and adding the equation obtained to the third equation of the system, we get

$$S_1 + S_2 = \frac{42^2}{2a} + \frac{9a}{2}.$$

On the interval $(0, +\infty)$ the function $f(a) = S_1 + S_2$ attains its minimum value for $a = 14$.

Answer. $a = 14$ km/h^2.

1.71. A free-falling body is known to cover 4.9 m in the first second and 9.8 m more than during the preceding second for every subsequent second. Suppose two bodies begin falling from the same height, one after the other, with an interval of 5 s. In what time will they be at a distance of 220.5 m from each other?

1.72. Two bodies begin moving simultaneously in the same direction from two points which are 20 m apart. One of the bodies, which is behind, moves with a uniform acceleration and covers 25 m in the first second and 1/3 m more than during the preceding second for every subsequent second; the other body, which moves with uniform deceleration, covers 30 m in the first second and 1/2 m less than during the preceding second for every subsequent second. In how many seconds will they meet?

1.73. Two particles, which are at a distance of 295 m from each other, simultaneously begin moving towards each other. The first particle moves uniformly at 15 m/s, and the second particle covers 1 m during the first second and 3 m more than during the preceding second for every subsequent second. Through what angle will the second hand of a clock move during the time which passes from the moment the particles begin moving until they meet?

1.74. Two bodies move towards each other from two points which are 390 m apart. The first body covers 6 m in the first second and 6 m more than during the preceding second for every subsequent second. The second body moves uniformly at 12 m/s, having started 5 s after the first body. How many seconds after the first start will they meet?

1.75. Two ships sail towards each other in a fog at the same speed v_0. At the distance of 4 km between them the captains reverse motion for a certain time period with an acceleration of 0.1 m/s^2. What is the greatest speed of the ships at which they will not collide?

1.76. A ball rolls along a foot-ball field at right angles to its side line. Assume that moving with uniform deceleration the ball rolls 4 m in the first second and 0.75 m less than during the preceding second for every subsequent second. A player who is at a distance of 10 m from the ball runs in the same direction as the ball in order to overtake it. Running with uniform acceleration, the player covers 3.5 m in the first second and 0.3 m more than during the preceding second for every subsequent second. How long does it take the player to overtake the ball and will he have time enough to do that before the ball crosses the side line if the player has to cover 23 m to the side line?

1.77. A car drives up a hill. It reaches point A and then passes 30 m in the first second and 2 m less than during the preceding second for every subsequent second. Nine seconds after the car reached point A, a bus starts from point B, which is 258 m from A, and drives towards the car. In the first second the bus covers 2 m and then it makes 1 m more than during the preceding second for every subsequent second. What distance did the bus cover before it met the car?

1.78. A motor-cyclist starts from point A and travels with a uniform acceleration of 12 km/h^2 (the initial speed being zero). Having attained the speed v km/h, he travels with that speed for 25 km and then slows down uniformly, the speed decreasing by 24 km/h, to a complete stop. Then he immediately reverses the motor-cycle and travels to point A at the constant speed v km/h. At what speed v will the motorcyclist cover the way back in the shortest time possible, from the stop to point A?

1.79. Two cars drive along a highway one after the other at a distance of 20 m from each other at 24 m/s. The drivers notice an obstacle, apply the brakes, and change to a uniformly decelerated motion with the accelerations a_1 and a_2 ($a_1 < 0$ and $a_2 < 0$) until they come to a complete stop. The driver of the front car began slowing down 2 s earlier than the driver of the rear car. The acceleration of the front car is $a_1 = -4$ m/s^2. The smallest distance the cars approached each other is 4 m. Which of the cars was the first to stop? Find the acceleration a_2 of the rear car.

1.80. A service lift goes down a tower 320 m high. First its speed is 20 m/s then it switches over instantaneously to 50 m/s. Some time after the lift begins to move, a stone is thrown from the top of the tower which performs a free fall and reaches the ground at the same time as the lift. When falling, the stone was always above the lift, their maximum distance being 60 m. At the moment the speed of the lift was switched over, the speed of the stone was higher than 25 m/s but lower than 45 m/s. How long after the lift began moving was the stone thrown down? Assume the acceleration of the free fall of the stone to be equal to 10 m/s^2.

1.81.* Two trains started simultaneously from points A and B towards each other. At first each of them travelled with uniform acceleration (the initial speeds of the trains are zero, the accelerations are different), and then, having attained a certain speed, they travelled uniformly. The ratio of the constant speeds of the trains is 4/3. At the moment of their meeting, the trains ran with equal speeds and arrived at points B and A at the same time. Find the ratio of their accelerations.

1.82*. Two trains started simultaneously towards each other from points A and B. At first each of them travelled with uniform acceleration and then, having attached a certain speed, they travelled uniformly. The ratio of the constant speeds of the trains is 5/4. At a certain moment their speeds became equal; by that moment one of them covered a distance $1\frac{1}{4}$ times larger than the other. The trains arrived at points B and A at the same time. What part of the distance had each train covered by the time their speeds became equal?

2. Problems on Work and Productivity

In problems involving work, the system of equations which must be formed on the basis of the hypotheses usually contains the following quantities: time t during which a job is performed, productivity N, i.e. work done in unit time, and work W done during time t.

The equation relating these three quantities has the form $W = N \cdot t$.

With obvious changes, problems involving pumping of liquids, which are often encountered in textbooks, can be referred to as problems on work. In such cases, it is convenient to consider the volume of the water pumped as the work done.

Example 2.1. Two pipes lead to a reservoir—an inlet pipe through which the reservoir is filled and an outlet pipe. Two hours more are needed to fill up the reservoir through the inlet pipe than to drain the water through the outlet pipe. Both valves were opened when the reservoir was a third full and it became empty 8 hours later. How many hours does it take the inlet pipe to fill up the reservoir if it operates alone and how many hours does it take the outlet pipe to empty the full reservoir if it operates alone?

Solution. Assume that V m^3 is the volume of the reservoir, x m^3/h is the productivity of the inlet pipe and y m^3/h is the productivity of the outlet pipe, V/x (h) is the time the inlet pipe needs to fill up the reservoir and V/y (h) is the time the outlet pipe needs to drain all the water from the reservoir. By the hypothesis

$$V/x - V/y = 2.$$

Since the productivity of the outlet pipe exceeds that of the inlet pipe ($x < y$), it follows that when both pipes are opened simultaneously, the water is drained and one-third of the water is drained during the time $\frac{V/3}{y-x}$ which is equal to 8 hours by the hypothesis.

Thus according to the hypothesis we obtain the following system of two equations in three unknowns:

$$V/x - V/y = 2,$$
$$V/(y-x) = 24.$$

We have to find V/x and V/y. Let us isolate a combination of the unknowns V/x in the problem, writing the system in the form

$$\frac{V}{x} - \frac{V/x}{y/x} = 2,$$
$$\frac{V/x}{y/x-1} = 24.$$

Introducing new unknowns $V/x = t$ and $y/x = k$, we get the following systems:

$$\begin{aligned} t - t/k &= 2, \\ t/(k-1) &= 24, \end{aligned} \quad \Rightarrow \quad \begin{aligned} kt - t - 2k &= 0, \\ t &= 24\ (k-1). \end{aligned}$$

Substituting $t = 24\ (k-1)$ into the first equation of the last system, we get an equation for k:

$$12k^2 - 25k + 12 = 0 \Rightarrow k_1 = 4/3;\ k_2 = 3/4.$$

The root $k_1 = 4/3$ satisfies the condition of the problem ($y > x$). From the second equation of the last system we find $t = 8$, i.e. the first pipe can fill up the reservoir in 8 h. Let us now find the time during which the second pipe can drain the water:

$$\frac{V}{y} = \frac{V}{x}\frac{x}{y} = \frac{V}{x} : \frac{y}{x} = 8 : \frac{4}{3} = 6.$$

Answer. 8 h and 6 h.

Example 2.2. Two excavators of different designs have to dig a trench. One excavator alone can do the job 3 h quicker than the other. The sum of the times needed by the two excavators working separately is $4\frac{4}{35}$ times larger than the time it takes the excavators to do the job if they work together. Find the time it takes each excavator to do the job if it works alone.

Solution. We introduce the following quantities as unknowns: V (m^3), the volume of the earth removed, N_1 (m^3/h) and N_2 (m^3/h), the productivity of the first and the second excavator respectively. V/N_1 is the time it takes the first excavator to dig the trench when it works alone, and V/N_2 is the time it takes the second excavator to do the job if it works alone. By the hypothesis, these two quantities are related as

$$V/N_1 + 3 = V/N_2,$$

and their sum $V/N_1 + V/N_2$ is $4\frac{4}{35}$ times larger than the time it takes the two excavators when they work together, i.e. than the time $A/(N_1 + N_2)$:

$$4\frac{4}{35}\cdot\frac{V}{N_1+N_2}=\frac{V}{N_1}+\frac{V}{N_2}.$$

The hypothesis can be written as the following system of two equations:

$$\frac{V}{N_1}+3=\frac{V/N_1}{N_2/N_1},$$

$$\frac{V}{N_1}+\frac{V/N_1}{N_2/N_1}=\frac{144}{35}\frac{V/N_1}{1+N_2/N_1}.$$

For the unknowns $t = V/N_1$ and $k = N_2/N_1$ the system assumes the form

$$t+3=\frac{t}{k},$$

$$t+\frac{t}{k}=\frac{144}{35}\frac{t}{1+k}.$$

Expressing k from the first equation in terms of t and substituting $k = t/(t + 3)$ into the second equation, we obtain

$$4t^2 + 12t - 315 = 0.$$

The positive root of this quadratic equation, $t = 15/2$ (h), is the time it takes the first excavator to dig the trench when it works alone. From the first equation of the last system we can find the value $k = 15/21$ corresponding to $t = 15/2$. Noting that the second required quantity is $V/N_2 = t/k$, we find that the time it takes the second excavator to do the job alone is 21/2 h.

Answer. 7.5 h and 10.5 h.

2.1. According to schedule, a team of lumbermen had to stock 216 m^3 firewood in several days. For the first three days the team worked according to the schedule, and then each day they chopped 8 m^3 of firewood more than was planned. Therefore, a day before the assigned date, they had a stock of 232 m^3 of wood. How many cubic metres of wood a day should the team have chopped according to plan?

2.2. A team of metal workers can do a certain job of machining parts 15 h quicker than a team of apprentices. If a team of apprentices works for 18 h on the job and then a team of metal workers continues with the task for 6 h, only 0.6 of the total job will be done. How much time will it take a team of apprentices to complete the job?

2.3. Two valves A and B fill up a tank. If valve A is switched on alone, then it takes 22 min more to fill up the tank than if valve B is switched on alone. When the two valves are opened, the tank is filled in an hour. How much time does it take each valve separately to fill up the tank?

2.4. There are two different exits from the cinema. Using them both, the spectators can leave the cinema hall in $3\frac{3}{4}$ min. If the spectators use only the larger exit, it will take them 4 min less to leave the hall than if they use only the smaller exit. How much time will it take the spectators to leave the cinema hall if they use each exit separately?

2.5. A certain job is being done by three computers of different makes. It takes the second computer, working alone, 2 min more to do the job than the first computer working alone. It takes the third computer, working alone, twice as much time as the first computer. The parts of the work being equivalent, the job can be divided between the three computers. In that case, working together, they can finish the job in 2 min 40 s. How long would it take each computer to do the whole job if they worked separately?

2.6. A can finish a job t days later than B and T days later than C; A and B, working together, can do the job in as many days as C working alone. How long does it take each of them to complete the job if they work separately. For what ratio of t and T does the problem have a solution?

2.7. It takes each of the three workers a certain time to do the assigned job, the third worker doing the job an hour quicker than the first worker. Working together they can finish the job in an hour. If the first worker works for an hour and then the second worker works for 4 hours, they will complete the job together. How long would it take each worker to complete the job himself?

2.8. Two workers received the same assignment: to make a certain amount of machine parts during a certain time period. The first of them did the job in time, and the second worker accomplished only 90 per cent of the task in that time, having left unfinished as many parts as the first worker could make in 40 min. If the second worker made three parts more per hour, he would accomplish 95 per cent of the assignment. How many parts was each worker supposed to manufacture?

2.9. Two workers did a job in ten days, the last two of which the first worker did not work. How many days would it take the first worker to complete the job alone if it is known that for the first seven days they accomplished 80% of the job working together?

2.10. Two teams of plasterers, working together, plastered an apartment house in 6 days. Some other time, they worked on a club and did three times as much work as they did when plastering the house. In the club they took turns working: first the first team worked and then the second team completed the job. The first team accomplished twice as much work as the second team. The job was completed in 35 days. How many days would it take the first team to plaster the apartment house if it is known that it would take the second team 14 days to do the job?

2.11. A team of three tractors (two tractors of make A and one of make B) plough a field of 400 hectares in 10 days when the three tractors work together. The tractor of make B can plough the field $8\frac{1}{3}$ days quicker than one tractor of make A can do the job. How many hectares a day can a tractor of make A and a tractor of make B plough if they work separately?

2.12*. Each of two workers has to machine the same number of parts. The first of them began working at once and finished the job 8 hours later. The second worker first spent more than 2 h adjusting the attachment and then, using it, finished the job 3 h earlier than the first worker. The second worker is known to machine as many parts an hour after he began working as the first worker machined by that time. How many times does the attachment increase the productivity of the machine-tool (i.e. the amount of the parts machined per hour)?

2.13. Two tractors plough a field divided into two equal parts. The tractors begin working at the same time, each on its part. Five hours after they ploughed half the field working together, it was found that it remained for the first tractor to plough 1/10 of its part, and the second tractor had to plough 4/10 more of its part. How long would it take the second tractor to plough the whole field?

2.14. A team of workers had to produce 360 parts. Manufacturing 4 parts more every day than was planned, the team finished the job a day ahead of schedule. How many days did it take the team to do the job?

2.15. Two typists had to do a certain job. The second typist began working an hour later than the first. Three hours after the first typist began working, they had to do $\frac{9}{20}$ of the whole assignment. When they finished, it was found that each of them did half the job. How many hours would it take each of them to do the whole job?

2.16. Three teams work with constant productivity laying railway track. The first and the third teams, working together, lay 15 km of track a month. When they all work together, the three teams can do twice as much in a month as the first and the second teams working together. Find how many km of track the third team lays in a month if it is known that the second and third teams, working together, laid a certain part of the track four times as fast as the second team could do the same job.

2.17. Two teams of dock workers are given a job of unloading a ship. The sum of the times it takes the first team and the second team to unload the ship, if they work separately, is equal to 12 h. Find those times if the difference constitutes 45 per cent of the time it would take the teams to unload the ship if they work together.

2.18. Two pipes are connected to a reservoir with a volume of 24 m^3. The outlet pipe only drains the water at the rate of 2 m^3/h, and the inlet pipe only fills the reservoir. When the two valves were opened, the reservoir was empty. When the reservoir was half full, the first pipe was closed and the second pipe kept filling the reservoir. As a result, the reservoir was filled in 28 h 48 min. What quantity of water does the second pipe supply per hour?

2.19. Two pumps transferred 64 m^3 of water. They began working at the same time and with the same capacity. After the first pump transferred 9 m^3 of water, it was stopped for 1 h 20 min. After that time, the capacity of the first pipe was increased by 1 m^3/h. Find the initial capacity of the pumps if the first pump transferred 33 m^3 of water and both pumps completed the job at the same time.

2.20. Two sections of a coal mine were operating for some time, then a third section went into operation. As a result, the productivity of the mine increased by one and a half times. What is the percentage of the productivity of the second section as compared to that of the first if it is known that the first and the third section produce as much coal in four months as the second section does in a whole year?

2.21. Two workers, one of whom began working a day and a half later than the other, worked independently and papered several rooms in 7 days, reckoning from the moment the first of them began working. If the job was entrusted to each worker separately, then it would take the first of them 3 days more to do it than the second. How many days would it take each of the workers to complete the job if he worked alone?

2.22. If we open two pipes simultaneously, the reservoir will be full in 2 h 24 min. Actually, first only the first pipe was opened for a quarter of the time it would take the second pipe to fill the reservoir. Then the first pipe was closed and the second pipe was opened for a quarter of the time it would take the first pipe to fill up the reservoir. After that, 11/24 of the volume of the reservoir remained to be filled. How much time does it take each pipe to fill the reservoir?

2.23. Two students began preparing for an examination at the same time and had to pass it on the same day. The first student had to read 240 pages and the second 420 pages. Each of them read the same number of complete pages every day, the first of them reading 12 pages less than the second per day. After they both read their material once, they had some time left for repetition, the first of them had 7 more days and the second, 5 days. What whole number of pages a day should each student read in order that each of them had 3 more days to repeat the material?

2.24. One of the ship compartments developed a leak and it became totally filled with water. Two pumps of equal capacity were switched on to evacuate the water. The leak was stopped 18 h later, the second pump was switched off, and another 12 h later the compartment was dry again. If it were impossible to stop the leak, then the two pumps, working together, would pump out half the water from the compartment in 10 hours of simultaneous operation. How long would it take the second pump to pump out half the water if it were impossible to stop the leak?

2.25. Three excavators participated in digging a pit with the volume of 340 m^3. The first excavator removes 40 m^3 of earth per hour, the second excavator takes out c m^3 less than the first, and the third one removes $2c$ m^3 more than the first. First the first and the second excavators worked simultaneously and removed 140 m^3 of earth. Then, the remaining part of the pit was dug by the first and the third excavators working together. Find the value of c $(0 < c < 15)$ for which the pit was dug in 4 h if the excavators worked without breaks.

2.26. Three excavators had to dig a pit each: the first and the second excavator a pit of 800 m^3 and the third one, a pit of 400 m^3. The first and the second excavator, working together, removed three times as much earth per hour as the third one; the first and the third excavator began working simultaneously and the second one began at the moment when the first excavator had removed 300 m^3 of earth. When the third excavator had completed 2/3 of its job, the second excavator had removed 100 m^3 of earth. The third excavator was the first to complete the job. How much earth did the first excavator remove by the time the third excavator completed its job?

2.27*. There are three pumps. The second pump transfers twice as much water per hour as the first one, and the third pump transfers 8 m^3 of water more per hour than the second pump. They simultaneously began filling two reservoirs, one with a volume of 600 m^3 and the other with a volume of 1680 m^3. The first pump filled up the smaller reservoir. First 240 m^3 of water were supplied into the larger reservoir by the second pump, and then, without losing time, the second pump was replaced by the third which filled up the reservoir. The larger reservoir was filled 6 h later than the smaller one. If from the very beginning the third pump alone pumped water into the larger reservoir, then it would be filled 5 h later than the smaller reservoir. How many cubic metres of water does the first pump transfer in an hour?

2.28*. Several identical harvester combines were allotted to gather in the harvest, which they could do in 24 h if they began working at the same time. But it so happened that they began working one after another at equal time intervals and then all of them worked till the job was completed. How long did it take to gather in the harvest if the first combine worked 5 times as long as the last one?

2.29*. Several pumps of the same capacity are switched on one after another at equal time intervals to fill a resorvoir. The last pump supplied V litres of water. How much water was supplied by the first pump if it is known that with a decrease in the supply of each pump by 10% (with the same intervals between switching) the time it takes to fill the reservoir will increase by 10%?

2.30. Three pumps simultaneously began pumping out the water each from its own reservoir. When the third pump emptied an αth of the volume of its reservoir $(\alpha < 1/2)$, the second pump had to pump out as much water as the first pump had drained; when the third pump had to pump out $(1 - \alpha)$th of the volume, it remained for the first pump to pump out as much water as did the second pump. The first pump empties the second reservoir during the time it takes the second pump to empty the first reservoir. Which pump worked longer than the others and by how many times? (The delivery of each pump is constant.) Investigate the dependence of the solution on the value of α.

2.31*. The reservoir was filled up by several pumps which were switched on one after another at certain time intervals. For a large part of the time the pumps worked together and they filled the second half of the reservoir t h quicker than the first half. How much quicker will the reservoir be filled if the intervals between the switchings are decreased n times with the same sequence of switchings? (The delivery of the pumps is constant.)

3. Problems on Per Cent Increment and Calculations of "Compound Interests"

The solution of problems on the per cent increase and the calculation of "compound interests" are based on the use of the following notions and formulas. Assume that a certain variable A, dependent on time t, has the value A_0 at the initial moment $t = 0$ and the value A_1 at a certain moment t_1. The *absolute increment* of the quantity A at the time t_1 is the difference $A_1 - A_0$, the *relative increment* of the quantity A at the time t_1 is the ratio $\frac{A_1 - A_0}{A_0}$, and the *per cent increment* of the quantity A at the time t_1 is the quantity

$$\frac{A_1 - A_0}{A_0} \cdot 100\%.$$

Designating the per cent increment of the quantity A as $p\%$, we get the following formula relating A_0, A_1, and p:

$$\frac{A_1 - A_0}{A_0} \cdot 100\% = p\%.$$

The notation of the last formula

$$A_1 = A_0 \left(1 + \frac{p}{100}\right) = A_0 + A_0 \frac{p}{100}$$

makes it possible to calculate the value of A_1, i.e. the value of A at the moment t_1, from the known value of A_0 and the assigned value of p.

Assume that for $t > t_1$ the quantity A has a per cent increment $p\%$. Then at the time moment $t_2 = 2t_1$ the value of the quantity $A_2 = A(t_2)$ is

$$A_2 = A_1 \left(1 + \frac{p}{100}\right) = A_0 \left(1 + \frac{p}{100}\right)^2.$$

At the time moment $t_3 = 3t_1$ the value of the quantity $A_3 = A(t_3)$ is

$$A_3 = A_2 \left(1 + \frac{p}{100}\right) = A_0 \left(1 + \frac{p}{100}\right)^3,$$

and at the time moment nt_1 it is

$$A_n = A_0 \left(1 + \frac{p}{100}\right)^n.$$

If at the time t_1 (at the "first stage") the quantity A changed by $p_1\%$, at the "second stage" (i.e. during the time $t_2 - t_1 = t_1$) by $p_2\%$, at the "third stage" (i.e. during the time $t_3 - t_2 = t_1$) by $p_3\%$ and so on, then at the time moment $t_n = nt_1$ the value of the quantity A can be calculated by the formula

$$A_n = A_0\left(1+\frac{p_1}{100}\right)\left(1+\frac{p_2}{100}\right)\dots\left(1+\frac{p_n}{100}\right).$$

Example 3.1. A factory operated for three years. The output of the factory for the second year of operation increased by $p\%$ and the next year it increased by 10% more than it increased in the second year. By how many per cent did the output increase in the second year if it is known that for two years it increased by 48.59%?

Solution. Let us designate the output of the factory for the first, second, and third year, respectively, as A_1, A_2, and A_3. By the hypothesis, in the second year the increase in per cent was $p\%$ and in the third year, $(p + 10)\%$. In accordance with the definition of the per cent increment, these conditions yield two equations:

$$\frac{A_2-A_1}{A_1}\cdot 100\% = p\%,\qquad \frac{A_3-A_2}{A_2}\cdot 100\% = (p+10)\ \%.$$

By the hypothesis, it is also known that in two years the production increased by 48.59%, i.e. in the third year the output of the factory was 48.59% higher than in the first year. This condition can be written as follows:

$$\frac{A_3-A_1}{A_1}\cdot 100\% = 48.59\%.$$

Let us write the equations obtained in the form of the following system:

$$A_2 = A_1\left(1+\frac{p}{100}\right),$$
$$A_3 = A_2\left(1+\frac{p+10}{100}\right),$$
$$A_3 = A_1\left(1+\frac{48.59}{100}\right).$$

Multiplying the first equation by the second, we get

$$A_3 = A_1\left(1+\frac{p}{100}\right)\left(1+\frac{p+10}{100}\right).$$

From this equation and from the third equation of the system we get an equation to find the unknown quantity p:

$$\left(1+\frac{p}{100}\right)\left(1+\frac{p+10}{100}\right) = 1+\frac{48.59}{100} \Rightarrow p^2+210p-3859=0.$$

The roots of the last quadratic equation are $p_1 = 17$, $p_2 = -227$. The first root suits the sense of the problem, and so $p_1 = 17$.

Answer. 17%.

3.1. A savings bank adds annually 3% from the sum of the account. In how many years will the deposited sum double?

3.2. The population of the town increases annually by 1/50 of the actual number of citizens. In how many years will the population triple?

3.3. A customer paid 2 roubles for a kilogram of one product and ten kilograms of another product. If after the seasonal change in prices the first product goes up in price by 15% and the other product becomes cheaper by 25%, then the customer will pay 1 rouble and 82 kopecks for the same amount of products. What is the cost of a kilogram of each product?

3.4. A customer deposited in a savings bank 1640 roubles at the beginning of a year and withdrew 882 roubles at the end of the year. A year later he had 882 roubles on his account. How many per cent a year does the savings bank add?

3.5. In a second-hand book-shop, the price of an antique collection costing 350 roubles was reduced twice by the same number of per cent. Find that number if it is known that after the price was reduced twice the collection cost 283 roubles and 50 kopecks.

3.6. During a year a factory twice increased the output by the same number of per cent. Find that number if it is known that at the beginning of the year the factory produced 600 articles per month and at the end of the year it produced 726 articles per month.

3.7. A savings bank added 6 roubles to the account of a customer over a year. Having added another 44 roubles, the customer left his deposit for one more year. At the end of that year, the savings bank added a new number of per cent to the customer's account which constituted now, with the extra per cent, 257 roubles and 50 kopecks. What sum of money did the customer deposit initially and how many per cent, 2 or 3, did the savings bank add every year?

3.8. On the first working day of a month a shop of radio goods sold 105 TV sets. Every subsequent working day the shop sold 10 more TV sets a day, and the monthly plan, 4000 TV sets, was fulfilled ahead of schedule, in a whole number of working days. After that, the shop sold 13 TV sets less per day than on the last day of fulfilling the plan. By how many per cent did the shop overfulfill the monthly plan of selling TV sets if there are 26 working days in a month?

3.9. A sum of money deposited at the beginning of a year is known to increase by a certain per cent (differing from bank to bank) by the end of the year. At the beginning of a year 5/6 of a certain amount of money was deposited in one bank and the remaining part of the money, in another bank. By the end of the year the sum of these deposits equalled 670 monetary units, and by the end of the next year, 749 monetary units. It was calculated that if from the very beginning, 5/6 of the initial sum of money had been deposited in the second bank and the remaining sum in the first bank, then by the end of the first year the deposits in the banks would have been equal to 710 monetary units. Assuming that from the very beginning the initial sum of money was deposited in the first bank, find the value of the deposit by the end of the second year.

3.10. The output of factory A constitutes 40.96% of that of factory B. The annual increase of production at factory A in per cent is 30% larger than the annual increase of production at factory B. What is the annual increase of production at factory A in per cent if on the fourth year of operation the output of factory A will be the same as of factory B?

3.11. A deposit of N roubles was made in a savings bank with a $p\%$ annual interest. At the end of each year the customer withdraws M roubles. How many years after a withdrawal of a corresponding sum will the remainder be thrice as large as the initial deposit?

3.12. At the initial moment there are N bacteria in a retort. By the end of each hour the number of bacteria increases by $p\%$ as compared to their amount at the beginning of that hour; in addition, at the beginning of each hour a portion containing n $(n < N)$ bacteria is taken from the retort. In how many hours will the amount of bacteria in the retort (after the withdrawal of the corresponding portion) become twice their inital amount?

4. Problems with Integral Unknowns

An integral value of the required unknown is usually an additional condition making it possible to choose it uniquely from a certain set of values satisfying the other conditions of the problem.

Example 4.1. A boy puts all his stamps into a new album. If he puts 20 stamps on each page, there will not be enough pages in the album and if he puts 23 stamps on each page, then at least one page will remain empty. If the boy is presented with one more album of that kind, in which there are 21 stamps on each page, he will have 500 stamps in all. How many pages are there in the album?

Solution. Assume that there are m pages in the album and the boy has N stamps. We can form a system of equations and inequalities of the problem:

The hypothesis	Equality, inequality
If the boy puts 20 stamps on each page, there will not be enough pages in the album	$20\,m < N$
If the boy puts 23 stamps on each page, then at least one page will remain empty	$23\,(m-1) \geqslant N$
If the boy is presented with one more album of that kind, in which there are 21 stamps on each page, he will have 500 stamps in all	$21\,m + N = 500$

Thus the hypothesis can be written as a system of one equation and two inequalities. Let us substitute the expression for N from the equation into each of the two inequalities. As a result we get a system of two inequalities:

$$20m < 500 - 21m, \quad 23\,(m-1) \geqslant 500 - 21m.$$

Taking into account that m is an integer, we find from the first inequality that $m \leqslant 12$ and from the second inequality that $m \geqslant 12$. Comparing these results, we obtain $m = 12$, *Answer*. There are 12 pages in the album.

Example 4.2. The teams taking part in a motor-rally have the same number of Volga cars and Moskvich cars, the total number of cars on each team being less than 7. If the number of Volga's remains the same on each team and that of Moskvich's increases by three times, then the total number of Moskvich's participating in the rally will exceed by 50 the total number of Volga's and the total number of cars on each team will exceed 12. Find the number of teams participating in the rally and the number of Volga cars and Moskvich cars on each team.

Solution. Let us designate the number of teams participating in the rally as N and the number of Volga's and Moskvich's on each team as m and n respectively. The following system of equations and inequalities corresponds to the conditions of the problem.

The hypothesis	Equations, inequalities
On each team the total number of cars is less than 7	$m+n<7$
If the number of Volga's remains the same on each team and that of Moskvich's increases by three times, then the total number of Moskvich cars will exceed by 50 the total number of Volga cars	$3nN-mN=50$
In that case the total number of cars on each team will exceed 12	$m+3n>12$

Let us investigate the inequalities $m+n<7$ and $m+3n>12$. Subtracting the first inequality from the second, we obtain $2n>5 \Rightarrow n>5/2$. Consequently, n can be equal to 3, 4, 5, ... It follows from the first inequality that n can be equal to 3, 4, 5.

Assume that $n=3$. Then it follows from the first inequality that m can assume the values 1, 2, 3; from the second inequality it follows that m can assume values exceeding 3. Consequently, $n \neq 3$.

Assume that $n=4$. It follows from the first inequality that m can assume the values 1, 2 and from the second inequality it follows that m can assume the values 1, 2, 3, ...

Assume that $n=5$. It follows from the first inequality that m can assume the only value equal to 1.

Thus the following three pairs of numbers satisfy the inequalities:

$$\{n=4,\ m=1\},\ \{n=4,\ m=2\},\ \{n=5,\ m=1\}.$$

Substituting these pairs of numbers into the remaining equation of the system, we get equations enabling us to find N:

$$11N=50,\quad 10N=50,\quad 14N=50.$$

Since by the hypothesis N must be an integer, it follows that $N=5$ and the pair $n=4$, $m=2$ is the only solution of the problem.

Answer. The number of teams is 5; the number of Volga cars on each team is 2; the number of Moskvich cars on each team is 4.

4.1. The students in a group of 30 passed an examination with the grades 1, 2, 3, 4. The sum of the grades they received is equal to 93, 2.0 grades being more than 4.0 grades and less than 3.0 grades. In addition, the number of 3.0 grades is divisible by 10 and the number of 4.0 grades is even. Find the number of grades of each kind the students received.

4.2. A test was given to a group of students. Among the grades the students received there are only 1, 2, 3 and 4. The same number of students received grades 1.0, 2.0, and 4.0, and there were more 3.0 grades than the other grades taken together. Less than 10 students received grades higher than 2.0. How many 2.0 grades and 4.0 grades did the students receive if not less than 12 students took the test?

4.3. There are five-storey and nine-storey houses on one block, the nine-storey houses being less in number than the five-storey ones. If we double the number of nine-storey houses, then the total number of houses will exceed 24, and if we double the number of five-storey houses, then the total number of houses will be less than 27. How many five-storey houses and nine-storey houses are there on the block?

4.4. There are Volga and Moskvich cars in the parking lot. Their total number is less than 30. If we double the number of Volga's and increase the number of Moskvich's by 27, then there will be more Volga's, and if we double the number of Moskvich cars without changing the number of Volga's, then there will be more Moskvich cars. How many Volga's and how many Moskvich's are there in the parking lot?

4.5. It is stated in the school paper that the percentage of students of a certain class who made progress in their studies during the second term, ranges between 2.9 and 3.1%. Determine the minimum possible number of students in such a class.

4.6*. The factory has to deliver 1100 articles to the customer. The articles must be packed in boxes of three kinds. A box of the first kind accommodates 70 articles, a box of the second kind accommodates 40 articles, and that of the third kind 25 articles. It costs 20 roubles to transport a box of the first kind, 10 roubles to transport a box of the second kind, and 7 roubles to transport a box of the third kind. What kind of boxes should the factory use for the cost of transportation to be the lowest? (An underloading of the boxes is not permitted.)

4.7*. A collective farm rented two excavators. The rent of the first excavator costs 60 roubles a day and its productivity in soft ground is 250 m^3 a day and in hard ground 150 m^3 a day. The rent of the second excavator costs 50 roubles a day and its productivity in soft ground is 180 m^3 a day and in hard ground 100 m^3 a day. The first excavator worked several full days and turned over 720 m^3 of earth. The second excavator also worked several full days and turned over 330 m^3 of earth. How many days did each excavator work if the collective farm paid 300 roubles for the rent?

4.8*. There are candies of two kinds in a bowl. The candies of the first kind exceeding by more than 20 those of the second kind. One candy of the first kind weighs 2 g and that of the second kind 3 g. Fifteen candies of the same kind were taken from the bowl, their weight constituting a fifth of the weight of all the candies in the bowl. Then another 20 candies of the other kind were taken from the bowl and the weight proved to be equal to that of the candies remaining in the bowl. How many candies of each kind were there in the bowl initially?

4.9. Several lorries are loaded in turn at point A (the time of loading is the same for all the lorries) and then transport the goods to point B where they are quickly unloaded and return to point A. The speeds of the lorries are equal, the speed of a loaded lorry being 6/7 of the speed of an empty one. Driver Petrov was the first to leave point A. On his return trip he met driver Ivanov, who was the last to leave point A, and arrived at A 6 min after their meeting. Here Petrov began loading at once and left for B and met Ivanov once again 40 minutes after the first meeting. Not less than 16 min and not more than 19 min passed from the time of the second meeting to Ivanov's arrival at A. Find the time it took the lorries to unload.

4.10. Rafts are sent from point A to point B at equal time intervals. The speeds of all the rafts relative to the river bank are constant and equal to each other. A pedestrian walking from A to B along the river bank covered a third of the distance from A to B by the moment the first raft was sent. Having arrived at B, the pedestrian at once left for A and met the first raft having covered more than 3/13 of the way from B to A and met the last raft having covered more than 9/10 of the way from B to A. The pedestrian arrived at point A when the seventh raft reached point B. From point A the pedestrian at once started for point B and arrived there at the same time as the last raft. The speed of the pedestrian is constant, the part of the river from A to B is rectilinear. How many rafts were sent from A to B?

Problems on writing numbers in decimal positional notation is one more type of problem on deriving equations with integral unknowns.

Example 4.3. The required three-place number ends with the digit 1. If we transfer that digit from the last place to the first one, without changing the order of the other two digits, then the resulting number will be less than the required number by 90. Find the number.

Solution. Let us designate the hundreds digit of the required three-place number as m and the tens digit as n. The required three-place number $mn1$ (the multiplication sign between m, n, and 1 is ommitted: m, n are digits of the decimal notation and $m \neq 0$) is an abbreviated notation of the number $m \cdot 10^2 + n \cdot 10 + 1$. The three-place number resulting from the transfer of 1 from the last place to the first place is $1 \cdot 10^2 + m \cdot 10 + n$. By the hypothesis, the last number is smaller than the required number by 90:

$$m \cdot 10^2 + n \cdot 10 + 1 = 1 \cdot 10^2 + m \cdot 10 + n + 90.$$

We have thus obtained an equation in two unknowns m and n and we know that m and n are digits of a decimal positional system of notation and that $m \neq 0$. The number of units appearing on the left-hand side must coincide with the number of units in the number appearing on the right-hand side and, therefore, $n = 1$. The equation now assumes the form

$$m \cdot 10^2 + 10 = 1 \cdot 10^2 + m \cdot 10 + 90.$$

Thus we find that $m = 2$.

Answer. The required number is 211.

Example 4.4. If we divide the two-digit number by the sum of its digits, we get 4 as a quotient and 3 as a remainder. Now if we divide that number by the product of its digits, we get 3 as a quotient and 5 as a remainder. Find the number.

Solution. Before proceeding with the solution of the problem, recall that if the number N is divisible by the number p and the number k is the quotient and the number r $(r < p)$ is the remainder, then the number N can be represented as

$$N = kp + r.$$

The solution of the problem is based on the use of this equation. Let us write the two-place number in the form $10 \cdot m + n$. The hypothesis leads us to a system of two equations:

$$\begin{cases} 10m + n = 4(m+n) + 3, \\ 10m + n = 3mn + 5 \end{cases} \Rightarrow \begin{cases} 6m = 3n + 3, \\ 10m + n = 3mn + 5 \end{cases} \Rightarrow \begin{cases} n = 2m - 1, \\ 10m + n = 3mn + 5. \end{cases}$$

Substituting $n = 2m - 1$ into the second equation of the system, we obtain an equation

$$2m^2 - 5m + 2 = 0,$$

whose solution is $m_1 = 2$, $m_2 = 1/2$. The condition of the problem (m and n are digits) is satisfied only by the first root $m = 2$. The first equation of the system yields $n = 3$.

Answer. The required number is 23.

4.11. What two-place number is smaller than the sum of the squares of its digits by 11 and larger than their doubled product by 5?

4.12. The sum of the digits of a two-place number is 12. If we add 36 to the required number, we get a number written by the same digits in the reverse order. Find the number.

4.13. The sum of the squares of the digits in a two-place number is 13. If we subtract 9 from that number, we get a number written by the same digits in the reverse order. Find the number.

4.14*. Having thought of an integral positive number smaller than 10, we add 5 to its notation from the right and subtract from the resulting number the square of the thought-of number. Then we divide the difference by the thought-of number and subtract the thought-of number. We have a unity as a remainder. Find the number.

4.15. A student was to multiply 72 by a two-place number in which there are thrice as many tens as ones; by mistake he reversed the digits in the second factor and obtained a product which is smaller than the actual one by 2592. What is the actual product equal to?

4.16*. The notation of a six-place number begins with 2. If we transfer that digit from the first place to the last one, without changing the order of the other five digits, the resulting number will be three times larger than the initial number. Find the initial number.

4.17. Find an integral positive number from the following data: if we add the digit 4 to its digital notation from the right, we obtain a number which is exactly divisible by a number exceeding the required number by 4, the quotient being a number smaller than the divisor by 27.

4.18*. The sum of all even two-place numbers was divided by one of them without a remainder. The quotient obtained differs from the divisor only by the order of the digits and the sum of its digits is 9. What two-place number is the divisor?

4.19. A positive digit was added to the right of the digital notation of a thought-of positive number. The square of the thought-of number was subtracted from the resulting number. The difference was found to be larger than the thought-of number as many times as the complement of the thought-of number to 11. Prove that this is possible if and only if the complement is equal to the thought-of number.

4.20. Find two two-place numbers possessing the following properties: if we add to the larger required number, from the right, first 0 and then the smaller number and add to the smaller number from the right first the larger number and then 0, then the first of the resulting five-place numbers, being divided by the second resulting number, gives 2 as a quotient and 590 as a remainder. It is known in addition that the sum of the doubled larger required number and the trebled smaller number is equal to 72.

4.21. An error occurred when two numbers were multiplied, one of which is larger than the other by 10: the tens digit in the product was decreased by 4. When the answer was verified by dividing the resulting product by the smaller factor, the quotient obtained was equal to 39 and the remainder to 22. Find the factors.

4.22*. Find two two-place numbers A and B from the following condition: if the digital notation of the number A is written in front of B and the resulting number is divided by B, the quotient will be equal to 121. Now if the digital notation of the number B is written in front of the number A and the resulting number is divided by A, then the quotient will be 84 and the remainder 14. Find A and B.

4.23*. The square of an integral positive prime number N is divided (with a remainder) by 3, the resulting incomplete quotient is divided (without a remainder) by 3, the quotient is again divided (with a remainder) by 3, and, finally, the resulting incomplete quotient is again divided by 3 with a remainder and gives 16 as a result. Find N.

4.24*. The denominator of a fraction is larger than the square of its numerator by unity. If we add 2 to both numerator and the denominator, the fraction will be larger than 1/4, and if we subtract 3 from the numerator and the denominator, the fraction will be smaller than 1/10. Find the fraction.

5. Problems on Concentration and Percentage

The solution of problems on concentration and percentage is based on the use of the following concepts and formulas.

Assume that we are given three different substances A, B, and C with the masses M_A, M_B, and M_C. The mass of the mixture consisting of these substances is equal to $M_A + M_B + M_C$.

The *mass concentration* of the substance A in the mixture is the quantity c_A which can be calculated by the formula

$$c_A = \frac{M_A}{M_A + M_B + M_C}.$$

Correspondingly, the mass concentrations of the substances B and C in that mixture can be calculated by the formulas

$$c_B = \frac{M_B}{M_A + M_B + M_C}, \quad c_C = \frac{M_C}{M_A + M_B + M_C}.$$

The mass concentrations c_A, c_B, and c_C are related as $c_A + c_B + c_C = 1$.

The *percentages* of the substances A, B, C in the given mixture are the quantities $p_A\%$, $p_B\%$, and $p_C\%$, respectively, which can be calculated by the formulas

$$p_A\% = c_A \cdot 100\%, \quad p_B\% = c_B \cdot 100\%, \quad p_C\% = c_C \cdot 100\%.$$

Similar formulas can be used to calculate the concentrations of substances in a mixture in cases when the number of various mixed substances (components) is equal to two, four, five, etc.

The *volume concentrations* of substances in a mixture can be found by the same formulas as mass concentrations with the only difference that the volumes of the components V_A, V_B, and V_C appear in the formulas instead of the masses of the components M_A, M_B, and M_C. When we speak of volume concentrations, we usually assume that when substances are mixed, the volume of the mixture is equal to the sum of the volumes of the components. This assumption is not a physical law but an agreement accepted when problems on volume concentration are being solved.

Example 5.1. A vessel of 6 litres capacity contains 4 litres of 70% solution of sulphuric acid. Another vessel of the same capacity contains 3 litres of 90% solution of sulphuric acid. How many litres of the solution must be transferred from the second vessel into the first to obtain in it an r% solution of sulphuric acid? Find all values of r for which the problem has a solution.

Solution. Let us designate as x (l) the volume of the 90% solution of sulphuric acid which is transferred from the second vessel to the first. This volume contains $\frac{9x}{10}$ (l) of pure (100%) sulphuric acid. The initial volume of pure acid in the first vessel was equal to $\frac{7}{10} \cdot 4$ (l). When x (l) of 90% solution of sulphuric acid were added to the first vessel, it contained

$$\frac{7}{10} \cdot 4 + \frac{9}{10} \cdot x$$

litres of pure acid. Using the definition of the percentage by volume, we find, in accordance with the hypothesis, that

$$\frac{\frac{7}{10} \cdot 4 + \frac{9}{10} x}{x + 4} \cdot 100\% = r\%.$$

Solving this equation, we find the value of the volume transferred to be

$$x = \frac{4(r - 70)}{90 - r}.$$

It remains to find the values of r for which the problem has a solution. It is evident from the hypothesis that the quantity of the solution being added cannot exceed 2 litres since the volume of the first vessel is equal to 6 l, i.e. $0 \leqslant x \leqslant 2$. Using the value obtained for x, we get restrictions imposed on r:

$$0 \leqslant \frac{4(r-70)}{90-r} \leqslant 2.$$

Solving this inequality (with due regard for the fact that $70 \leqslant r \leqslant 90$), we obtain $70 \leqslant r \leqslant 76\frac{2}{3}$.

Answer. $\frac{4(r-70)}{90-r}$ (l), the problem is solvable for $70 \leqslant r \leqslant 76\frac{2}{3}$.

5.1. There is a piece of copper and tin alloy with the total mass of 12 kg containing 45% copper. How much pure tin must be added to the alloy for the resulting new alloy to contain 40% copper?

5.2. There are two bars of copper and tin alloys. The first weighs 3 kg and contains 40% copper and the second weighs 7 kg and contains 30% copper. What must be the weights of the pieces of those bars in order to obtain 8 kg of an alloy containing r% copper after the bars are smelted together?

5.3. Fresh fruit contains 72% water and dry fruit contains 20% water. How much dry fruit can be obtained from 20 kg of fresh fruit?

5.4. Sea water contains 5% salt by weight. How many kilograms of fresh water must be added to 40 kg of sea water for the content of the salt in the solution to be 2%?

5.5. Two vessels contain a solution of different concentrations, the first vessel containing m litres less than the second. The same quantity of n litres is taken simultaneously from each vessel and the solution taken from the first vessel is poured into the second and that taken from the second vessel is poured into the first. After that, the concentrations of solutions in both vessels became the same. How many litres of solution are there in each vessel?

5.6*. Identical pieces were cut off from two bars with different percentages of copper weighing m kg and n kg. Each cut-off piece was alloyed with the remainder of the other bar, and then the percentage of copper in both alloys became the same. Find the weight of each cut-off piece.

5.7. Two kinds of cast iron with different percentages of chromium were alloyed. If we take five times as much of one kind of cast iron as the other, then the percentage of chromium in the alloy will be twice that in the smaller of the alloyed parts. Now if we take the same amount of both kinds, then the alloy will contain 8% chromium. Find the percentage of chromium in each kind of cast iron.

5.8. Given three different iron compounds. Each cubic centimetre of the first compound contains 3/20 g less iron than each cubic centimetre of the second compound, and each cubic centimetre of the third compound contains 10/9 times more iron than each cubic centimetre of the first compound. The volume of a piece of the third compound, containing 1 g of iron, is larger by 4/3 cm^3 than that of a piece of the second compound also containing 1 g of iron. What volume of the third compound contains 1 g of iron?

Hint. Use the formula $m = \rho V$, relating the mass, the density, and the volume of a substance.

5.9. The percentages of alcohol in three solutions form a geometric progression. If we mix the first, the second, and the third solution in the ratio 2 : 3 : 4 by weight, we obtain a solution containing 32% alcohol. Now if we mix them in the ratio 3 : 2 : 1, we get a solution containing 22% alcohol. Find the percentage of alcohol in each solution.

5.10*. There is a solution of sodium chloride of four different concentrations in the laboratory. If we mix the first, the second, and the third solution in the weight ratio 3 : 2 : 1, we get a 15% solution. The second, the third, and the fourth solution, taken in equal proportions, give a 24% solution when mixed, and, finally, a solution made of equal parts by weight of the first and the third solution, has a concentration of 10%. What concentration will result from mixing the second and the fourth solutions in the proportion of 2 : 1?

5.11. Three identical test tubes are half filled with alcohol solutions. When the content of the third tube was divided in half and each half was poured into each of the first two tubes, the volume concentration of alcohol in the first tube became smaller by 20% of its value and that in the second tube increased by 10% of its value. By how many times did the initial quantity of alcohol in the first test tube exceed the initial quantity of alcohol in the second test tube? (The change in the volume upon mixing the solutions may be ignored.)

5.12. There are two aqueous salt solutions. To obtain a mixture containing 10 g of salt and 90 g of water, we must take twice as much by mass of the first solution as the second. A week later, 200 g of water evaporated from each kg of the first and the second solution and, to obtain the same mixture as before, we must take four times as much, by mass, of the first solution as the second. How many grams of salt were there initially in 100 g of each solution?

5.13. There are two aqueous solutions of substances A and B differing in their weight ratios of substances A and B and water. There is as much of the substance A in the first solution as water and one and a half as much of substance B as substance A. In the second solution there is half as much of substance B as substance A and twice as much of substance B as water. What quantity of each solution must we take and how much water must we add to obtain 37 kg of a new solution in which there is the same quantity of substance A as substance B and twice as much water as substance A?

5.14. Pure water and an acid solution of constant concentration are simultaneously fed into an empty reservoir through two pipes. When the reservoir is filled up, it contains a 5% acid solution. If the feeding of water is stopped the moment when the reservoir is half full, then the filled up reservoir would contain a 10% acid solution. Which pipe feeds the liquid quicker and by how many times?

5.15. Two pipes, operating simultaneously, feed 100 l of a liquid per minute into a tank. There are two acid solutions, one strong and the other weak. If we mix 10 l of each solution and 20 l of water, we get 40 l of a 20% solution. It is also known that if we feed the weak solution into an empty tank through the first pipe and the strong solution through the second pipe for an hour, we can obtain 30% acid solution. Find the concentration (in per cent) of the resulting acid solution if we feed, into the initially empty tank, the strong solution through the first pipe and the weak solution through the second pipe for an hour. (The volume is assumed to remain constant when the acid and the water are mixed.)

5.16. There are three bars of various alloys of gold and silver. It is known that the amount of gold in 2 g of the alloy in the third bar is the same as in 1 g of the first bar and 1 g of the second bar taken together. The weight of the third bar is equal to the total weight of a part of the first bar which contains 10 g of gold and a part of the second bar which contains 80 g of gold. The third bar is four times as heavy as the first bar and contains 75 g of gold. How many grams of gold are there in the first bar?

5.17. There are two alloys of zinc, copper, and tin. The first alloy is known to contain 40% tin and the second 20% copper. The percentage of zinc in the first and the second alloy is the same. If we alloy 150 kg of the first alloy and 250 kg of the second, we get a new alloy which contains 30% zinc. How many kilograms of tin are there in the new alloy?

5.18. There are three alloys. The first alloy contains 30% nickel and 70% copper, the second contains 10% copper and 90% manganese and the third alloy contains 15% nickel, 25% copper, and 60% manganese. They must be used to obtain a new alloy which contains 40% manganese. Find the highest and the lowest percentage of copper the new alloy can contain.

Example 5.2. A vessel contains M kg of a $p\%$ salt solution. We pour off a kg of the solution from the vessel and add a kg of water and stir the solution. We repeat the procedure n times. Find the law according to which the concentration of the salt in the vessel changes, i.e. the concentration of the salt after n procedures.

Solution. The initial quantity of salt in the solution is $\frac{p}{100}\cdot M$ (kg). After a kg of the mixture were poured off, the mixture contains

$$\frac{p}{100}M-\frac{p}{100}a=\frac{p}{100}M\left(1-\frac{a}{M}\right)$$

of salt, and after a kg of water were added, the concentration of the mixture becomes

$$c_1=\frac{p}{100}\left(1-\frac{a}{M}\right).$$

After another a kg of the mixture were poured off (which had the concentration c_1), the mixture contained

$$\frac{p}{100}M\left(1-\frac{a}{M}\right)-c_1a=\frac{p}{100}M\left(1-\frac{a}{M}\right)^2$$

of salt, and after another a kg of water were added, the concentration of the mixture becomes

$$c_2=\frac{p}{100}\left(1-\frac{a}{M}\right)^2.$$

After n pourings, the concentration of salt in the solution can be found from the formula

$$c_n=\frac{p}{100}\left(1-\frac{a}{M}\right)^n,$$

which is the formula for a general term of an infinitely decreasing geometric progression. The factor $1-a/M$, which is the common ratio of the progression, shows how many times the concentration decreases after each successive pouring.

Answer. After n pourings, the concentration of salt is $\frac{p}{100}\times\left(1-\frac{a}{M}\right)^n$.

5.19. One litre of a 12% (by mass) salt solution was poured off from the bottle and one litre of water was added. Then one more litre was poured off and again water was added. As a result the bottle contained 3% (by mass) salt solution. Find the capacity of the bottle.

5.20. There are two tanks, the first filled with pure glycerine and the second with water. We take two three-litre buckets, ladle glycerine out of the first tank by the first bucket and water out of the second tank by the second bucket and then pour glycerine from the first bucket into the second tank and water from the second bucket into the first tank. After stirring the mixtures, we ladle the mixture out of the first tank by the first bucket and the mixture out of the second tank by the second bucket and then pour the mixture from the first bucket into the second tank and that from the second bucket into the first tank. As a result, half the volume of the first tank was occupied by pure glycerine. Find the volumes of the tanks if it is known that their total volume is ten times as large as that of the first tank.

5.21. Two and a half litres of 96% acid solution were poured off from the vessel and 2.5 l of 80% solution of the same acid were added, then again 2.5 l were poured off and 2.5 l of 80% acid solution were added. As a result, the vessel contained 89% acid solution. Find the capacity of the vessel.

5.22. Each of the two vessels contains V l of pure acid. We pour off a l of acid from the first vessel and add a l of water and repeat the procedure once again. Then we pour off $2a$ l of acid from the second vessel and add $2a$ l of water and repeat the procedure once again. As a result, the concentration of acid in the first vessel proves to be 25/16 times as high as that in the second vessel. Find the fraction of a litres in the volume of the vessel.

5.23. Gold dust which is not panned contains k% of pure gold. After each panning, p% of the impurities are washed off and q% of gold are lost. How many pannings must be carried out for the percentage of pure gold in the gold dust to be not less than r?

Chapter 11

Plane Geometry

1. Triangles

Criteria of the equality of triangles. Two triangles are equal if one of the following conditions is fulfilled:

(1) two sides and the angle between them of one triangle are respectively equal to two sides and the angle between them of the other triangle;

(2) two angles and the adjacent side of one triangle are equal to two angles and the adjacent side of the other triangle;

(3) three sides of one triangle are equal to three sides of the other triangle.

Each of the conditions (1)-(3) defines a triangle, i.e. by means of the sine and cosine theorems all other parameters of a triangle can be calculated from any of the conditions (1)-(3).

Formulas for calculating the area of a triangle.

$$S=\frac{1}{2}ah_a=\frac{1}{2}bh_b=\frac{1}{2}ch_c,$$

$$S=\sqrt{p(p-a)(p-b)(p-c)} \quad (\textit{Hero's formula})$$

$$S=\frac{1}{2}ab\sin\gamma,$$

$$S=\frac{abc}{4R},$$

$$S=pr.$$

The sides and the angles of a triangle are related by the formulas

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}=2R \quad (\textit{the sine theorem})$$

$$\left.\begin{aligned} a^2&=b^2+c^2-2bc\cos\alpha\\ b^2&=a^2+c^2-2ac\cos\beta\\ c^2&=b^2+a^2-2ab\cos\gamma \end{aligned}\right\} \quad (\textit{the cosine theorem})$$

where a, b, and c are the sides of the triangle, h_a, h_b, and h_c are the altitudes of the triangle dropped to the sides a, b and c, respectively, α, β, and γ are the interior angles of the triangle lying opposite the sides a, b, and c, respectively, $p=\frac{1}{2}(a+b+c)$ is a semi-perimeter, R is the radius of the circle circumscribed about the triangle, and r is the radius of the circle inscribed into the triangle.

Lines in a triangle. The *median* of a triangle is a line segment connecting the vertex of the triangle and the midpoint of the opposite side.

The main properties of the medians.

(1) A median of a triangle is a locus of points which are the midpoints of the line segments which are contained within the triangle and are parallel to the side to which the median is drawn.

(2) The medians of a triangle meet at one point and are divided by that point in the ratio 2 : 1, reckoning from the vertex of the triangle.

(3) A median divides a triangle into two triangles of equal areas.

(4) Assume that AM, BN, CL are medians of the triangle ABC (Fig. 11.1), and O is the point of intersection of the medians. The areas

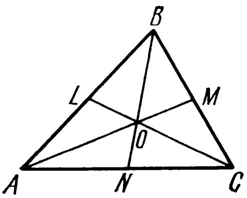

Fig. 11.1

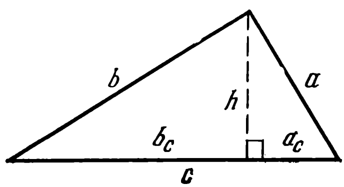

Fig. 11.2

of the triangles ABO, BCO, and ACO are equal to one another and to one-third of the area of the triangle ABC.

An *altitude* of a triangle is a segment of the perpendicular dropped from the vertex of the triangle to the opposite side or to its extension.

A *bisector* of a triangle is a segment of the bisector of the interior angle of the triangle between the vertex of the triangle and the point at which the bisector of the interior angle cuts the opposite side.

The main properties of a bisector.

(1) Three bisectors of a triangle meet at one point which lies inside the triangle and is the centre of a circle inscribed into the triangle.

(2) A bisector of an angle of a triangle is the locus of points which are equidistant from the sides of the angle.

(3) A bisector of the angle of a triangle divides the side of the triangle into parts which are proportional to the adjacent sides.

Some properties of medians, bisectors, and altitudes of triangles of a special kind.

(1) An altitude drawn from the vertex of an isosceles triangle is a bisector and a median at the same time.

(2) In an equilateral triangle an altitude, a bisector, and a median drawn from the same vertex of the triangle coincide; the centre of a circle inscribed in an equilateral triangle coincides with the centre of a circle circumscribed about the triangle and that point is known as the *centre of the triangle*.

(3) In a right triangle the legs a, b and the hypotenuse c are related as

$$a^2 + b^2 = c^2 \quad \text{(the \textit{Pythagorean theorem})}.$$

(4) A leg of a right triangle is the mean proportional between a hypotenuse and the projection of the leg onto the hypotenuse (Fig. 11.2):

$$b_c : b = b : c, \quad a_c : a = a : c$$

(5) An altitude of a right triangle drawn from the vertex of the right angle is the mean proportional between the projections of the legs onto the hypotenuse:

$$b_c : h = h : a_c$$

(6) The centre of a circle circumscribed about a right triangle lies at the midpoint of the hypotenuse; the radius of the circumsribed circle is equal to half the hypotenuse (and also to the median drawn from the vertex of the right angle).

Problems 1.1-1.18 can be solved with the use of the sine and cosine theorems. The conditions of the problems are such that the desired result can be obtained by direct calculations.

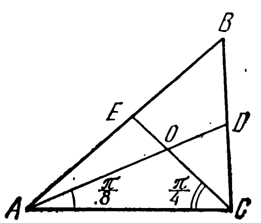

Fig. 11.3

Example 1.1. Medians AD and CE are drawn in the triangle ABC. It is known that $|AD| = 5, \angle DAC = \pi/8, \angle ACE = \pi/4$. Find the area of $\triangle ABC$.

Solution. Assume that O is the point of intersection of the medians of the triangle ABC (Fig. 11.3). To solve the problem, we use the following properties of medians:

(1) the point of intersection divides medians in the ratio 2 : 1 (reckoning from the vertex);

(2) the area of the triangle whose sides are a side of the given triangle and segments of the medians (i.e. $\triangle AOC$) is equal to 1/3 of the area of the given triangle (i.e. $\triangle ABC$).

Thus, to find the area of $\triangle ABC$, it is sufficient to find the area of $\triangle AOC$. By the hypothesis, two angles are known in $\triangle AOC$, and by virtue of (1) the length of the side AO is also known: $|AO| = 10/3$. By the sine theorem we have for $\triangle AOC$

$$\frac{|CO|}{\sin DAC} = \frac{|AO|}{\sin ACE} \Rightarrow \frac{|CO|}{\sin(\pi/8)} = \frac{10/3}{\sin(\pi/4)} \Rightarrow$$

$$|CO| = \frac{10}{3} \cdot \frac{\sin(\pi/8)}{\sin(\pi/4)}.$$

Since the sum of the angles in a triangle is equal to π, we have $\angle AOC = 5\pi/8$. Using the formula for calculating the area of a triangle $S = \frac{1}{2}ab \sin \gamma$, we obtain

$$S_{\triangle AOC} = \frac{1}{2} |AO|\, |CO| \sin AOC$$

$$= \frac{1}{2} \cdot \frac{10}{3} \cdot \frac{10}{3} \frac{\sin(\pi/8)}{\sin(\pi/4)} \sin \frac{5\pi}{8} = \frac{50}{9} \frac{\sin(\pi/8)}{\sin(\pi/4)} \sin\left(\frac{\pi}{2} + \frac{\pi}{8}\right)$$

$$= \frac{50}{9} \frac{\sin(\pi/8)\cos(\pi/8)}{\sin(\pi/4)} = \frac{25}{9} \frac{2\sin(\pi/8)\cos(\pi/8)}{\sin(\pi/4)} = \frac{25}{9}.$$

According to property (2) of medians, we have

$$S_{\triangle ABC}=3S_{\triangle AOC}=\frac{25}{3}.$$

Answer. 25/3.

1.1. The base of a triangle is 12 cm, one of the base angles is 120°, the side opposite that angle is 28 cm. Find the third side.

1.2. Find the bisector of the angle BAC of the triangle ABC if $|AB| = c$, $|AC| = b$, $\angle BAC = \alpha$.

1.3. In the isosceles triangle ABC ($|AB| = |BC|$) the altitude $|AE| = 12$, the base $|AC| = 15$. Find the area of the triangle.

1.4. Given is the acute triangle ABC: $|AB| = c$; the median from the vertex B is $|BD| = m$. The angle BDA is acute and equal to β. Find the area of the triangle ABC.

1.5. Find whether in a triangle with sides equal to 4, 5, 6 cm there is an angle smaller than 22.5°.

1.6. Given in the right triangle ABC: $\angle A = \alpha$, $|AB| = a$. An altitude BE is dropped from the vertex B of the right angle. A median ED is drawn in the triangle BEA. Find the area of $\triangle AED$.

1.7. In the right triangle ABC, with right angle at B, $\angle A = \alpha$, $|AB| = c$. A point D is taken on the extension of the hypotenuse AC (in the direction of the point C) such that $|AD| = r$. Find the area of $\triangle BCD$.

1.8. In the right triangle ABC an altitude BE is dropped from the vertex of the right angle B. A perpendicular is erected at the point C to AC along which a segment CD, equal to r, is laid off. Find the area of $\triangle CED$ if $\angle A = \alpha$, $|AB| = c$.

1.9. The base angle α of an isosceles triangle is such that $\alpha > 45°$ and the area is equal to S. Find the area of the triangle whose vertices are the feet of the altitudes of the given triangle.

1.10. The base of a triangle is 20 cm, the medians of the lateral sides are 24 cm and 18 cm. Find the area of the triangle.

1.11. Given two equilateral triangles with sides $\frac{2}{\sqrt[4]{3}}$, the second triangle resulting from the first upon a rotation through an angle of 30° about its centre. Calculate the area of the common part of the triangles.

1.12. In the triangle ABC, $\angle A = \angle B = \alpha$, $|AB| = a$, AH is an altitude, BE is a bisector (the point H lies on the side BC, the point E on the side AC). Find the area of $\triangle CHE$.

1.13. The bisector AD of $\angle BAC$ and the bisector CF of $\angle ACB$ are drawn in $\triangle ABC$ (the point D lies on the side BC and the point F on the side AB). Find the ratio of the areas of the triangles ABC and AFD if it is known that $|AB| = 21$, $|AC| = 28$ and $|CB| = 20$.

1.14. The base of an isosceles triangle is equal to b, the base angle is equal to α. A straight line cuts the extension of the base at a point M at the angle β and bisects the lateral side of the triangle which is the nearest to M. Find the area of the quadrilateral which the straight line cuts off from the given triangle.

1.15. An altitude BD and a bisector BE are drawn in the triangle ABC from the vertex B. It is known that the length of the side $|AC| = 1$, and the magnitudes of the angles BEC, ABD, ABE, BAC form an arithmetic progression. Find the length of the side BC.

1.16. The vertex angle of an isosceles triangle is equal to 2α. A straight line cutting an altitude at the distance c from the vertex makes an angle β with the extension of the base. Find the area of the triangle which the straight line cuts off from the given triangle.

1.17*. Find the area of the triangle if the lengths of its two sides are equal to 1 and $\sqrt{15}$ cm respectively and the length of the median of the third side is 2 cm.

1.18. In the triangle ABC the bisector of $\angle A$ cuts the side BC at a point M, and the bisector of $\angle B$ cuts the side AC at a point P, with $|AM| = |BP|$. The bisectors meet at a point O. It is known that $\triangle BOM$ is similar to $\triangle AOP$, $|BO| = (1+\sqrt{3})\,|OP|$, $|BC| = 1$. Find the area of the triangle ABC.

The hypotheses of problems 1.19-1.21 do not include quantities which have lengths. It is convenient to introduce an auxiliary quantity a which has length (say, a side of a triangle) and then solve the problem, with the extended condition. In the expression for the required quantities a will be cancelled out and the resulting expression will depend only on the quantities given in the hypothesis.

1.19. The base angle of an isosceles triangle is equal to α. In what ratio is the area of that triangle divided by a straight line which divides its base in the ratio 2 : 1 and makes an acute angle β with the smaller part of the base?

1.20. Given in $\triangle ABC$: $\angle ACB = 60°$, $\angle ABC = 45°$. A point K is taken on the extension of AC beyond the vertex C such that $|AC| = |CK|$. A point M is taken on the extension of BC beyond the vertex C such that the triangle with vertices C, M, and K is similar to the initial one. Find $|BC| : |MK|$ if it is known that $|CM : |MK| < 1$.

1.21. In $\triangle ABC$ the angle B is equal to $\pi/4$ and the angle C is equal to $\pi/3$. Circles meeting at points P and Q are constructed on the medians BM and CN as diameters. The chord PQ cuts the midline MN at point F. Find the ratio of the length of the segment NF to that of the segment FM.

Some problems can be solved by introducing an auxiliary unknown for which, by the hypothesis, it is necessary to derive and solve an equation. A linear dimension of an angle can be taken as an auxiliary unknown. It must be so chosen that the quantities given in the hypothesis and the auxiliary unknown define the triangle uniquely).

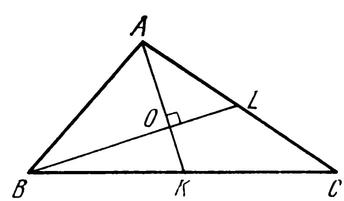

Fig. 11.4

Example 1.2. Find the area of the triangle ABC if $|AC| = 3$, $|BC| = 4$ and the medians AK and BL are mutually perpendicular.

Solution. Let us introduce an angle $\alpha = \angle BCA$ as an unknown quantity (Fig. 11.4). The triple of the given data (the lengths of the sides AC and BC and the angle α) defines the triangle ABC, and all the other parameters of the triangle can be expressed in terms of them. In addition, if the magnitude of the angle α was known, the area of $\triangle ABC$ could be found from the formula

$$S=\frac{1}{2}\,|BC|\,|AC|\sin\alpha.$$

We derive a trigonometric equation for finding the angle α using the condition of mutual perpendicularity of the medians AK and BL. By the cosine theorem, we find $|BA|$ for $\triangle ABC$:

$$|BA|^2=|BC|^2+|CA|^2-2\,|BC|\,|CA|\cos\alpha,$$
$$|BA|^2=4^2+3^2-2\cdot 4\cdot 3\cos\alpha,$$
$$|BA|^2=25-24\cos\alpha.$$

By the cosine theorem, we find the length of the median BL in the triangle BCL (BL being a median, it follows that $|CL|=|LA|=3/2$):

$$|BL|^2=|BC|^2+|CL|^2-2\,|BC|\,|CL|\cos\alpha,$$
$$|BL|^2=4^2+\left(\frac{3}{2}\right)^2-2\cdot 4\cdot\frac{3}{2}\cos\alpha,$$
$$|BL|^2=\frac{73}{4}-12\cos\alpha.$$

Similarly, using the cosine theorem, we find

$$|KA|^2=13-12\cos\alpha$$

for the triangle KCA.

According to the properties of medians we have

$$|BO|=\frac{2}{3}\,|BL|=\sqrt{\frac{73}{9}-\frac{16}{3}\cos\alpha},$$
$$|OA|=\frac{2}{3}\,|KA|=\sqrt{\frac{52}{9}-\frac{16}{3}\cos\alpha}$$

(O is the point of intersection of medians). By the hypothesis BOA is a right triangle and, consequently,

$$|BA|^2=|BO|^2+|OA|^2\Rightarrow$$

$$25-24\cos\alpha=\frac{73}{9}-\frac{16}{3}\cos\alpha+\frac{52}{9}-\frac{16}{3}\cos\alpha\Rightarrow\cos\alpha=\frac{5}{6}.$$

Knowing now the cosine of the angle α, we can find the area of the triangle ABC:

$$S_{\triangle ABC}=\frac{1}{2}\,|BC|\,|AC|\sin\alpha=\frac{1}{2}\,|BC|\,|AC|\sqrt{1-\cos^2\alpha}$$
$$=\frac{1}{2}\cdot 4\cdot 3\sqrt{1-\frac{25}{36}}=\sqrt{11}.$$

Answer. $\sqrt{11}$.

It is convenient to take a linear dimension as an auxiliary unknown in problems 1.22-1.27, an angle in problems 1.28-1.32, and introduce two auxiliary unknowns in problems 1.33-1.35.

1.22. In triangle ABC the altitudes $|CD| = 7$ and $|AE| = 6$. Point E divides the side BC so that $|BE| : |EC| = 3 : 4$. Find the length of the side AB.

1.23. Find the area of an isosceles triangle if the altitude drawn to the base is equal to 10 and that drawn to a lateral side is equal to 12.

1.24. In an isosceles right triangle the medians drawn to the legs are equal to l. Find the area of the triangle.

1.25. In a regular triangle ABC with side a points E and D are the midpoints of the sides BC and AC respectively. Point F lies on the segment DC, the segments BF and DE meet at a point M. Find the length of the segment MF if it is known that the area of the quadrilateral $ABMD$ constitutes 5/8 of the area of the triangle ABC.

1.26. In a triangle with an angle of 120° the lengths of the sides form an arithmetic progression. Find the lengths of all sides of the triangle if the greatest of them is 7 cm.

1.27. The lengths of two sides of an isosceles triangle and the length of the altitude drawn to the base form a geometric progression. Find the tangent of the base angle of the triangle if it is known to exceed 2.

1.28. In a right triangle the ratio of the product of the lengths of the bisectors of the interior acute angles to the square of the length of the hypotenuse is 1/2. Find the acute angles of the triangle.

1.29. In the triangle ABC the length of the side AC is equal to b, the length of the side BA is equal to c, and the bisector of the interior angle A meets the side BC at a point D such that $|DA| = |DB|$. Find the length of the side BC.

1.30. The chord AB subtends the arc of a circle equal to 120°. Point C lies on that arc and point D lies on the chord AB; $|AD| = 2$, $|BD| = 1$, $|DC| = \sqrt{2}$. Find the area of the triangle ABC.

1.31. Given a triangle ABC. A median AM is drawn from the vertex A and a median BP from the vertex B. It is known that $\angle APB$ is equal to $\angle BMA$, the cosine of $\angle ACB$ is equal to 0.8, and $|BP| = 1$ cm. Find the area of $\triangle ABC$.

1.32. Two identical regular triangles ABC and CDE with side 1 lie on the plane so that they have only one point C in common and the angle $BCD < \pi/3$. Point K is the midpoint of the side AC, point L is the midpoint of the segment CE, point M is the midpoint of the segment BD. The area of the triangle KLM is equal to $\sqrt{3}/5$. Find the length of the segment BD.

1.33. A right triangle MNC is inscribed in a right isosceles triangle ABC with $\angle B = 90°$ so that $\angle MNC = 90°$, point N lies on the hypotenuse AC and point M on the side AB. In what ratio must the point N divide the hypotenuse AC for the area of $\triangle MNC$ to constitute 3/8 of the area of $\triangle ABC$?

1.34. A rectangle $MNKB$ is inscribed in an isosceles right triangle ABC with the right vertex angle B so that two sides MB and KB of the rectangle lie on the legs and the vertex N lies on the hypotenuse AC. In what ratio must the point N divide the hypotenuse for the area of the rectangle to constitute 18% of the area of the triangle?

1.35. In the isosceles triangle ABC ($|AB| = |BC|$) the median AD and the bisector CE are perpendicular. Find the angle ADB.

Problems 1.36-1.41 can be solved with the use of the common properties of triangles and various formulas for calculating their areas.

Example 1.3. A point D is taken on the side AB of the triangle ABC between points A and B so that $|AD| : |AB| = \alpha$ $(\alpha < 1)$; a point E is taken on the side BC between points B and C so that $|BE| : |BC| = \beta$ $(\beta < 1)$. A straight line which is parallel to the side AC and cuts the side AB at a point F is drawn through the point E. Find the ratio of the areas of the triangles BDE and BEF.

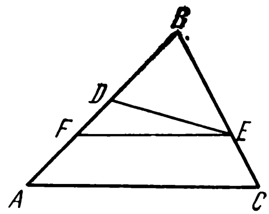

Fig. 11.5

Solution. Assume that the area of $\triangle ABC$ is equal to S. The triangle BEF is similar to $\triangle ABC$ since $FE \parallel AC$ (Fig. 11.5). Since the areas of similar triangles are related as the squares of the respective sides, we have

$$\frac{S_{\triangle BEF}}{S} = \frac{|BE|^2}{|BC|^2} = \beta^2 \Rightarrow S_{\triangle BEF} = S\beta^2.$$

The areas of the triangles BDE and ABC can be expressed in terms of the sides and the angles of those triangles by the formulas

$$S_{\triangle BDE} = \frac{1}{2} |BD|\,|BE| \sin B, \qquad S = \frac{1}{2} |AB|\,|BC| \sin B,$$

from which it follows that

$$\frac{S_{\triangle BDE}}{S} = \frac{|BD|}{|AB|} \cdot \frac{|BE|}{|BC|} = \frac{|BD|}{|AB|} \beta.$$

By the hypothesis, $|AD| = |AB|\alpha$ and since

$$|BD| = |AB| - |AD| = |AB| - |AB|\alpha = |AB|(1 - \alpha),$$

it follows that

$$|BD| / |AB| = 1 - \alpha.$$

Thus we have

$$\frac{S_{\triangle BDE}}{S} = (1-\alpha)\beta \Rightarrow S_{\triangle BDE} = S(1-\alpha)\beta.$$

We have to find the ratio $S_{\triangle BDE} : S_{\triangle BEF}$. Substituting $S_{\triangle BEF} = S\beta^2$ and $S_{\triangle BDE} = S(1 - \alpha)\beta$ into this ratio, we get

$$\frac{S_{\triangle BDE}}{S_{\triangle BEF}} = \frac{1-\alpha}{\beta}.$$

Answer. $(1 - \alpha) : \beta$.

1.36. In triangle ABC a straight line is drawn from the vertex A cutting the side BC at a point D which lies between the points B and C, with $|CD| : |BC| = \alpha$ $(\alpha < 1/2)$. A point E is taken on the side BC between the points B and D and a straight line which is parallel to the side AC and cuts the side AB at a point F is drawn through it. Find the ratio of the areas of the trapezoid $ACEF$ and the triangle ADC if it is known that $|CD| = |DE|$.

1.37. Points E, F, M lie on the sides AB, BC, and AC of the triangle ABC, respectively. The segment AE constitutes 1/3 of the side AB, the segment BF constitutes 1/6 of the side BC, and the segment AM constitutes 2/5 of the side AC. Find the ratio of the area of $\triangle EFM$ to that of $\triangle ABC$.

1.38. Points P, Q, and R are taken on the extensions of the medians AK, BL, and CM of the triangle ABC such that $|KP| = \frac{1}{2}|AK|$, $|LQ| = \frac{1}{2}|BL|$ and $|MR| = \frac{1}{2}|CM|$. Find the area of the triangle PQR if the area of the triangle ABC is equal to unity.

1.39. Given triangle ABC whose area is unity. Points P, Q, and R are taken on the medians AK, BL, and CN of the triangle ABC, respectively, so that

$$\frac{|AP|}{|PK|} = 1, \quad \frac{|BQ|}{|QL|} = \frac{1}{2}, \quad \frac{|CR|}{|RN|} = \frac{5}{4}.$$

Find the area of the triangle PQR.

1.40. The triangle ABC does not have obtuse angles. A point D is taken on the side AC of the triangle such that $|AD| = \frac{3}{4}|AC|$. Find the angle BAC if it is known that the straight line BD divides the triangle ABC into two similar triangles.

1.41. Points P and Q divide the sides BC and CA of the triangle ABC in the ratio

$$\frac{|BP|}{|PC|} = \alpha; \quad \frac{|CQ|}{|QA|} = \beta.$$

Assume that O is the intersection point of the straight lines AP and BQ. Find the ratio of the area of the quadrilateral $OPCQ$ to that of the given triangle.

2. Quadrilaterals

The parallelogram. A quadrilateral whose opposite sides are pairwise parallel is known as a *parallelogram*. A parallelogram possesses the following main properties:

(1) the opposite sides of a parallelogram are equal;

(2) the opposite angles of a parallelogram are equal;

(3) the diagonals of a parallelogram are bisected by the intersection point;

(4) the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of all its sides.

The area of a parallelogram can be calculated by the formula

$$S = ah_a, \quad S = ab \sin \alpha,$$

where a, b are the sides of the parallelogram, h_a, is the altitude of the parallelogram drawn to the side a, α is an angle of the parallelogram.

The rhombus. A parallelogram whose all sides are equal is known as a *rhombus*. Being a parallelogram of a special kind, a rhombus has all the properties of a parallelogram. In addition, a rhombus possesses the following special properties:

(1) the diagonals of a rhombus are mutually perpendicular;

(2) the diagonals of a rhombus are the bisectors of its interior angles.

The area of a rhombus can be calculated by the same formulas as the area of a parallelogram. In addition, the area of a rhombus can be calculated by the formula

$$S = \frac{1}{2} d_1 d_2,$$

where d_1 and d_2 are diagonals of the rhombus.

The rectangle and the square. A parallelogram whose all angles are right angles is known as a *rectangle*. The area of a rectangle can be calculated by the formula

$$S = ab,$$

where a and b are adjacent sides of the rectangle.

A rectangle whose all sides are equal is known as a *square*. A square has all the properties of a parallelogram, a rhombus, and a rectangle. The area of a square can be calculated by the formula

$$S = a^2,$$

where a is a side of the square.

The trapezoid. A quadrilateral whose two sides are parallel and the other two sides are nonparallel is known as a *trapezoid*. The area of a trapezoid with bases a and b and the altitude h can be calculated by the formula

$$S = \frac{a+b}{2} h.$$

The line segment which connects the midpoints of the nonparallel sides of a trapezoid is called a *median* of the trapezoid. The median of a trapezoid possesses the following properties:

(1) the median of a trapezoid is parallel to the bases and is equal to half their sum;

(2) the median divides the altitude of a trapezoid into two equal segments.

In problems 2.1-2.17 the required quantity can be found by direct calculations.

Example 2.1. Given a trapezoid $PQRN$ with bases PN and QR, in which $|PN| = 8$, $|QR| = 4$, $|PQ| = \sqrt{28}$, $\angle RNP = 60°$.

A straight line passes through the point R and divides the trapezoid into two figures of equal areas. Find the length of the segment of that line which is in the interior of the trapezoid.

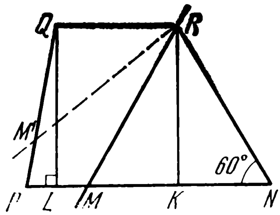

Fig. 11.6

Solution. Assume that the line dividing the trapezoid into two figures of equal areas cuts the base PN of the trapezoid at a point M (Fig. 11.6). We shall verify the validity of this statement somewhat later.

Let us drop altitudes from the points Q and R to the base PN of the trapezoid. Since $|LK| = |QR| = 4$ and $|PN| = 8$, the lengths of the segments $|PL|$ and $|KN|$ satisfy the equality $|PL| + |KN| = 4$. We introduce the designation $|PL| = x$, and then we have $|KN| = 4 - x$. Since the length of the nonparallel side PQ is known, we obtain from the triangle PQL by the Pythagorean theorem

$$|QL| = \sqrt{28 - x^2}.$$

We express $|KR|$ from the triangle KRN as follows:

$$|KR| = (4 - x) \tan 60° = \sqrt{3}\,(4 - x);$$

QL and KR are altitudes of the trapezoid, and we find from the equation $|QL| = |KR|$ that

$$\sqrt{28 - x^2} = \sqrt{3}\,(4 - x) \Rightarrow 28 - x^2 = 3\,(4 - x)^2$$
$$\Rightarrow x^2 - 6x + 5 = 0 \Rightarrow x = 5,\ x = 1.$$

By the geometrical meaning of the problem, from the two values of the unknown x the value $x = 1$ is suitable and, consequently,

$$|PL| = 1, \quad |KN| = 3, \quad |KR| = 3\sqrt{3}.$$

We can now calculate the area of the trapezoid $PQRN$:

$$S_{PQRN} = \frac{4+8}{2}\, 3\sqrt{3} = 18\sqrt{3}.$$

Since by the hypothesis the straight line RM divides the trapezoid into two parts of equal areas and by the assumption the line RM cuts the base of the trapezoid, we have

$$S_{\Delta MRN} = 9\sqrt{3}.$$

Knowing the altitude of the triangle MRN (the length of the line segment RK), we can calculate the base of the triangle MRN:

$$9\sqrt{3} = \frac{1}{2}\,|KR|\,|MN| \Rightarrow |MN| = 6.$$

Since $|PN| = 8$, $|MN| = 6$, our assumption that the line passing through the point R cuts the base PN of the trapezoid proved to be correct. If the calculations carried out led to the inequality $|MN| > |PN|$, that would mean that the line passing through the point R cuts one of the nonparallel sides, PQ, of the trapezoid.

Let us now calculate the required length of the segment MR. We have found that $|KN| = 3$, $|MK| = |MN| - |KN| = 3$. By the Pythagorean theorem we find from the right triangle MRK that $|MR| = \sqrt{|MK|^2 + |RK|^2} = \sqrt{9 + 27} = 6$.

Note that if the straight line cut one of the nonparallel sides, PQ, of the trapezoid at a point M', (Fig. 11.6), we could find the length of the segment $M'R$ from the triangle $M'QR$. To calculate that segment, it would be necessary first to find the angle PQR of the trapezoid (which angle is also an angle of the triangle $M'QR$ being considered), and then, from the known angle PQR, the area of the triangle $S_{\Delta M'QR} = \frac{1}{2} S_{PQRN}$ and the base QR to find successively the length $|M'Q|$ and, by the cosine theorem, the length $|M'R|$.

Answer. The length of the segment is equal to 6.

2.1. Find the diagonal and the area of an isosceles trapezoid if its bases are 3 cm and 5 cm and one of the nonparallel sides is 7 cm.

2.2. Find the area of an isosceles trapezoid whose bases are equal to 12 cm and 20 cm and the diagonals are mutually perpendicular.

2.3. In the trapezoid $ABCD$ the length of the base AD is 2 m and that of the base BC is 1 m. The lengths of the nonparallel sides AB and CD are equal to 1 m. Find the length of the diagonal of the trapezoid.

2.4. One of the angles of a trapezoid is 30° and the nonparallel sides, when extended, meet at right angles. Find the smaller of the nonparallel sides of the trapezoid if its median is 10 cm and the smaller base is 8 cm.

2.5. In the trapezoid $ABCD$ the length of the smaller base BC is 3 m, the lengths of the nonparallel sides AB and CD are equal to 3 m. The diagonals of the trapezoid make an angle of 60°. Find the length of the base AD.

2.6. The larger base of the trapezoid is 5 cm, one of the nonparallel sides is 3 cm. One of the diagonals is known to be perpendicular to the given nonparallel side and the other bisects the angle between the given nonparallel side and the base. Find the area of the trapezoid.

2.7. Given in the isosceles trapezoid $ABCD$: $|AC| = a$, $\angle CAD = \alpha$. Find the area of the trapezoid.

2.8. Given a square in which another square is inscribed whose vertices lie on the sides of the first square, and the angles between the sides of the squares are equal to 60°. What part of the given square does the area of the inscribed square constitute?

2.9. Given an isosceles trapezoid $ABCD$. It is known that $|AD| = 10$, $|BC| = 2$, $|AB| = |CD| = 5$. The bisector of the angle BAD cuts the extension of the base BC at a point K. Find the length of the bisector of the angle B in the triangle ABK.

2.10. The bases of an isosceles trapezoid are equal to a and b and the angle the diagonal makes with the base is equal to α. Find the length of the segment connecting the point of intersection of the diagonals with the midpoint of one of the nonparallel sides of the trapezoid.

2.11. In the trapezoid $ABCD$, where AD is the base, diagonals AC and BD are drawn which meet at a point O. It is known that the length of the diagonal AC is equal to l and the angles AOB, ACB, ACD, BDC, ADB form an arithmetic progression (in the order they are written). Find the length of the base AD.

2.12. Given an isosceles trapezoid $ABCD$ in which $|AB| = |CD| = 3$, $|AD| = 7$, $\angle BAD = 60°$. A point M is located on the diagonal BD so that $|BM| : |MD| = 3 : 5$. Which of the sides of the trapezoid, BC or CD, is cut by the extension of the segment AM?

2.13. Calculate the area of the common part of two rhombi, the diagonals of the first rhombus being equal to 2 and 3, and the second rhombus being obtained by a rotation of the first through 90° about its centre.

2.14. In the square $ABCD$ with the area 1 the side AD is extended beyond the point D and a point O is taken on the extension at the distance of 3 from the point D. Two rays are drawn from the point O. The first ray cuts the segment CD at a point M and the segment AB at a point N, the length of the segment ON being equal to a. The second ray cuts the segment CD at a point L and the segment BC at a point K, with $\angle BKL = \alpha$. Find the area of a polygon $BKLMN$.

2.15. The bisectors of four angles are drawn in a parallelogram with sides a and b and an angle α. Find the area of the quadrilateral bounded by the bisectors.

2.16. The length of the lateral side AB of the parallelogram $ABCD$ is equal to a; the length of the perpendicular dropped from the point of intersection of the diagonals to the base is equal to h; the angle between the larger diagonal BD and the base AD is equal to α. Find the area of the parallelogram.

2.17. The bases in the trapezoid $ABCD$ are: $|AD| = 16$, $|BC| = 9$. A point M is chosen on the extension of BC such that $|CM| = 3.2$. In what ratio does the straight line AM divide the area of the trapezoid $ABCD$?

Problems 2.18-2.29 can be solved by introducing an auxiliary unknown (or several unknowns) for which an equation (a system of equations, respectively) is set up from the hypothesis. An angle or an unknown linear dimension can be taken as an auxiliary unknown.

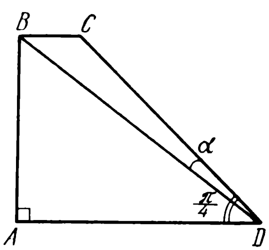

Fig. 11.7

Example 2.2. In the convex quadrilateral $ABCD$ the vertex angles A and B are right angles, the vertex angle D is equal to $\pi/4$, $|BC| = 1$, and the length of the diagonal BD is 5. Find the area of the quadrilateral.

Solution. We designate the angle BDC as α (Fig. 11.7). Since the sum of the interior angles of any quadrilateral is equal to 2π, and three angles of the quadrilateral $ABCD$ are known from the hypothesis, it follows that $\angle C = 3\pi/4$. Let us consider the triangle BDC. By means of the sine theorem, we find the angle α of the triangle BDC:

$$\frac{|BC|}{\sin\alpha}=\frac{|BD|}{\sin C}\Rightarrow\frac{1}{\sin\alpha}=\frac{5}{\sqrt{2}/2}$$

$$\Rightarrow \sin\alpha=\frac{\sqrt{2}}{10}\Rightarrow\alpha=\arcsin\frac{\sqrt{2}}{10}$$

(α is an acute angle).

In the triangle BDC: $\angle DBC=\frac{\pi}{4}-\arcsin\frac{\sqrt{2}}{10}$.

In the triangle ABD: $\angle ADB=\frac{\pi}{4}-\arcsin\frac{\sqrt{2}}{10}$.

From the right triangle ADB we obtain

$$|AD|=5\cos\left(\frac{\pi}{4}-\arcsin\frac{\sqrt{2}}{10}\right)$$

$$=5\left[\cos\frac{\pi}{4}\cos\left(\arcsin\frac{\sqrt{2}}{10}\right)+\sin\frac{\pi}{4}\sin\left(\arcsin\frac{\sqrt{2}}{10}\right)\right]$$

$$=5\left[\frac{\sqrt{2}}{2}\sqrt{\frac{98}{100}}+\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{10}\right]=4,$$

$$|AB|=\sqrt{|BD|^2-|AD|^2}=\sqrt{5^2-4^2}=3.$$

Since by the hypothesis the angles A and B are right angles, it follows that $AD \parallel BC$ and the quadrilateral $ABCD$ is a trapezoid (AD and BC are the bases, AB is an altitude), whence we have

$$S_{\text{tr}}=\frac{|BC|+|AD|}{2}\,|AB|=\frac{15}{2}.$$

Answer. 15/2.

2.18. Given a square with vertices A, B, C, D and a point O lying outside the square. It is known that $|OA|=|OB|=5$ and $|DO|=\sqrt{13}$. Find the area of the square.

2.19. In the convex quadrilateral $ABCD$ the bisector of the angle ABC cuts the side AD at a point M, and a perpendicular dropped from the vertex A to the side BC cuts BC at a point N so that $|BN|=|NC|$ and $|AM|=2|MD|$. Find the sides and the area of the quadrilateral $ABCD$ if its perimeter is equal to $5+\sqrt{3}$, $\angle BAD=90°$ and $\angle ABC=60°$.

2.20. In the trapezoid $ABCD$ the angles A and D at the base AD are equal to 60° and 90° respectively. Point N lies on the base BC and $|BN|:|NC|=3:2$. Point M lies on the base AD, and the straight line $|MN|$ is parallel to one of the nonparallel sides, AB, and divides the area of the trapezoid in half. Find the ratio $|AB|:|BC|$.

2.21. The length of the median of the trapezoid is 5 cm, and the length of the segment connecting the midpoints of the bases is 3 cm.

The angles at the larger base of the trapezoid are 30° and 60°. Find the area of the trapezoid.

2.22. The sum of the acute angles of a trapezoid is 90°, the altitude is 2 cm, and the bases are 12 cm and 16 cm. Find the nonparallel sides of the trapezoid.

2.23. In the rhombus $ABCD$ with side a the vertex angle A is equal to $\pi/3$. Points E and F are the midpoints of the sides AB and CD respectively. Point K lies on the side BC, the segments AK and EF meet at a point M. Find the length of the segment MK if it is known that the area of the quadrilateral $MKCF$ constitutes 3/8 of the area of the rhombus $ABCD$.

2.24. In the right-angled trapezoid $ABCD$ the angles A and D are right angles, the side AB is parallel to the side CD; $|AB| = 1$, $|CD| = 4$, $|AD| = 5$. A point M is taken on the side AD such that the angle CMD is twice the angle BMA. In what ratio does the point M divide the side AD?

2.25. The length of the diagonal BD of the trapezoid $ABCD$ is equal to m, and the length of one of the nonparallel sides, AD, is equal to n. Find the length of the base CD if it is known that the lengths of the base, the diagonal, and one of the nonparallel sides of the trapezoid drawn from the vertex C are equal to one another.

2.26. In the trapezoid $ABCD$ the diagonals AC and DB are mutually perpendicular, $\angle BAC = \angle CDB$. The extensions of the nonparallel sides AB and DC meet at a point K forming an angle AKD equal to 30°. Find the area of the triangle AKD if the area of the trapezoid is equal to S.

2.27. In an isosceles trapezoid with bases a and b $(a > b)$ the diagonals are the bisectors of the angles at the larger base. Find the area of the trapezoid.

2.28. In the trapezoid $ABCD$ the diagonal AC is perpendicular to one of the nonparallel sides, CD, and the diagonal DB is perpendicular to the other nonparallel side, AB. Line segments BM and CN are laid off on the extensions of the nonparallel sides AB and CD beyond the smaller base BC so that a new trapezoid $BMNC$ results which is similar to the trapezoid $ABCD$. Find the area of the trapezoid $ABCD$ if the area of the trapezoid $AMND$ is S and the sum of the angles CAD and BDA is 60°.

2.29. Given a parallelogram $ABCD$ with sides $|AB| = 2$ and $|BC| = 3$. Find the area of the parallelogram if it is known that the diagonal AC is perpendicular to the segment BE which connects the vertex B and the midpoint E of the side AD.

When solving problems 2.30-2.36, use is made of the special properties of polygons and triangles following from the hypotheses.

Example 2.3. In a trapezoid with bases a and b a straight line, which is parallel to the base, is drawn through the point of intersection of the diagonals. Find the length of the segment of that line between the nonparallel sides of the trapezoid.

Solution. Assume that the base BC of the trapezoid $ABCD$ is equal to a, and the base AD is equal to b (Fig. 11.8), AC and BD are diagonals, O is the point of their intersection, BN is an altitude of the trapezoid, M is the point of intersection of the altitude BN and the required segment KL. By the hypothesis, $KL \parallel BC$, and, consequently, the triangle ABD is similar to the triangle KBO, and the triangle ABC

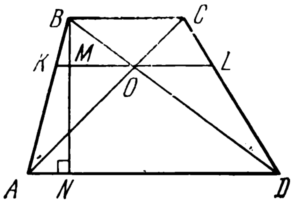

Fig. 11.8

is similar to the triangle AKO. Since the altitudes of similar triangles are proportional to the sides to which they are drawn, we have

$$\frac{|KO|}{|AD|}=\frac{|BM|}{|BN|}, \quad \frac{|KO|}{|BC|}=\frac{|MN|}{|BN|}.$$

As a consequence of these equations and the hypothesis, we obtain

$$\frac{|KO|}{|AD|}+\frac{|KO|}{|BC|}=\frac{|BM|}{|BN|}+\frac{|MN|}{|BN|} \Rightarrow |KO|\left(\frac{|BC|+|AD|}{|AD|\,|BC|}\right)$$
$$=\frac{|BM|+|MN|}{|BN|} \Rightarrow |KO|\,\frac{a+b}{ab}=1 \Rightarrow |KO|=\frac{ab}{a+b}.$$

Similarly, from the similarity of two pairs of triangles $\triangle DOL \sim \triangle DBC$ and $\triangle OCL \sim \triangle ACD$, we find that $|OL| = \frac{ab}{a+b}$, and, therefore, $|KL| = |KO| + |OL| = \frac{2ab}{a+b}$.

Answer. $\frac{2ab}{a+b}$.

2.30. Given a parallelogram $ABCD$ with area 1. A straight line is drawn through the midpoint M of the side BC and the vertex A which cuts the diagonal BD at a point O. Find the area of the quadrilateral $OMCD$.

2.31. The following points are taken on the sides of a convex quadrilateral $ABCD$ whose area is equal to 1: K on AB; L on BC; M on CD; N on AD. We also have

$$\frac{|AK|}{|KB|}=2, \quad \frac{|BL|}{|LC|}=\frac{1}{3}, \quad \frac{|CM|}{|MD|}=1, \quad \frac{|DN|}{|NA|}=\frac{1}{5}.$$

Find the area of the hexagonal $AKLCMN$.

2.32. A, B, C, D are successive vertices of a parallelogram. Points E, F, P, H lie on the sides AB, BC, CD, AD, respectively.

The segment AE is equal to 1/3 of the side AB, the segment BF is equal to 1/3 of the side BC, and the points P and H bisect the sides on which they lie. Find the ratio of the area of the quadrilateral $EFPH$ to that of the parallelogram $ABCD$.

2.33. In the convex quadrilateral $ABCD$ the length of the segment connecting the midpoints of the sides AB and CD is equal to 1. The straight lines BC and AD are perpendicular. Find the length of the segment connecting the midpoints of the diagonals AC and BD.

2.34. The lengths of the diagonals of a rhombus and the length of its side form a geometric progression. Find the sine of the angle between the side of the rhombus and its larger diagonal if it is known to be larger than 1/2.

2.35. In the convex quadrilateral $KLMN$ points E, F, G, H are, respectively, the midpoints of the sides KL, LM, MN, NK. The area of the quadrilateral $EFGH$ is equal to Q, $\angle HEF = \pi/6$, $\angle EFH = \pi/2$. Find the lengths of the diagonals of the quadrilateral $KLMN$.

3. The Circumference and the Circle

A *circumference* is a set of all points of a plane which are at a given positive distance from a certain given point of the plane, which is called the *centre of the circumference*. A *circle* * consists of a circumference and its interior points.

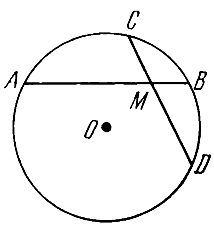

Fig. 11.9

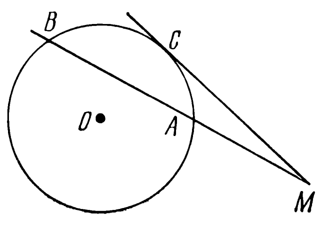

Fig. 11.10

A line segment connecting two points of a circle is called a *chord*. Chords possess the following properties:

(1) The diameter bisecting the chord is perpendicular to it.

(2) Equal chords of a circle are equidistant from its centre; chords, which are equidistant from the centre of a circle are equal.

(3) If two chords AB and CD are drawn through the point M which lies in the interior of a circle (Fig. 11.9), then the products of the segments of the chords are equal: $|AM|\,|MB| = |CM|\,|MD|$.

Theorem on a tangent and a secant. If a tangent MC and a secant MA are drawn from a point M (Fig. 11.10) lying outside the circle, then the product of the secant by its exterior part is equal to the square of the tangent: $|MC|^2 = |MA|\,|MB|$.

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* When no ambiguity arises, it is used as a synonym of a circumference.

Lengths and areas.
The length of a circle of radius R: $L = 2\pi R$.
The area of a circle of radius R: $S = \pi R^2$.
The length of an arc of a circle of radius R with the central angle α (in radians): $l = R\alpha$.
The area of a sector of a circle of radius R with the central angle α (in radians): $s = \frac{1}{2}R^2\alpha$.

The properties of lines in tangent and intersecting circles.
(1) The line of centres of two tangent circles passes through the point of tangency.
(2) The common interior tangent of two externally tangent circles is perpendicular to their line of centres.
(3) The common tangent of two internally tangent circles is perpendicular to their line of centres.
(4) The common chord of two intersecting circles is perpendicular to their line of centres and is bisected by their point of intersection.

Problems 3.1-3.9 can be solved by direct calculations based on the properties of lines in circles.

Example 3.1. The common chord of two intersecting circles can be seen from their centres at the angles of 90° and 60°. Find the radii of the circles if the distance between their centres is equal to $\sqrt{3} + 1$.

Solution. Assume that O_1 and O_2 are the centres of the circles, AB is the common chord, K is the point of intersection of the line of

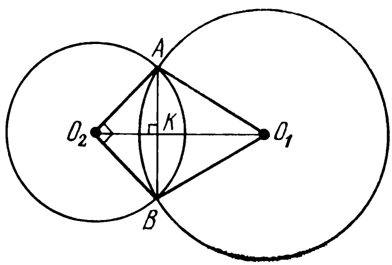

Fig. 11.11

centres O_1O_2 and the common chord AB (Fig. 11.11); the angle AO_1B is 60°, and the angle AO_2B is 90°. Let us consider the triangle AO_1B. The triangle is isosceles ($|AO_1| = |BO_1|$) and $O_1K \perp AB$, i.e. O_1K is the altitude, the median and the bisector of the triangle AO_1B. By the hypothesis, the angle AO_1B is 60° and, consequently, the angle AO_1K is 30°. Similarly, we find for the triangle AO_2B that the angle AO_2K is 45°. Let us consider the triangle O_1AO_2. In this triangle we know two angles (AO_1K and AO_2K), which are equal to 30° and 45° respectively, and the segment O_1O_2 which is equal to $\sqrt{3} + 1$. The sides O_1A and AO_2 of the triangle are the required radii.

Since the sum of the angles of a triangle is equal to 180°, the angle O_1AO_2 is 105° and by the sine theorem we have for the triangle O_1AO_2

$$\frac{|O_1A|}{\sin 45^\circ}=\frac{\sqrt{3}+1}{\sin 105^\circ}, \quad \frac{|O_2A|}{\sin 30^\circ}=\frac{\sqrt{3}+1}{\sin 105^\circ}. \qquad (*)$$

But $\sin 105^\circ = \sin (90^\circ + 15^\circ) = \cos 15^\circ$. By the formula $1 + \cos\alpha = 2\cos^2\frac{\alpha}{2}$, we can set $\alpha = 30^\circ$ and calculate $\cos 15^\circ$:

$$2\cos^2 15^\circ = 1+\cos 30^\circ \Rightarrow 2\cos^2 15^\circ = \frac{2+\sqrt{3}}{2}$$

$$\Rightarrow \cos 15^\circ = \frac{\sqrt{2+\sqrt{3}}}{2};$$

from equations (*) we find that

$$|O_1A| = \frac{\sqrt{2}}{2}\,\frac{(\sqrt{3}+1)\cdot 2}{\sqrt{2+\sqrt{3}}} = \frac{\sqrt{2}(\sqrt{3}+1)}{\sqrt{2+\sqrt{3}}} = \frac{\sqrt{2}(\sqrt{3}+1)}{\sqrt{(\sqrt{3}+1)^2/2}} = 2,$$

$$|O_2A| = \frac{(\sqrt{3}+1)\cdot 2}{2\sqrt{2+\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{2+\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{(\sqrt{3}+1)^2/2}} = \sqrt{2}.$$

Answer. 2 and $\sqrt{2}$.

3.1. Three circles lie on a plane so that each of them externally touches the other two. Two of them have radius 3, the third, radius 1. Find the area of the triangle ABC, where A, B, and C are points of tangency of the circles.

3.2. Given two externally tangent circles of radii R and r. Find the length of the segment of the external tangent between the points of tangency.

3.3. Two circles whose radii are equal to 4 and 8 meet at right angles. Find the length of their common tangent.

3.4. The centres of four circles lie at the vertices of a square with side a. The radii of all the circles are also equal to a. Find the area of the part of the plane which is common for all the circles.

3.5. A point O is taken outside the right angle with vertex C on the extension of its bisector such that $|OC| = \sqrt{2}$. A circle of radius 2 with centre at the point O is constructed. Find the area of the figure bounded by the sides of the angle and the arc of the circle contained between them.

3.6. Points A and B are taken on the straight line, which passes through the centre O of the circle of the radius of 12 cm, such that $|OA| = 15$ cm and $|AB| = 5$ cm. Tangents to the circle are drawn from the points A and B whose points of tangency lie on the same side of the straight line OAB. Find the area of the triangle ABC, if C is the point of intersection of those tangents.

3.7. Given two nonintersecting circles of radii R and $2R$. Common tangents are drawn to them and meet at a point A of the segment connecting the centres of the circles. The distance between the centres of the circles is equal to $2R\sqrt{3}$. Find the area of the figure bounded by the segments of the tangents contained between the points of tangency and the larger arcs of the circles connecting the points of tangency.

3.8. A tangent AB is drawn to a circle from a point A which lies on the extension of the diameter KL of the circle in the direction of the point L (B is the point of tangency). The tangent AB mak s an angle α with the diameter KL. Find the area of the figure formed by the sides of the angle and the arc LB if the radius of the circle is R.

3.9. Two circles of radii 5 cm and 3 cm are internally tangent. A chord of the larger circle touches the smaller circle and is divided by the point of tangency in the ratio 3 : 1. Find the length of the chord.

Problems 3.10-3.24 can be solved by introducing an auxiliary unknown for which an equation is set up from the hypothesis.

Example 3.2. Two circles of radii R and r are externally tangent. Find the radius of the third circle which is between them and touches those circles and their external tangent.

Solution. Assume that O_1, O_2, and O_3 are, respectively, the centres of the circles of radii R, r and the required circle; M_1M_2 is the common external tangent of the given circles (Fig. 11.12). We designate

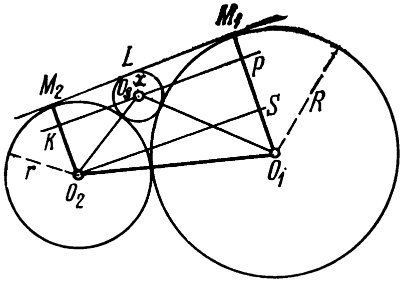

Fig. 11.12

as L the point of tangency of the required circle and the straight line M_1M_2. Through the centre O_3 of the sought-for circle we draw a straight line which is parallel to the line M_1M_2 (P an l K are the points of intersection of that line and the segments O_1M_1 and O_2M_2). The straight line PK is perpendicular to the lines O_1M_1, O_2M_2 and O_3L. We designate the radius of the required circle as x. In the right triangle O_1PO_3 the length of the hypotenuse $|O_1O_3| = R + x$, the length of the leg $O_1P = R - x$. By the Pythagorean theorem, $|PO_3| = 2\sqrt{Rx}$. Similarly, from the right triangle O_3KO_2 we find that $|O_3K| = 2\sqrt{rx}$.

Through the centre of the smaller of the circles of radii R and r we draw a straight line which is parallel to the common external tangent (in Fig. 11.12 the circle with centre O_2 has a smaller radius than the circle with centre O_1, and O_2S is the drawn straight line). From the right triangle O_1SO_2 in which $|O_1O_2| = R + r$, $|O_1S| = R - r$, we find that $|O_2S| = 2\sqrt{Rr}$. The line segments SO_2, PK, M_1M_2 are parallel to one another since each of them is perpendicular to the parallel straight lines O_1M_1 and O_2M_2 and $|SO_2| = |PK| = |M_1M_2|$.

The equality $|SO_2| = |PK| = |PO_3| + |O_3K|$ yield an equation making it possible to find the unknown x:

$$2\sqrt{Rx}+2\sqrt{rx}=2\sqrt{Rr} \Rightarrow Rx+2x\sqrt{Rr}+rx=Rr$$

$$\Rightarrow x(R+2\sqrt{Rr}+r)=Rr \Rightarrow x=\frac{Rr}{R+2\sqrt{Rr}+r}=\frac{Rr}{(\sqrt{R}+\sqrt{r})^2}.$$

Answer. $\dfrac{Rr}{(\sqrt{R}+\sqrt{r})^2}$.

Example 3.3. AOB is a sector of a circle of radius r. The angle AOB is equal to α ($\alpha < \pi$). Find the radius of the circle which lies inside the sector and touches the chord AB, the arc AB, and the bisector of $\angle AOB$.

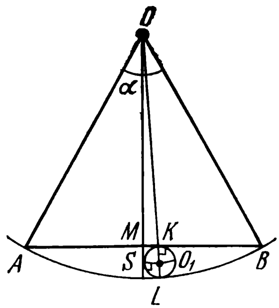

Fig. 11.13

Solution. Assume that OM is the bisector of the angle AOB, O_1 is the centre of the required circle, S is the point of tangency of the required circle and the bisector OM, K is the point of tangency of the required circle and the chord AB (Fig. 11.13). The segment OM is the bisector of the angle AOB and, since $\triangle AOB$ is an isosceles triangle, it follows that OM is at the same time an altitude of the triangle AOB. The quadrilateral $SMKO_1$ is a square since $|SO_1| = |KO_1|$, and the angles O_1SM, SMK, and MKO_1 are right angles. By the theorem on a straight line passing through the centres of two tangent circles, the centres O and O_1 of the circles and the point of tangency L of those circles lie on the same straight line OL.

We designate the radius of the sought-for circle as $|O_1K| = x$. The diagonal MO_1 of the square MSO_1K is equal to $\sqrt{2}\,x$. From the triangle OMB we find, in accordance with the hypothesis, that $|OM| = r\cos\frac{\alpha}{2}$. In the triangle OMO_1 we have: $|MO_1| = \sqrt{2}\,x$, $|OM| = r\cos\frac{\alpha}{2}$, $|OO_1| = r - x$, $\angle OMO_1 = 135°$. The cosine theorem gives for the triangle OMO_1 an equation for the unknown x:

$$(r-x)^2=2x^2+r^2\cos^2\frac{\alpha}{2}-2x\sqrt{2}\,r\cos\frac{\alpha}{2}\cos 135°,$$

$$(r-x)^2=2x^2+r^2\cos^2\frac{\alpha}{2}+2rx\cos\frac{\alpha}{2}$$

$$\Rightarrow x^2+2r\left(\cos\frac{\alpha}{2}+1\right)x+r^2\left(\cos^2\frac{\alpha}{2}-1\right)=0$$

$$\Rightarrow x_{1,2} = -2r\cos^2\frac{\alpha}{4} \pm 2r\cos\frac{\alpha}{4} \Rightarrow$$

$$x_{1,2} = 2r\cos\frac{\alpha}{4}\left(-\cos\frac{\alpha}{4} \pm 1\right).$$

Since the quantity x must be positive, and only the first of the roots obtained is positive

$$x_1 = 2r\cos\frac{\alpha}{4}\left(1-\cos\frac{\alpha}{4}\right) = 4r\cos\frac{\alpha}{4}\sin^2\frac{\alpha}{8},$$

it is the first root which gives the value of the radius of the circle in question.

Answer. $4r\cos\frac{\alpha}{4}\sin^2\frac{\alpha}{8}$.

3.10. When two circles of radius 32 with centres O_1 and O_2 intersect, they divide the line segment O_1O_2 into three equal parts. Find the radius of the circle which internally touches the two given circles and the segment O_1O_2.

3.11. Given two intersecting circles of the same radius R. The distance between the centres of the circles $|O_1O_2| = l$. Find the area of the circle which internally touches the two circles and the straight line O_1O_2.

3.12. Two circles whose radii are R_1 and R_2 intersect. The distance between their centres is equal to d. Find the radius of the circle which touches the given circles and their common tangent.

3.13. A circle of the radius of 6 cm lies in the interior of a semicircle of the radius of 24 cm and touches the midpoint of the diameter of the semicircle. Find the radius of the smaller circle which touches the given circles, the semicircle and the diameter of the semicircle.

3.14. Given a circle of radius r with centre at a point O. From a point A of the segment OA, which meets the circle at a point M, a secant is drawn to the circle which cuts the circle at points K and P; the point K lies between the points A and P. The angle MAK is equal to $\pi/3$. The length of the segment OA is a. Find the radius of the circle touching the segments AM, AK and the arc MK.

3.15. A secant OM and a tangent MC are drawn from the point M to a circle of radius of 3 cm with centre at a point O, the tangent touching the circle at a point C. Find the radius of the circle which touches the given circle and the straight lines MC and OM and lies in the interior of the triangle OMC if $|OM| = 5$ cm.

3.16. Given a segment with an arc of 120° and an altitude h. A rectangle is inscribed into it whose base is 4 times the altitude. Find the sides of the rectangle.

3.17. A rectangle $KMPT$ lies in a circular sector OAB whose central angle is equal to $\pi/4$. The side KM of the rectangle lies on the radius OA, the vertex P lies on the arc AB, and the vertex T lies on the radius OB. The side KT is 3 cm longer than the side KM. The area of the rectangle $KMPT$ is 18 cm². Find the length of the radius.

3.18. Two secants AKC and ALB are drawn from a point A, which is at the distance of 5 cm from the centre of a circle of the radius of 3 cm, the angle between the secants being equal to 30° (K, C,

L, and B are the points of intersection of the secants and the circle). Find the area of $\triangle AKL$ if the area of $\triangle ABC$ is 10 cm^2.

3.19. Given two identical intersecting circles. The ratio of the distance between their centres to the radius is $2m$. A third circle is externally tangent to both circles and their common tangent. Find the ratio of the area of the common part of the first two circles to that of the third circle.

3.20. A square is inscribed into a circular sector, bounded by the radii OA and OB, with central angle α ($\alpha < \pi/2$), so that its two adjacent vertices lie on the radius OA, the third vertex lies on the radius OB and the fourth vertex lies on the arc AB. Find the ratio of the areas of the square and the sector.

3.21. Two mutually perpendicular intersecting chords AB and CD are drawn in a circle. It is known that

$$|AB| = |BC| = |CD|.$$

Find what is larger, the area of the circle or that of the square with side AB.

3.22. Two circles with the radii of $\sqrt{5}$ cm and $\sqrt{2}$ cm intersect. The distance between the centres of the circles is 3 cm. A straight line which is drawn through a point A (one of the intersection points) and cuts the circle at points B and C ($B \neq C$) is such that $|AB| = |AC|$. Find the length of the segment AB.

4. Triangles and Circles

A triangle whose all vertices lie on a circle is said to be *inscribed in the circle* and the circle is said to be *circumscribed about the triangle.* The centre of the circle circumscribed about the triangle lies on the intersection of the perpendiculars drawn to the midpoints of the sides of the triangle.

The radius of a circle circumscribed about a triangle can be calculated by the formula

$$R = \frac{1}{2}\,\frac{a}{\sin\alpha} = \frac{1}{2}\,\frac{b}{\sin\beta} = \frac{1}{2}\,\frac{c}{\sin\gamma}$$

or by the formula

$$R = abc/4S,$$

where a, b, c are the sides of the triangle, α, β, γ are the angles of the triangle which lie opposite the sides a, b, and c respectively, S is the area of the triangle.

A circle which touches all the sides of a triangle is said to be *inscribed in the triangle*. The centre of the circle inscribed into the triangle lies on the intersection of the bisectors of the interior angles of the triangle.

The radius of a circle inscribed in a triangle can be calculated by the formula

$$r = S/p,$$

where $p = \frac{1}{2}(a + b + c)$ is a semi-perimeter of the triangle.

Problems 4.1-4.36 can be solved by direct calculations using the properties of triangles inscribed in a circle and of circles inscribed in triangles.

Example 4.1. A point D is taken on the side AC of an acute triangle ABC such that $|AD| = 1$, $|DC| = 2$ and $|BD|$ is an altitude of the triangle ABC. A circle of radius 2, which passes through the points A and D, touches at the point D a circle circumscribed about the triangle BDC. Find the area of the triangle ABC.

Solution. Assume that O_1 is the centre of the circle of radius 2 which passes through the points A and D, O_2 is the centre of the circle which is circumscribed about the triangle BDC (Fig. 11.14). Since BD

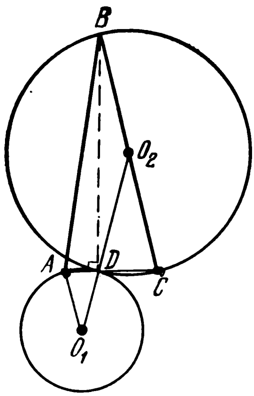

Fig. 11.14

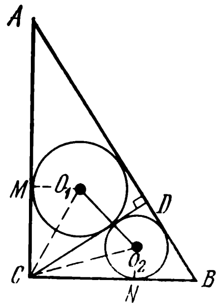

Fig. 11.15

is an altitude of the triangle ABC, the triangle BDC is a right-angled triangle and, consequently, the centre of the circle circumscribed about the triangle BDC lies at the midpoint of the hypotenuse BC.

Let us consider the triangle AO_1D. The triangle is isosceles, and by the hypothesis $|AD| = 1$, $|AO_1| = |O_1D| = 2$. We use the cosine theorem to find the angle ADO_1 of that triangle:

$$\angle ADO_1 = \arccos \frac{1}{4}.$$

Since by the hypothesis, the circles with centres O_1 and O_2 touch each other at the point D, the line of centres O_1O_2 passes through the point of tangency and $|O_1O_2| = |O_1D| + |DO_2|$. The angles ADO_1 and CDO_2 are vertical and, consequently, equal, i.e. $CDO_2 = \arccos \frac{1}{4}$. The triangle DO_2C is isosceles since DO_2 and O_2C are radii of the circle and, consequently, $\angle O_2CD = \arccos \frac{1}{4}$.

We know that in the right triangle BCD the leg $|DC| = 2$ and $\angle O_2CD = \arccos\frac{1}{4}$. From these data we find the hypotenuse BC:

$$\frac{|DC|}{|BC|} = \cos\left(\arccos\frac{1}{4}\right) \Rightarrow \frac{|DC|}{|BC|} = \frac{1}{4} \Rightarrow |BC| = 8.$$

We have thus found three parameters in the triangle ABC which make it possible to calculate the area of the triangle:

$$S_{\triangle ABC} = \frac{1}{2}|AC|\,|BC| \sin DCB$$

$$= \frac{1}{2}\cdot 3\cdot 8 \sin\left(\arccos\frac{1}{4}\right) = 12\cdot\frac{\sqrt{15}}{4} = 3\sqrt{15}.$$

Answer. $3\sqrt{15}$.

Example 4.2. In the right triangle ABC with an acute angle of 30° an altitude CD is drawn from the vertex of the right angle C. Find the distance between the centres of the circles inscribed into the triangles ACD and BCD if the smaller leg of the triangle ABC is 1.

Solution. Assume that O_1 and O_2 are the centres of the circles inscribed into the right triangles ACD and BCD respectively; the angle CAB is 30°, $|BC| = 1$ (Fig. 11.15). From the right triangle ABC we find that

$$\frac{|BC|}{|AC|} = \tan 30° \Rightarrow |AC| = \sqrt{3}, \quad \angle ABC = 60°.$$

From the known data $|AC| = \sqrt{3}$ and $\angle A = 30°$, we find from the right triangle ACD that

$$|CD| = \sqrt{3}/2; \quad |AD| = 3/2.$$

From the known data $|BC| = 1$ and $\angle B = 60°$, we find from the right triangle CDB that

$$|BD| = 1/2, \quad |DC| = \sqrt{3}/2.$$

Let us calculate the areas and semi-perimeters of the triangles ACD and CDB:

$$S_{\triangle ACD} = \frac{1}{2}|AD|\,|DC| = \frac{3\sqrt{3}}{8}, \quad p_{\triangle ACD} = \frac{3(1+\sqrt{3})}{4},$$

$$S_{\triangle BCD} = \frac{1}{2}|CD|\,|BD| = \frac{\sqrt{3}}{8}, \quad p_{\triangle BCD} = \frac{3+\sqrt{3}}{4}.$$

Next, from the formula $S = pr$ we calculate the radii of the circles inscribed into the triangles ACD and CDB:

$$r_1 = \frac{S_{\triangle ACD}}{p_{\triangle ACD}} = \frac{\sqrt{3}}{2(1+\sqrt{3})},$$

$$r_2 = \frac{S_{\triangle CDB}}{p_{\triangle CDB}} = \frac{1}{2(1+\sqrt{3})}.$$

We draw the radii O_1M and O_2N to the points of tangency of the circles with the sides AC and BC respectively and consider the right triangles O_1MC and O_2NC. In the triangle O_1MC we know the length of the side $|MO_1| = r_1 = \sqrt{3}/2(1+\sqrt{3})$ and the angle $MCD_1 = 30°$. (The centre of the circle which is inscribed in the triangle lies on the bisector of the interior angle of the triangle: $\angle MCO_1 = \angle DCO_1 = \frac{1}{2}\angle MCD = 30°$.)

From the right triangle MCO_1 we find that

$$|MO_1|/|O_1C| = \sin 30° \Rightarrow$$
$$|O_1C| = 2|MO_1| = 2r_1 = \sqrt{3}/(1+\sqrt{3}).$$

Similarly, from the triangle NCO_2 we find that

$$|NO_2|/|O_2C| = \sin 15°.$$

Calculating $\sin 15°$ by the formula $1 - \cos 30° = 2\sin^2 15°$ we find, that

$$|O_2C| = \frac{|NO_2|}{\sin 15°} = \frac{|NO_2|}{(\sqrt{2-\sqrt{3}})/2} = \frac{2r_2}{\sqrt{2-\sqrt{3}}}$$
$$= \frac{1}{(1+\sqrt{3})\sqrt{2-\sqrt{3}}}.$$

Let us consider the triangle O_1CO_2. In this triangle we know two sides $|O_1C|$ and $|O_2C|$ and the angle between them which is equal to 45°. By the cosine theorem we can find the third side of the triangle, $|O_1O_2|$, which is the sought-for distance between the centres of the circles:

$$|O_1O_2|^2 = |O_1C|^2 + |O_2C|^2 - 2|O_1C|\,|O_2C|\cos 45°$$
$$= \frac{3}{(1+\sqrt{3})^2} + \frac{1}{(1+\sqrt{3})^2(2-\sqrt{3})}$$
$$-2\frac{\sqrt{3}}{1+\sqrt{3}}\,\frac{1}{(1+\sqrt{3})\sqrt{2-\sqrt{3}}}\,\frac{\sqrt{2}}{2}$$
$$= \frac{3}{(1+\sqrt{3})^2} + \frac{2+\sqrt{3}}{(1+\sqrt{3})^2} - \frac{\sqrt{3}\sqrt{2}\sqrt{2+\sqrt{3}}}{(1+\sqrt{3})^2}$$
$$= \frac{5+\sqrt{3}-\sqrt{3}\sqrt{4+2\sqrt{3}}}{(1+\sqrt{3})^2} = \frac{5+\sqrt{3}-\sqrt{3}\sqrt{(1+\sqrt{3})^2}}{(1+\sqrt{3})^2}$$
$$= \frac{5+\sqrt{3}-\sqrt{3}(1+\sqrt{3})}{(1+\sqrt{3})^2} = \frac{2}{(1+\sqrt{3})^2} = \frac{1}{2+\sqrt{3}} = 2-\sqrt{3},$$
$$|O_1O_2| = \sqrt{2-\sqrt{3}}.$$

Answer. $\sqrt{2-\sqrt{3}}$.

4.1. Find the length of the circle inscribed in an isosceles right triangle with the hypotenuse equal to c.

4.2. Given an isosceles triangle with the length of the lateral side b and the base angle α. Find the distance from the centre of the circumscribed circle to that of the inscribed circle.

4.3. Given a circular sector of radius R with the central angle α. Find the radius of the circle inscribed in the sector.

4.4. Two chords of lengths a and b are drawn from the same point of the circle. Find the radius of the circle if the distance between the midpoints of the given chords is $c/2$.

4.5. A semicircle is constructed on the base of an equilateral triangle as a diameter which bisects the triangle. The side of the triangle is a long. Find the area of the part of the triangle which lies outside the semicircle.

4.6. Two points are given on one of the sides of the angle α, the distances between the points and the other side being equal to b and c. Find the radius of the circle which passes through those two points and touches the other side of the angle.

4.7. The regular triangle ABC with side equal to 3 is inscribed in a circle. A point D lies on the circle, the length of the chord AD being equal to $\sqrt{3}$. Find the lengths of the chords BD and CD.

4.8. The vertex C of the right angle of the right-angled triangle ABC with legs equal to 3 and 4 is connected with the midpoint D of the hypotenuse AB. Find the distance between the centres of the circles inscribed in the triangles ACD and BCD.

4.9. A circle of radius R passes through the vertices A and B of $\triangle ABC$ and touches the straight line AC and a point A. Find the area of $\triangle ABC$ knowing that $\angle ABC = \beta$, $\angle ACB = \alpha$.

4.10. In a right triangle ABC the angle A is a right angle, the angle B is 30°, and the radius of the inscribed circle is $\sqrt{3}$. Find the distance from the vertex C to the point of tangency of the inscribed circle and the leg AB.

4.11. A circle touches the side BC and the extensions of the other two sides of the triangle ABC. Find the radius of the circle if $|AB| = c$, $\angle BAC = \alpha$, $\angle ABC = \beta$.

4.12. A circle of radius R is inscribed in the triangle ABC and touches the side AC at a point D, the side AB at a point E and the side BC at a point F. The length of the segment AD is equal to R, and that of the segment DC is equal to a. Find the area of $\triangle BEF$.

4.13. A circle is inscribed in a regular triangle with side equal to a. Another circle is drawn from the vertex by a radius which is equal to half its side. Find the area of the common part of the circles.

4.14. A circle is inscribed in an isosceles triangle with base a and the base angle α. In addition, a second circle is constructed which touches the base, one of the lateral sides of the triangle and the first circle inscribed in it. Find the radius of the second circle.

4.15. A circle is inscribed in $\triangle ABC$ with sides $|BC| = a$, $|AC| = 2a$ and the angle $C = 120°$. Another circle is drawn through the points of tangency of that circle with the sides AC and BC and through the vertex B. Find the radius of the second circle.

4.16. In the triangle ABC the length of the side AB is equal to 4, $\angle CAB$ is equal to $\pi/6$, and the radius of the circumscribed circle is

3. Prove that the length of the altitude dropped from the vertex C to the side AB is less than 3.

4.17. In the triangle ABC the lateral sides AB and BC are equal to a, and $\angle ABC = 120°$. A circle is inscribed in the triangle ABC which touches the side AB at a point D. A second circle has the point B as its centre and passes through the point D. Find the area of the part of the inscribed circle which lies inside the second circle.

4.18. A semicircle is inscribed in the acute triangle ABC so that its diameter lies on the side AB and the arc touches the sides AC and BC. Find the radius of the circle which touches the arc of the semicircle and the sides AC and BC of the triangle if $|AC| = b$, $|BC| = a$, $\angle ACB = \alpha$.

4.19. A circle of radius $1 + \sqrt{2}$ is circumscribed about an isosceles right triangle. Find the radius of the circle which touches the legs of the triangle and is internally tangent to the circle circumscribed about it.

4.20. In the right triangle ABC with legs $|AB| = 3$, $|BC| = 4$ a circle is drawn through the midpoints of the sides AB and AC which touches the side BC. Find the length of the segment of the hypotenuse AC which lies in the interior of the circle.

4.21. The lengths of the sides $|AB| = 21$ and $|BC| = 15$ in $\triangle ABC$ are given and a bisector BD of $\angle ABC$ is drawn. Find the radius of the circle inscribed in the triangle ABD knowing that $\cos BAC = 5/7$.

4.22. An equilateral triangle ABC is inscribed in a circle of radius R. The side BC is divided into three equal parts and a straight line is drawn through the point of division which is closest to C. The straight line passes through the vertex A and cuts the circle at a point D. Find the perimeter of the triangle ABD.

4.23. In the triangle ABC $|AB| = \sqrt{14}$, $|BC| = 2$. A circle passes through the point B, through the midpoint D of the segment BC, through a point E on the segment AB, and touches the side AC. Find the ratio in which the circle divides the segment AB if DE is the diameter of the circle.

4.24. A circle is circumscribed about the triangle ABC with sides $|AB| = 10\sqrt{2}$, $|AC| = 20$ and $\angle B = 45°$. A tangent is drawn to the circle through the point C which cuts the extension of the side AB at a point D. Find the area of the triangle BCD.

4.25. Given a triangle ABC and the lengths of the sides $|AB| = 6$, $|BC| = 4$, $|AC| = 8$. The bisector of the angle C cuts the side AB at a point D. A circle drawn through the points A, D, C cuts the side BC at a point E. Find the area of the triangle ADE.

4.26. In the triangle ABC $\angle BAC = \alpha$, $\angle BCA = \beta$. $|AC| = b$. A point D is taken on the side BC such that $|BD| = 3|DC|$. A circle is drawn through the points B and D which touches the side AC or its extension beyond the point A. Find the radius of the circle.

4.27. In the triangle ABC $\angle A = 120°$, $|AC| = 1$, $|BC| = \sqrt{7}$. A point M is taken on the extension of the side CA such that BM is an altitude of the triangle ABC. Find the radius of the circle which passes through the points A and M and touches at the point M the circle passing through the points M, B, and C.

4.28. A circle is circumscribed about the triangle ABC with sides $|AB| = 5(1 + \sqrt{3})$, $|BC| = 5\sqrt{6}$, $|AC| = 10$. A tangent to the circle is drawn through the point C and a straight line parallel to the side AC is drawn through the point B. The tangent and the straight line meet at a point D. Find the area of the quadrilateral $ABDC$.

4.29. A circle is inscribed in the triangle ABC with sides $|AB| = 10$, $|BC| = 20$ and the angle C equal to 30°. A tangent to the circle is drawn through the point M of the side AC which is at the distance of 10 from the vertex A. Assume that K is a point of intersection of the tangent and the straight line which passes through the vertex B parallel to the side AC. Find the area of the quadrilateral $ABKM$.

4.30. Given in the triangle BCD: $|BC| = 4$, $|CD| = 8$, $\cos BCD = 3/4$. A point A is chosen on the side CD such that $|CA| = 2$. Find the ratio of the area of the circle circumscribed about the triangle BCD to that of the circle inscribed into the triangle ABD.

4.31. In the triangle one of whose angles is equal to the difference between the other two, the length of the smaller side is 1, and the sum of the areas of the squares constructed on the other two sides is twice that of the circle circumscribed about the triangle. Find the length of the larger side of the triangle.

4.32. In the triangle ABC the length of the side BC is 2 cm, the length of the altitude drawn from the vertex C to the side AB is $\sqrt{2}$ cm, and the radius of the circle circumscribed about the triangle ABC is $\sqrt{5}$ cm. Find the lengths of the sides AB and AC of the triangle if $\angle ABC$ is known to be acute.

4.33. The triangle ABC, whose angle B is equal to $2\alpha < \pi/3$, is inscribed in a circle of radius R. The diameter of the circle bisects the angle B; the tangent to the circle at the point A cuts the extension of the side BC at a point M. Find the area of the triangle ABM.

4.34. Given a regular triangle ABC with side a. A circle passes through the centre of the triangle and touches the side AB at its midpoint M. The straight line drawn from the vertex A touches the circle at a point E. Find the area of the triangle AEM.

4.35. A circle with centre at O is circumscribed about the triangle ABC ($\angle A > \pi/2$). The point F is the midpoint of the larger of the arcs subtended by the chord BC. Let us designate the point of intersection of the side BC with the extension of the radius AO as E and with the chord AF as P. Assume that AH is altitude of $\triangle ABC$. Find the ratio of the area of the quadrilateral $OEPF$ to that of the triangle APH if it is known that the radius of the circumscribed circle $R = 2\sqrt{3}$, $|AE| = \sqrt{3}$ and $|EH| = 3/2$.

4.36. Given an isosceles triangle MNP in which $|MN| = |NP| = l$, $\angle MNP = \beta$. A circle with centre on the side MP touches the sides MN and NP. The tangent to that circle cuts the side MN at a point Q and the side NP at a point R. It is known that $|MQ| = n$. Find the area of the triangle QNR.

The hypotheses of problems 4.37-4.44 do not include quantities which have lengths. To solve these problems, it is necessary to introduce an auxiliary quantity a which has length (say, a side of a triangle) and solve the problem with the extended condition.

The expressions for the required quantities do not contain a.

Example 4.3. In the right triangle ABC the legs AC and BC are equal. Find the ratio of the areas of the inscribed and the circumscribed circle.

Solution. Let us designate the leg AC of the triangle ABC as a. Since by the hypothesis the legs are equal, we have $|BC| = |AC| = a$ and $\angle A = \angle B = 45°$. By the Pythagorean theorem, we find the hypotenuse:

$$|AB| = \sqrt{a^2+a^2} = a\sqrt{2}.$$

The radius of the circle circumscribed about the right triangle is equal to half the hypotenuse:

$$R = a\sqrt{2}/2.$$

The radius of the circle inscribed in the triangle can be calculated by the formula

$$r = \frac{S}{p}, \text{ where } S = \frac{1}{2}a^2, \quad p = \frac{a+a+a\sqrt{2}}{2} = \frac{a(2+\sqrt{2})}{2},$$

and, consequently, $r = \dfrac{a}{2+\sqrt{2}}$.

Let us calculate the ratio of the areas of the inscribed and the circumscribed circle:

$$\frac{\pi r^2}{\pi R^2} = \frac{\left(\frac{1}{2+\sqrt{2}}\right)^2}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1}{(1+\sqrt{2})^2}.$$

Answer. $1/(1+\sqrt{2})^2$.

4.37. A circle is circumscribed about a right triangle. Another circle of the same radius touches the legs of that triangle so that one of the points of tangency is the vertex of the triangle. Find the ratio of the area of the triangle to the area of the common part of the two given circles.

4.38. Angles B and C are given in $\triangle ABC$. A circle is drawn through the midpoint O of the side AB and the vertex A which touches the side BC. Find the ratio of the radius of the circle to the length of the side BC.

4.39. Find the ratio of the radii of the inscribed and the circumscribed circle for an isosceles triangle with the base angle α.

4.40. Given a right triangle with the acute angle α. Find the ratio of the radii of the circumscribed and the inscribed circle and determine the value of α for which this ratio is the least.

4.41. In the isosceles triangle ABC, inscribed in a circle, $|AB| = |BC|$ and $\angle BAC = \alpha$. The straight line drawn from the vertex C and making an angle $\alpha/4$ with AC lies in the interior of the triangle and cuts the circle at a point E. That straight line meets the bisector of the angle BAC at a point F. The vertex A of the triangle is connected with the point E by a line segment. Find the ratio of the areas of the triangles AFC and AEC.

4.42. Given a right triangle ABC with the right vertex angle C. The angle CAB is equal to α. The bisector of $\angle ABC$ cuts the leg AC at a point K. A circle which cuts the hypotenuse AB at a point M is constructed on the side BC as a diameter. Find $\angle AMK$.

4.43. Angles B and C are given in $\triangle ABC$. The bisector of $\angle BAC$ cuts the side BC at a point D and the circle circumscribed about $\triangle ABC$ at a point E. Find the ratio $|AE| : |DE|$.

4.44. The circle inscribed in $\triangle ABC$ touches its sides AC and BC at points M and N respectively and cuts the bisector BD at points P and Q. Find the ratio of the areas of the triangles PQM and PQN if $\angle A = \pi/4$ and $\angle B = \pi/3$.

Problems 4.45-4.66 can be solved by introducing an auxiliary unknown (or several auxiliary unknowns) for which, by the hypothesis, an equation (or a system of equations) is set up. It is often convenient to choose as an auxiliary unknown the quantities which, together with the quantities of the hypothesis, give a collection of elements which define the triangles uniquely (see Example 4.5).

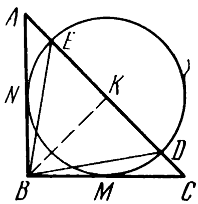

Fig. 11.16

Example. 4.4. In the isosceles triangle ABC the angle B is a right angle, $|AB| = |BC| = 2$. The circle touches both legs at their midpoints and intercepts a chord DE on the hypotenuse. Find the area of $\triangle BDE$.

Solution. Let us designate the lengths of the segments CD, DE and EA as x, y and z respectively (Fig. 11.16). We designate the points of tangency of the legs and the circle as M and N. Then, by the theorem on a tangent and a secant, we have

$$|CM|^2 = |CE|\,|CD|, \qquad (*)$$
$$|AN|^2 = |DA|\,|EA|.$$

By the hypothesis, the circle touches the legs at their midpoints, and this means that $|CM| = |AN| = 1$. With the aid of the unknowns we have introduced we can write equation $(*)$ as a system of equations

$$x(x+y) = 1,$$
$$z(z+y) = 1.$$

Subtracting the second equation from the first, we get

$$x^2 - z^2 + y(x-z) = 0 \Rightarrow (x-z)(x+y+z) = 0 \Rightarrow x = z.$$

i.e. $|CD| = |EA|$. Adding an equation $(2x+y)^2 = 8$, which is the notation of the Pythagorean theorem for $\triangle ABC$, to the equation $x(x+y) = 1$, we obtain a system of two equations in two unknowns:

$$\begin{matrix} x(x+y)=1, \\ (2x+y)^2=8 \end{matrix} \Rightarrow \begin{matrix} x=\sqrt{2}-1, \\ y=2. \end{matrix}$$

We can find the required area of $\triangle BDE$ as follows: The altitude BK of $\triangle ABC$ drawn to the hypotenuse AC is at the same time an altitude of $\triangle BDE$ and, therefore,

$$\frac{S_{\triangle\ BDE}}{S_{\triangle\ ABG}}=\frac{\frac{1}{2}\ |DE|\ |BK|}{\frac{1}{2}\ |CA|\ |BK|}=\frac{|DE|}{|CA|}.$$

On the other hand.

$$S_{\triangle\ ABC}=\frac{1}{2}\ |BC|\ |BA|=2;$$

$$|DE|=y=2,\ \ |CA|=\sqrt{|BC|^2+|BA|^2}=2\sqrt{2},$$

and, consequently,

$$\frac{S_{\triangle\ BDE}}{2}=\frac{2}{2\sqrt{2}}\Rightarrow S_{\triangle\ BDE}=\sqrt{2}.$$

Answer. $\sqrt{2}$.

Example 4.5. The radius of the circle inscribed into an isosceles triangle is four times as small as that of a circle circumscribed about that triangle. Find the angles of the triangle.

Solution. Assume that a isthe length of the base AC of the triangle ABC, α is an acute base angle (Fig. 11.17). Using the parameters we have introduced, we calculate the radii of the inscribed and the circumscribed circle. Since the triangle ABC is isosceles, the bisector of the angle ABC ($|AB| = |BC|$) is at the same time a median and an altitude of the triangle ABC and the angle ODA is a right angle, $|AD| = a/2$, $\angle OAD = \alpha/2$ (O is the centre of the inscribed circle and OD is the radius of the inscribed circle). From $\triangle AOD$ we find that

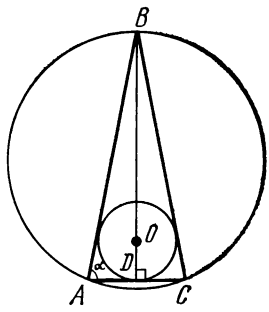

Fig. 11.17

$$r=|OD|=\frac{a}{2}\tan\frac{\alpha}{2}.$$

We find the radius of the circumscribed circle by the sine theorem:

$$\frac{|AC|}{\sin(\angle ABC)}=2R,$$

where $\angle ABC = 180° - 2\alpha$, $|AC| = a$, and, consequently,

$$R=\frac{a}{2\sin 2\alpha}.$$

By the hypothesis, the radius of the circumscribed circle is four times as large as the radius of the inscribed circle:

$$\frac{R}{r}=\frac{\dfrac{a}{2\sin 2\alpha}}{\dfrac{a}{2}\tan\dfrac{\alpha}{2}}=4.$$

The last equation is a trigonometric equation for finding the angle α:

$$\frac{1}{\sin 2\alpha}=4\tan\frac{\alpha}{2}\Rightarrow \tan\frac{\alpha}{2}\sin 2\alpha=\frac{1}{4}$$

$$\Rightarrow \frac{\sin\dfrac{\alpha}{2}\,2\sin\alpha\cos\alpha}{\cos\dfrac{\alpha}{2}}=\frac{1}{4}\Rightarrow \frac{4\sin^2\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}\cos\alpha}{\cos\dfrac{\alpha}{2}}=\frac{1}{4}$$

$$\Rightarrow \cos\alpha\, 2\sin^2\frac{\alpha}{2}=\frac{1}{8}\Rightarrow \cos\alpha\,(1-\cos\alpha)=\frac{1}{8}$$

$$\Rightarrow 8\cos^2\alpha-8\cos\alpha+1=0.$$

Introducing the designation $\cos\alpha=y$, we get a quadratic equation for the unknown y:

$$8y^2-8y+1=0,$$

$$y_{1,2}=\frac{1}{2}\pm\frac{\sqrt{2}}{4},$$

$$\cos\alpha=\frac{1}{2}\pm\frac{\sqrt{2}}{4}.$$

Thus, the trigonometric equation we have set up possesses two solutions:

$$\alpha_1=\arccos\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right),\quad \alpha_2=\arccos\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right).$$

Each value of α is associated with an angle at the vertex of the isosceles triangle:

$$\pi-2\arccos\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right),\quad \pi-2\arccos\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right).$$

Answer. $\pi-2\arccos\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right),\ \pi-2\arccos\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right).$

4.45. The perimeter of a right triangle is 24 cm and the area is 24 cm^2. Find the area of the circumscribed circle.

4.46. One of the legs of a right triangle is 15 cm and the projection of the other leg onto the hypotenuse is 16 cm. Find the radius of the circle inscribed in that triangle.

4.47. Find the angles of a right triangle knowing that the ratio of the radius of the circle circumscribed about the triangle to the radius of the inscribed circle is 5 : 2.

4.48. The length of the base of an isosceles triangle is 4 cm. The length of the lateral side is divided by the point of tangency of the circle inscribed in the triangle in the ratio 3 : 2, reckoning from the vertex. Find the perimeter of the triangle.

4.49. The area of the right triangle is 6 cm^2, and the radius of an inscribed circle, which touches one of the legs, is 3 cm. Find the sides of the triangle.

4.50. Each of the two circles with centres on the medians of an isosceles triangle drawn to the lateral sides touches lateral side and the base. Calculate the distance between the centres of the circles if the length of the base of the triangle is 2 dm and the lengths of the medians are equal to $\sqrt{6}$ dm.

4.51. The area of the triangle ABC is $15\sqrt{3}$ cm^2. The angle BAC is equal to 120°, $\angle ABC$ is larger than $\angle ACB$. The distance from the vertex A to the centre of the circle inscribed in $\triangle ABC$ is 2 cm. Find the length of the median of $\triangle ABC$ drawn from the vertex B.

4.52*. A circle is constructed on the leg BC of the right triangle ABC as a diameter, the circle cutting the hypotenuse at a point D so that $|AD| : |DB| = 1 : 4$. Find the length of the altitude dropped from the vertex C of the right angle to the hypotenuse if the length of the leg BC is known to be equal to 10.

4.53. A circle is inscribed in a right triangle. The point of tangency with the circle divides one of the legs of the triangle into segments 6 cm and 10 cm long, reckoning from the vertex of the right angle. Find the area of the triangle.

4.54. In the triangle ABC the bisector AP of the angle A is divided by the centre O of the inscribed circle in the ratio $|AO| : |OP| = \sqrt{3} : 2\sin\frac{5\pi}{18}$. Find the angles B and C if the angle A is known to be equal to $5\pi/9$.

4.55. In the triangle ABC the ratio of the bisector AE to the radius of the inscribed circle is $\sqrt{2} : (\sqrt{2} - 1)$. Find the angles B and C if $\angle A$ is known to be equal to $\pi/3$.

4.56. An altitude BD is drawn from the vertex B of the isosceles triangle ABC to its base AC. The length of each of the lateral sides AB and BC of $\triangle ABC$ is 8 cm. A median DE is drawn in $\triangle BCD$. A circle is inscribed in $\triangle BDE$ which touches the side BE at a point K and the side DE at a point M. The length of the segment KM is 2 cm. Find $\angle BAC$.

4.57. The point of intersection of the medians of a right triangle lies on the circle inscribed in the triangle. Find the acute angles of the triangle.

4.58. Find the cosine of the base angle of an isosceles triangle if the point of intersection of its altitudes is known to lie on the circle inscribed in the triangle.

4.59. In the triangle ABC the bisector AK is perpendicular to the median BM and $\angle ABC = 120°$. Find the ratio of the area of $\triangle ABC$ to the area of the circle circumscribed about the triangle.

4.60. A circle is circumscribed about an isosceles triangle ABC A chord of length m, cutting the base BC at a point D is drawn through the vertex A. Given a ratio $|BD|:|DC| = k$ and an angle A $(A < \pi/2)$. Find the radius of the circle.

4.61. A circle is inscribed into an isosceles triangle ABC with base AC, the circle touching the lateral side AB at a point M. A perpendicular ML is drawn through the point M to the side AC of the triangle ABC (the point L is the foot of the perpendicular). Find the angle BCA if the area of the triangle ABC is known to be equal to 1 and the area of the quadrilateral $LMBC$ to S.

4.62. Given an acute angle α and a point M in its interior which is at the distances of a and $2a$ from the sides of the angle. Find the radius of the circle which passes through the point M and touches the sides of the angle.

4.63. A right angle is given on a plane. A circle with centre lying in the interior of the angle touches one of its sides, cuts the other side at points A and B and cuts the bisector of the angle at points C and D. The length of the chord CD is $\sqrt{7}$ and that of the chord AB is $\sqrt{6}$. Find the radius of the circle.

4.64*. A point D is taken on the side AC in the triangle ABC such that the circles inscribed in the triangles ABD and BCD touch each other. It is known that $|AD| = 2$, $|CD| = 4$, $|BD| = 5$. Find the radii of the circles.

4.65. The point D lies on the side AC of the triangle ABC. A circle of radius $2/\sqrt{3}$, inscribed in the triangle ABD, touches the side AB at a point M, and the circle of radius $\sqrt{3}$ inscribed into the triangle BCD touches the side BC at a point N. It is known that $|BM| = 6$, $|BN| = 5$. Find the lengths of the sides of the triangle ABC.

4.66. The point D lies on the side AC of the triangle ABC. A circle l_1 inscribed in $\triangle ABD$ touches the segment BD at a point M; a circle l_2 inscribed in $\triangle BCD$ touches it at a point N. The ratio of the radii of the circles l_1 and l_2 is 7/4. It is known that $|BM| = 3$, $|MN| = |ND| = 1$. Find the lengths of the sides of $\triangle ABC$.

5. Polygons and Circles

A polygon whose all vertices lie on a circle is said to be *inscribed in the circle* and the circle is said to be *circumscribed about the polygon.* A circle which touches all sides of the polygon is said to be *inscribed in the polygon.*

The area of a regular polygon with n angles (n-gons) S_n, the side a_n of the n-gon, the perimeter P_n, and the radii of the circumscribed and the inscribed circle R and r are related as

$$S_n = \frac{1}{2} P_n r,$$

$$a_n = 2R \sin \frac{180^\circ}{n},$$

$$S_n = \frac{1}{2} R^2 n \sin \frac{360^\circ}{n}.$$

Theorems on quadrilaterals and circles.

1. To inscribe a circle in a convex quadrilateral, it is necessary and sufficient that the sums of the opposite sides of the quadrilateral be equal.

2. To circumscribe a circle about a convex quadrilateral, it is necessary and sufficient that the sum of the opposite interior angles of the quadrilateral be equal to 180°.

The required quantities in problems 5.1-5.13 can be found by direct calculations with the use of the properties of polygons and circles inscribed in them or circumscribed about them.

Example 5.1. Given a trapezoid $ABCD$, one of whose legs, AB, is perpendicular to the bases. A circle with centre at a point O is inscribed in the trapezoid. A circle with centre at a point O_1 is drawn through the points A, B, and C. Find the diagonal AC if $|OO_1| = 1$ cm, and the smaller base BC of the trapezoid is 10 cm.

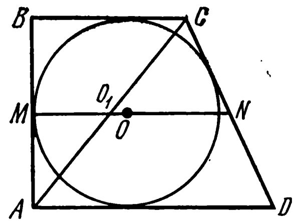

Fig. 11.18

Solution. Assume that MN is the median of the trapezoid (Fig. 11.18). The circle which passes through the points A, B, C is the circle circumscribed about the right triangle ABC (the anlge B is a right angle), and its centre O_1 lies at the midpoint of the hypotenuse AC. On the other hand, the median MN of the trapezoid cuts the diagonal AC of the trapezoid at its middle. Consequently, the point O_1 is the point of intersection of the diagonal AC of the trapezoid and its median MN.

Let us determine where the point O, the centre of the circle inscribed in the trapezoid $ABCD$, lies. Since the circle in question touches two parallel straight lines BC and AD, its centre is a point which is equidistant from those straight lines. The set of points equidistant from the two bases of the trapezoid, its median, and, consequently, the centre of the circle inscribed in the trapezoid also lies on the median MN. In the triangle ABC, the side AB exceeds the side BC since AB is equal to the diameter of the inscribed circle and the side BC is smaller than the diameter. Since the larger angle of a triangle lies opposite the larger side ($\angle BCA > \angle CAB$), and the sum of the angles BCA and CAB is 90°, it follows that the angle CAB is smaller than 45°. The circle with centre at O touches the sides of the right angle BAD, the point O lies on the bisector of that angle and, consequently, $\angle BCO = 45°$.

Thus, two angles ($\angle BAC$ and $\angle BAO$) have a side in common and $\angle BAC < \angle BAO$. Hence we can infer that the ray AO_1 lies between the sides of the angle BAO, i.e. the point O_1 lies between the points M and O.

We can now find the radius of the circle inscribed in the trapezoid $ABCD$. The segment MO_1 is a median of the triangle ABC and, consequently,

$$|MO_1| = \frac{1}{2}|BC| = 5 \text{ cm}$$

The length of the segment MO (the radius of the circle inscribed in the trapezoid) is

$$|MO| = |MO_1| + |O_1O| = 6 \text{ cm}.$$

Since the length of a leg, AB, of the trapezoid is equal to the diameter of the circle ($|AB| = 2|MO| = 12$),we obtain from the right triangle ABC, applying the Pythagorean theorem,

$$|AC| = \sqrt{|AB|^2+|BC|^2} = \sqrt{12^2+10^2} = \sqrt{244} = 2\sqrt{61}.$$

Answer. $2\sqrt{61}$ cm.

5.1. Calculate the area of an isosceles trapezoid if its altitude is h and one of the legs can be seen from the centre of the circumscribed circle at an angle of 60°.

5.2. The side AB of the rectangle $ABCD$ inscribed in a circle is equal to a. From the endpoint K of the diameter KP, which is parallel to the side AB, the side BC can be seen at an angle of β. Find the radius of the circle.

5.3. An isosceles trapezoid $ABCD$ is circumscribed about a circle of radius r; E, and K are the points of tangency of the circle and nonparallel sides of the trapezoid. The angle between the base AB and a leg, AD, of the trapezoid is 60°. Prove that EK is parallel to AB and find the area of the trapezoid $ABEK$.

5.4. In the parallelogram $ABCD$ with an angle A equal to 60° the bisector of the angle B is drawn which cuts the side CD at a point E. A circle of radius r is inscribed in the triangle ECB. Another circle is inscribed in the trapezoid $ABED$. Find the distance between the centres of the circles.

5.5. There are two circles in a parallelogram. One of them, of radius 3, is inscribed in the parallelogram, and the other touches two sides of the parallelogram and the first circle. The distance between the points of tangency, which lie on the same side of the parallelogram, is equal to 3. Find the area of the parallelogram.

5.6. A rhombus $ABCD$ with side $1+\sqrt{5}$ and an acute angle of 60° contains a circle which is inscribed in a triangle ABD. A tangent, drawn from the point C to the circle, cuts the side AB at a point E. Find the length of the segment AE.

5.7. Each of the two circles with centres lying on the diagonals of an isosceles trapezoid touches one of its nonparallel sides and the larger base. Calculate the distance between the centres of the circles if the length of the non-parallel sides of the trapezoid is 4 cm and the lengths of the bases are 6 cm and 3 cm.

5.8. The bases of the trapezoid $ABCD$ are $|AD| = 39$ cm and $|BC| = 26$ cm, and the nonparallel sides are $|AB| = 5$ cm and $|CD| = 12$ cm. Find the radius of the circle which passes through the points A and B and touches the side CD or its extension.

5.9. In the trapezoid $ABCD$ with bases AD and BC the length of a leg, AB, is 2 cm. The bisector of the angle BAD cuts the straight line BC at a point E. A circle is inscribed into the triangle ABE which touches the side AB at a point M and the side BE at a point N. The length of the segment MN is 1 cm. Find the angle BAD.

5.10. The side BC of the quadrilateral $ABCD$ is a diameter of a circle circumscribed about the quadrilateral. Calculate the length of the side AB if $|BC| = 8$, $|BD| = 4\sqrt{2}$, $\angle DCA : \angle ACB = 2:1$.

5.11. In the convex quadrilateral $ABCD$ the length of the side AB is $10\frac{3}{10}$, the length of the side AD is 14, the length of the side CD is 10. The angle DAB is known to be acute, the sine of the angle DAB being equal to 3/5 and the cosine of the angle ADC to $-3/5$. A circle with centre at a point O touches the sides AD, AB and BC. Find the length of the segment BO.

5.12. A pentagon $ABCDE$ (the vertices are designated in a successive order) is inscribed in a circle of unit radius. It is known that $|AB| = \sqrt{2}$, $\angle ABE = 45°$, $\angle EBD = 30°$ and $|BC| = |CD|$. What is the area of the pentagon equal to?

5.13. A circle of radius r is inscribed in a rhombus $CDEF$ in which $\angle DCF = \gamma$. A tangent to the circle cuts the side CD at a point A and the side CF at a point B. It is known that $|AD| = m$. Find the area of the triangle ABC.

In problems 5.14-5.16 it is necessary to introduce an auxiliary quantity, having length, and solve the problem with the extended condition. In the required quantities the auxiliary parameter will be cancelled out and the required quantities will depend only on the quantities given in the hypothesis.

5.14. An isosceles trapezoid is inscribed in a circle so that a diameter of the circle serves as a base of the trapezoid. Find the ratio of the areas of the circle and the trapezoid if the obtuse angle of the trapezoid is equal to α.

5.15. Given an isosceles trapezoid in which a circle is inscribed and about which a circle is circumscribed. The ratio of the altitude of the trapezoid to the radius of the circumscribed circle is $\sqrt{\frac{2}{3}}$. Find the angles of the trapezoid.

5.16. A trapezoid with the base angles equal to α and β is circumscribed about a circle. Find the ratio of the area of the trapezoid to that of the circle.

Problems 5.17-5.32 can be solved by introducing an auxiliary unknown (or several auxiliary unknowns) for which an equation (a system of equations respectively) is set up which corresponds to the hypothesis.

Example 5.2. The side AB of the square $ABCD$ is equal to 1 and is a chord of a certain circle, all the other sides of the square lying outside of the circle. The length of the tangent CK drawn from the vertex C to the circle is equal to 2. Find the radius of the circle.

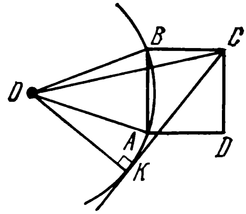

Fig. 11.19

Solution. Assume that O is the centre of the circle whose radius we have to find (Fig. 11.19). We designate the sought-for radius as R. Let us consider the triangle AOB which is isosceles ($|AO| = |OB| = R$) and $|AB| = 1$. By the cosine theorem we find the angle ABO: $\angle ABO = \arccos \frac{1}{2R}$.

Let us consider the triangle CBO in which $|CB| = 1$, $|BO| = R$ and $\angle CBO = \frac{\pi}{2} + \arccos\frac{1}{2R}$. By the cosine theorem we find the length of the side CO:

$$|CO|^2 = |CB|^2 + |BO|^2 - 2|CB|\,|BO|\cos(\angle CBO)$$

$$\Rightarrow |CO|^2 = 1 + R^2 - 2R\cos\left(\frac{\pi}{2} + \arccos\frac{1}{2R}\right)$$

$$\Rightarrow |CO|^2 = 1 + R^2 + 2R\sin\left(\arccos\frac{1}{2R}\right)$$

$$\Rightarrow |CO|^2 = 1 + R^2 + \sqrt{4R^2 - 1}. \qquad (*)$$

Next we consider the triangle COK (OK is a radius drawn to the point of tangency and, consequently, $\angle CKO = \pi/2$). By the Pythagorean theorem we have the following equation for this triangle:

$$|CO|^2 = |CK|^2 + |OK|^2.$$

Since $|CK| = 2$ by the hypothesis and $|OK| = R$, it follows tha

$$|CO|^2 = 4 + R^2. \qquad (**)$$

From (*) and (**) we obtain an equation for R:

$$1 + R^2 + \sqrt{4R^2 - 1} = 4 + R^2 \Rightarrow \sqrt{4R^2 - 1} = 3 \Rightarrow R = \frac{\sqrt{10}}{2}.$$

Answer. $\sqrt{10}/2$.

Example 5.3. The circle touches the sides AB and AD of the rectangle $ABCD$ and cuts the side DC at a single point F and the side BC at a single point E. Find the area of the trapezoid $AFCB$ if $|AB| = 32$, $|AD| = 40$ and $|BE| = 1$.

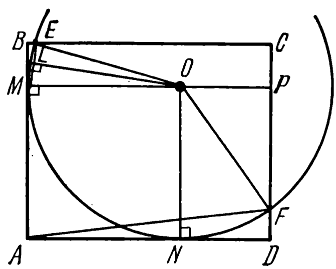

Fig. 11.20

Solution. Assume that O is the centre of the circle; M and N are the points of tangency of the circle and the sides AB and AD (Fig. 11.20); OL is an altitude of the isosceles triangle MEO. We introduce the radius R of the circle and $\alpha = \angle BME$ as the unknowns.

In the right triangle MBE we have $|BE| = 1$ by the hypothesis. We express the sides of the triangle MBE in terms of the angle α and $|BE| = 1$:

$$|MB| = \cot\alpha, \quad |ME| = \frac{1}{\sin\alpha}.$$

T e quadrilateral $AMON$ formed by the radii NO and MO, drawn o the points of tangency, and the parts of the sides AM and AN of the rectangle is a square since

$$|NO| = |MO| = R, \ \angle NAM = \pi/2, \ \angle AMO = \pi/2,$$
$$\angle ANO = \pi/2,$$

and, consequently, $|AM| = |AN| = R$. The equality $|AM| + |MB| = |AB|$ yields the first equation relating R and α:

$$R + \cot\alpha = 32.$$

Let us consider the triangle OME. In that triangle

$$|ME| = \frac{1}{\sin\alpha} = 2\,|ML|, \ |MO| = |OE| = R, \ \angle\, OME = \frac{\pi}{2} - \alpha.$$

In the right triangle MLO (OL is the altitude of $\triangle MOE$)

$$\frac{|ML|}{R} = \cos(\angle OME) \Rightarrow \frac{|ML|}{R} = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha.$$

But we have found that $|ML| = \frac{1}{2}\,|ME| = \frac{1}{2\sin\alpha}$, and, consequently, the equation $\frac{|ML|}{R} = \sin\alpha$ can be written as

$$\frac{1}{2R\sin\alpha} = \sin\alpha.$$

The last equation is the second equation for R and α. Thus, the final system of equations for R and α has the form

$$R + \cot\alpha = 32, \quad R = \frac{1}{2\sin^2\alpha}.$$

Excluding R from the system, we get a trigonometric equation for the angle α:

$$\frac{1}{2\sin^2\alpha} + \cot\alpha = 32,$$

which, with the aid of the equation $1 + \cot^2\alpha = \frac{1}{\sin^2\alpha}$, can be reduced to the form

$$\cot^2\alpha + 2\cot\alpha + 1 = 64 \Rightarrow \cot\alpha = 7$$

(α is an acute angle). From the first equation of the system we obtain $R = 25$.

Next we drop a perpendicular from the point O to the side DC and consider a right triangle OPF. In that triangle $|OF| = R = 25$

$$|OP| = |MP| - |MO| = |AD| - R = 40 - 25 = 15,$$

By the Pythagorean theorem we find that

$$|FP| = \sqrt{|OF^2| - |OP|^2} = \sqrt{25^2 - 15^2} = 20.$$

Since $\cot \alpha = 7$, it follows that in the triangle MBE the side $|MB| = 7$. Since $|PC| = |MB| = 7$ and $|FC| = |FP| + |PC|$, it follows that $|FC| = 20 + 7 = 27$.

We have found the base $|FC| = 27$ in the required trapezoid $AFBC$, and the second base $|AB|$ and the altitude $|AD|$ are known from the hypothesis and, consequently, the area of the trapezoid $AFBC$ is

$$S_{AFBC} = \frac{|AB| + |FC|}{2}\,|AD| = \frac{32+27}{2}\cdot 40 = 1180.$$

Answer. 1180.

5.17. Given a circle of radius r and a right trapezoid circumscribed about it whose smaller base is equal to $\frac{3}{2}r$. Find the area of the trapezoid.

5.18. A rectangle $ABCD$ lies in a semicircle so that its side AB lies on the diameter which bound the semicircle, and the vertices C and D lie on the arc which bounds the semicircle. The radius of the semicircle is 5 cm long. Find the lengths of the sides of the rectangle $ABCD$ if its area is 24 cm^2 and the length of the diagonal exceeds 8 cm.

5.19. A parallelogram is circumscribed about a circle of radius R. The area of the quadrilateral with vertices at the points of tangency of the circle and the parallelogram is S. Find the lengths of the sides of the parallelogram.

5.20. The length of the median of an isosceles trapezoid is 10. It is known that a circle can be inscribed in the trapezoid. The median of the trapezoid divides it into two parts the ratio of whose areas is 7/13. Find the length of the altitude of the trapezoid.

5.21. A quadrilateral $ABCD$ is inscribed into a circle with the radius of 6 cm and centre at a point O. Its diagonals AC and BD are mutually perpendicular and meet at a point K. Points E and F are the midpoints of AC and BD respectively. The segment OK is 5 cm long, and the area of the quadrilateral $OEKF$ is 12 cm^2. Find the area of the quadrilateral $ABCD$.

5.22. A circle with centre at O is inscribed in a trapezoid $ABCD$ with bases BC and AD and with the nonparallel sides AB and CD. Find the area of the trapezoid if $\angle DAB$ is a right angle and $|OC| = 2$, $|OD| = 4$.

5.23. The bisector AE of the angle A cuts the quadrilateral $ABCD$ into an isosceles triangle ABE ($|AB| = |BE|$) and a rhombus $AECD$. The radius of the circle circumscribed about the triangle ECD is 1.5 times as large as the radius of the circle inscribed in the triangle ABE. Find the ratio of the perimeters of the triangles.

5.24. A circle is inscribed in a trapezoid $ABCD$ with the base $|AD| = 40$, the vertex angles A and D equal to 60° and the nonparallel sides $|AB| = |CD| = 10$ so that it touches both bases AD and BC and the side AB. A tangent is drawn to the circle through the point M of the base AD which is at the distance of 10 from the vertex D. The tangent cuts the base BC at a point K. Find the ratio of the area of the trapezoid $ABKM$ to that of the trapezoid $MDCK$.

5.25. A circle constructed on the base AD of a trapezoid $ABCD$ as a diameter passes through the midpoints of the nonparallel sides AB and CD of the trapezoid and touches the base BC. Find the angles of the trapezoid.

5.26. A, B, C, D are successive vertices of a rectangle. A circle passes through A and B and touches the side CD at its midpoint. A straight line drawn through D touches the same circle at a point E and then cuts the extension of the side AB at a point K. Find the area of the trapezoid $BCDK$ if it is known that $|AB| = 10$ and $|KE| : |KA| = 3 : 2$.

5.27. In the quadrilateral $ABCD$ the side AB is equal to the side BC, the diagonal AC is equal to the side CD, and the angle ABC is equal to the angle ACD. The radii of the circles inscribed in the triangles ABC and ACD are related as 3 : 4. Find the ratio of the areas of the triangles.

5.28. A circle is inscribed in a rhombus $ABCD$ in which $|AB| = l$ and $\angle BAD = \alpha$. A tangent to the circle cuts the side AB at a point M and the side AD at a point N. It is known that $|MN| = 2a$. Find the lengths of the line segments MB and ND.

5.29. In the rectangle $ABCD$ the side BC is half the side CD in length. A point E lies in the interior of the rectangle, and $|AE| = \sqrt{2}$, $|CE| = 3$, $|DE| = 1$. Calculate the cosine of $\angle CDE$ and the area of the rectangle $ABCD$.

5.30. Given a parallelogram $ABCD$ and the lengths of its side $|AB| = \sqrt{2}$ and the diagonals $|BD| = 2$. A circle of radius $\sqrt{2}$ with centre at the point B, which lies in the plane of the parallelogram, meets a second circle of radius 1 passing through the points A and C. The tangents passing through one of the points of intersection of the circles are known to be mutually perpendicular. Find the length of the diagonal AC.

5.31. A circle is inscribed in an isosceles trapezoid. The distance between the centre of the circle and the point of intersection of the diagonals of the trapezoid is related to the radius as 3 : 5. Find the ratio of the perimeter of the trapezoid to the length of the inscribed circle.

Chapter 12

Solid Geometry

Common properties of straight lines and planes. The plane α and the straight line a which does not belong to α are said to be *parallel* if they have no points in common.

The criterion of parallelism of a straight line and a plane. If a straight line is parallel to another straight line, which lies in a plane, then either the given straight line and the plane are parallel or the straight line belongs to the plane.

Theorems on a plane and a straight line which is parallel to a plane.

(1) If a plane contains a straight line, which is parallel to another plane, and cuts that plane, then the line of intersection of the planes is parallel to the given straight line.

(2) If an arbitrary plane is drawn through each of two parallel straight lines and the planes intersect, then the line of their intersection is parallel to each of the given straight lines.

(3) If two intersecting planes are parallel to a given straight line, the line of their intersection is also parallel to the given straight line.

Two planes α and β are *parallel* if they have no points in common.

The criterion of parallelism of two planes. If two intersecting straight lines of one plane are respectively parallel to two straight lines of another plane, then the planes are parallel.

Theorem on parallel planes.

(1) If two parallel planes are cut by a third plane, then the lines of their intersection are parallel.

A straight line and a plane are said to be *mutually perpendicular* if the straight line is perpendicular to every straight line belonging to the plane. The straight line which is perpendicular to a plane is said to be a *perpendicular* to that plane.

The criterion of perpendicularity of a straight line and a plane. If a straight line is perpendicular to each of two intersecting straight lines which lie in a plane, then the straight lines and the plane are mutually perpendicular.

Theorem on the perpendicularity of a straight line and a plane.

(1) Two different perpendiculars to a plane are parallel.

(2) If one of two parallel straight lines is perpendicular to a plane, then the other one is also perpendicular to the plane.

(3) A straight line which is perpendicular to one of two parallel planes is also perpendicular to the other plane.

(4) Two planes perpendicular to the same straight line are parallel.

The criterion of perpendicularity of planes. If a plane contains a perpendicular to another plane, then it is perpendicular to that plane.

Theorems on mutually perpendicular planes.

(1) If two planes are mutually perpendicular, then the straight

line which belongs to one plane and perpendicular to the line of intersection of the planes is also perpendicular to the other plane.

(2) If two planes are mutually perpendicular and a perpendicular, passing through the line of intersection of the planes, is drawn to one of the planes, then the perpendicular entirely lies in the other plane.

A straight line which cuts a plane but is not perpendicular to it is said to be *oblique to that plane*.

Theorem on three perpendiculars. For a straight line belonging to a plane to be perpendicular to an oblique line, it is necessary and sufficient that the straight line be perpendicular to the projection of the oblique line onto the plane.

An *angle between an oblique line and a plane* is the angle between the oblique line and its orthogonal projection onto the plane. Two noncoinciding half-planes, which have a straight line as a common boundary and which bound the half-plane, are called a *dihedral angle*. The straight line which is a common boundary of the two half-planes is called an *edge* of the dihedral angle. The half-plane whose boundary coincides with the edge of the dihedral angle and which divides the dihedral angle into two equal dihedral angles is called a *bisecting plane*. The angle resulting from the intersection of a dihedral angle and a plane which is perpendicular to its edge is called a *plane* dihedral angle.

1. Polyhedrons

A prism. A polyhedron whose two faces are equal n-gons which lie in parallel planes, and the other n faces which do not lie in these planes are parallel is called an *n-gonal prism*. A pair of equal n-gons are said to be the *bases* of the prism. The other faces of the prism are said to be its *lateral faces*, and their union is called the *lateral surface* of the prism. The sides of the faces of a prism are called *edges* and the ends of the edges are called the *vertices* of the prism. The edges which do not lie at the base of a prism are said to be *lateral edges*.

A prism whose lateral edges are perpendicular to the planes of the bases is called a *right prism*. A segment of the perpendicular to the planes of the bases of the prism, whose ends belong to those planes, is called an *altitude* of the prism. A right prism whose base is a regular polygon is said to be a *regular prism*.

The lateral area of a prism can be calculated by the formula

$$S_{\text{lat}} = P_n \mid A_1A_2 \mid,$$

where P_n is the perimeter of the perpendicular section of the prism and $\mid A_1A_2 \mid$ is the length of a lateral edge.

The volume of an oblique prism can be calculated by the formula

$$V = S_n \mid A_1A_2 \mid,$$

where S_n is the area of the perpendicular section of the prism and $\mid A_1A_2 \mid$ is the length of a lateral edge, or by the formula

$$V = S_{\text{base}}H,$$

where S_{base} is the area of the base of the prism and H is the altitude.

A *parallelepiped* is a prism whose bases are parallelograms. All six faces of a parallelepiped are parallelograms.

The properties of a parallelepiped.

(1) The midpoint of a diagonal of a parallelepiped is its centre of symmetry.

(2) The opposite faces of a parallelepiped are pairwise equal and parallel.

(3) All four diagonals of a parallelepiped meet at one point and are bisected by it.

A parallelepiped whose lateral edges are perpendicular to the plane of its base is said to be a *right* parallelepiped. A right parallelepiped whose bases are rectangles is said to be a *rectangular* parallelepiped. All the faces of a rectangular parallelepiped are rectangles.

The volume of a rectangular parallelepiped is

$$V = abc,$$

where a, b, c are the lengths of three edges of the rectangular parallelepiped drawn from the same vertex.

A rectangular parallelepiped with equal edges is called a *cube*. All the faces of a cube are equal squares. The volume of a cube is

$$V = a^2.$$

Example 1.1. Find the lateral area of a regular triangular prism whose altitude is h, if the straight line joining the centre of the upper base and the middle of the side of the lower base is at an angle α to the plane of the base.

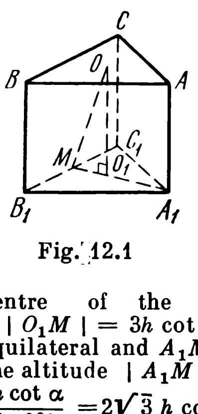

Fig. 12.1

Solution. Assume that $ABCA_1B_1C_1$ is a regular triangular prism whose bases are regular triangles ABC and $A_1B_1C_1$, OO_1 is an altitude ($|OO_1| = h$), M is the midpoint of the segment B_1C_1, O and O_1 are the centres of the triangles serving as the upper and lower bases (Fig. 12.1). Let us consider the triangle O_1OM. By the hypothesis $\angle OMO_1 = \alpha$, $\angle OO_1M$ is a right angle, $|OO_1| = h$.

From the right triangle O_1OM we find that $|O_1M| = h \cot \alpha$. Since O_1 is the centre of the triangle $A_1B_1C_1$, it follows that $|A_1M| = 3|O_1M| = 3h \cot \alpha$. By the hypothesis the triangle $A_1B_1C_1$ is equilateral and A_1M is an altitude, a median, and a bisector. From the altitude $|A_1M|$ of the triangle we find the side $|A_1B_1| = \frac{3h \cot \alpha}{\sin 60^\circ} = 2\sqrt{3}\, h \cot \alpha$. The lateral area of the regular triangular prism $ABCA_1B_1C_1$ is equal to the product of the perimeter of the base by the altitude of the prism:

$$S = 3\,|A_1B_1|\, h = 6\sqrt{3}\, h^2 \cot \alpha.$$

Answer. $6\sqrt{3}\, h^2 \cot \alpha$.

1.1. Given a cube $ABCDA_1B_1C_1D_1$ with edge a. Find the angle between the diagonal A_1C and the edge A_1D_1.

1.2. The distance between the nonintersecting diagonals of two ad acent lateral faces of a cube is d. Find its volume.

1.3. Find the volume of the parallelepiped if all its faces are rhombi whose sides are a long and the acute angles are equal to α.

1.4. Find the volume of a regular quadrangular prism if its diagonal makes an angle α with a lateral face and the side of the base is b.

1.5. The nonintersecting diagonals of two adjacent lateral faces of a rectangular parallelepiped are at the angles α and β to the plane of its base. Find the angle between the diagonals.

1.6. In an oblique triangular prism the lateral edges are 8 cm long; the sides of the perpendicular section are related as 9:10:17, and its area is 144 cm^2. Find the lateral area of the prism.

1.7. In a rectangular parallelepiped the angle between the diagonal of the base and its side is α. The diagonal of the parallelepiped is d and makes an angle φ with the plane of the base. Find the lateral area of the parallelepiped.

1.8. A rhombus with side a and an acute angle α serves as the base of a quadrangular prism, and the lateral edges of the prism are equal to b and make an angle β with the plane of the base of the prism. Find the volume of the prism.

1.9. The angles formed by the diagonal of a rectangular parallelepiped and its faces which meet at one of its vertices are equal to α, β, γ. Prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$.

A pyramid. A polyhedron, one of whose faces is an arbitrary polygon and the other faces are triangles which have a vertex in common, is called a *pyramid*. The polygon is the *base* of the pyramid and the other faces (triangles) are the *lateral faces* of the pyramid.

The sides of the faces of a pyramid are the *edges* of the pyramid. The edges belonging to the base of the pyramid are called the *base edges* and all the other edges are the *lateral edges*. The common vertex of all the triangles (lateral faces) is the *vertex* of the pyramid.

The *altitude* of a pyramid is a segment of the perpendicular drawn from the vertex of the pyramid to the plane of the base (the vertex of the pyramid and the foot of the perpendicular are the endpoints of the segment).

A regular pyramid. A pyramid is said to be *regular* if its base is a regular polygon and the orthogonal projection of the vertex of the pyramid coincides with the centre of the polygon which serves as the base of the pyramid. All the lateral edges of a regular pyramid are equal; all the lateral faces are equal isosceles triangles. The altitude of a lateral face of a regular pyramid, drawn from its vertex, is called an *apothem* of the pyramid.

A triangular pyramid whose base is a triangle is called a *tetrahedron*. A tetrahedron is said to be *regular* if all its edges are equal.

The lateral area of a regular pyramid is

$$S = \frac{1}{2} Ph,$$

where P is the perimeter of the base and h is the apothem.

The volume of a pyramid can be calculated by the formula

$$V = \frac{1}{3} SH,$$

where S is the area of the base of the pyramid and H is the altitude.

A truncated pyramid. A polyhedron whose vertices are the vertices of the base of a pyramid and the vertices of its section by a plane which is parallel to the base of the pyramid is called a *truncated pyramid*. The bases of a truncated pyramid are homothetic polygons. The centre of the homothety is the vertex of the pyramid. The perpendicular drawn to the planes of the bases, with its endpoints lying on the planes of the bases of the pyramid, is the *altitude* of the truncated pyramid. The lateral faces of a truncated pyramid are trapezoids.

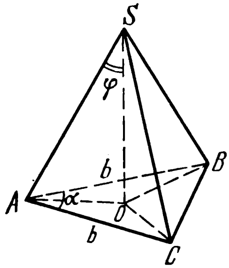

Fig. 12.2

A truncated pyramid is said to be *regular* if it is a part of a regular pyramid. The lateral faces of a regular truncated pyramid are equal equilateral trapezoids. The altitude of each of these trapezoids is an *apothem* of the regular truncated pyramid.

The lateral area of a regular truncated pyramid is

$$S = \frac{1}{2}(P+p)h,$$

where P and p are the perimeters of the bases and h is the apothem.

The volume of a truncated pyramid is

$$V = \frac{1}{3}H(S_1 + \sqrt{S_1S_2} + S_3),$$

where H is its altitude and S_1 and S_2 are the base areas.

Example 1.2. The base of the pyramid is an isosceles triangle whose equal sides are b and the angle between them is α. Find the volume of the pyramid if each lateral edge of the pyramid makes an angle φ with the altitude of the pyramid.

Solution. Assume that $SABC$ is a given pyramid, SO is the altitude of the pyramid, $|AB| = |AC|$, $\angle A = \alpha$, $\angle ASO = \angle BSO = \angle CSO = \varphi$ (Fig. 12.2). Let us consider the triangles ASO, BSO, and CSO. All of them are right triangles (SO is the altitude of the pyramid which is perpendicular to the plane of $\triangle ABC$ and, consequently, SO is perpendicular to the straight lines AO, BO, CO belonging to the plane $\triangle ABC$), SO is a common side of the triangles, and the vertex angles S are equal to φ by the hypothesis. Consequently, all these triangles are equal and equal sides lie opposite equal angles of these triangles: $|AO| = |BO| = |CO|$. Thus we find that O which is a point equidistant from all the vertices of $\triangle ABC$, is the centre of the circle circumscribed about $\triangle ABC$.

In the isosceles $\triangle ABC$ we know the lateral side $|AB| = b$ and the angle α at the vertex A. The radius of the circle circumscribed about $\triangle ABC$ is $b/2\cos(\alpha/2)$. We know the leg $|AO| = b/2\cos(\alpha/2)$ of the right $\triangle ASO$ and the acute vertex angle S equal to φ. We find the other leg SO which is the altitude of the pyramid:

$$|SO| = |AO| \cot \varphi = \frac{b}{2\cos(\alpha/2)} \cot \varphi.$$

Let us now find the volume of the pyramid $SABC$:

$$V_{SABC} = \frac{1}{3} S_{\triangle ABC} |SO|$$

$$= \frac{1}{3} \cdot \frac{1}{2} b^2 \sin \alpha \frac{b}{2\cos(\alpha/2)} \cot \varphi = \frac{1}{6} b^3 \sin \frac{\alpha}{2} \cot \varphi$$

Answer. $\frac{1}{6} b^3 \sin \frac{\alpha}{2} \cot \varphi$.

1.10. Find the volume of a regular triangular pyramid whose lateral edge is l and makes an angle α with the plane of the base.

1.11. Find the total surface area of a regular triangular pyramid whose plane angle at the base of a lateral face is α and the radius of the circle inscribed in the base is r.

1.12. The lateral faces of a triangular pyramid are right triangles, and the lateral edges are equal to a. Find the angle between a lateral edge and an altitude. Calculate the volume of the pyramid.

1.13. The base of the pyramid is an isosceles triangle with base a and lateral side b. The lateral faces make dihedral angles equal to α with the base. Find the altitude of the pyramid.

1.14. The altitude of a regular triangular pyramid is H, and the dihedral angle at the lateral edge is α. Find the volume of the pyramid.

1.15. Find the volume of a regular triangular pyramid knowing the edge angle α and the distance a from a lateral face to the opposite vertex.

1.16. A lateral edge of a regular triangular pyramid is equal to a, the angle between the lateral faces is 2φ. Find the length of a side of the base.

1.17. The base of a pyramid is a right triangle with the hypotenuse c and the acute angle α. Each lateral face of the pyramid makes an angle β with the base. Find the lateral area of the pyramid.

1.18. The sides of the base of a triangular pyramid are equal to a, b, and c. All the edge angles are right angles. Calculate its volume.

1.19. The lateral edges of a triangular pyramid have the same length l. Of the three plane angles these edges form at the vertex of the pyramid, two angles are equal to α and the third is equal to β. Find the volume of the pyramid.

1.20. The dihedral angle at the base of a regular triangular pyramid is α. Find the dihedral angle between the lateral faces.

Hint. In this and in the subsequent problem introduce an auxiliary parameter, the length a of an edge of the pyramid.

1.21. In a regular triangular pyramid the dihedral angle at the lateral edge is α. Find the edge angle of the pyramid.

1.22. The edges of the bases of a regular truncated triangular pyramid are equal to a and b respectively. Find the altitude of the pyramid if all the lateral faces make an angle α with the plane of the base.

1.23. In the triangular pyramid $SABC$ the edge SA is perpendicular to the plane of the face ABC, the dihedral angle with the edge SC

is $\pi/4$, $|SA| = |BC| = a$ and $\angle ABC$ is a right angle. Find the length of the edge AB.

1.24. All the faces of the triangular pyramid are equal isosceles triangles and the altitude of the pyramid coincides with the altitude of one of its lateral faces. Find the volume of the pyramid if the distance between the larger opposite edges is unity.

1.25. Find the volume of the tetrahedron each face of which is a triangle with sides a, b, c, where a, b, c are different numbers.

Example 1.3. The base of the pyramid is a rectangle whose area is Q. Two lateral faces of the pyramid are perpendicular to the plane of the base and the other two make the angles α and β with the plane of the base. Find the volume of the pyramid.

Solution. Assume that $SABCD$ is a given pyramid whose base is a rectangle $ABCD$ of area Q (Fig. 12.3). Since all the lateral faces of the pyramid have a point S in common (the vertex of the pyramid), only the adjacent lateral faces can be perpendicular to the base of the pyramid (the faces BSC and DSC in Fig. 12.3). Next, since the faces BSC and DSC are perpendicular to the plane of the base and have a point S in common, they intersect along a straight line which passes through that point and is perpendicular to the plane of the base, it follows that the altitude of the pyramid coincides with the lateral edge SC.

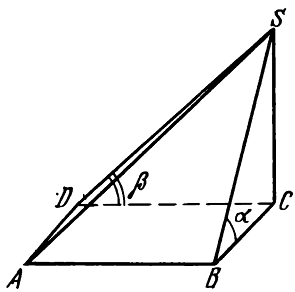

Fig. 12.3

Since the base of the pyramid is a rectangle, the straight line AD is perpendicular to the straight line DC and the straight line AB is perpendicular to the straight line BC, and they both (AD and BC) are perpendicular to the altitude of the pyramid SC. The segments DC and BC are orthogonal projections of the segments DS and BS onto the plane of the base of the pyramid and, according to the theorem on three perpendiculars, we have $AB \perp BS$ and $AD \perp DS$. Thus it turns out that $\angle SBC$ is a plane dihedral angle formed by the planes $ABCD$ and ASB, and $\angle SDC$ is a plane dihedral angle formed by the planes $ABCD$ and ASD. By the hypothesis one of these angles (say, $\angle SBC$) is equal to α and the other ($\angle SDC$) to β.

Assume that $|BC| = x$, and $|DC| = y$. From the right triangle BSC we have $|SC| = x \tan\alpha$, and from the right triangle DSC we have $|SC| = y \tan\beta$. It follows that $x \tan\alpha = y \tan\beta$, and by virtue to the hypothesis $xy = Q$.

From the resulting equations we find that $y = \sqrt{Q \tan\alpha \cot\beta}$ and, consequently, $|SC| = y \tan\beta = \sqrt{Q \tan\alpha \tan\beta}$. It is easy to verify that if we set $\angle SBC = \beta$, $\angle SDC = \alpha$, then, as before, the altitude of the pyramid $|SC|$ is defined by the same expression.

Let us find the volume of the pyramid V_{SABCD}:

$$V_{SABCD} = \frac{1}{3} S_{ABCD} |SC| = \frac{1}{3} Q \sqrt{Q \tan\alpha \tan\beta},$$

Answer. $\frac{1}{3} Q \sqrt{Q \tan \alpha \tan \beta}$.

1.26. Find the volume of a regular quadrangular pyramid the side of whose base is a and the dihedral angle between the lateral faces is α.

1.27. Find the volume of a regular quadrangular pyramid whose lateral edge is l and the dihedral angle between two adjacent lateral faces is β.

1.28. The base of a quadrangular pyramid is a rhombus whose larger diagonal is equal to d and the acute angle is α. All the lateral faces make an angle β with the plane of the base. Find the lateral area of the pyramid.

1.29. The edge angle of a regular quadrangular pyramid is α and the altitude is h. Find the volume of the pyramid.

1.30. The altitude of a regular quadrangular pyramid is H and the volume is V. Find the lateral area Q.

1.31. In a regular quadrangular pyramid the plane vertex angle is α. Find the angle between the opposite lateral edges.

1.32. In a regular quadrangular pyramid the dihedral angle at a lateral edge is 2α. Find the base dihedral angle.

1.33. The base of a pyramid is a rectangle, its two lateral faces make angles α and β with the base respectively. Find the volume of the pyramid if the length of the larger lateral edge is l.

1.34. In the quadrangular pyramid $SABCD$ the planes of the lateral faces SAB, SBC, SCD, and SAD make the angles of 60°, 90°, 45°, and 90° with the plane of the base, respectively. The base $ABCD$ is an equilateral trapezoid, $|AB| = 2$, the base area is 2. Find the surface area of the pyramid.

1.35. Find the volume and the lateral area of a regular hexagonal pyramid if a lateral edge l and the diameter d of the circle inscribed in the base of the pyramid are given.

1.36. The edge angle of a regular hexagonal pyramid is equal to the angle between the lateral edge and the plane of the base. Find it.

1.37. The dihedral angle at a lateral edge of a regular hexagonal pyramid is φ. Find the edge angle of the pyramid.

1.38. Find the volume of a regular pyramid whose base is a regular pentagon and the lateral faces are regular triangles with side a.

1.39. In a regular n-gon pyramid the lateral faces make an angle α with the plane of the base. At what angle are the lateral edges of the pyramid inclined to the plane of the base?

1.40. The plane angle at the vertex of a regular n-gon pyramid is α. Find the dihedral angle θ between two adjacent lateral faces.

1.41. Find the volume of a regular truncated quadrangular pyramid the side of whose smaller base is b, that of the larger base is a, and the lateral face is at an angle of 60° to the plane of the larger base.

2. Sections of Polyhedrons

To construct a section of a polyhedron by a plane means to indicate the points of intersection of a secant plane and the edges of the polyhedron and connect the points by line segments belonging to the faces of the polyhedron. The points of intersection of the plane of the section with the edges of the polyhedron will be the vertices and the line segments belonging to the faces, the sides of the polyhedron resulting from the section of the polyhedron by the plane.

To construct a section of a polyhedron by a plane we must indicate, in the plane of each intersected face of the polyhedron, two points belonging to the section, connect them by a straight line, and find the points of intersection of the straight line and the edges of the polyhedron. The plane of the section of a polyhedron can be defined by different conditions. Let us consider several simple typical methods of defining the section of a cube.

Example 2.1. Construct the section of the cube $ABCD\ A_1B_1C_1D_1$, with an edge a, by a plane passing through the midpoints of the edges AB, BC, and CC_1.

Solution. Two points M and N (the midpoints of the edges AB and BC respectively) (Fig. 12.4), belonging to the section, lie on the

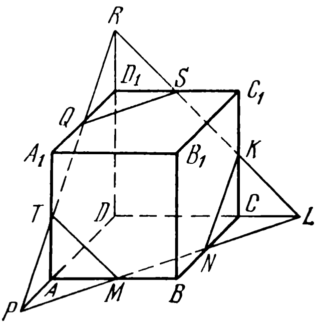

Fig. 12.4

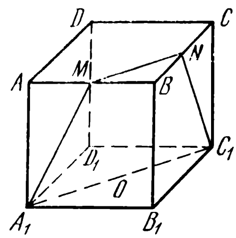

Fig. 12.5

same face. Draw a straight line through M and N until it meets the extensions of AD and DC respectively at points P and L. It is easy to find from the triangles MBN and NLC that $|LC|=|NC|=a/2$. The points L and K (the midpoint of the edge CC_1) lie in the plane of the face DD_1C_1C. We draw a straight line through the points L and K. Taking into account that $|CK| = a/2$, we find from the triangles LCK and KC_1S that $|SC_1| = a/2$, i.e. the point S lies in the middle of the edge D_1C_1. The straight line LK cuts the extension of the edge DD_1 at a point R. By analogy with the aforesaid we can show that $|D_1R| = a/2$. Since the points P and R lie in the plane of the face A_1ADD_1, the straight line PR will cut the sides of the square A_1ADD_1 at points T and Q, the point T being the midpoint of the edge AA_1 and the point Q, the midpoint of the edge A_1D_1.

We have thus obtained six points (M, N, K, S, Q, and T) belonging to the plane of the section and lying on the faces of the cube. Connecting the pairs of points M and T, N and K, S and Q, we obtain the required hexagon of the section.

Example 2.2. Construct the section of the cube $ABCDA_1B_1C_1D_1$ by a plane passing through the midpoints of the edges AB and BC and the centre of the square $A_1B_1C_1D_1$.

Solution. In this problem, two points, M and N (Fig. 12.5) belong to the upper face and the third point, O, belongs to the face of the lower base which is parallel to it. It is easy to verify that in the given problem the method of constructing the section described in the preceding problem cannot serve the purpose. The same is true of the problems in which the straight line connecting two given points of the section proves to be parallel to the edge of the polyhedron. In those cases, the following *theorems* are used in constructing the sections.

(1) If two planes are parallel and are cut by a third plane, then the lines of intersection of the parallel planes by the third plane are parallel.

(2) If two intersecting planes are parallel to the same straight line, then the line of their intersection is parallel to the straihgt line.

(3) If a plane and a straight line are parallel and a plane drawn through the straight line cuts the given plane, then the line of intersection of the planes is parallel to the given straight line.

The required plane of section passes through the point O and the straight line MN which is parallel to the plane $A_1B_1C_1D_1$. By theorem (3), the plane of section cuts the plane of the face $A_1B_1C_1D_1$ along a straight line which is parallel to the straight line MN. Since MN is parallel to the straight line AC (as a median of the triangle ABC), and $AC \parallel A_1C_1$, it follows that the diagonal A_1C_1 of the square is the line of intersection of the plane of the section and that of the face $A_1B_1C_1D_1$. The points A_1 and M belong to the face A_1ABB_1 and the points N and C_1 belong to the face BCC_1B_1 and, consequently, a quadrilateral A_1MNC_1 results from the section of the cube by a plane.

In the two examples considered above the points defining the section belonged to the surface of the cube. There are problems, however, in which the points defining the section belong to different faces or one of the points lies in the interior of the polyhedron. In those cases, in order to solve the problem, it is necessary to carry out auxiliary constructions which would make it possible to reduce the solution of the problem to the scheme of construction of a section described above. An auxiliary plane is often constructed for the purpose, which contains a straight line belonging to the plane of the section and a straight line belonging to the plane of a face. In that auxiliary plane the point of intersection of the straight lines is sought and thus one more point is found which lies in the plane of a lateral face.

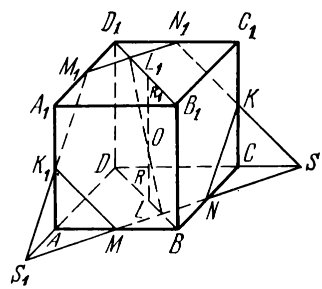

Fig. 12.6

Example 2.3. Given a cube $ABCDA_1B_1C_1D_1$ with lateral edges AA_1, BB_1, CC_1, DD_1. Find the area of the section of the cube by a plane P which passes through the centre of the cube and the midpoints of the edges AB and BC if the edge of the cube is equal to 1.

Solution. Assume that M and N are the midpoints of the edges AB and BC respectively (Fig. 12.6). The plane P passes through the points M and N and, consequently, cuts the face $ABCD$ along the straight line MN. To construct the section of the cube by the plane P, we construct an auxiliary section of the cube by a diagonal plane Q

which passes through the vertices B, D, D_1, B_1 of the cube. The diagonal section of the cube is a rectangle with sides $|BB_1| = 1$ and $|BD| = \sqrt{2}$. The planes P and Q intersect along a straight line passing through the point L and the point O (the centre of the cube) which also belongs to the plane Q, with $|BL| = \frac{1}{4}|BD|$. Using the equality of the triangles LRO and L_1R_1O, it is easy to prove that $|L_1D_1| = \frac{1}{4}|B_1D_1|$. We have thus proved that the plane P passes through the point L_1 belonging to the upper base of the cube and $|L_1D_1| = \frac{1}{4}|B_1D_1|$.

Since the planes $ABCD$ and $A_1B_1C_1D_1$ are parallel and the plane P cuts them both, the lines of intersection of these planes by the plane P are parallel to each other. Drawing a straight line M_1N_1 through the point L_1, parallel to the diagonal A_1C_1, we get two points (M_1 and N_1) belonging to the plane of the section P and the edges of the cube, with M_1N_1 being the median of the triangle $A_1C_1D_1$, $|M_1N_1| = \frac{1}{2}|A_1C_1|$ and $|D_1N_1| = |N_1C_1|$.

Let us extend the edge DC beyond the point C. Since the straight lines MN and DC belong to the plane of the lower base of the cube and are not parallel, they meet at a certain point S. It follows from the equality of the triangles MBN and NSC that $|SC| = |MB|$. On the other hand, the point S, belonging to the plane P, also belongs to the face DCC_1D_1 of the cube. We have thus obtained two points (S and N_1) which belong both to the plane P and to the plane of the face DCC_1D_1. The straight line passing through the points S and N_1 cuts the edge CC_1 of the cube at a point K. It follows from the equality of the isosceles triangles CSK and KN_1C_1 that $|SC| = |CK| = |KC_1| = |N_1C_1|$. Connecting the points N and K belonging to the plane P and the plane of the face BCC_1B_1 we get one more side of the polygonal section.

By analogy, extending the edge AD of the cube beyond the point A, we obtain a point S_1 which is the point of intersection of the straight lines MN and AD. Next, connecting the points S_1 and M_1, we get a point K_1 which is the point of intersection of the plane P and the edge AA_1, with $|A_1K_1| = |A_1M_1| = |AK_1|$.

We have thus obtained a hexagon $MNKN_1M_1K_1$ in the section of the cube by the plane P. It follows from the equality of the triangles MBN, NCK, KC_1N_1, $N_1D_1M_1$, $M_1A_1K_1$ and K_1AM that the sides of the hexagon are equal and the length of its side is $\sqrt{2}/2$. Since the triangles NCS, SCK and NCK are equal (they are all right-angled and $|NC| = |CS| = |CN|$), the triangle NSK is equilateral, $\angle SNK = 60°$ and, consequently, $\angle MNK = 120°$. We can prove by analogy that all the other angles of the hexagon $MNKN_1M_1K_1$ are equal to 120° and, consequently, the hexagon is regular. The area of a regular hexagon with side $\sqrt{2}/2$ is equal to $3\sqrt{3}/4$.

Answer. $3\sqrt{3}/4$.

To find the area of the section of a polyhedron, it is also convenient to use, in a number of cases, the *property of an orthogonal projection* *of a plane polygon*:

$$s = S \cos \alpha,$$

where S is the area of the polygon, s is the area of its orthogonal projection onto a plane P, α is the angle between the plane of the polygon and the plane P.

2.1. A plane is drawn through the midpoint of the diagonal of a cube perpendicular to it. Find the area of the figure resulting from the section of the cube by the plane if the edge of the cube is equal to a.

2.2. In the cube $ABCDA_1B_1C_1D_1$ ($AA_1 \parallel BB_1 \parallel CC_1 \parallel DD_1$) a plane is drawn through the midpoints of the edges DD_1 and D_1C_1 and the vertex A. Find the angle between that plane and the face $ABCD$.

2.3. In the cube $ABCDA_1B_1C_1D_1$ ($AA_1 \parallel BB_1 \parallel CC_1 \parallel DD_1$) the plane P passes through the diagonal A_1C_1 and the midpoint of the edge DD_1. Find the distance from the midpoint of the edge CD to the plane P if the edge of the cube is equal to 4.

2.4. Given a cube $ABCDA_1B_1C_1D_1$ ($AA_1 \parallel BB_1 \parallel CC_1 \parallel DD_1$). Find the distance from the vertex A to the plane passing through the vertices A_1, B, D if the edge of the cube is equal to a.

2.5. Given a cube $ABCDA_1B_1C_1D_1$ ($AA_1 \parallel BB_1 \parallel CC_1 \parallel DD_1$). Points M and N are taken on the extensions of the edges AB and BB_1 respectively, the points being such that $|AM| = |B_1N| = \frac{1}{2}|AB|$, ($|BM| = |BN| = \frac{3}{2}|AB|$). Where, on the edge CC_1, must the point P lie for a pentagon to result in the section of the cube by a plane drawn through the points M, N and P?

2.6. Assume that M and N are the midpoints of the edges AA_1 and CC_1 of the cube $ABCDA_1B_1C_1D_1$, and a point P is taken on the extension of the edge D_1D beyond the point D such that $|DP| = 1/2$ m. A plane is drawn through the points M, N and P. Find the area of the section if the edge of the cube is equal to 1 m.

2.7. The length of the edge of the cube $KLMNK_1L_1M_1N_1$ ($KK_1 \parallel LL_1 \parallel MM_1 \parallel NN_1$) is equal to 1. A point A is taken on the edge MM_1 such that the length of the segment AM is equal to 3/5. A point B is taken on the edge K_1N_1 such that the length of the segment K_1B is equal to 1/3. A plane α is drawn through the centre of the cube and the points A and B. The point P is the projection of the vertex N onto the plane α. Find the length of the segment BP.

2.8. A point F is taken on the edge BB_1 of the cube $ABCDA_1B_1C_1D_1$ such that $|B_1F| = \frac{1}{3}|BB_1|$, a point E is taken on the edge C_1D_1 such that $|D_1E| = \frac{1}{3}|C_1D_1|$. What is the largest value that the ratio $\frac{|AP|}{|PQ|}$ can assume, where the point P lies on the ray DE and the point Q on the straight line A_1F?

Example 2.4. The altitude of a right prism is equal to 1. The base is a rhombus with side equal to 2 and an acute angle equal to 30°. A secant plane with the angle of inclination to the plane of the base equal to 60° is drawn through the side of the base. Find the area of the section.

Solution. Assume that $ABCDA_1B_1C_1D_1$ is the given prism (see Fig. 12.7) and that the secant plane passes through the edge A_1B_1 of the base, $|A_1B_1| = 2$, $\angle B_1A_1D_1 = 30°$, $|AA_1| = 1$. Depending on the linear dimensions of the prism, the plane of the section passing through the edge A_1B_1 cuts either the lateral face DCC_1D_1 of the prism or the face $ABCD$ of the upper base. Let us assume (and then prove) that the plane of the section cuts the face $ABCD$ of the base along the straight line MN. The line MN is parallel to the edge A_1B_1 (the planes $ABCD$ and $A_1B_1C_1D_1$ are parallel and, consequently, the lines of intersection of these two planes by a third plane, a secant plane, are parallel to each other) and $|MN| = |AB| = |A_1B_1|$.

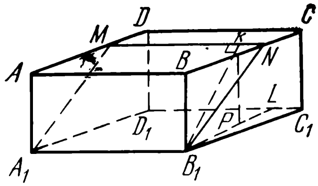

Fig. 12.7

We draw a perpendicular B_1K from the point B_1 to the straight line A_1B_1 belonging to the plane A_1MNB_1 and a perpendicular B_1L to the straight line A_1B_1 belonging to the plane $A_1B_1C_1D_1$. By construction, the angle KB_1L is a plane dihedral angle formed by the secant plane and the plane of the base. By the hypothesis, $\angle KB_1L = 60°$.

Let us drop a perpendicular KP from the point K to the plane $A_1B_1C_1D_1$. By the theorem on three perpendiculars, the point P belongs to the straight line B_1L. Let us consider the triangle B_1KP. This is a right triangle (the angle P is a right angle), and $|KP| = 1$ (KP is the altitude of a right prism) and $\angle KB_1P = 60°$. From the triangle B_1KP we find that

$$|B_1K| = 2/\sqrt{3}, \quad |B_1P| = 1/\sqrt{3}.$$

Let us consider the quadrilateral A_1MNB_1. As we have shown above, $A_1B_1 \parallel MN$ and $|A_1B_1| = |MN| = 2$. A quadrilateral whose opposite sides are equal and parallel is a parallelogram. The segment B_1K is an altitude of the parallelogram A_1MNB_1 since by construction] $B_1K \perp A_1B_1$. The area of the parallelogram is

$$S_{A_1MNB_1} = |A_1B_1| \cdot |B_1K| = 4/\sqrt{3}.$$

Now we have only to show that the plane of the section indeed cuts the upper base of the prism and not its lateral face. By the hypothesis the base of the prism is a rhombus with side equal to 2 and an acute angle of 30°. From the right triangle B_1C_1L in which $|B_1C_1| = 2$ and $\angle B_1C_1L = 30°$ we find the altitude of the rhombus $|B_1L| = 1$.

Assume that the secant plane cuts the lateral face DCC_1D_1 along the straight line MN. Having constructed a plane dihedral angle with edge A_1B_1, we get a triangle B_1KL, the point K lying on a lateral face and the segment KL being a part of the altitude of the prism, i.e. $|KL| < 1$. From the triangle B_1KL we find, however, that $|KL| = \sqrt{3} > 1$. The contradiction we have established proves that the secant plane cannot cut the lateral face DCC_1D_1.

It should be pointed out in conclusion that the size of the acute angle of the rhombus was necessary only to prove that the plane of the section cuts the upper base of the prism and not its lateral face and was of no use when we sought the area of the section. We can consider a more general problem assuming the acute angle of the rhombus to be equal to α. In that case the area of the section is equal to $4/\sqrt{3}$ for all angles α satisfying the inequality $\sin\alpha \geqslant 1/(2\sqrt{3})$.

Answer. $4/\sqrt{3}$.

2.9. A plane is drawn through the vertices A, C, and D_1 o a rectangular parallelepiped $ABCDA_1B_1C_1D_1$ which makes an angle of 60° with the plane of the base. The sides of the base of the parallelepiped are equal to 4 and 3. Find the volume of the parallelepiped.

2.10. The altitude of a right triangular prism is H. The plane drawn through the median of the lower base and through the side of the upper base which is parallel to it makes an angle α with the plane of the base. Find the area of the section.

2.11. The base of a right prism is an equilateral trapezoid with bases a and b ($a > b$) and an acute angle α. The plane passing through the larger base of the upper trapezoid and the smaller base of the lower trapezoid makes an angle β with the plane of the lower base. Find the volume of the prism.

2.12. In a regular quadrangular prism a section is drawn through a side of the base at an angle α to the plane of the base. Find the angle between the diagonal of the section and the side of the base.

2.13. In a regular triangular prism a plane is drawn through a side of the lower base and the opposite vertex of the upper base which makes a dihedral angle of 45° with the plane of the base. The area of the section is S. Find the volume of the prism.

2.14. A plane is drawn through a vertex of a regular quadrangular prism so that a rhombus with an acute angle α results in the section. Find the angle of inclination of that plane to the plane of the base of the prism.

2.15. A side of the base of a regular quadrangular prism is a. A plane is drawn through the diagonal of the lower base and a vertex of the upper base which cuts two adjacent lateral faces of the prism along straight lines which make an angle α. Find the volume of the prism.

2.16. The base of a right prism is a right triangle with hypotenuse c and an acute angle of 30°. A plane is drawn through the hypotenuse of the lower base and the vertex of the right angle of the upper base which makes an angle of 45° with the plane of the base. Find the volume of the triangular pyramid which the plane cuts off from the prism.

2.17. The altitude of a right prism is 1 m, its base is a rhombus with side equal to 2 m and an acute angle of 30°. A secant plane is drawn through a side of the base which is at an angle of 60° to the plane of the base. Find the area of the section.

2.18. The base of a right triangular prism is an isosceles triangle with the lateral side a and the base angle α. A plane is drawn through the base of the triangle in the interior of the prism at an angle φ. Find the area of the section knowing that the section is a triangle.

2.19. The base of a right prism is an equilateral triangle. A plane is drawn through one of the sides of the base and the opposite vertex at an angle φ to the plane of the base. The area of the section is S. Find the volume of the prism.

2.20. The lateral edge of the triangular prism $ABCA_1B_1C_1$ is l. The base of the prism is a regular triangle with side b, and the straight line which passes through the vertex B_1 and the centre of the base ABC is perpendicular to the bases. Find the area of the section which passes through the edge BC and the midpoint of the edge AA_1.

2.21. Each edge of a regular hexagonal prism is equal to 1. Find the area of the section which passes through a side of the base and the larger diagonal of the prism.

2.22. The lateral edge of a right prism is equal to a. Its base is a right triangle whose smaller angle is equal to α. A section is drawn through the smaller leg of the base and the midpoint of the opposite lateral edge which makes an angle β with the plane of the base. Find the area of the section.

2.23. A section which is drawn through the side a of the base of a regular triangular prism at an angle α to it divides a lateral edge into parts in the ratio $m : n$, reckoning from the upper base. Find the volume of the resulting parts and the area of the section.

2.24. The base of the right prism $ABCA_1B_1C_1$ is an isosceles right triangle ABC with legs $|AB| = |BC| = 1$. A plane is drawn through the midpoints of the edges AB and BC and a point P lying on the extension of the edge BB_1 beyond the point B. Find the area of the resulting section if $|BP| = 1/2$ and $|BB_1| = 1$.

2.25. Given a rectangular parallelepiped $ABCDA_1B_1C_1D_1$ with the area of the base S and the altitude h. A secant plane is drawn through the vertex A_1 of the upper base $A_1B_1C_1D_1$ which cuts the lateral edge BB_1 at a point B_2, the lateral edge CC_1 at a point C_2 and the lateral edge DD_1 at a point D_2. Find the volume of the part of the parallelepiped which lies under the secant plane, if it is known that $|CC_2| = c$.

2.26. Given a right triangular prism $ABCA_1B_1C_1$ (AA_1, BB_1, CC_1 are lateral edges) in which $|AC| = 6$, $|AA_1| = 8$. A plane is drawn through the vertex A which cuts the edges BB_1 and CC_1 at points M and N respectively. Find the ratio in which the plane divides the volume of the prism if it is known that $|BM| = |MB_1|$, and AN is the bisector of the angle CAC_1.

2.27. In the rectangular parallelepiped $ABCDA_1B_1C_1D_1$ ($ABCD$ and $A_1B_1C_1D_1$ are the bases, $AA_1 \parallel BB_1 \parallel CC_1 \parallel DD_1$) the lengths of the edges are: $|AB| = a$, $|AD| = b$, $|AA_1| = c$. Assume that O is the centre of the base $ABCD$, O_1 is the centre of the base $A_1B_1C_1D_1$ and S is a point which divides the segment O_1O in the ratio 1 : 3, i.e. $|O_1S| : |SO| = 1 : 3$. Find the area of the section of the given parallelepiped by a plane passing through the point S parallel to the diagonal AC_1 of the parallelepiped and the diagonal BD of its base.

2.28. The lower base of the rectangular parallelepiped $ABCDA_1B_1C_1D_1$ is a square $ABCD$. Find the largest possible magnitude of the angle between the straight line BD_1 and the plane BDC_1.

Example 2.5. In a regular quadrangular pyramid a secant plane is drawn through a side of the base a which divides in half the dihedral angle α at the base of the pyramid. Find the area of the section.

Solution. Assume that $SABCD$ is a regular quadrangular pyramid with vertex S (Fig. 12.8), and the plane of the section passes through the edge AB of the base. By the hypothesis the given pyramid is regular and its base is a square ($AB \parallel DC$); consequently, the side AB of the base is parallel to the plane DSC. The plane of the section $ABMN$ passing through the straight line AB cuts the plane of the lateral face DSC along a straight line which is parallel to the line AB ($MN \parallel AB$). Consequently, the quadrilateral $ANMB$ is a trapezoid.

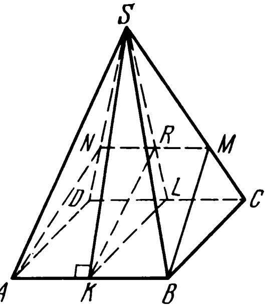

Fig. 12.8

We construct an auxiliary secant plane passing through the midpoint of the edge AB (point K), the midpoint of the edge DC (point L) and the vertex S of the pyramid. The plane KSL cuts the lateral faces of the pyramid along the apothems, with $SK \perp AB$, $KL \perp AB$, and, consequently, the angle SKL is a plane dihedral angle with edge AB equal to α. By the hypothesis $\angle SKR = \angle RKL$ and these angles are equal to $\alpha/2$.

Let us consider the triangle KSL in which $\angle SKL = \angle SLK = \alpha$, $|KL| = a$, KR is the bisector of the angle SKL. By the sine theorem we find the lateral side of the isosceles triangle KSL:

$$\frac{|KS|}{\sin\alpha} = \frac{|KL|}{\sin(180^\circ - 2\alpha)} \Rightarrow |KS| = \frac{a}{2\cos\alpha}.$$

Let us find the segments into which the point R divides the side SL. We designate $|SR| = x$. Then $|RL| = \frac{a}{2\cos\alpha} - x$. By the property of a bisector we have

$$\frac{|SR|}{|KS|} = \frac{|RL|}{|KL|} \Rightarrow \frac{x}{\frac{a}{2\cos\alpha}} = \frac{\frac{a}{2\cos\alpha} - x}{a} \Rightarrow |SR| = x$$

$$= \frac{a}{2\cos\alpha\,(2\cos\alpha + 1)}.$$

By the sine theorem, we find the length of the bisector KR from the triangle KSR:

$$|KR| = \frac{a\sin\alpha}{\sin\frac{3\alpha}{2}}.$$

Since $KR \perp AB$, it follows that KR is an altitude of the trapezoid $ABNM$ whose base AB is known to be equal to a from the hypothesis. To find the second base (MN), let us consider the triangle DSC and a triangle MSN which is similar to it ($MN \parallel DC$). The pyramid $SABCD$ being regular, the apothem SL of the lateral face is the altitude of the triangle DSC, and the segment SR of the apothem is the altitude of the triangle MSN. The sides of similar triangles are proportional to the altitudes drawn to them:

$$\frac{|MN|}{|SR|}=\frac{|DC|}{|SL|}\Rightarrow\frac{|MN|}{\frac{a}{2\cos\alpha\,(2\cos\alpha+1)}}=\frac{a}{\frac{a}{2\cos\alpha}}$$

$$\Rightarrow|MN|\,(2\cos\alpha+1)=a\Rightarrow|MN|=\frac{a}{2\cos\alpha+1}.$$

The area of the trapezoid $ABNM$ is

$$S_{ABNM}=\frac{|AB|+|MN|}{2}\,|KR|=\frac{a+\frac{a}{2\cos\alpha+1}}{2}\,\frac{a\sin\alpha}{\sin\frac{3\alpha}{2}}$$

$$=\frac{2a^2(\cos\alpha+1)\sin\alpha}{2\,(2\cos\alpha+1)\sin\frac{3\alpha}{2}}=\frac{4a^2\cos^3\frac{\alpha}{2}}{(1+2\cos\alpha)^2}.$$

Answer. $\frac{4a^2\cos^3\frac{\alpha}{2}}{(1+2\cos\alpha)^2}$.

2.29. The lateral edge of a regular triangular pyramid is equal to a and is at an angle α to the plane of the base. Find the plane of the section of the pyramid by a plane passing through a vertex of the base and through the median of the opposite lateral face.

2.30. Given a regular triangular pyramid $SABC$ in which $|AB|=a$ and a dihedral angle formed by adjacent lateral faces is equal to α. Find the area of the section of the pyramid by a plane passing through the vertex A and the bisector of the angle SBA.

2.31. Given a regular triangular pyramid with a lateral edge a long and an edge angle α. Find the area of the section which passes through the side AB of the base and is perpendicular to the lateral edge SC.

2.32. The side of the base of a regular triangular pyramid is a; a lateral edge is b. Find the area of the section of the pyramid by a plane which passes through the centre of the base and is parallel to two nonintersecting edges of the pyramid.

2.33. Given a regular triangular pyramid with a lateral edge l. A plane is drawn through a side of the base and the midpoint of the opposite lateral edge which makes an angle α with the plane of the base. Find the area of the section.

2.34. In a regular triangular pyramid with the side of the base equal to a and the lateral edge to $2a$ a plane is drawn through the midpoint of the lateral edge at right angles to it. Find the area of the section.

2.35. A lateral edge of a regular truncated quadrangular pyramid is equal to the side of the smaller base and to a. The angle between the lateral edge and the side of the larger base is α. Find the area of the diagonal section of the truncated pyramid.

2.36. The area of the section of a regular tetrahedron is shaped as a square and is equal to m^2. Find the surface area of the tetrahedron.

2.37. A regular triangular pyramid is cut by a plane which is perpendicular to the base and divides the two sides of the base in half. Find the volume of the cut-off pyramid if the side of the base of the initial pyramid is a and the base dihedral angle is 45°.

2.38. In the regular triangular pyramid $SABC$ the plane passing through the side AC at right angles to the edge SB cuts off a pyramid S_1ABC whose volume is one and a half times as small as the volume of the pyramid $SABC$. Find the lateral area of the pyramid $SABC$ if $|AC| = a$.

2.39. A right prism has an equilateral triangle as its base. The plane drawn through one of its sides at an angle α to the base cuts off a triangular pyramid of volume v from the prism. Find the area of the section.

2.40. In the triangular pyramid $SABC$ a point D is taken on the side AC such that $|AC| = 3|DC|$; a point E is taken on the side BC such that $|BC| = 3|CE|$. Find the area of the section of the pyramid by the plane which passes through the points D and E parallel to the edge SC, if it is known that $|SA| = |SB|$, $|SC| = a$, $|AC| = |BC| = b$, $\angle ACB = \alpha$.

2.41. A plane is drawn in a triangular truncated pyramid through a side of the upper base parallel to the opposite lateral edge. In what ratio does the plane divide the volume of the pyramid if the respective sides of the bases are related as 1 : 2?

2.42. All the edges of the triangular pyramid $SABC$ are equal. A point M is taken on the edge SA such that $|SM| = |MA|$; a point N is taken on the edge SB such that $|SN| = \frac{1}{3}|SB|$. A plane is drawn through the points M and N which is parallel to the median AD of the base ABC. Find the ratio of the volume of the pyramid which is cut off from the initial pyramid by the drawn plane to that of the pyramid $SABC$.

2.43. A regular triangular prism is inscribed into a regular triangular pyramid with the plane angle α so that the lower base of the prism lies on the base of the pyramid and the upper base coincides with the section of the pyramid by a plane passing through the upper base of the prism. The length of a lateral edge of the prism is equal to the length of a side of the base of the prism. Find the ratio of the volumes of the prism and the pyramid.

2.44. The angle between a lateral edge and the plane of the base of a regular triangular pyramid $SABC$ is 60°. A plane is drawn through the point A which is perpendicular to the bisector of the angle S of the triangle BSC. In what ratio does the line of intersection of that plane and the plane BSC divide the area of the face BSC?

2.45. Given a regular triangular pyramid $SABC$. A point M is taken on the extension of the edge AB such that

$$|AM| = |AB| \quad (|MB| = 2|AB|).$$

A point N is taken on the edge SB such that $|SN| = |NB|$. Where should a point P be on the apothem SD of the face SBC for a triangle to result in the section of the pyramid by a plane drawn through the points M, N and P?

2.46. A plane cuts the lateral edges SA, SB and SC of the triangular pyramid $SABC$ at points K, L and M respectively. In what ratio does the plane divide the volume of the pyramid if it is known that

$$|SK| : |KA| = |SL| : |LB| = 2,$$

and the median SN of the triangle SBC is bisected by that plane?

2.47. In the regular quadrangular pyramid $SABCD$ a plane is drawn through the midpoints of the sides AB and AD of the bases, parallel to the lateral edge SA. Find the area of the section knowing the side of the base a and the lateral edge b.

2.48. Given a regular quadrangular pyramid with a lateral edge l. The plane of the section passes through the diagonal of the base and the midpoint of the lateral edge and makes an angle α with the plane of the base. Find the area of the section.

2.49. In the regular quadrangular pyramid $SABCD$ a side of the base is equal to 4. A section is drawn through the side CD of the base of the pyramid which cuts the face SAB along the median of the triangle SAB. The area of the section is equal to 18. Find the volume of the pyramid $SABCD$.

2.50. In the regular quadrangular pyramid $SABCD$ the plane drawn through the side AD at right angles to the face SBC divides that face into two parts of equal areas. Find the area of the total surface of the pyramid if $|AD| = a$.

2.51. The altitude of a regular quadrangular pyramid makes an angle of 30° with a lateral face. A plane is drawn through a side of the base of the pyramid, at right angles to the opposite face. Find the ratio of the volumes of the polyhedrons resulting from the section of the pyramid by that plane.

2.52. Given a regular quadrangular pyramid $SABCD$ with vertex S. A plane is drawn through the midpoints of the edges AB, AD and CS. In what ratio does the plane divide the volume of the pyramid?

2.53. The area of a lateral face of a regular hexagonal pyramid is S. Calculate the area of the section passing through the midpoint of the altitude of the pyramid parallel to the lateral face.

3. Solids of Revolution

A cylinder. *A right circular cylinder* (or simply a *cylinder*) is a solid resulting from a rotation of a quadrilateral about the axis passing through one of its sides. When a polygonal line consisting of the sides of a rectangle, not lying on the axis of revolution, is rotated about the same axis a solid results which is known as the *surface of the cylinder*. The circles obtained as a result of a rotation of the sides adjacent to the side belonging to the axis of revolution are known as the *bases of the cylinder*.

A solid obtained as a result of a rotation of a side of a rectangle which is not adjacent to the side belonging to the axis of revolution is known as the *lateral surface* of the cylinder. A perpendicular drawn to the planes of the bases of a cylinder whose ends coincide with the centres of the bases of the cylinder is known as the *altitude* of the cylinder.

The volume of a cylinder can be calculated by the formula

$$V = \pi R^2 H.$$

The area of the total and the lateral surface of a cylinder can be calculated by the formulas

$$S_{\text{lat}} = 2\pi RH,$$
$$S_{\text{tot}} = 2\pi RH + 2\pi R^2,$$

where R is the radius of the base and H is the altitude of the cylinder.

A cone. *A right circular cone* (or simply a *cone*) is a solid resulting from a rotation of a right triangle about the axis containing its leg. A solid resulting from a rotation, about the same axis, of a polygonal line consisting of the hypotenuse and a leg, not belonging to the axis of revolution, is known as the *surface of the cone*. A solid resulting from a rotation of the hypotenuse is the *lateral surface* of the cone and a circle resulting from a rotation of a leg is the *base of the cone*. The radius of the circle is known as the *radius of the base* of the cone.

The leg of the triangle belonging to the axis of revolution is the *altitude* of the cone, and the hypotenuse of the right triangle is the *generatrix* of the cone.

The volume of a cone can be calculated by the formula

$$V_{\text{cone}} = \frac{1}{3}\pi R^2 H.$$

The area of the lateral surface of a cone can be calculated by the formula

$$S_{\text{lat}} = \pi RL,$$

where R is the radius of the base, H is the altitude, and L is the generatrix of the cone.

A frustum of a cone. A part of a cone bounded by its base and the section which is parallel to the plane of the base is known as a *frustum of a cone* or a *truncated cone*. The *bases* of a truncated cone are homothetic circles with the centre of homothety at the vertex of the cone.

A frustum of a cone can be obtained by rotating an isosceles trapezoid about its axis of symmetry.

Nonparallel sides of a trapezoid are known as *generatrices* of truncated cones; the circles resulting from a rotation of the bases of a trapezoid are the *bases* of a truncated cone.

The volume of a truncated cone can be calculated by the formula

$$V = \frac{1}{3}\pi H\,(R_1^2 + R_1R_2 + R_2^2),$$

where H is the altitude, R_1 and R_2 are the radii of the upper and lower bases of the truncated cone.

The area of the lateral surface of a truncated cone can be calculated by the formula

$$S_{\text{lat}} = \pi\,(R_1 + R_2)\,L,$$

where L is the generatrix of the truncated cone.

A sphere. The set of all points of space which are at a given positive distance R from a given point of space O is a *sphere*. The given point O is the *centre of the sphere*.

A sphere can also be defined as a solid resulting from a rotation of a semicircle about an axis containing the diameter of the semicircle.

The segment OM (M is an arbitrary point of the sphere) is the *radius of the sphere*. The line segment connecting any two points of the sphere and passing through its centre is a *diameter of the sphere*. A diameter of a sphere is equal to double its radius.

The set of all points of space which are at a distance from a given point O which does not exceed the given distance R is called a ball*. A ball can also be defined as a solid resulting from a rotation of a semicircle about an axis containing a diameter of the semicircle.

The volume of a sphere of radius R can be calculated by the formula

$$V = \frac{4}{3}\pi R^3.$$

The area of a sphere of radius R can be calculated by the formula

$$S = 4\pi R^2.$$

A section of a sphere by a plane passing through the centre of the sphere is called a *larger circle*. A *tangent plane* to a sphere (a ball) is a plane which has a single point in common with the sphere. That point is called a *point of tangency* of the sphere and the plane. For a plane to be tangent to a sphere, it is necessary and sufficient that the plane be perpendicular to the radius of the sphere and pass through its end-point.

A straight line belonging to the tangent plane to a sphere and passing through the point of tangency is known as a *straight line which is tangent to the sphere*.

Example 3.1. A point M lying on the circle of the lower base of a cylinder and a point N lying on the circle of the upper base are connected by a line segment passing through the midpoint of the altitude of the cylinder. Find the volume of the cylinder if the length of the segment MN is a and the angle of inclination of the line MN to the plane of the base of the cylinder is α.

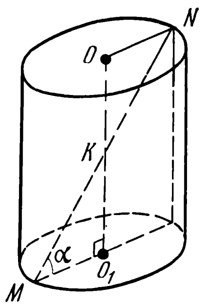

Fig. 12.9

Solution. Assume that OO_1 is the altitude of the cylinder (Fig. 12.9), K is the point of intersection of the altitude of the cylinder and the segment MN, and, by the hypothesis, $|OK| = |KO_1|$. Let us consider the triangles KON and KO_1M. These are right triangles ($KO \perp ON$ and $KO_1 \perp O_1M$), and the angles OKN and O_1KM are vertical and $|OK| = |KO_1|$; consequently, these triangles are equal (by a side and two angles). It follows from the equality of the triangles that

$$|KO_1| = |OK|; \quad |KM| = |KN| = \frac{1}{2}|MN| = \frac{a}{2}.$$

The radius MO_1 of the base of the cylinder is the projection of the segment MN, and the angle KMO_1 which is the angle between the straight line MN and the plane of the base of the cylinder is equal to

* Sometimes, when there is no ambiguity, a ball will be used as a synonym of a sphere.

α by the hypothesis. From the right triangle MKO_1 we find that

$$|MO_1| = \frac{a}{2}\cos\alpha,\quad |KO_1| = \frac{a}{2}\sin\alpha = \frac{|OO_1|}{2}.$$

Let us calculate the volume of the cylinder:

$$V_{\text{cyl}} = S_{\text{base}}H = \pi\,|MO_1|^2\cdot|OO_1| = \frac{\pi a^3}{4}\cos^2\alpha\sin\alpha.$$

Answer. $\frac{\pi a^3}{4}\sin\alpha\cos^2\alpha$.

Example 3.2. The angle at the vertex of the axial section of a right circular cylinder is α. A plane is drawn through its vertex at an angle β ($\beta < \alpha/2$) to the axis of the cone. Find the angle between two generatrices of the cone along which the plane cuts its surface.

Solution. Assume that SO is the altitude of the cone, MSN is a secant plane making an angle α with the altitude SO, MN is a chord of the circle of the base (Fig. 12.10). Since by the hypothesis the angle at the vertex of the axial section of the cone is α, the angle between any generatrix of the cone (generatrices SM and SN, in particular) and the altitude of the cone is equal to $\alpha/2$. Let us consider the right triangle MSO (MOS is a right angle). Designating the radius of the base of the cone as r, we obtain

$$|SM| = \frac{r}{\sin\frac{\alpha}{2}},\quad |SO| = r\cot\frac{\alpha}{2}.$$

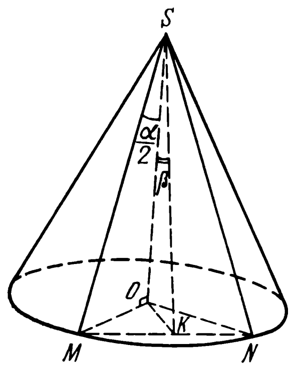

Fig. 12.10

Let us construct an angle between the plane SMN and the altitude SO. For that purpose, we drop a perpendicular OK in the plane of the base of the cone from the point O to the chord MN. By the property of a chord, $|MK| = |NK|$. The straight line MN is perpendicular to the altitude SO of the cone and to the straight line OK and, consequently, it is perpendicular to the plane of the triangle KSO. The plane MSN passes through the perpendicular MN to the plane KSO. Consequently, the planes of the triangles KSN and KSO are mutually perpendicular and the ray SK is an orthogonal projection of the ray SO onto the plane MSN. Since by definition the angle between an inclined line (SO) and a plane (MSN) is the angle between the inclined line and its orthogonal projection onto the plane, the angle KSO is precisely the angle whose size is β. From the right triangle KSO we find that

$$|SK| = \frac{|SO|}{\cos\beta} = \frac{r\cot\frac{\alpha}{2}}{\cos\beta}.$$

As was indicated above, $KO \perp MN$ and $SO \perp MN$ and, consequently, according to the theorem on three perpendiculars, $SK \perp MN$. The triangle MSN is isosceles ($|MS| = |NS|$) and SK is an altitude, a median and a bisector of that triangle. From the right triangle MSK we find the angle MSK:

$$\cos(\angle MSK) = \frac{|SK|}{|SM|} = \frac{r\cot\frac{\alpha}{2}}{\cos\beta\,\dfrac{r}{\sin\frac{\alpha}{2}}} = \frac{\cos\frac{\alpha}{2}}{\cos\beta}$$

$$\Rightarrow \angle MSK = \arccos\left(\frac{\cos\frac{\alpha}{2}}{\cos\beta}\right),$$

$$\angle MSN = 2\angle MSK = 2\arccos\left(\frac{\cos\frac{\alpha}{2}}{\cos\beta}\right).$$

Answer. $2\arccos\left(\dfrac{\cos\frac{\alpha}{2}}{\cos\beta}\right)$.

3.1. Being developed, the lateral surface of a cylinder is a rectangle whose diagonal is equal to d and makes an angle α with the base. Find the volume of the cylinder.

3.2. The area of the section of a cylinder by a plane which is perpendicular to the generatrix is equal to s_1, and the area of the axial section is equal to s_2. Find the lateral area and the volume of the cylinder.

3.3. The radius of the base of a cone is r and the vertex angle in the development of its lateral surface is 90°. Find the volume of the cone.

3.4. The lateral surface of a cone is a folded quarter of a circle. Find the total surface of the cone if the area of its axial section is S.

3.5. The area of the lateral surface of a right circular cone is S; the distance from the centre of the base to the generatrix is r. Find the volume of the cone.

3.6. The area of the lateral surface of a cone is related to that of the base as 2 : 1. The area of its axial section is S. Find the volume of the cone.

3.7. The radius of the base of a cone is r, and the area of its lateral surface is equal to the sum of the areas of the base and the axial section. Find the volume of the cone.

3.8. The altitude of a cone is equal to the diameter of its base. Find the ratio of the area of its base to that of the lateral surface.

3.9. A plane, drawn through the vertex of a cone, cuts the base along a chord whose length is equal to the radius of the base. Find the ratio of the volumes of the resulting parts of the cone.

3.10. Two perpendicular generatrices of a right circular cone divide the circle of the base in the ratio 1 : 2. Find the volume of the cone if its altitude is h.

3.11. A plane is drawn through two generatrices of a cone which make an angle α. Find the ratio of the area of the section to the total surface of the cone if the angle between the generatrices of the cone and the plane of the base is β.

3.12. The generatrix of a cone is l and makes an angle β with the altitude of the cone. Find the area of the section of the cone by a plane which passes through its vertex and makes an angle α with its altitude.

3.13. Two planes are drawn through the vertex of a cone. One of them makes an angle α with the plane of the base of the cone and cuts the base along a chord a long, and the other makes an angle β with the plane of the base and cuts the base along a chord b long. Find the volume of the cone.

3.14. The areas of parallel sections of a sphere lying on the same side of the centre are equal to S_1 and S_2, and the distance between the sections is d. Find the area of the section of the ball which is parallel to the sections S_1 and S_2 and bisects the distance between them.

3.15. A sphere of radius R is inscribed in a dihedral angle of 60°. Find the radius of the sphere inscribed in that angle and touching the given sphere if it is known that the straight line connecting the centres of the two spheres makes an angle of 45° with an edge of the dihedral angle.

3.16. Two equal spheres of radius r touch each other and the faces of the dihedral angle which is equal to α. Find the radius of the sphere which touches the faces of the dihedral angle and the two given spheres.

3.17. Four equal spheres of radius r are externally tangent to each other so that each of them touches the other three. Find the radius of the circle which touches all the four spheres and contains them in its interior.

3.18. Two spheres touch a plane P at points A and B and lie on different sides of the plane. The distance between the centres of the sphere is equal to 10. A third sphere touches the two given spheres and its centre O lies in the plane P. It is known that

$$|AO| = |OB| = 2\sqrt{10}, \quad |AB| = 8.$$

Find the radius of the third sphere.

3.19. Three spheres touch the plane of the triangle ABC at its vertices and each sphere touches the other two. Find the radii of the spheres if the side AB is c long and the angles adjacent to it are α and β.

3.20. Two spheres of radii r and R lie on a plane without meeting each other. The distance between the centres of the spheres is ρ. Find the minimal possible radius of the sphere which would lie on that plane and touch the given spheres.

4. Combinations of Polyhedrons and Solids of Revolution

A prism and a sphere. A sphere which touches all the faces of a prism is said to be *inscribed into the prism*. A prism is said to be *inscribed into a sphere* if all its vertices lie on the surface of the sphere.

In problems dealing with combinations of a prism (a parallelepiped and a cube, in particular) and a sphere solution, as a rule, begins with a geometric construction showing where the centre of the sphere is. When seeking the centre of the sphere inscribed into the prism, use is made of the theorem stating that the centre of a sphere inscribed into a prism is the point of intersection of the bisecting planes of all dihedral angles of the prism, and the centre of the sphere circumscribed about the prism is the point of intersection of all planes which pass through the midpoints of the edges of the prism and are perpendicular to them.

Example 4.1. Three spheres of radius r touch the lower base of a regular triangular prism, each sphere touching the other two spheres and two lateral faces of the prism. The fourth sphere touches each of these three spheres, all the lateral faces and the upper base of the prism. Find the altitude of the prism.

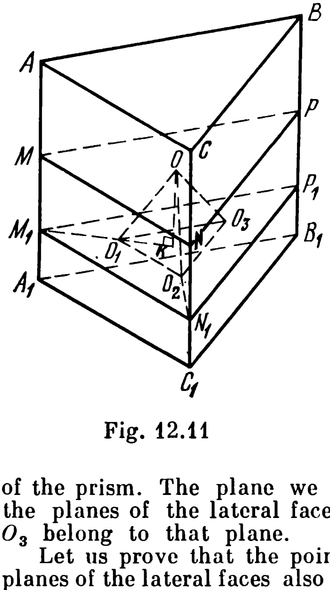

Fig. 12.11

Solution. Assume that O_1, O_2 and O_3 are the centres of the spheres of radius r, ABC and $A_1B_1C_1$ are equilateral triangles which are the upper and the lower base of the prism respectively (Fig. 12.11). Since the prism $ABCA_1B_1C_1$ is regular, its lateral edges AA_1, BB_1, and CC_1 are perpendicular to the planes of the bases, and the planes of the lateral faces are also perpendicular to the planes of the bases.

We draw a plane through the centre O_1 of the sphere, parallel to the plane of the base of the prism. An equilateral triangle $M_1N_1P_1$ results in the section of the prism by the plane, the side of the triangle being equal to the side of the base of the prism. The plane we have constructed is perpendicular to the planes of the lateral faces of the prism and the points O_2 and O_3 belong to that plane.

Let us prove that the points of tangency of the spheres and the planes of the lateral faces also belong to the plane of the triangle $M_1N_1P_1$. We drop a perpendicular from the point O_1 to the plane A_1ACC_1. Since the sphere touches that plane, the perpendicular gets into the point of tangency and its length is r. On the other hand, since the planes $M_1N_1P_1$ and A_1ACC_1 are mutually perpendicular and the point O_1 belongs to the plane $M_1N_1P_1$, the perpendicular drawn from the point O_1 to the plane A_1ACC_1 entirely belongs to the plane $M_1N_1P_1$ and, consequently, the foot of the perpendicular (point K in Fig. 12.12) belongs to the line of intersection of these mutually perpendicular planes (see the theorem on mutually perpendicular planes).

We can prove by analogy that the point of tangency of the sphere with the centre O_1 and the plane A_1ABB_1 also belongs to the plane $M_1N_1P_1$, and the points of tangency of the spheres with centres O_2 and O_3 and the planes of the lateral faces belong to the plane $M_1N_1P_1$. Since the plane $M_1N_1P_1$ passes through the centres of the three pairwise tangent spheres, the lines of centres of these spheres (the segments O_1O_2, O_2O_3, O_1O_3) belong to the plane $M_1N_1P_1$. We have thus obtained, in the section of the prism by the plane $M_1N_1P_1$, an equilateral triangle $M_1N_1P_1$ which contains in its interior three circles of radius r, each of which touches the other two circles and two sides of the triangle (Fig. 12.12).

Let us consider the triangle $M_1N_1P_1$. We draw radii O_1K and O_3L to the points of tangency of the circles and the side M_1P_1. Since $O_1K \perp M_1P_1$, $O_3L \perp M_1P_1$ and $|O_1K| = |O_3L| = r$, it follows that

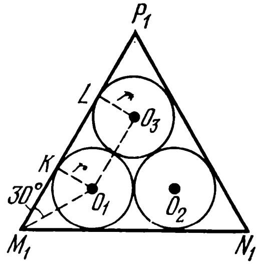

Fig. 12.12

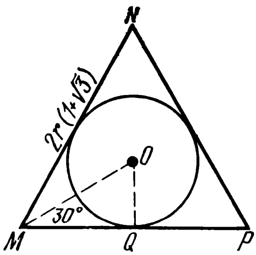

Fig. 12.13

the quadrilateral O_1O_3KL is a rectangle and $|KL| = |O_1O_3| = 2r$. Since the circle with centre O_1 touches the sides of the triangle M_1N_1 and M_1P_1, the point O_1 lies on the bisector of the angle $N_1M_1P_1$ and $\angle O_1M_1K = 30°$. In the right triangle M_1O_1K we know the angle $O_1MK = 30°$ and the leg $|O_1K| = r$. We find the other leg:

$$|M_1K| = r\sqrt{3}.$$

Similarly, from $\triangle LO_3P_1$ we find $|LP_1| = r\sqrt{3}$. Thus the side of the triangle $M_1N_1P_1$ is equal to $2r(1 + \sqrt{3})$.

We draw a plane through the centre O of the fourth sphere which is parallel to the plane of the base of the prism. The section of the prism by that plane is an equilateral triangle MNP (Fig. 12.13) and the section of the sphere with centre O is a circle of a large diameter inscribed into the triangle MNP (the points of tangency of the sphere with centre O and the planes of the lateral faces belong to the plane MNP, which fact can be proved in the same way as in the case of the construction of the section $M_1N_1P_1$). Since the planes of the triangle $M_1N_1P_1$ and MNP are parallel, the triangles $M_1N_1P_1$ and MNP are equal and, consequently, the side of the triangle MNP is equal to $2r(1 + \sqrt{3})$. The radius of the circle inscribed in the triangle MNP and, consequently, the radius of the sphere with centre O are equal to

$$|OQ| = r\left(1 + \frac{\sqrt{3}}{3}\right).$$

The required altitude of the prism is constituted by the distance from the upper base to the plane MNP, the distance between the planes MNP and $M_1N_1P_1$ and the distance from the plane $M_1N_1P_1$ to the lower base of the prism. By the hypothesis, the sphere with centre

O touches the upper base of the prism, and the distance l_1 between the upper base and the plane MNP, containing the point O, is equal to the radius of the sphere with centre at point O:

$$l_1 = r\left(1+\frac{\sqrt{3}}{3}\right).$$

It also follows from the hypothesis that three spheres (with centres O_1, O_2, O_3) lie on the lower base of the prism and the distance l_2 between the lower base and the plane $M_1N_1P_1$, containing the points O_1, O_2, O_3, is equal to the radii of the spheres:

$$l_2 = r.$$

Now we have only to find the distance between the planes MNP and $M_1N_1P_1$. Let us consider a polyhedron with vertices O, O_1, O_2, O_3. It is a pyramid whose base is an equilateral triangle $O_1O_2O_3$ with side $2r$. The lateral edges OO_1, OO_2, OO_3 are equal to one another (since all the spheres are pairwise tangent) and are equal to $r\left(2+\frac{\sqrt{3}}{3}\right)$. We draw an altitude OK of the regular triangular pyramid $OO_1O_2O_3$ (Fig. 12.11). The segment OK is perpendicular to the plane $M_1N_1P_1$ and to the plane MNP which is parallel to it, and, consequently, the length of the segment is the distance between the planes. Since in a regular triangular pyramid the foot of the altitude coincides with the centre of the base, it follows that $|O_1K| = 2r/\sqrt{3}$. By the Pythagorean theorem we find from the right triangle O_1OK that

$$|OK| = \sqrt{|OO_1|^2-|O_1K|^2} = \frac{1}{3}r\sqrt{27+12\sqrt{3}}.$$

The altitude of the pyramid

$$H = l_1+l_2+|OK| = \frac{1}{3}r\left(6+\sqrt{3}+\sqrt{27+12\sqrt{3}}\right).$$

Answer. $\frac{1}{3}r\left(6+\sqrt{3}+\sqrt{27+12\sqrt{3}}\right)$.

4.1. The base of a regular prism is a square with side a and its altitude is H. Find the radius of the circumscribed sphere.

4.2. A regular triangular prism is circumscribed about a sphere and a sphere is circumscribed about the prism. Find the ratio of the surfaces of the spheres.

4.3. A regular hexagonal prism is circumscribed about a sphere of radius R. Find its surface area.

4.4. A sphere touches the lateral edges of a regular right hexagonal prism whose base lies outside of the sphere. Find the ratio of the lateral area of the prism included in the sphere to the surface area of the sphere which lies outside of the prism.

4.5. A sphere is inscribed into a cube with edge a. Find the radius of another sphere which touches three faces of the cube and the first sphere.

4.6. A cube is inscribed into a hemisphere of radius R so that its four vertices lie on the base of the hemisphere and the other four belong to the spherical surface of the half-ball. Calculate the volume of the cube.

4.7. Given a cube with bases $ABCD$ and $A_1B_1C_1D_1$, where $AA_1 \parallel BB_1 \parallel CC_1 \parallel DD_1$. A ball of radius $R = 1/2$ is inscribed into the angle A. Find the radius of the sphere inscribed into the angle C and touching the given sphere, provided that the edge of the cube is equal to 3/2.

4.8. Given a cube with bases $ABCD$ and $A_1B_1C_1D_1$. A point E is the midpoint of the edge C_1D_1, a point F is the midpoint of the edge B_1C_1. Find the radius of the sphere drawn through the points E, F, A, C if the edge of the cube is equal to a.

A pyramid and a sphere. A sphere is said to be *inscribed in a pyramid* if it touches all the faces of the pyramid. The centre of the sphere inscribed in the pyramid is the point of intersection of the bisecting planes of all dihedral angles of the pyramid.

A sphere is said to be *circumscribed about a pyramid* if all the vertices of the pyramid lie on its surface. If a ball is circumscribed about a pyramid, then its centre is the point of intersection of all the planes which are drawn through the midpoints of the edges of the pyramid at right angles to the edges.

In problems dealing with combinations of a pyramid and a sphere, it is necessary, as a rule, to begin a solution with a geometric construction as a result of which a point is found which is the centre of the sphere. In addition, it is often convenient to construct an auxiliary section of the pyramid and the sphere dividing the combination of the pyramid and the sphere into two symmetric parts as a result of which the solution of the space geometry problem can sometimes be reduced to that of a plane geometry problem (such a technique is used in Example 4.3).

Example 4.2. The base of a pyramid is an equilateral triangle with side a. The altitude of the pyramid passes through the midpoint of one of the edges of the base and is equal to $3a/2$. Find the radius of the ball circumscribed about the pyramid.

Solution. Assume that S is the vertex of the pyramid, ABC is an equilateral triangle lying at the base of the pyramid (Fig. 12.14), SK is the altitude of the pyramid (and also the altitude of the triangle ASB), and by the hypothesis $|AK| = |KB|$. The triangles ASK and BSK are equal (they are both right triangles, SK is the common side and $|AK| = |KB|$), and, consequently, the triangle ASB is isosceles. By definition, a pyramid is inscribed into a ball if all the vertices of the pyramid belong to the surface of the ball and the centre of the ball is a point which is equidistant from all the vertices of the pyramid. Let us find the locus of points which are equidistant from all the vertices of the pyramid.

The locus of points which are equidistant from the three vertices A, B and C is the perpendicular O_1M to the plane of the equilateral triangle ABC drawn from the centre O_1 (Fig. 12.14). The locus of points which are equidistant from the vertices A, S, B is the perpendicular

O_2N to the plane of the isosceles triangle ASB drawn from the point O_2 which is the centre of the circle circumscribed about the triangle ASB.

Let us prove that two perpendiculars, O_1M and O_2N intersect. By the hypothesis, the segment SK is perpendicular to the plane of the triangle ABC, $SK \perp AB$, K is the midpoint of the segment AB and, consequently, KC is an altitude, a median and a bisector of the equilateral triangle ABC ($KC \perp AB$). Thus the segment AB is perpendicular to two different straight lines SK and KC and, consequently, it is perpendicular to the plane passing through the points S, K and C (see theorem on mutually perpendicular straight lines and planes). The planes ASB and ABC are perpendicular to the plane SKC, since each of them includes the straight line AB which is perpendicular to the plane SKC.

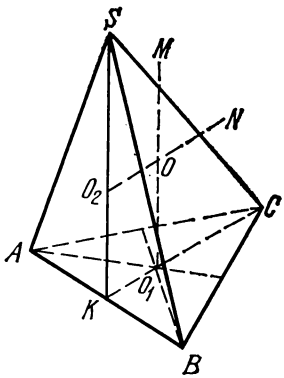

Fig. 12.14

The point O_1 belongs to the line of intersection of the planes ABC and SKC, and the point O_2 to the line of intersection of the planes ASB and SKC. The perpendicular O_1M to the plane ABC entirely belongs to the plane SKC and the perpendicular O_2N to the plane ASB also belongs to the plane SKC (according to the theorem on two mutually perpendicular planes and a perpendicular te one of the planes passing through tho lines of their intersection).

We have thus proved that the straight lines O_1M and O_2N belong to the same plane, meet at a point O and the quadrilateral KO_2OO_1, whose all vertices belong to the plane SKC, is a rectangle. The point O is a point which is equidistant from the points A, B, C, S is the centre of the ball circumscribed about the pyramid. The radius of the circle circumscribed about the triangle ASB is equal to $5a/6$, and $|O_2K| = |SK| - |SO_2| = \frac{3a}{2} - \frac{5a}{6} = \frac{2a}{3}$. In the equilateral triangle ABC the distance from the centre O_1 to the vertex C is equal to $|O_1C| = \frac{2}{3}|KC| = \frac{a\sqrt{3}}{3}$. Since $|O_2K| = |O_1O|$ (KO_2OO_1 is a rectangle), from the right triangle OO_1C we obtain, by the Pythagorean theorem, the length of the segment OC which is equal to the required radius of the ball:

$$R = |OC| = \sqrt{|O_1O|^2 + |O_1C|^2}$$

$$= \sqrt{\left(\frac{2a}{3}\right)^2 + \left(\frac{a\sqrt{3}}{3}\right)^2} = \frac{a\sqrt{7}}{3}.$$

Answer. $\frac{a\sqrt{7}}{3}$.

4.9. A regular tetrahedron is inscribed in a sphere of radius R. Find the volume of the tetrahedron.

4.10. The base of the pyramid is a regular triangle with side equal to 6 cm. One of the lateral edges is perpendicular to the plane of the base and is equal to 4 cm. Find the radius of the sphere circumscribed about the pyramid.

4.11. The side of the base of a regular triangular pyramid is a and the edge angle of the pyramid is α. Find the radius of the sphere inscribed into the pyramid.

4.12. The lateral edges and two sides of the base of a triangular pyramid are equal to a and the angle between the equal sides of the base is α. Find the radius of the sphere circumscribed about the pyramid.

4.13. The edge of a regular tetrahedron is a. Find the radius of the sphere which touches the lateral edges of the tetrahedron at the vertices of the base.

4.14. The edge of a regular tetrahedron is a. Find the radius of the sphere which touches the lateral faces of the tetrahedron at the points lying on the sides of the base.

4.15. Find the radius of the sphere which touches the base and the lateral edges of a regular triangular pyramid the side of whose base is a and the base dihedral angle is α.

4.16. The faces of a regular truncated triangular pyramid touch a sphere. Find the ratio of the surface of the sphere to the total surface of the pyramid if the lateral faces of the pyramid make an angle α with the plane of the base.

4.17. A given regular truncated triangular pyramid can contain a sphere which touches all the faces and a sphere which touches all the edges. Find the sides of the base of the pyramid if the lateral edges are equal to b.

4.18. The edge of a regular tetrahedron is a. Find the radius of the sphere which touches all the edges of the tetrahedron.

4.19. The side of the base of a regular triangular pyramid is a, the lateral edge is b. Find the radius of the sphere which touches all the edges of the pyramid.

4.20. A sphere is inscribed in a regular triangular pyramid with the plane vertex angle β. Find the ratio of the volumes of the sphere and the pyramid.

4.21. The edge of a regular tetrahedron $SABC$ is a. Find the radius of the sphere inscribed in a trihedral angle formed by the faces of the tetrahedron with the vertex at the point S which touches the plane drawn through the midpoints of the edges SA, SC and AB.

4.22. A sphere is inscribed in a regular triangular pyramid. Find the angle of inclination of the lateral sides of the pyramid to the plane of the base, knowing that the ratio of the volume of the pyramid to that of the sphere is $27\sqrt{3}/(4\pi)$.

4.23. In a regular triangular pyramid the angle between the lateral edges and the altitude drawn to the base is α. Find the ratio of the volume of the pyramid to that of the circumscribed ball.

4.24. In a regular triangular pyramid the dihedral angle between the plane of the base and a lateral face is α. Find the ratio of the volume of the sphere inscribed into the pyramid to the volume of the pyramid.

4.25. Given a sphere of radius R into which a regular triangular pyramid $SABC$ is inscribed whose base dihedral angle is α. Find the side of the base of the pyramid.

4.26. Four equal spheres lie inside a regular tetrahedron with edge a so that each sphere touches the other three spheres and three faces of the tetrahedron. Find the radius of the spheres.

4.27. In the tetrahedron $SABC$ the dihedral angles at the edges AB, AC, and SB are right angles and the dihedral angles at the edges SA, and BC are equal to 15°. Find the radius of the ball inscribed into the tetrahedron if $|BC| = 2$.

4.28. A plane is drawn through the side of the base of a regular triangular pyramid and the centre of the ball inscribed into it. In what ratio does the plane divide the volume of the pyramid if its lateral edge is 3.5 times as large as the side of the base?

4.29. In the regular triangular pyramid $SABC$ the side of the base ABC is b and the altitude of the pyramid is $b\sqrt{2}$. The sphere inscribed into the pyramid touches SBC at a point K. Find the area of the section of the pyramid by the plane which passes through the point K and the edge SA.

4.30. The base of a pyramid is an equilateral triangle with side a. One of the lateral edges of the pyramid is also equal to a and the other two are equal to b. Find the radius of the sphere circumscribed about the pyramid.

4.31. The base of the pyramid is an isosceles triangle whose equal sides are b long; the lateral faces corresponding to them are perpendicular to the plane of the base and the angle between them is α. The angle between the third lateral face and the plane of the base is also equal to α. Find the radius of the ball inscribed into the pyramid.

4.32. A sphere lies on the base of a regular triangular pyramid with the altitude H and the radius of the ball inscribed in the base equal to r. The ball touches the base at its centre. Find the radius of the sphere if the plane drawn through the vertex of the pyramid and the midpoints of two sides of the base touches the sphere.

4.33. In the triangular pyramid $SABC$ the face SAC is perpendicular to the face ABC, $|SA| = |SC| = 1$, and the vertex angle B of the triangle ABC is a right angle. The ball touches the plane of the base of the pyramid at the point B and the face SAC at the point S. Find the radius of the ball.

4.34. Three identical spheres are in the interior of a regular triangular pyramid with the length of the edge of the base equal to a and the base dihedral angle equal to 60°. Each sphere touches the other two spheres, the plane of the base and two lateral faces. Find the radii of the spheres.

4.35. A sphere of radius 1 is in the interior of a regular triangular pyramid. At the point which bisects the altitude of the pyramid it is externally tangent to a hemisphere. The hemisphere rests on a circle inscribed in the base of the pyramid; the sphere touches the lateral faces of the pyramid. Find the lateral area of the pyramid and the dihedral angle between the lateral faces of the pyramid.

4.36. A regular triangle with side a lies in the plane P. Its medians divide it into four triangles and three regular pyramids with the altitude a are constructed on three of them, adjacent to the vertices as the bases (the three pyramids lie on the same side of P). Find the radius of the sphere which lies between the pyramids and touches both the plane P and the three pyramids.

Example 4.3. A sphere is inscribed in a regular quadrangular pyramid. The distance from the centre of the sphere to the vertex of the pyramid is a and the angle of inclination of the lateral face to the plane of the base is α. Find the volume of the pyramid. *B*

Solution. Assume that $SABCD$ (Fig. 12.15) is a regular quadrangular pyramid ($ABCD$ is a square, SK is the altitude of the pyramid, K is the centre of the square, the lateral faces ara equal isosceles triangles). Let us find a point O which is the centre of the sphere inscribed into the pyramid.

We draw an altitude from the vertex A of the triangle ASB to the base SB. Next we draw an altitude from the vertex C of the triangle BSC to the base SB. Since the triangles ASB and BSC are equal, one and the same point M serves as the feet of the altitudes and $|AM| = |MC|$. By construction, the angle AMC is a plane dihedral angle formed by the planes of the lateral faces ASB and BSC. In the triangle AMC the bisector of the angle AMC is a median and an altitude and cuts the base AC at a point K.

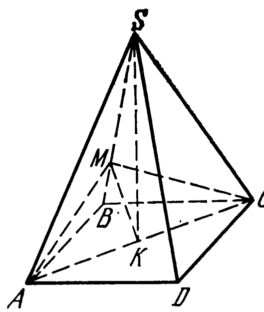

Fig. 12.15

In our case, the point S belongs to an edge of the dihedral angle and the point K is the bisector of the dihedral angle and, consequently, the altitude SK of the pyramid belongs to the bisecting plane of the dihedral angle with edge SB. Constructing the disecting plane of the dihedral angle with edge SC, we can prove by analogy that the altitude SK belongs to that bisecting plane as well. Since two different planes intersect along the same straight line and the line SK belongs to both planes, the line of intersection of two different bisecting planes contains the altitude SK of the pyramid. The centre of the ball inscribed into the pyramid lies at the intersection of the bisecting planes of all the dihedral angles of the pyramid and, consequently, in our case (when the pyramid is regular) on the altitude of the pyramid.

Let us draw a plane through the altitude SK of the pyramid and the midpoints L and P of the opposite sides of the square (Fig. 12.16). Since $\triangle DSC$ is isosceles, it follows that SP is a median and an altitude of $\triangle DSC$ and $SP \perp DC$. The segment LP is also perpendicular to DC and, consequently, the angle LPS is a plane dihedral angle formed by the lateral face DSC and the plane of the base $ABCD$ which, by the hypothesis, is equal to α. We can prove by analogy that $\angle SLP = \alpha$. The centre of the sphere inscribed in the pyramid lies in the bisecting plane of the dihedral angle with edge DC. The point of intersection of this plane and the altitude SK is the centre of the ball inscribed in the pyramid (the point O in Fig. 12.16).

The planes DSC and $ABCD$ are perpendicular to the plane LSP since they contain the straight line DC which is perpendicular to the plane LSP. We drop a perpendicular from the point O belonging to the plane LSP to the plane DSC. The foot of this perpendicular (point N) coincides with the point of tangency of the ball and the plane DSC. On the other hand, according to the theorem on two mutually perpendicular planes (DSC and LSP) and a perpendicular to one of the planes, the straight line ON entirely belongs to the plane LSP. We have thus proved that the point of tangency of the sphere and the plane of the lateral face DSC belongs to the plane LSP.

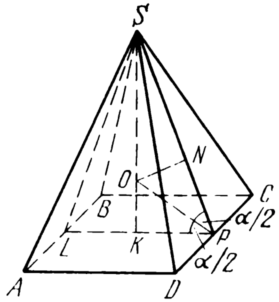

Fig. 12.16

We can prove by analogy that the points of tangency of the ball with the planes $ABCD$ and ASB also belong to the plane LSP.

Let us consider the triangles OSN and PSK. These triangles are similar (they are right-angled and have an angle in common). It follows from the similarity of the triangles that $\angle SON = \alpha$ and, by the hypothesis, $|SO| = a$. From the triangle OSN we find that $|ON| = a\cos\alpha$. The altitude of the triangle is $|SK| = |SO| + |OK| = a(1+\cos\alpha)$. From the triangle SKP we find that

$$|KP| = |SK|\cot\alpha = a(1+\cos\alpha)\cot\alpha.$$

In the triangle LSP we have $|LP| = 2|KP| = 2a(1+\cos\alpha)\times \cot\alpha = |AB|$. We have thus found the quantities which we need to calculate the volume of the pyramid:

$$V_{SABCD} = \frac{1}{3}|SK|\,S_{ABCD} = \frac{1}{3}a(1+\cos\alpha)\,4a^2(1+\cos\alpha)^2\cot^2\alpha$$

$$= \frac{4}{3}a^3(1+\cos\alpha)^3\cot^2\alpha.$$

Answer. $\frac{4}{3}a^3(1+\cos\alpha)^3\cot^2\alpha$

4.37. Find the surface of the sphere circumscribed about a regular quadrangular pyramid if the side of the base of the pyramid is a and the lateral edge of the pyramid makes an angle α with the plane of the base.

4.38. A regular quadrangular pyramid is inscribed in a sphere. The radius of the sphere is R, the edge angle is α. Find the lateral surface of the pyramid.

4.39. A pyramid whose base is a square is inscribed in a sphere of radius R. Two lateral faces are perpendicular to the base. The larger lateral edge makes an angle α with the side of the base intersecting it. Find the lateral surface of the pyramid.

4.40. The base of the pyramid is a square with side a. The altitude of the pyramid passes through the midpoint of one of the edges of the base and is equal to $a\sqrt{3}/2$. Find the radius of the sphere circumscribed about the pyramid.

4.41. A regular quadrangular is inscribed in a sphere of radius R. Find the volume of the pyramid if the radius of the circle circumscribed about its base is r.

4.42. A sphere of radius R is inscribed into a pyramid whose base is a rhombus with an acute angle α. The lateral faces of the pyramid make an angle φ with the plane of the base. Find the volume of the pyramid.

4.43. A pyramid whose base is a square is inscribed in a sphere of radius R. One of the lateral edges of the pyramid is perpendicular to the plane of the base and the larger lateral edge makes an angle α with the base. Find the lateral surface of the pyramid.

4.44. The base area of a regular quadrangular pyramid is Q and the base dihedral angle is α. The pyramid is cut by a plane which passes through the centre of the inscribed sphere parallel to the base. Find the area of the section of the pyramid.

4.45. Given a pyramid $SABCD$ whose base is a rhombus $ABCD$. The side of the base is a, $|SA| = |SC| = a$, $|SB| = |SD|$, $\angle BCD = 2\alpha$. Find the radius of the inscribed sphere.

4.46. The centre of the sphere circumscribed about a regular quadrangular pyramid is at the distance a from a lateral face and at the distance b from a lateral edge. Find the radius of the sphere.

4.47. A sphere whose radius is $a/(2\sqrt{3})$ is inscribed in a regular quadrangular pyramid $SABCD$ with vertex S and the side of the base a. A plane P which makes an angle of 30° with the plane of the base touches the sphere and meets the plane of the base, without meeting the base itself, along a line which is parallel to a side of the base. Find the area of the section of the pyramid by the plane P.

4.48. A pyramid whose lateral edges are equal to c is inscribed into a sphere. Its base is a rectangle whose sides subtend arcs of α and β radians in the sections of the ball by the planes of the lateral faces. Find the radius of the circumscribed sphere.

4.49. The lateral edges of a regular quadrangular pyramid $SABCD$ (S is the vertex) are equal to a and the sides of its base are equal to $a/\sqrt{2}$. Find the distance from the centre of the ball inscribed into the pyramid $SABCD$ to the plane which passes through the diagonal BD of the base $ABCD$ and the midpoint E of the edge SA.

4.50. A regular quadrangular pyramid contains a sphere of radius 2. The ball touches the lateral faces of the pyramid and is externally tangent to a hemisphere resting on the sphere inscribed in the base of the pyramid. The point of tangency of the ball and the hemisphere is at the distance from the base of the pyramid which is equal to one-third of the altitude of the pyramid. Find the volume of the pyramid and the dihedral angle at the lateral edge of the pyramid.

4.51. Given a regular quadrangular pyramid $SABCD$ with the side of the base a and the lateral edge b ($b > a$). A sphere with centre at a point O lies above the plane of the base $ABCD$, touches that plane at the point A and, in addition, touches the lateral edge SB. Find the volume of the pyramid $OABCD$.

4.52. Given a sphere of radius R into which a regular hexagonal truncated pyramid is inscribed the plane of whose lower base passes through the centre of the sphere and a lateral edge makes an angle of 60° with the plane of the base. Find the volume of the pyramid.

4.53. Given a sphere of radius R about which a regular hexagonal pyramid is circumscribed whose lateral face makes an angle α with the plane of the base. Find the lateral surface and the volume of the pyramid.

4.54. Given a sphere in which a rectangular prism is inscribed, whose base is a regular triangle, and the altitude of the prism is equal to the side of the base. Find the ratio of the volume of the prism to that of a regular hexagonal pyramid inscribed in the sphere, the lateral edge of the pyramid being equal to double the side of the base.

4.55. Find the ratio of the volume of a regular n-gonal pyramid to that of the sphere inscribed into it knowing that the circles circumscribed about the base and the lateral faces of the pyramid are equal.

Combinations of solids of revolution. A sphere is said to be *inscribed into a right circular cone* if it touches the base of the cone at its centre and is tangent to the lateral surface of the cone along a circle. A right circular cone is said to be *inscribed in a sphere* if its vertex and the circle serving as its base lie on the surface of the sphere.

A sphere is said to be *inscribed in a right circular cylinder* if the sphere touches the bases of the cylinder at their centres and is tangent to the lateral surface of the cylinder along the circumference of the large circle of the sphere which is parallel to the bases. A right circular cylinder is said to be *inscribed in a sphere* if the circles serving as the bases of the cylinder lie on the surface of the sphere.

A cone is said to be *inscribed in a cylinder* if the base of the cone coincides with one of the bases of the cylinder and the vertex of the cone coincides with the centre of the other base.

It is often convenient to solve problems dealing with combinations of solids of revolution by constructing an auxiliary section which divides the combination of the solids of revolution into two symmetric parts. It is, as a rule, convenient to construct the auxiliary section so that it passes through the axis of a cylinder (or the axis of a cone), depending on the kind of the problem, and the centre of a sphere. As a result, a rectangle is obtained in the section of the cylinder and an isosceles triangle in the section of the cone, and the section of the ball is a circle with the radius equal to that of the sphere. Thus, for instance, if by the hypothesis a ball is inscribed in a cone, then the axial section of the cone is an isosceles triangle and a circle inscribed in it; if a sphere is circumscribed about a cylinder, then the section is a rectangle circumscribed about a circle.

Example 4.4. The generatrix of a cone is l and makes an angle α with the altitude of the cone. A plane is drawn through two generatrices of the cone which make an angle β. Find the distance from that plane to the centre of the ball inscribed into the cone.

Solution. Assume that S is the vertex of the cone, O is the foot of the altitude of the cone, SA and SB are the generatrices of the cone which make an angle β (Fig. 12.17). From the right triangle SOB ($\angle SOB$ is a right angle, $|SB| = l$, $\angle BSO = \alpha$) we find the radius of the circle lying at the base of the cone and the altitude of the cone:

$$|BO| = l \sin \alpha, \quad |SO| = l \cos \alpha.$$

From the isosceles triangle ASB ($\angle ASB = \beta$, $|AS| = |SB| = l$) we find the length of the chord AB which is cut by the plane ASB from the base of the cone:

$$|AB| = 2l \sin \frac{\beta}{2}.$$

The distance from the centre of the base of the cone to the midpoint of the segment AB (point N) can be found from the triangle NOB by the Pythagorean theorem:

$$|NO| = \sqrt{|BO|^2 - |NB|^2} = l \sqrt{\sin^2 \alpha - \sin^2 \frac{\beta}{2}}.$$

We draw a plane through the midpoint of the chord AB (point N) and the altitude SO of the cone. The section of the cone by the plane is an isosceles triangle MSM_1 with the lateral side l and the vertex

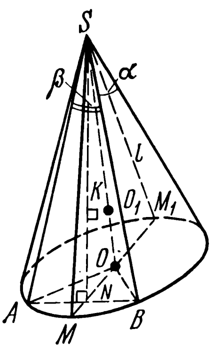

Fig. 12.17

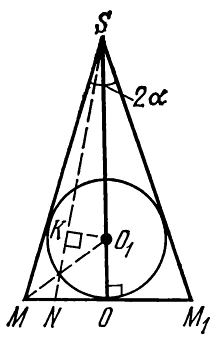

Fig. 12.18

angle 2α; the section of the ball, inscribed into the cone, is a circle with the radius equal to the radius of the ball inscribed in the isosceles triangle MSM_1, and the planes MSM_1 and ASB intersect along a straight line NS (Fig. 12.18).

Let us prove that the distance from the centre O_1 of the circle to the line SN (i.e. the length of the segment O_1K) is the required distance from the centre of the sphere to the plane ASB.

Let us consider Fig. 12.17. The chord AB of the circle serving as the base of the cone is perpendicular to the altitude of the cone and is perpendicular to the radius OM passing through the midpoint of the chord AB (point N). Consequently, the chord AB is perpendicular to the plane MSM_1 since it is perpendicular to two nonparallel straight lines (SO and MM_1) belonging to the plane. We thus have a plane MSM_1 and a straight line AB which is perpendicular to it. Since the plane ASB passes through the straight line AB, which is perpendicular to the plane MSM_1, the plane ASB itself is perpendicular to the plane MSM_1, i.e. the two planes are mutually perpendicular.

The centre of the sphere inscribed into the cone belongs to the plane MSM_1. If we drop a perpendicular from the centre of the sphere to the plane ASB, then, according to the theorem on two mutually perpendicular planes, that perpendicular entirely belongs to the plane MSM_1, i.e. the foot of the perpendicular (point K) lies on the line of intersection SN of the planes as can be seen from Fig. 12.17.

Thus, seeking the distance from the centre of the sphere to the plane ASB reduces to seeking the length of the segment $|KO_1|$ shown in Fig. 12.18 from the problem of plane geometry. Let us find the radius of the circle inscribed in $\triangle MSM_1$:

$$|OO_1| = |MO| \cdot \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = |BO| \cdot \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$
$$= l \sin\alpha \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

Let us consider the triangles KSO_1 and NSO. These triangles are similar (they are right-angled and have a common angle NSO). Let us calculate the hypotenuse SO_1 of the triangle KSO_1:

$$|SO_1| = |SO| - |O_1O| = \left(\cos\alpha - \sin\alpha \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right).$$

The hypotenuse SN of the triangle SON can be calculated by the Pythagorean theorem:

$$|SN| = \sqrt{|NO|^2 + |SO|^2}$$
$$= l\sqrt{\sin^2\alpha - \sin^2\frac{\beta}{2} + \cos^2\alpha} = l\cos\frac{\beta}{2}.$$

The similitude of the triangles KSO_1 and NSO yields equations

$$\frac{|KO_1|}{|NO|} = \frac{|SO_1|}{|SN|} \Rightarrow |KO_1|$$
$$= \frac{l\left(\cos\alpha - \sin\alpha\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right)\sqrt{\sin^2\alpha - \sin^2\frac{\beta}{2}}}{\cos\frac{\beta}{2}}.$$

Answer. $\dfrac{l\sqrt{\sin^2\alpha - \sin^2\frac{\beta}{2}}\left(\cos\alpha - \sin\alpha\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right)}{\cos\frac{\beta}{2}}.$

4.56. A sphere is inscribed into a cone. Find the volume of the ball if the generatrix of the cone is l and makes an angle α with the plane of the base.

4.57. Given a sphere of radius R about which a truncated cone is circumscribed whose generatrix makes an angle α with the plane of the larger base. Find the volume and the lateral area of the truncated cone.

4.58. A sphere is inscribed into a cone, the surface of the ball being equal to that of the base of the cone. Find the vertex angle in the axial section of the cone.

4.59. A sphere is inscribed in a cone. The radius of the circle along which the cone and the ball are tangent is r. Find the volume of the cone if the angle between the altitude and the generatrix of the cone is α.

4.60. A cone is inscribed in a sphere whose surface area is S. The angle between the generatrix of the cone and the plane of the base is α. Find the area of the total surface of the cone.

4.61. A sphere is inscribed in a cone. Prove that the ratio of the total surface of the cone to the surface of the sphere is equal to the ratio of their volumes.

4.62. A sphere is inscribed in a cone whose generatrices make an angle α with the plane of the base. Find the ratio of the volume of the sphere to that of the cone.

4.63. A sphere is inscribed in a right circular cone. The ratio of the volumes of the cone and the sphere is equal to 2. Find the ratio of the total surface of the cone to the surface of the sphere.

4.64. The altitude of a cylinder is equal to that of a cone. The lateral surface of the cylinder is related to that of the cone as 3 : 2. In addition, it is known that the generatrix of the cone makes an angle α with the plane of the base. Find the ratio of the volume of the cylinder to that of the cone.

4.65. A sphere of area s is inscribed into a truncated cone with the lateral area S. Find the angle between the generatrix and the plane of the base of the cone.

4.66. The vertex angle of the axial section of a right circular cone is α, and the radius of the base of the cone is R. Find the volume of the sphere with the centre at the vertex of the cone which divides the volume of the cone in half.

4.67. Three identical spheres of radius r are in the interior of a right circular cone, with the vertex angle of 60° in the axial section of the cone, so that each sphere touches the other two spheres, the lateral surface of the cone and the plane of the base. Find the radius of the base of the cone.

4.68. Three spheres of radius r lie on the base of a right circular cone. A fourth sphere of the same radius lies on the first three. Each of the four spheres touches the lateral surface of the cone and the three other spheres. Find the altitude of the cone.

4.69. Find the vertex angle in the axial section of a cone circumscribed about four equal spheres located so that each of them touches the other three.

4.70. A sphere of radius r is inscribed into a cone. Find the volume of the cone if it is known that the plane which touches the sphere and is perpendicular to one of the generatrices of the cone is at the distance d from the vertex of the cone.

4.71. A sphere is inscribed in a truncated cone in which the radii of the upper and lower bases are R and r. Find the radius of the second ball which touches the first sphere, the lateral surface of the truncated cone and the upper base.

4.72. Two cones have the altitudes h_1 and h_2 and a common base of radius R, and their vertices lie on different sides of the plane of the base. A ball is inscribed into the surface formed by the lateral surfaces of those cones. Find the radius of the other ball which touches both the lateral surface of the first cone (along an integral circle) and the first ball.

4.73. A sphere touches the base of a cone at its centre. The surface of the sphere cuts the lateral surface of the cone along two circles one of which has a radius equal to that of the sphere and lies in the plane which is parallel to the base of the cone. The radius of the base is known to be 4/3 times as large as the radius of the sphere. Find the ratio of the volume of the sphere to that of the cone.

4.74. A right circular cylinder is circumscribed about a sphere of radius R. A point C lies in the interior of the cylinder on its axis and is at the distance $\frac{3}{4}R$ from the lower base. A plane P is drawn through that point, which has only one common point with the circle of the lower base. A right circular cone is inscribed into the sphere whose base lies in the plane P and the vertex is above that plane. Find the volume of the cone.

4.75. A right circular cone has the radius of the base r and an angle α in the axial section. Two identical balls of radius R touch each other, the lateral surface of the cone (externally) and the plane of the base of the cone. Find the area of the triangle whose vertices are the centres of the balls and the centre of the base of the cone.

4.76. Three identical spheres are inscribed into a right circular cylinder with the radius of the base $r = 1$ and the altitude $H = 12/(3 + 2\sqrt{3})$ so that the spheres touch the upper base of the cylinder, its lateral surface and pairwise touch one another. Find the volume of a right circular cone whose base coincides with the lower base of the cylinder and which touches all the three spheres.

4.77. Given three identical right circular cones with the angle α ($\alpha < 2\pi/3$) in the axial section and the radius of the base r. The bases of the cones lie in the same plane and pairwise externally touch one another. Find the radius of the sphere which touches all the three cones and the plane passing through their vertices.

Chapter 13

The Method of Coordinates

1. Vectors in Coordinates

An ordered triple of noncoplanar vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is called a *basis* in the set of all vectors. Every vector $\mathbf{a}$ can be uniquely represented in the form

$$\mathbf{a} = X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3, \tag{1}$$

the numbers X, Y, Z are called the *coordinates* of the vector a in the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Representation (1) is also called a *resolution of the vector* $\mathbf{a}$ *into components* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and is written as $\mathbf{a} = (X, Y, Z)$.

The basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is said to be *rectangular* if the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are pairwise perpendicular and have a unit length (a vector of a unit length is called a *unit vector*). In that case the following designations are accepted:

$$\mathbf{e}_1 = \mathbf{i}, \quad \mathbf{e}_2 = \mathbf{j}, \quad \mathbf{e}_3 = \mathbf{k}.$$

If a point A has coordinates (a_1, a_2, a_3) relative to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and a point B has coordinates (b_1, b_2, b_3), then

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3). \tag{2}$$

The actions on vectors defined by their coordinates relative to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are carried out according to the following rules: if $\mathbf{a} = (X_1, Y_1, Z_1)$, $\mathbf{b} = (X_2, Y_2, Z_2)$, then

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (X_1 + X_2, \ Y_1 + Y_2, \ Z_1 + Z_2), \\ \mathbf{a} - \mathbf{b} &= (X_1 - X_2, \ Y_1 - Y_2, \ Z_1 - Z_2), \end{aligned} \tag{3}$$

$$\lambda\mathbf{a} = (\lambda X_1, \lambda Y_1, \lambda Z_1), \tag{4}$$

where λ is a certain number.

The *length of the vector* $\mathbf{a} = (X_1 + Y_1 + Z_1)$, designating as $|\mathbf{a}|$ can be calculated from the formula

$$|\mathbf{a}| = \sqrt{X_1^2 + Y_1^2 + Z_1^2}.$$

Example 1.1. Given the vectors

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}, \ \mathbf{b} = -3\mathbf{j} - 2\mathbf{k}, \ \mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Find the coordinates of the vector $\mathbf{a} - \frac{1}{2}\mathbf{b} + \mathbf{c}$.

Solution. By the hypothesis

$$\mathbf{a} = (2, 3, 0), \mathbf{b} = (0, -3, -2), \mathbf{c} = (1, 1, -1).$$

Using rules (3), (4), we get

$$\mathbf{a} - \frac{1}{2}\mathbf{b} + \mathbf{c} = \left(2 - 0 + 1,\ 3 + \frac{3}{2} + 1,\ 0 + 1 - 1\right).$$

Answer. $\mathbf{a} - \frac{1}{2}\mathbf{b} + \mathbf{c} = \left(3,\ \frac{11}{2},\ 0\right)$.

1.1. Given the vectors $\mathbf{a} = (-3, -1, 2)$, $\mathbf{b} = (4, 0, 6)$, $\mathbf{c} = (5, -2, 7)$. Find the coordinates of the vectors (a) $2\mathbf{a}$, (b) $-\mathbf{a} + 3\mathbf{c}$.

1.2. Given three vectors: $\mathbf{a} = (2, 4)$, $\mathbf{b} = (-3, 1)$, $\mathbf{c} = (5, -2)$. Find the coordinates of the vectors

(a) $= 2\mathbf{a} + 3\mathbf{b} - 5\mathbf{c}$, (b) $\mathbf{a} + 24\mathbf{b} + 14\mathbf{c}$, (c) $2\mathbf{a} - \frac{1}{2}\mathbf{b}$, (d) $5\mathbf{c}$.

Example 1.2. Given three vectors: $\mathbf{a} = (5, 3)$, $\mathbf{b} = (2, 0)$ $\mathbf{c} = (4, 2)$. Find nonzero numbers α, β, and γ such that

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0.$$

Solution. Using rules (3), (4), we write

$$(5\alpha, 3\alpha) + (2\beta, 0) + (4\gamma, 2\gamma) = (0, 0),$$

or, taking into account the uniqueness of the representation of a vector in terms of base vectors, we write a system of equations relative to the unknowns α, β, γ:

$$5\alpha + 2\beta + 4\gamma = 0,$$
$$3\alpha + 2\gamma = 0,$$

from which we can find the expression for α and γ in terms of β: $\alpha = 2\beta$, $\gamma = -3\beta$. Since $\beta \neq 0$, we can assume, for example, that $\beta = 1$ and obtain $\alpha = 2$, $\beta = 1$, $\gamma = -3$.

Answer. $\alpha = 2$, $\beta = 1$, $\gamma = -3$.

Example 1.3. Points $A(1, 1)$, $B(0, 3)$, $C(-1, -1)$ are vertices of the triangle ABC. Find the coordinates of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$. Prove that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$.

Solution. We find the coordinates of the vector $\overrightarrow{AB}$ applying formula (2): $X = 0 - 1 = -1$, $Y = 3 - 1 = 2$, i.e. $\overrightarrow{AB} = (-1, 2)$. By analogy we find that $\overrightarrow{BC} = (-1, -4)$, $\overrightarrow{CA} = (2, 2)$ and $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = (-1 -1 + 2, 2 - 4 + 2) = 0$.

1.3. Given vectors $\mathbf{a} = (1, 5, 3)$, $\mathbf{b} = (6, -4, -2)$, $\mathbf{c} = (0, -5, 7)$ and $\mathbf{d} = (-20, 27, -35)$. Find numbers α, β, γ such that

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \mathbf{d} = \mathbf{0}.$$

1.4. Given a tetrahedron $OABC$. In the basis consisting of the edges $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ find the coordinates

(a) of the vector $\overrightarrow{DE}$, where D and E are the midpoints of the edges $\overrightarrow{OA}$ and $\overrightarrow{BC}$,

(b) of the vector $\overrightarrow{OF}$, where F is the point of intersection of the medians of the base ABC.

1.5. In the tetrahedron $OABC$ the median AL of the face ABC is divided by a point M in the ratio $|\overrightarrow{AM}| : |\overrightarrow{ML}| = 3 : 7$. Find the coordinates of the vector $\overrightarrow{OM}$ in the basis formed by the edges $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$.

1.6. Express the vector $\mathbf{c}$ in terms of the vectors $\mathbf{a}$ and $\mathbf{b}$ in each of the following cases:

(a) $\mathbf{a} = (4, -2)$, $\mathbf{b} = (3, 5)$, $\mathbf{c} = (1, -7)$;
(b) $\mathbf{a} = (5, 4)$, $\mathbf{b} = (-3, 0)$, $\mathbf{c} = (19, 8)$;
(c) $\mathbf{a} = (-6, 2)$, $\mathbf{b} = (4, 7)$, $\mathbf{c} = (9, -3)$.

1.7. Find the coordinates of the vector $\overrightarrow{PQ}$ from the coordinates of the points P and Q:

(a) $P(2, -3, 0)$, $Q(-1, 2, -3)$;

(b) $P\left(\frac{1}{2}, -\frac{4}{3}, \frac{5}{6}\right)$, $Q\left(-\frac{3}{5}, 0, \frac{2}{3}\right)$.

1.8. Given four points: $A(0, 2)$, $B(3, 1)$, $C(-5, 3)$, $D(2, 4)$. Find the coordinates of a point Q such that

$$\overrightarrow{QA}+\overrightarrow{QB}+\overrightarrow{QC}+\overrightarrow{QD}=0.$$

1.9. A vector $\overrightarrow{AB} = \mathbf{a}$ is laid off from the point A. Find the coordinates of the point B in each of the following cases:

(a) $A(0, 0)$, $\mathbf{a} = (-2, 1)$,
(b) $A(-1, 5)$, $\mathbf{a} = (1, -3)$,
(c) $A(2, 7)$, $\mathbf{a} = (-2, -5)$.

1.10. On the abscissa axis find a point M which is at the distance 5 from the point $A(3, -3)$.

1.11. On the axis of ordinates find a point M which is equidistant from the points $A(1, -4, 7)$ and $B(5, 6, -5)$.

1.12. Find the coordinates of the point M which lies on the Ox axis and is equidistant from the points $A(1, 2, 3)$ and $B(-3, 3, 2)$.

1.13*. Find the coordinates of the centre of gravity of the triangle ABC if the points A, B, and C have the following coordinates:

(a) $A(0, 0)$, $B(0, 3)$, $C(5, 0)$,
(b) $A(0, 0)$, $B(2, 5)$, $C(-1, 7)$,
(c) $A(1, 3)$, $B(3, 6)$, $C(-2, 5)$.

The *condition of collinearity* of two vectors $\mathbf{a} = (X_1, Y_1, Z_1)$ and $\mathbf{b} = (X_2, Y_2, Z_2)$ has the form

$$X_1 = \lambda X_2,\ Y_1 = \lambda Y_2,\ Z_1 = \lambda Z_2,$$

or, when λ is removed, the form

$$\frac{X_1}{X_2} = \frac{Y_1}{Y_2} = \frac{Z_1}{Z_2}.$$

1.14. In each of the following cases find the value of k for which the vector $\mathbf{a} + k\mathbf{b}$ is collinear with the vector $\mathbf{c}$:

(a) $\mathbf{a} = (2, 3)$, $\mathbf{b} = (3, 5)$, $\mathbf{c} = (-1, 3)$;
(b) $\mathbf{a} = (1, 0)$, $\mathbf{b} = (2, 2)$, $\mathbf{c} = (3, -5)$;
(c) $\mathbf{a} = (3, -2)$, $\mathbf{b} = (1, 1)$, $\mathbf{c} = (0, 5)$.

1.15. Using the condition of collinearity of two vectors, find whether the following vectors are collinear:

(a) $\mathbf{a} = \left(\frac{3}{7}, \frac{1}{2}, -\frac{3}{4}\right)$ and $\mathbf{b} = \left(\frac{2}{7}, \frac{1}{3}, -\frac{1}{2}\right)$;

(b) $\mathbf{c} = \left(-\frac{3}{2}, 6, \frac{4}{3}\right)$ and $\mathbf{d} = \left(\frac{9}{8}, -\frac{9}{2}, -1\right)$.

1.16. For what values of X and Y are the vectors $\mathbf{a} = (X, -2, 5)$ and $\mathbf{b} = (1, Y, -3)$ collinear?

1.17. Given four points: $A(-2, -3, 8)$, $B(2, 1, 7)$, $C(1, 4, 5)$, and $D(-7, -4, 7)$. Prove that the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear

1.18. A segment with the endpoints $A(3, -2)$ and $B(6, 4)$ is divided into three equal parts. Find the coordinates of the points of division.

1.19. Find the coordinates of the endpoints of the segment which is divided into three equal parts by the points $C(2, 0, 2)$ and $D(5, -2, 0)$.

1.20. Given the vertices of a triangle: $A(1, 0, 2)$, $B(1, 2, 2)$, and $C(5, 4, 6)$. A point L divides the segment $\overrightarrow{AC}$ in the ratio $1:3$; CE is a median drawn from the vertex C. Find the coordinates of the point of intersection of the straight lines BL and CE.

1.21. For what values of α and β are the vectors $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} + \alpha\mathbf{k}$ and $\mathbf{b} = \beta\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$ collinear?

The *scalar product* of the vectors $\mathbf{a}$ and $\mathbf{b}$ designated as $\mathbf{a}\cdot\mathbf{b}$ or $\mathbf{ab}$ is the product $|\mathbf{a}|\cdot|\mathbf{b}|\cos(\widehat{\mathbf{a}, \mathbf{b}})$. If the vectors $\mathbf{a} = (X_1, Y_1, Z_1)$ and $\mathbf{b} = (X_2, Y_2, Z_2)$ are defined by their coordinates in a rectangular basis, then their scalar product can be calculated from the formula

$$\mathbf{ab} = X_1X_2 + Y_1Y_2 + Z_1Z_2. \tag{5}$$

The cosine of the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ can be calculated from the formula

$$\cos(\widehat{\mathbf{a}, \mathbf{b}}) = \frac{X_1X_2 + Y_1Y_2 + Z_1Z_2}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}\cdot\sqrt{X_2^2 + Y_2^2 + Z_2^2}}. \tag{6}$$

The *condition of* mutual *perpendicularity* of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the following:

$$X_1X_2 + Y_1Y_2 + Z_1Z_2 = 0.$$

Example 1.4. Given two vectors: $\mathbf{a} = (5, 2)$, $\mathbf{b} = (7, -3)$. Find a vector $\mathbf{c}$ which satisfies the conditions $\mathbf{ac} = 38$, $\mathbf{bc} = 30$.

Solution. Assume $\mathbf{c} = (X, Y)$. Then, by virtue of (5), we have

$$5X + 2Y = 38,$$
$$7X - 3Y = 30.$$

Solving this system for X and Y, we get $X = 6$, $Y = 4$.

Answer. $\mathbf{c} = (6, 4)$.

1.22. Given vectors $\mathbf{a} = (4, -2, -4)$ and $\mathbf{b} = (6, -3, 2)$. Calculate: (a) $\mathbf{ab}$, (b) $(2\mathbf{a} - 3\mathbf{b})(\mathbf{a} + 2\mathbf{b})$, (c) $(\mathbf{a} - \mathbf{b})^2$, (d) $|2\mathbf{a} - \mathbf{b}|$.

1.23. Given a vector $\mathbf{a} = (-6, 8)$. Find the coordinates of a unit vector which is collinear with the vector $\mathbf{a}$ and

(a) has the same direction as the vector $\mathbf{a}$,

(b) is of the opposite direction to the vector $\mathbf{a}$.

1.24*. The vectors $\mathbf{a} = (-12, 16)$, $\mathbf{b} = (12, 5)$ are drawn from the same point. Find the coordinates of the vector which, being laid off from the same point, bisects the angle between the vectors.

1.25. Knowing that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 1$, $|\mathbf{c}| = 4$ and $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, calculate $\mathbf{ab} + \mathbf{bc} + \mathbf{ca}$.

1.26. Calculate the lengths of the vectors

(a) $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, (b) $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

1.27. The vector is 3 long. Calculate the coordinates of the vector if they are known to be equal.

1.28. Calculate the length of the vector $2\mathbf{a} + 3\mathbf{b}$ if

$$\mathbf{a} = (1, 1, -1), \ \mathbf{b} = (2, 0, 0).$$

1.29. Given the following vectors: $\mathbf{a} = (1, 1, -1)$, $\mathbf{b} = (5, -3, -3)$ and $\mathbf{c} = (3, -1, 2)$. Find the vectors, collinear with the vector $\mathbf{c}$, whose lengths are equal to that of the vector $\mathbf{a} + \mathbf{b}$.

1.30*. The vectors $\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{k}$ and $\overrightarrow{BC} = (-1, 0, -2)$ are the sides of the triangle ABC. Find the length of the median AM.

Example 1.5. Calculate the angle between the vectors $\mathbf{a} = (-1, 2, -2)$ and $\mathbf{b} = (6, 3, -6)$.

Solution. By formula (6)

$$\cos(\widehat{\mathbf{a}, \mathbf{b}}) = \frac{(-1)\cdot 6 + 2\cdot 3 + (-2)\cdot(-6)}{\sqrt{1+4+4}\ \sqrt{36+9+36}} = \frac{12}{3\cdot 9} = \frac{4}{9}.$$

Answer. $(\widehat{\mathbf{a}, \mathbf{b}}) = \arccos\frac{4}{9}$.

1.31. Calculate the angle between the following vectors:

(a) $\mathbf{a} = (6, -2, -3)$, $\mathbf{b} = (5, 0, 0)$;

(b) $\mathbf{a} = (2, -4, 5)$, $\mathbf{b} = (0, 2, 0)$;

(c) $\mathbf{a} = (-2, 6, -3)$, $\mathbf{b} = (0, 0, -3)$;

(d) $\mathbf{a} = (-4, -6, 2)$, $\mathbf{b} = (4, 0, 0)$;

(e) $\mathbf{a} = (3, -2, 6)$, $\mathbf{b} = (0, -5, 0)$;

(f) $\mathbf{a} = (4, -5, -2)$, $\mathbf{b} = (0, 0, 2)$,

1.32*. What angle do the vectors $\mathbf{a} = (2, 3)$, $\mathbf{b} = (-2, 5)$, $\mathbf{c} = (-5, 1)$, and $\mathbf{d} = (-1, 1)$ make with the unit vector $\mathbf{i}$?

1.33. Calculate the cosine of the angle between the vectors $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ if $\mathbf{a} = (1, 2, 1)$ and $\mathbf{b} = (2, -1, 0)$.

1.34*. Calculate the cosines of the angles which the following vectors make with the base vectors:

(a) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, (b) $\mathbf{b} = -3\mathbf{j} - \mathbf{k}$, (c) $\mathbf{c} = -5\mathbf{i}$, (d) $\mathbf{d} = 3\mathbf{j} + 4\mathbf{k}$.

1.35. Calculate the coordinates of the vector $\mathbf{p}$ which is collinear with the vector $\mathbf{q} = (3, -4)$ if the vector $\mathbf{p}$ is known to make an obtuse angle with the vector $\mathbf{i}$ and $|\mathbf{p}| = 10$.

1.36. The vector $\mathbf{b}$ is collinear to the vector $\mathbf{a} = (6, 8, -15/2)$ and makes an acute angle with the unit vector $\mathbf{k}$. Knowing that $|\mathbf{b}| = 50$, find its coordinates.

1.37. Calculate the angle between the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$ and find the lengths of the diagonals of the parallelogram constructed on these vectors as on sides.

1.38*. The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are equal in length and pairwise make equal angles. Find the coordinates of the vector $\mathbf{c}$ if $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

1.39. A straight line makes equal angles with the edges of a right trihedral angle. Find the angles.

1.40. The vectors $\overrightarrow{AB} = (3, -2, 2)$ and $\overrightarrow{BC} = (-1, 0, -2)$ are adjacent sides of a parallelogram. Find the angle between its diagonals.

Example 1.6. Calculate the coordinates of the unit vector $\mathbf{a}$ if it is known to be perpendicular to the vectors $\mathbf{b} = (1, 1, 0)$ and $\mathbf{c} = (0, 1, 1)$.

Solution. Assume that the vector $\mathbf{a}$ has the coordinates X, Y, Z. Then, by the hypothesis,

$$X^2 + Y^2 + Z^2 = 1. \qquad (*)$$

From the condition of perpendicularity of the vector $\mathbf{a}$ to the vectors $\mathbf{b}$ and $\mathbf{c}$ we get an equations

$$X + Y = 0 \text{ and } Y + Z = 0.$$

Substituting X and Z expressed in terms of Y in equation (*), we obtain $Y = \pm 1/\sqrt{3}$; consequently, there are two vectors which satisfy the hypothesis:

$$\mathbf{a}_1 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right),$$

$$\mathbf{a}_2 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

1.41. For what value of Z are the vectors $\mathbf{a} = (6, 0, 12)$ and $\mathbf{b} = (-8, 13, Z)$ perpendicular?

1.42. For what X and Y is the vector $\mathbf{a} = X\mathbf{i} + Y\mathbf{j} + 2\mathbf{k}$ perpendicular to the vector $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and the scalar product of the vectors $\mathbf{a}$ and $\mathbf{c} = \mathbf{i} + 2\mathbf{j}$ is equal to 4?

1.43. The vector $\mathbf{c}$ is perpendicular to the vectors $\mathbf{a} = (2, 3, -1)$ and $\mathbf{b} = (1, -2, 3)$ and satisfies the condition $\mathbf{c} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = -6$. Find the coordinates of $\mathbf{c}$.

1.44. Calculate the coordinates of the vector $\mathbf{c}$ which is perpendicular to the vectors $\mathbf{a} = 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and makes an obtuse angle with the unit vector $\mathbf{j}$ if $|\mathbf{c}| = \sqrt{7}$.

1.45*. Find the coordinates of the vector $\mathbf{a} = (X, Y, Z)$ which makes equal angles with the vectors $\mathbf{b} = (Y, -2Z, 3X)$, $\mathbf{c} = (2Z, 3X, -Y)$ if $\mathbf{a}$ is perpendicular to the vector $\mathbf{d} = (1, -1, 2)$, $|\mathbf{a}| = 2\sqrt{3}$ and the angle between the vector $\mathbf{a}$ and the unit vector $\mathbf{j}$ is obtuse.

1.46. In the parallelogram $ABCD$ we know the coordinates of three vertices: A (3, 1, 2), B (0, -1, -1), C (-1, 1, 0). Find the length of the diagonal BD.

1.47. Prove that the points A (1, -1, 1), B (1, 3, 1), C (4, 3, 1), D (4, -1, 1) are the vertices of a triangle. Calculate the lengths of its diagonals and the coordinates of their point of intersection.

1.48. Prove that the points A (2, 4, -4), B (1, 1, -3), C (-2, 0, 5) and D (-1, 3, 4) are the vertices of a parallelogram and calculate the angle between its diagonals.

1.49. Find the cosine of the angle φ between the diagonals AC and BD of a parallelogram if its three vertices are known: A (2, 1, 3), B (5, 2, -1) and C (-3, 3, -3).

1.50. A triangle is defined by the coordinates of its vertices A (3, -2, 1), B (3, 1, 5), C (4, 0, 3). Calculate the lengths of the medians AA_1 and BB_1, the distance between the origin and the centre of gravity of the triangle ABC, and the angles of the triangle.

1.51. Calculate the coordinates of the vertex C of the equilateral triangle ABC if A (1, 3) and B (3, 1).

1.52. Calculate the coordinates of the vertices C and D of the square $ABCD$ if A (2, 1) and B (0, 4).

1.53. Given points B (1, -3) and D (0, 4) which are the vertices of the rhombus $ABCD$. Calculate the coordinates of the vertices A and C if $\angle BAD = 60°$.

1.54. Given the vertices of a triangle: A (1, -1, -3), B (2, 1, -2) and C (-5, 2, -6). Calculate the length of the bisector of its interior angle A.

1.55. Given the coordinates of three points: A (3, 3, 2), B (1, 1, 1) and C (4, 5, 1). Calculate the coordinates of the point D which belongs to the bisector of the angle ABC and is at the distance $\sqrt{870}$ from the vertex B.

1.56. Calculate the work done by the force $\mathbf{F} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ in transferring a particle from the position A (-1, 2, 0) to the position B (2, 1, 3).

1.57. Given three forces: $\mathbf{M} = (3, -4, 2)$, $\mathbf{N}$ (2, 3, -5) and $\mathbf{P} = (-3, -2, 4)$ applied to the same point. Calculate the work done by the resultant of these forces when their point of application moves rectilinearly and is displaced from the point A (5, 3, -7) to B (4, 1, -4).

1.58. Find the lengths of the sides and the angles of a triangle with vertices $A(-1, -2, 4)$, $B(-4, -2, 0)$ and $C(3, -2, 1)$.

1.59. Given the coordinates of the vertices of a triangle: $A(1, 1, 1)$, $B(2, 4, 2)$ and $C(8, 3, 3)$. Determine whether the triangle is right-angled or obtuse.

1.60. The vertices of a triangle are at the points $A(2, -3, 0)$, $B(2, -1, 1)$ and $C(0, 1, 4)$. Find the angle φ formed by the median DB and the base AC.

1.61.* In the triangle ABC a point H is the point of intersection of the altitudes. It is known that $\overrightarrow{AB} = (6, -2)$, $\overrightarrow{AC} = (3, 4)$. Find the coordinates of the vector $\overrightarrow{AH}$.

1.62. Prove that the triangle ABC, whose vertices are at the points $A(1, 0, 1)$, $B(1, 1, 0)$, $C(1, 1, 1)$, is right-angled. Find the distance from the origin to the centre of the circle circumscribed about the triangle.

1.63. A triangular pyramid is defined by the coordinates of its vertices:

$$A(3, 0, 1),\ B(-1, 4, 1),\ C(5, 2, 3),\ D(0, -5, 4).$$

Calculate the length of the vector $\overrightarrow{AG}$, if G is the point of intersection of the medians of the face BCD.

1.64*. The volume of a right triangular prism $ABCA_1B_1C_1$ is equal to 3. Calculate the coordinates of the vertex A_1 if the coordinates of the vertices of one of the bases of the prism are known:

$$A(1, 0, 1),\ B(2, 0, 0),\ C(0, 1, 0).$$

1.65. Points A and B given in a Cartesian rectangular system of coordinates Oxy on the curve $y = x^2$ such that $\overrightarrow{OA}\cdot\mathbf{i} = 1$ and $\overrightarrow{OB}\cdot\mathbf{i} = -2$. Find the length of the vector $12\cdot\overrightarrow{OA} - 3\cdot\overrightarrow{OB}$.

1.66. A point $A(x_1, y_1)$ with the abscissa $x_1 = 1$ and a point $B(x_2, y_2)$ with the ordinate $y_2 = 11$ are given in a Cartesian rectangular system of coordinates Oxy on the curve $y = x^2 - 2x + 3$ lying in the first quarter. Find the scalar product of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

2. Problems on Analytic Notation of Lines on a Plane and Surfaces in Space

A line on a plane in the Cartesian rectangular system of coordinates Oxy can be defined by one of the equations (1)-(7):

$$Ax + By + C = 0. \tag{1}$$

The equation of a straight line which passes through the point $M_0(x_0, y_0)$ and is perpendicular to the vector $\mathbf{n} = (A, B)$:

$$A(x - x_0) + B(y - y_0) = 0. \tag{2}$$

The equation of a straight line which passes through the point $M_0(x_0, y_0)$ and is parallel to the vector $\mathbf{a} = (m, n)$:

$$\frac{x - x_0}{m} = \frac{y - y_0}{n}. \tag{3}$$

Intercept equation of a straight line:

$$\frac{x}{a}+\frac{y}{b}=1, \tag{4}$$

where a and b are the x and y intercepts, respectively.
The equation of a straight line with a slope k:

$$y = kx + b. \tag{5}$$

The equation of a straight line passing through a given point M_0 (x_0, y_0) with the given slope k:

$$y - y_0 = k(x - x_0). \tag{6}$$

The equation of a straight line passing through two points M_1 (x_1, y_1) and M_2 (x_2, y_2):

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}. \tag{7}$$

The *angle between the straight lines* l_1 and l_2 is the smaller of the two adjacent angles formed by those lines. The angle between the lines l_1 and l_2 with the slopes k_1 and k_2 can be calculated from the formula

$$\tan(\widehat{l_1, l_2})=\left|\frac{k_2-k_1}{1+k_1k_2}\right|, \quad k_1\cdot k_2 \neq -1. \tag{8}$$

If $l_1 \perp l_2$ or $l_1 \parallel l_2$, then, respectively,

$$k_1\cdot k_2 = -1, \quad k_1 = k_2. \tag{9}$$

The distance from the point M_0 (x_0, y_0) to the straight line l, defined by the equation $Ax + By + C = 0$, can be found from the formula

$$\rho(M_0, l)=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}. \tag{10}$$

Example 2.1. The straight line l passes through the point $M_0(x_0, y_0)$ at right angles to the vector $\mathbf{n} = (A, B)$. Write the equation of the line l if M_0 $(-1, 2)$ and $\mathbf{n} = (2, 2)$.

Solution. According to formula (2), we have $2(x + 1) + 2(y - 2) = 0$. Removing the brackets and collecting terms, we obtain an equation $x + y - 1 = 0$.

Example 2.2. Write the equation of a straight line which passes through the point M_0 $(-1, 2)$ parallel to the vector $\mathbf{a} = (3, -1)$.

Solution. According to formula (3) we have $\frac{x+1}{3}=\frac{y-2}{-1}$, or $x+3y-5=0$.

Example 2.3. Given a straight line l: $-2x+y-1=0$, and a point M $(-1, 2)$. It is required:

(1) to calculate the distance $\rho(M, l)$ from the point M to the line l;

(2) to write the equation of a straight line l' which passes through the point M at right angles to the given straight line l;

(3) to write the equation of a straight line l'' which passes through the point M parallel to the given line l.

Solution. (1) According to (10) we have

$$\rho(M, l) = \frac{|(-2)(-1)+1\cdot 2-1|}{\sqrt{4+1}} = \frac{3}{\sqrt{5}}.$$

(2) Applying the first formula (9) for $k_1 = 2$, we get $k_2 = -1/2$. According to (6) we have $x + 2y - 3 = 0$.

(3) Applying now the second formula (9) and formula (6), we get $y - 2 = 2(x + 1)$, i.e. $y = 2x + 4$.

2.1. The straight line l passes through two points: $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$. Write the equation of the line l for:

(a) $M_1(1, 2)$, $M_2(-1, 0)$;

(b) $M_1(1, 1)$, $M_2(1, -2)$;

(c) $M_1(2, 2)$, $M_2(0, 2)$.

2.2*. Derive the equation of a straight line which passes through the point $M(8, 6)$ and cuts off a triangle with area 12 from a coordinate angle.

2.3. Write the equation of a straight line, which is parallel to two given straight lines l_1 and l_2 and is equidistant from l_1 and l_2, if:

(a) l_1: $3x-2y-1=0$,

l_2: $\dfrac{x-1}{2}=\dfrac{y+3}{3}$;

(b) l_1: $3x-15y-1=0$,

l_2: $\dfrac{x+1/2}{5}=\dfrac{y+1/2}{1}$.

2.4. The triangle ABC is defined by the coordinates of its vertices: $A(1, 2)$, $B(2, -2)$, $C(6, 1)$.

Write the equations of the straight lines containing:

(1) the side AB;

(2) the altitude CD; calculate the length of $h = |CD|$.

(3) Find the angle φ between the altitude CD and the median BM.

(4) Write the equation of the bisectors l_1 and l_2 of the interior and exterior angles at the vertex A.

2.5. A ray of light emanates from the point $M(5, 4)$ at the angle $\varphi = \arctan 2$ to the axis Ox and is reflected from it. Write the equation of the incident and reflected rays (the equations of the straight lines containing those rays).

2.6. Write the equation of a straight line which passes through the point $M(2, 1)$ at the angle $\pi/4$ to the straight line $2x + 3y + 4 = 0$.

2.7*. Two vertices of the triangle ABC are at the points $A(-1, -1)$ and $B(4, 5)$, and the third vertex lies on the straight line $y = 5(x - 3)$. The area of the triangle is 9.5. Find the coordinates of the vertex C.

2.8. Given three points: $A(2, 1)$, $B(3, 1)$, $C(-4, 0)$ which are the vertices of an equilateral trapezoid $ABCD$. Calculate the coordinates of the point D if $\overrightarrow{AB} = k\overrightarrow{CD}$.

The *equation of a circle* with centre at the point C_0 (x_0, y_0) and radius R has the form

$$(x - x_0)^2 + (y - y_0)^2 = R^2.$$

Example 2.4. Derive an equation of a circle which passes through the points A (2, 0), B (5, 0) and touches the Oy axis.

Solution. Assume that the unknown centre C_0 of the circle has the coordinates (x_0, y_0). Then from the condition of tangency of the circle and the Oy axis we infer that the abscissa x_0 of the centre is equal to the radius R. Since the points A (2, 0) and B (5, 0) lie on the circle, their coordinates satisfy the equation of the circle. Using the indicated conditions, we get the following system of equations:

$$(x_0 - 2)^2 + y_0^2 = R^2,$$
$$(x_0 - 5)^2 + y_0^2 = R^2,$$
$$x_0 = R,$$

which has two solutions: $x_0 = 7/2$, $y_0 = \pm\sqrt{10}$; $R = 7/2$.

Answer. $\left(x-\frac{7}{2}\right)^2+(y-\sqrt{10})^2=\frac{49}{4}$ and $\left(x-\frac{7}{2}\right)^2+$ $+(y+\sqrt{10})^2=\frac{49}{4}$.

2.9. Derive an equation of a circle inscribed into a triangle whose sides lie on the straight lines $x = 0$, $y = 0$, $3x + 4y - 12 = 0$.

2.10. Given a circle $x^2 + y^2 = 4$. Derive an equation of a straight line l, which is parallel to the abscissa axis and cuts the circle at points M and N such that $|MN| = 1$.

2.11*. A square $ABCD$ is inscribed into a circle $x^2 + y^2 = 169$. Find the coordinates of the vertices B, C, and D if A (5, —12).

2.12*. Given a circle $x^2 + y^2 = 9$. Derive an equation of the circle, which passes through the origin, the point A (1, 0) and touches the given circle.

2.13. Derive an equation of the circle which passes through the point A (2, 1) and touches the coordinate axes.

In the rectangular system of coordinates $Oxyz$ the plane α can be defined by an equation belonging to one of the following kinds:

The *general equation of a plane*:

$$Ax + By + Cz + D = 0. \tag{11}$$

The equation of a plane, which passes through the point M_0 (x_0, y_0, z_0) at right angles to the vector $\mathbf{n} = (A, B, C)$:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{12}$$

The *angle between two planes* α_1 and α_2 is the smallest of the dihedral angles formed by those planes. The cosine of that angle can be calculated from the formula

$$\cos(\widehat{\alpha_1, \alpha_2}) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\,\sqrt{A_2^2+B_2^2+C_2^2}}, \quad (13)$$

where $\mathbf{n}_1 = (A_1, B_1, C_1)$ and $\mathbf{n}_2 = (A_2, B_2, C_2)$ are vectors which are perpendicular to the planes α_1 and α_2 respectively.

Example 2.5. Derive an equation of the plane if it is known that the point N (3, 5, 2) is the foot of the perpendicular drawn from the origin to that plane and belongs to the plane.

Solution. It follows from the hypothesis that the vector $\overrightarrow{ON}$ is perpendicular to the required plane, where O is the origin, and N is a point belonging to the plane, and $\overrightarrow{ON} = (3, 5, 2)$. According to (12) the equation of the plane which passes through the point N at right angles to the vector $\overrightarrow{ON}$ has the form

$$3(x-3) + 5(y-5) + 2(z-2) = 0,$$

or

$$3x + 5y + 2z - 38 = 0.$$

Example 2.6. Find the angle between the plane passing through the points M (0, 0, 0), N (1, 1, 1), K (3, 2, 1) and the plane passing through the points M (0, 0, 0), N (1, 1, 1), D (3, 1, 2).

Solution. According to formula (13), to calculate the cosine of the angle between the planes, it is necessary to find the coordinates of the vectors which are perpendicular to those planes. Assume that the vector $\mathbf{n}_1 = (A_1, B_1, C_1)$ is perpendicular to the first plane. Then $\mathbf{n}_1 \perp \overrightarrow{MN}$ and $\mathbf{n}_1 \perp \overrightarrow{NK}$, and, consequently, $\mathbf{n}_1 \cdot \overrightarrow{MN} = 0$, $\mathbf{n}_1 \cdot \overrightarrow{NK} = 0$. Writing these equations in the coordinate form, we obtain a system of equations

$$\begin{aligned} A_1 + B_1 + C_1 &= 0, \\ 3A_1 + 2B_1 + C_1 &= 0, \end{aligned} \quad (*)$$

whose solutions are the unknown coordinates of the vector $\mathbf{n}_1$. Since system (*) consists of only two equations, one of the unknowns, say, C_1, can be taken as a free unknown. Assuming it to be equal to 1, we get $\mathbf{n}_1 = (1, -2, 1)$. Reasoning by analogy, we find that the vector $\mathbf{n}_2$, which is perpendicular to the second plane, has coordinates (—1/2, —1/2, 1). Substituting the coordinates obtained into expression (13), we find that the cosine of the required angle is

$$\cos\varphi = \frac{-1/2 + 1 + 1}{\sqrt{6}\,\sqrt{3/2}} = \frac{1}{2},$$

and, consequently, the angle between the planes is 60°.

Example 2.7. Given a plane $2x+2y-z+4=0$ and a straight line l which passes through the points A (2, 1, 1) and B (—3, 4, 0). Calculate the coordinates of the point of intersection of the line l and the given plane.

Solution. Let us calculate the coordinates of the vector:

$$\overrightarrow{AB} = (-3, -2, 4-1, 0-1) = (-5, 3, -1).$$

Assume that the point M (x_0, y_0, z_0) is the point of intersection of the given plane and the straight line passing through the points A and B. This means that, first, the coordinates of the point M satisfy the equation of the given plane, i.e. are related as

$$2x_0 + 2y_0 - z_0 + 4 = 0, \qquad (*)$$

and, second, the vector $\overrightarrow{AM}$ is collinear with the vector $\overrightarrow{AB}$:

$$\overrightarrow{AM} = k\overrightarrow{AB},$$

whence it follows that

$$\begin{aligned} x_0 - 2 &= -5k, \\ y_0 - 1 &= 3k, \\ z_0 - 1 &= -k. \end{aligned} \qquad (**)$$

Solving the system of equations (*), (**), we find the coordinates of the point M: $x_0 = -13$, $y_0 = 10$, $z_0 = -2$.

Answer. (—13, 10, —2).

2.14. Derive an equation of a plane if it is known to pass through the origin and to be perpendicular to the vector $\mathbf{n} = (-6, 3, 6)$.

2.15. The straight line l passes through the points A and B. Derive an equation of the plane which passes through the point A at right angles to the line l, for each of the following cases:

(a) A (1, 2, 3), B (4, 6, 9);
(b) A (—1, 2, 3), B (3, —2, 1);
(c) A (1, 0, —3), B (2, —1, 1).

2.16. Find the angle between the planes

$$\text{(a)}\ \begin{aligned} 2x + y - z &= 2, \\ x + 2y - z &= 1; \end{aligned} \quad \text{(b)}\ \begin{aligned} 3x - y - 2z &= 1, \\ 2x + 3y - z &= 2; \end{aligned}$$

$$\text{(c)}\ \begin{aligned} x - y + 3z &= 2, \\ -x - 3y + z &= 2. \end{aligned}$$

2.17*. Given a plane $x - y + 2z - 1 = 0$ and a straight line which passes through the points A (2, 3, 0) and B (0, 1, 1). Calculate the sine of the angle between the straight line AB and the given plane.

2.18. The straight line is defined by the points A (1, —1, 1) and B (—3, 2, 1). Find the angle between the straight line AB and the plane:

(a) $6x + 2y - 3z - 7 = 0$; (b) $5x - y + 8z = 0$.

2.19. Calculate the distance between the plane $15x - 10y + 6z - 190 = 0$ and the origin.

2.20. Calculate the distance:

(a) from the point (3, 1, —1) to the plane $22x + 4y - 20z - 45 = 0$;

(b) from the point (4, 3, —2) to the plane $3x - y + 5z + 1 = 0$;

(c) from the point (2, 0, —1/2) to the plane $4x - 4y + 2z + 17 = 0$.

2.21. Calculate the altitude (h_S) of a pyramid with vertices S (0, 6, 4), A (3, 5, 3), B (—2, 11, —5), C (1, —1, 4).

2.22. Derive an equation of a plane which passes at the distance of 6 units from the origin and intercepts line segments related as $a : b : c = 1 : 3 : 2$ on the coordinate axes.

2.23. Find the coordinates of the point of intersection of the plane $2x - y + z = 0$ and the straight line which passes through the given points A (—1, 0, 2) and B (3, 1, 2).

2.24. Find the point of intersection:

(a) of a straight line which is the line of intersection of two planes $3x - 4y = 0$ and $y - 3z = 6$ with the plane $2x - 5y - z - 2 = 0$;

(b) of the straight line $\frac{x-7}{5} = \frac{y-4}{1} = \frac{z-5}{4}$ and the plane $3x - y + 2z - 5 = 0$.

The *equation of a sphere* with centre at a point C_0 (x_0, y_0, z_0) and radius R has the form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2. \tag{14}$$

If the centre of the sphere coincides with the origin, the equation assumes the form

$$x^2 + y^2 + z^2 = R^2. \tag{15}$$

Example 2.8. Derive an equation of a sphere which passes through the given point A (1, —1, 4) and touches the coordinate planes.

Solution. Since the required sphere touches the coordinate planes and the centre of the sphere is in the part of space for each point of which $x > 0$, $y < 0$, $z > 0$ (since the point A (1, —1, 4) lies in that part of space), the coordinates of the centre are $(R, -R, R)$. On the other hand, since the point A belongs to the sphere, its coordinates satisfy equation (14):

$$(1 - R)^2 + (-1 + R)^2 + (4 - R)^2 = R^2,$$

whence it follows that

$$R^2 - 6R + 9 = 0, \text{ or } (R - 3)^2 = 0, \text{ i.e. } R = 3.$$

Answer. $(x - 3)^2 + (y + 3)^2 + (z - 3)^2 = 9.$

2.25*. Calculate the distance from the plane $2x + 2y - z + 15 = 0$ to the sphere $x^2 + y^2 + z^2 - 4 = 0$.

2.26. Given a sphere $x^2 + y^2 + z^2 - 25 = 0$ and a straight line l which passes through the point A (2, 1, 1) parallel to the vector $\mathbf{a} = (2, -4, -1)$. Calculate the coordinates of the points of intersection of the straight line l and the sphere.

2.27. Find the set of points of space the sum of the squares of whose distances from two given points A (2, 3, —1) and B (1, —1, 3) has the same value m^2.

3. Using the Method of Coordinates to Solve Geometrical Problems

Geometrical problems presented in this section can be solved by means of introducing a Cartesian system of coordinates on a plane or in space. The problems given below can also be solved by methods of elementary geometry. These solutions, however, require, as a rule, the use of nontrivial artificial techniques.

Example 3.1. In the isosceles triangle ABC ($|AB| = |BC| = 8$) a point E divides the lateral side AB in the ratio $3:1$ (reckoning from the vertex B). Calculate the angle between the vectors $\overrightarrow{CE}$ and $\overrightarrow{CA}$ if $|CA| = 12$.

Solution. Let us introduce a system of coordinates Oxy as indicated in Fig. 13.1 ($|OA| = |OC|$) according to the property of an iso-

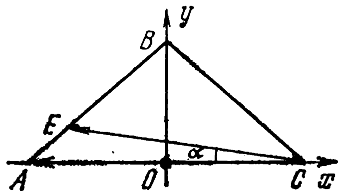

Fig. 13.1

sceles triangle. From the triangle OBC we find that

$$|OB| = \sqrt{|BC|^2 - |OC|^2} = 2\sqrt{7}.$$

Since $\overrightarrow{AE} = \frac{1}{4}\overrightarrow{AB}$, we have $\overrightarrow{CE} = \overrightarrow{CA} + \frac{1}{4}\overrightarrow{AB}$ and the coordinates of the vectors $\overrightarrow{CA}$ and $\overrightarrow{CE}$ are

$$\overrightarrow{CA} = (-12, 0), \quad \overrightarrow{AB} = (6, 2\sqrt{7}), \quad \overrightarrow{CE} = \left(-\frac{21}{2}, \frac{\sqrt{7}}{2}\right)$$

respectively. Substituting the coordinates obtained into the formula of a scalar product of vectors, we obtain

$$\cos\alpha = \frac{(-12)\cdot(-21/2)}{12\sqrt{(21/2)^2 + (\sqrt{7}/2)^2}} = \frac{3\sqrt{7}}{8}.$$

Answer. The angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CE}$ is equal to $\arccos\frac{3\sqrt{7}}{8}$.

3.1. In the isosceles triangle ABC ($|AB| = |BC| = 15$) a point E divides the side BC in the ratio $1:4$ (reckoning from the vertex B). Calculate the angle between the vectors $\overrightarrow{AE}$ and $\overrightarrow{AC}$ if $|AC| = 20$.

3.2. In the right triangle ABC the angle B is a right angle, ($|AB| = 3$, $|BC| = 4$). Calculate the angle between the medians AM and BD.

3.3. In a right triangle with legs AB and BC ($|AB| = 8$, $|BC| = 6$) a straight line AD divides BC in the ratio $|BD| : |DC| = 4 : 5$. Calculate the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

3.4. A straight line AD drawn in a right triangle with legs BC and BA ($|BC| = 4$, $|BA| = 3$) divides the side BC in the ratio $|BD| : |DC| = 3 : 5$. Calculate the angle between the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

3.5*. Given a right isosceles triangle ABC with a right vertex angle B, BS is its altitude, K is the midpoint of the altitude BS, and M is the intersection point of the straight line AK and the side BC. Find the ratio in which the point M divides the segment BC.

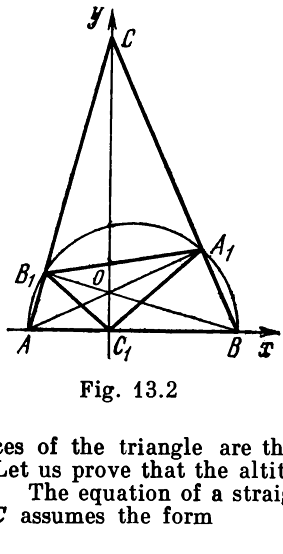

Fig. 13.2

Example 3.2. Prove that if the feet of the altitudes of the triangle ABC are connected by line segments, a triangle results for which those altitudes are bisectors (*Ptolemy's theorem*).

Solution. Let us drop the altitudes of the triangle from its vertices: $AA_1 \perp BC$, $BB_1 \perp AC$ and $CC_1 \perp AB$; we designate the point of intersection of the altitudes as O. Next we choose a system of coordinates such that its origin coincides with the point C_1 and the Ox axis passes through the vertex B (Fig. 13.2). Then the Oy axis will pass through the vertex C. Assume that the coordinates of the vertices of the triangle are the following: $A\ (-a, 0)$, $B\ (b, 0)$, $C\ (0, c)$. Let us prove that the altitude C_1C is the bisector of the angle $A_1C_1B_1$.

The equation of a straight line passing through the points A and C assumes the form

$$y = \frac{c}{a} x + c. \qquad (*)$$

The equation of a straight line which passes through the points B and O at right angles to the straight line AC assumes the form

$$y = -\frac{a}{c} x + \frac{ab}{c} \qquad (**)$$

(to obtain the last equation, we have used the relation between the slopes of two mutually perpendicular straight lines: $k_1 \cdot k_2 = -1$). Solving the system of equations (*) and (**), we find the coordinates of the point of intersection of those straight lines (point B_1):

$$B_1 \left(\frac{a(ab - c^2)}{a^2 + c^2}, \ \frac{ac(a+b)}{a^2 + c^2} \right).$$

By analogy, writing the equations of the straight lines passing through the pairs of points B, C and A, O, we find the coordinates of the point A_1:

$$A_1\left(\frac{b(c^2-ab)}{b^2+c^2}, \frac{bc(a+b)}{b^2+c^2}\right).$$

Writing the equations of the straight lines passing through the pairs of points A_1, C_1 and B_1, C_1, we find the slopes of those lines:

$$k_{A_1C_1}=\frac{c(a+b)}{c^2-ab}, \quad k_{B_1C_1}=\frac{c(a+b)}{ab-c^2},$$

whence it follows that $k_{B_1C_1} = -k_{A_1C_1}$. Since a slope is a tangent of the angle of inclination of a straight line to the positive direction of the Ox axis, we obtain

$$\angle BC_1B_1 = \pi - \angle BC_1A_1,$$

whence it follows that $\angle AC_1B_1 = \angle BC_1A_1$, and since the straight line C_1C is perpendicular to the straight line AB, it follows that $\angle B_1C_1C = \angle A_1C_1C$, i.e. the altitude C_1C of the triangle ABC is indeed the bisector of the triangle $A_1B_1C_1$.

We can prove by analogy that the other two altitudes of the triangle ABC are bisectors of the respective angles of the triangle $A_1B_1C_1$

3.6. An arbitrary point P is taken on the altitude CC_1 of the triangle ABC. The straight lines AP and BP cut the sides BC and CA at points A_1 and B_1 respectively. Prove that the ray C_1P is the bisector of the angle $A_1C_1B_1$.

3.7*. Given a right triangle ABC with legs a and b, $\angle C = 90°$. Derive an equation of the set of points M for which

$$|MA|^2 + |MB|^2 = 2|MC|^2.$$

3.8. Given a point M in the plane of the rectangle $ABCD$. Prove that

$$|MA|^2 + |MC|^2 = |MB|^2 + |MD|^2.$$

3.9. A circle is inscribed into a rhombus with an angle of 60°. The distance from the centre of the circle to the nearest vertex is equal to 1. Prove that the following equation holds for any point P of the circle:

$$|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2 = 11.$$

3.10. Prove that the sum of the squares of the distances from the point M, taken on the diameter of a certain circle, to the ends of any chords which are parallel to that diameter is constant.

3.11. A square $ABCD$ is circumscribed about a circle. Perpendiculars AA_1, BB_1, CC_1, and DD_1 are drawn from the vertices of the square to an arbitrary straight line which touches the circle. Prove that

$$|AA_1|\,|CC_1| = |BB_1|\,|DD_1|.$$

3.12. Given an equilateral triangle ABC and a circle, passing through the vertices A and B, whose centre D is symmetric with respect to the vertex C about the straight line AB. Prove that if M is an arbitrary point of the circle, then a right triangle can be formed by the segments MA, MB and MC.

3.13. A rectangle $ABCD$ is inscribed into a circle. From an arbitrary point P of the circle perpendiculars are drawn to the straight lines AB, BC, CD, and DA, which cut those lines at points K, L, M, and N, respectively. Prove that the point N is an orthocentre of the triangle KLM.

3.14. A circle is inscribed into a square. Prove that the sum of the squares of the distances from a point of the circle to the vertices of the square does not depend on the choice of a point of the circle. Find that sum.

3.15. A circle is circumscribed about a square. Prove that the sum of the squares of the distances from the points of the circle to the vertices of the square does not depend on the choice of the points on the circle. Find that sum.

If a problem deals with a cube or a right parallelepiped, then the most convenient is a system of coordinates whose origin is at one of the vertices of the lower base of those bodies, and the coordinate axes pass through the edges drawn from that vertex.

Example 3.3. The length of an edge of the cube $ABCDA_1B_1C_1D_1$ is equal to 1. A point E taken on the edge AA_1 is such that the length of the segment AE is 1/3. A point F taken on the edge BC is such that the length of the segment BF is 1/4. A plane α is drawn through the centre O_1 of the cube and the points E and F. Find the distance ρ from the vertex B_1 to the plane α.

Solution. We choose a system of coordinates such that its origin coincides with the vertex A, and the axes Ox, Oy, and Oz pass through the edges AB, AD, and AA_1, respectively. In that system of coordinates

$$F\left(1, \frac{1}{4}, 0\right), \quad E\left(0, 0, \frac{1}{3}\right),$$

$$O_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

Let us derive an equation of the secant plane α. Assume that the vector $\mathbf{n} = (n_1, n_2, n_3)$ is perpendicular to the required plane. Since the vectors

$$\overrightarrow{EF} = \left(-1, -\frac{1}{4}, \frac{1}{3}\right),$$

$$\overrightarrow{EO_1} = \left(-\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\right)$$

belong to the required plane, we can use the condition of perpendicularity of the pairs of vectors $\mathbf{n}$, $\overrightarrow{EF}$ and $\mathbf{n}$, $\overrightarrow{EO_1}$ and get the following system of equations for n_1, n_2, n_3:

$$-n_1 - \frac{n_2}{4} + \frac{n_3}{3} = 0,$$

$$-\frac{n_1}{2} + \frac{n_2}{4} + \frac{n_3}{2} = 0.$$

Assuming n_3 to be a free unknown, we obtain $n_1 = \frac{5}{9} n_3$ and $n_2 = -\frac{8}{9} n_3$. Assuming $n_3 = 9$, we get a vector $\mathbf{n} = (5, -8, 9)$ as the vector perpendicular to the required plane. The equation of the plane which passes through the point E $(0, 0, 1/3)$ at right angles to the vector $\mathbf{n} = (5, -8, 9)$ has the form

$$5x - 8y + 9z - 3 = 0.$$

The coordinates of the point B_1 in the chosen Cartesian system of coordinates are $(1, 0, 1)$. Let us calculate the distance from the point B_1 $(1, 0, 1)$ to the plane

$$5x - 8y + 9z - 3 = 0.$$

Assume that M (x_0, y_0, z_0) is a point of the foot of the perpendicular to the given plane which passes through the point B_1. Let us calculate the coordinates of the point M. Since the point belongs to the plane, the coordinates of that point must satisfy the equation of the plane:

$$5x_0 - 8y_0 + 9z_0 - 3 = 0. \qquad (*)$$

On the other hand, the vector $\overrightarrow{B_1M}$ is perpendicular to the given plane and, consequently, the vector $\overrightarrow{BM}$ is collinear with the vector $\mathbf{n}$:

$$\overrightarrow{B_1M} = k\mathbf{n}.$$

The last equation in the coordinate form yields the following three equations:

$$\begin{aligned} x_0 - 1 &= 5k, \\ y_0 &= -8k, \\ z_0 - 1 &= 9k. \end{aligned} \qquad (**)$$

Solving the system of equations $(*)$, $(**)$, we find the coordinates of the point M:

$$x_0 = \frac{115}{170}, \quad y_0 = -\frac{88}{170}, \quad z_0 = \frac{71}{170}$$

and the length of the vector $|\overrightarrow{B_1M}| = \frac{11}{\sqrt{170}}$ which is precisely the required distance from the point B_1 to the plane.

Answer. $11/\sqrt{170}$.

3.16. The length of an edge of the cube $ABCDA_1B_1C_1D_1$ is equal to 1. A point E taken on the edge BC is such that the length of the segment BE is $1/4$. A point F taken on the edge C_1D_1 is such that the length of the segment FD_1 is $2/5$. A plane α is drawn through the centre of the cube and the points E and F. Find the distance from the vertex A_1 to the plane α.

3.17. The length of an edge of the cube $ABCDA_1B_1C_1D_1$ is equal to 1. A point E taken on the edge AB is such that the length of the segment BE is 2/5. A point F taken on the edge CC_1 is such that the length of the segment FC is 2/3. A plane α is drawn through the centre of the cube and the points E and F. Find the distance from the vertex A to the plane α.

3.18. The length of an edge of the cube $KLMNK_1L_1M_1N_1$ is equal to 1. A point A taken on the edge MM_1 is such that the length of the segment AM is 3/5. A point B taken on the edge K_1N_1 is such that the length of the segment K_1B is 1/3. A plane α is drawn through the centre of the cube and the points A and B. A point P is the projection of the vertex N onto the plane α. Find the length of the segment BP.

3.19. The length of an edge of the cube $KLMNK_1L_1M_1N_1$ is equal to 1. A point A taken on the edge KL is such that the length of the segment AL is 3/4. A point B taken on the edge MM_1 is such that the length of the segment MB is 3/5. A plane α is drawn through the centre of the cube and the points A and B. Find the length of the segment BP, where the point P is the projection of the vertex N onto the plane α.

3.20. The length of an edge of the cube $KLMNK_1L_1M_1N_1$ is equal to 1. A point A taken on the edge KL is such that the length of the segment KA is 1/4. A point B taken on the edge MM_1 is such that the length of the segment M_1B is 2/5. A plane α is drawn through the centre of the cube and the points A and B. A point P is the projection of the vertex K_1 onto the plane α. Find the length of the segment AP.

3.21. Given a cube $ABCDA_1B_1C_1D_1$; a point K is the midpoint of the edge AA_1, L is the centre of the face CC_1D_1D. Find the angle between the planes BKL and AD_1C.

3.22*. Find the area of the section of the cube $ABCDA_1B_1C_1D_1$ by a plane passing through the vertex A and the midpoints of the edges B_1C_1 and D_1C_1. The edge of the cube is equal to a.

3.23*. In the cube $ABCDA_1B_1C_1D_1$ with an edge a a point K is the midpoint of the edge AB, and a point L is the midpoint of the edge DD_1. Find the sides of the triangle A_1KL and the ratio in which the volume of the cube is divided by a plane passing through the vertices of the triangle.

3.24. In the cube $ABCDA_1B_1C_1D_1$ with edge a the midpoints of the edges AA_1, A_1B_1, B_1C_1, C_1C, CD, DA, and AA_1 are consecutively connected. Prove that the figure obtained is a regular hexagon and find its area.

3.25. The length of an edge of the cube $ABCDA_1B_1C_1D_1$ is a. E and F are the midpoints of the edges BC and B_1C_1 respectively. We consider the triangles whose vertices are the points of intersection of the planes, which are parallel to the bases of the cube, and the straight lines A_1E, DF, AD_1. Find:

(a) the area of the triangle whose plane passes through the midpoint of the edge AA_1;

(b) the minimum possible value of the area of the triangles being considered.

3.26. A sphere is inscribed in a cube. Prove that the sum of the squares of the distances from each point of the sphere to the vertices of the cube does not depend on the choice of the point. Find that sum.

Example 3.4. The base of the pyramid $SABC$ is an equilateral triangle ABC whose side is $4\sqrt{2}$. The lateral edge SC is perpendicular to the plane of the base and is 2 long. Calculate the angle and the distance between the skew straight lines one of which passes through the point S and the midpoint of the edge BC and the other passes through the point C and the midpoint of the edge AB.

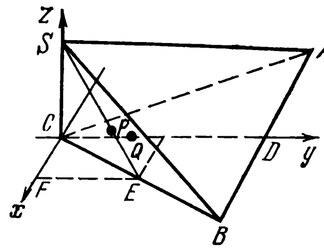

Fig. 13.3

Solution. We introduce a rectangular system of coordinates assuming the point C to be the origin and the straight line CD to be the axis of ordinates (D is the midpoint of the edge AB), the straight line CS to be the applicate axis and the straight line which belongs to the plane of the triangle ABC and is perpendicular to the straight line CD to be the abscissa axis, and the line segment whose length is 1 to be the unit length (Fig. 13.3). In that system of coordinates the vectors $\overrightarrow{CD}$ and $\overrightarrow{SE}$ (E is the midpoint of the side CD) have the following coordinates:

$$\overrightarrow{CD}=\left(0,\ \frac{\sqrt{3}}{2}\,|CB|,\ 0\right)=(0,\ 2\sqrt{6},\ 0),$$

$$\overrightarrow{SE}=\left(\frac{|AB|}{4},\ \frac{|CD|}{2},\ -|CS|\right)=(\sqrt{2},\ \sqrt{6},\ -2).$$

Therefore,

$$\cos\left(\widehat{\overrightarrow{CD},\ \overrightarrow{SE}}\right)=\frac{\overrightarrow{CD}\cdot\overrightarrow{SE}}{|\overrightarrow{CD}|\cdot|\overrightarrow{SE}|}=\frac{12}{2\sqrt{6}\cdot\sqrt{12}}=\frac{\sqrt{2}}{2},$$

and, consequently, the required angle is 45°. Assume that PQ is a common perpendicular to the straight lines SE and CD ($P\in SE$, $Q\in CD$). Then there are numbers α and β such that $\overrightarrow{SP}=\alpha\cdot\overrightarrow{SE}$, $\overrightarrow{CQ}=\beta\cdot\overrightarrow{CD}$. It is clear that

$$\overrightarrow{PQ}=\overrightarrow{PS}+\overrightarrow{SC}+\overrightarrow{CQ}=-\alpha\overrightarrow{SE}-\overrightarrow{CS}+\beta\overrightarrow{CD},$$

or, in the coordinate form,

$$\overrightarrow{PQ}=(-\alpha\sqrt{2};\ -\alpha\sqrt{6}+\beta\cdot 2\sqrt{6};\ 2\alpha-2).$$

Since $PQ\perp CD$ and $PQ\perp SE$, it follows that $\overrightarrow{PQ}\cdot\overrightarrow{CD}=0$, $\overrightarrow{PQ}\cdot\overrightarrow{SE}=0$. The last two vector equations in the coordinate form look like

$$(-\alpha\sqrt{6}+\beta\cdot 2\sqrt{6})\cdot 2\sqrt{6}=0,$$

$$-\alpha\sqrt{2}\cdot\sqrt{2}+(-\alpha\sqrt{6}+\beta\cdot 2\sqrt{6})\cdot\sqrt{6}+(2\alpha-2)(-2)=0,$$

or

$$\alpha = 2\beta,$$
$$-3\alpha + 3\beta + 1 = 0,$$

whence $\alpha = 2/3$, $\beta = 1/3$. Thus

$$\overrightarrow{PQ} = \left(-\frac{2\sqrt{2}}{3}, 0, -\frac{2}{3}\right), \quad |PQ| = \sqrt{\frac{8}{9}+\frac{4}{9}} = \frac{2}{\sqrt{3}}.$$

Answer. The angle is equal to $\pi/4$, the required distance is equal to $2/\sqrt{3}$.

3.27. The base of the pyramid $SABC$ is an isosceles right triangle ABC, the length of whose hypotenuse AB is equal to $4\sqrt{2}$. The lateral edge SC of the pyramid is perpendicular to the plane of the base and is 2 long. Find the angle and the distance between the skew lines one of which passes through the point S and the midpoint of the edge AC and the other passes through the point C and the midpoint of the edge AB.

3.28. The base of the pyramid $HPQR$ is an equilateral triangle PQR whose side is $2\sqrt{2}$ long. The lateral edge HR is perpendicular to the plane of the base and is 1 long. Find the angle and the distance between the skew lines one of which passes through the point H and the midpoint of the edge QR and the other passes through the point R and the midpoint of the edge PQ.

3.29. The base of the pyramid $HPQR$ is an isosceles right triangle PQR the length of whose hypotenuse PQ is equal to $2\sqrt{2}$. The lateral edge HR of the pyramid is perpendicular to the plane of the base and it is 1 long. Find the angle and the distance between the skew lines one of which passes through the point H and the midpoint of the edge PR and the other passes through the point R and the midpoint of the edge PQ.

3.30. All the edges of a regular prism $ABCA_1B_1C_1$ are a long. We consider line segments with their endpoints lying on the diagonals BC_1 and CA_1 of the lateral faces, the segments being parallel to the plane ABB_1A_1.

(a) One of the segments is drawn through the point M of the diagonal BC_1 such that $|BM| : |BC_1| = 1 : 3$. Find its length.

(b) Find the least length of all the segments under consideration.

3.31. The side of the base $ABCD$ of a regular pyramid $SABCD$ is a long, and a lateral edge is $2a$ long. We consider line segments with their endpoints lying on the diagonal BD of the base and on the lateral edge SC, the segments being parallel to the plane of the face SAD. Find the least length of all the segments under consideration.

4. Geometrical Problems Which Can Be Solved by the Methods of Vector Algebra

This section contains problems which can be solved by the methods of vector algebra. These methods are based on the property of uniqueness of the resolution of a vector on a plane into components with respect to two noncollinear vectors and on the property of uniqueness of the resolution of a vector in space into components with respect to three noncoplanar vectors.

The problems presented below can be conventionally divided into two types: "direct" problems and "inverse" problems. In direct problems we postulate that three points belong to the same straight line (in a plane case) or four points belong to the same plane (in space). In such problems it is usually required to establish or verify certain relations between the lengths of segments.

In inverse problems it is required, as a rule, to establish the relations between the lengths of segments under which certain three points A, B, C belong to the same straight line or certain four points A, B, C, D belong to the same plane, and sometimes it is required to establish the fact that certain straight lines meet at one point.

The solution of direct problems in a plane case is based on the verification of the vector formula

$$\overrightarrow{AB} = k\overrightarrow{BC}, \tag{1}$$

whose satisfaction for a certain real k signifies that three points A, B, C lie on the same straight line, or on the verification of the formula

$$\overrightarrow{OC} = \alpha\overrightarrow{OA} + (1 - \alpha)\,\overrightarrow{OB},$$

where A, B, C are points belonging to the same straight line and O is an arbitrary point.

When solving a number of problems on a plane, use is also made of the following properties of noncollinear vectors:

(1) If two vectors $\mathbf{a}$ and $\mathbf{b}$ are noncollinear, then the equality $\alpha\mathbf{a} + \beta\mathbf{b} = 0$ yields an equality $\alpha = \beta = 0$.

(2) If the vectors $\mathbf{a}$ and $\mathbf{b}$ are noncollinear, then the equality $\mathbf{c} = \alpha_1\mathbf{a} + \beta_1\mathbf{b} = \alpha_2\mathbf{a} + \beta_2\mathbf{b}$ yields equalities $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ (the property of uniqueness of a resolution of a vector into components with respect to two noncollinear vectors).

Example 4.1. Given a parallelogram $ABCD$. The straight line l cuts the straight lines AB, AC, and AD at points B_1, C_1, and D_1 respectively. Prove that if $\overrightarrow{AB_1} = \lambda_1\overrightarrow{AB}$, $\overrightarrow{AD_1} = \lambda_2\overrightarrow{AD}$, $\overrightarrow{AC_1} = \lambda_3\overrightarrow{AC}$, then

$$\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

(a direct problem).

Solution. Assume that $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AD} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{a} + \mathbf{b}$ (Fig. 13.4). Then $\overrightarrow{AB_1} = \lambda_1\mathbf{a}$, $\overrightarrow{AD_1} = \lambda_2\mathbf{b}$ and $\overrightarrow{AC_1} = \lambda_2\,(\mathbf{a} + \mathbf{b})$. Since three points A_1, B_1, C_1 lie on the same straight line l, the following equality holds true:

$$\overrightarrow{B_1C_1} = k\overrightarrow{B_1D_1}, \tag{*}$$

but

$$\overrightarrow{B_1C_1} = \overrightarrow{AC_1} - \overrightarrow{AB_1} = (\lambda_3 - \lambda_1)\,\mathbf{a} + \lambda_2\mathbf{b},$$
$$\overrightarrow{B_1D_1} = \overrightarrow{AD_1} - \overrightarrow{AB_1} = -\,\lambda_1\mathbf{a} + \lambda_2\mathbf{b}.$$

Substituting the resolution of the vectors $\overrightarrow{B_1C_1}$ and $\overrightarrow{B_1D_1}$ into components with respect to the noncollinear vectors **a** and **b** into relation

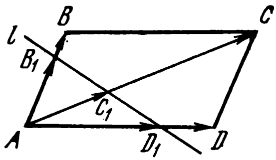

Fig. 13.4

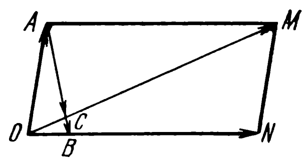

Fig. 13.5

(*), we obtain

$$(\lambda_3 - \lambda_1)\,\mathbf{a} + \lambda_2\mathbf{b} = k\lambda_3\mathbf{b} - k\lambda_1\mathbf{a}.$$

By virtue of the uniqueness of a resolution of a vector into components with respect to two noncollinear vectors **a** and **b** we obtain a system

$$\lambda_3 - \lambda_1 = -k\lambda_1,$$
$$\lambda_2 = k\lambda_3.$$

Excluding the coefficient k, we find the relation between λ_1, λ_2 and λ_3:

$$\lambda_1\lambda_3 + \lambda_2\lambda_3 = \lambda_1\lambda_2.$$

Dividing the last relation term-by-term by $\lambda_1\lambda_2\lambda_3$, we have

$$\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2},$$

and that is what we wished to prove.

Let us consider an example of an "inverse" problem.

Example 4.2. Points B and C are taken on the side ON of the parallelogram $AMNO$ and on its diagonal OM and are such that

$$\overrightarrow{OB} = \frac{1}{n}\overrightarrow{ON}, \quad \overrightarrow{OC} = \frac{1}{n+1}\overrightarrow{OM}.$$

Prove that the points A, B, and C lie on the same straight line.

Solution. Let us express the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of the vectors $\overrightarrow{ON}$ and $\overrightarrow{OA}$ (Fig. 13.5):

$$\overrightarrow{AC} = \frac{1}{n+1}\overrightarrow{OM} - \overrightarrow{OA}, \quad \overrightarrow{AB} = \frac{1}{n}\overrightarrow{ON} - \overrightarrow{OA}.$$

Since $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{ON}$ and, consequently,

$$\frac{1}{n+1}\overrightarrow{OM} = \frac{1}{n+1}(\overrightarrow{OA} + \overrightarrow{ON}),$$

it follows that

$$\overrightarrow{AC} = \frac{1}{n+1}(\overrightarrow{OA} + \overrightarrow{ON}) - \overrightarrow{OA} = \frac{1}{n+1}\overrightarrow{ON} - \frac{n}{n+1}\overrightarrow{OA}.$$

Comparing the resolution of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ into components with respect to the noncollinear vectors $\overrightarrow{ON}$ and $\overrightarrow{OA}$, we obtain

$$\overrightarrow{AB}=\lambda\overrightarrow{AC}, \text{ where } \lambda=\frac{n+1}{n}.$$

Since the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear and have a common origin, the three points A, B, C lie on the same straight line.

4.1. Points L, M and N lying on the same straight line are taken on the straight lines BC, CA and AB respectively, which define the triangle ABC. Prove that if

$$\overrightarrow{BL} = \alpha\overrightarrow{LC}, \overrightarrow{CM} = \beta\overrightarrow{MA}, \overrightarrow{AN} = \gamma\overrightarrow{NB},$$

then $\alpha\beta\gamma = -1$ (*Menelaas's theorem*).

4.2. Given a triangle MNP. Points A, B and C taken on the straight lines MN, NP, PM are such that $\overrightarrow{MA} = \alpha\overrightarrow{AN}$, $\overrightarrow{NB} = \beta\overrightarrow{BP}$, $\overrightarrow{PC} = \gamma\overrightarrow{CM}$. Prove that if $\alpha\beta\gamma = -1$, then the points A, B, C lie on the same straight line (*inverse Menelaas's theorem*).

4.3. The straight lines a and b are parallel. Arbitrary points A_1, A_2, A_3 are taken on the line a and arbitrary points B_1, B_2, B_3 are taken on the line b. Points C_1 C_2, C_3 taken on the line segments A_1B_1, A_2B_2, A_3B_3 are such that

$$|A_1C_1| = \alpha |A_1B_1|, |A_2C_2| = \alpha |A_2B_2|,$$
$$|A_3C_3| = \alpha |A_3B_3|.$$

Prove that the points C_1, C_2, C_3 lie on the same straight line.

4.4. Points C_1, C_2, C_3 divide the line segment AB into four equal parts; D is an arbitrary point. Express the vectors $\overrightarrow{DC_1}$, $\overrightarrow{DC_2}$, $\overrightarrow{DC_3}$ in terms of the vectors $\overrightarrow{DA} = \mathbf{a}$, $\overrightarrow{DB} = \mathbf{b}$.

4.5. Given three points M, A, B, and a fourth point C is such that $\overrightarrow{AB} = 3\overrightarrow{AC}$. Express the vector $\overrightarrow{MC}$ in terms of the vectors $\overrightarrow{MA}$ and $\overrightarrow{MB}$.

4.6. Three points A, B, M are taken on a plane. A point C taken on the line segment AB is such that $|AC| : |CD| = k$. Express the vector $\overrightarrow{MC}$ in terms of $\overrightarrow{MA}$ and $\overrightarrow{MB}$.

Example 4.3. If the point A of the intersection of the diagonals of the quadrilateral $MNPQ$ and the midpoints B and C of its opposite sides MN and PQ lie on the same straight line, then $MNPQ$ is a trapezoid or a parallelogram (Fig. 13.6).

Solution. Assume that $\overrightarrow{AM} = \mathbf{a}$, $\overrightarrow{AN} = \mathbf{b}$. Then $\overrightarrow{AP} = k\mathbf{a}$ and $\overrightarrow{AQ} = l\mathbf{b}$. Since B is the midpoint of the segment MN, we have

$$\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AM} + \frac{1}{2}\overrightarrow{AN} = \frac{1}{2}(\mathbf{a}+\mathbf{b}).$$

Similarly,

$$\overrightarrow{AC}=\frac{1}{2}\overrightarrow{AP}+\frac{1}{2}\overrightarrow{AQ}=\frac{1}{2}(k\mathbf{a}+l\mathbf{b}).$$

By the hypothesis the points A, B, C lie on the same straight line

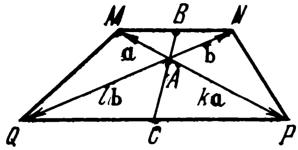

Fig. 13.6

and, therefore, there is a number m such that $\overrightarrow{AC} = m\overrightarrow{AB}$, i.e.

$$\frac{m}{2}(\mathbf{a}+\mathbf{b})=\frac{1}{2}(k\mathbf{a}+l\mathbf{b}),$$

or

$$\frac{m-k}{2}\mathbf{a}+\frac{m-l}{2}\mathbf{b}=0,$$

whence it follows that $m = k = l$. Then

$$\overrightarrow{MN} = \mathbf{b} - \mathbf{a}, \quad \overrightarrow{PQ} = l\mathbf{b} - k\mathbf{a} = -k(\mathbf{a} - \mathbf{b}),$$

i.e. $\overrightarrow{PQ} = k\overrightarrow{MN}$. Consequently, $PQ \parallel MN$, i.e. $MNPQ$ is a trapezoid or a parallelogram.

4.7. The point of intersection of the medians of a quadrilateral coincides with that of its diagonals. Prove that the quadrilateral is a parallelogram.

4.8. Prove that the midpoints of the bases of a trapezoid and the point of intersection of the extensions of its nonparallel sides belong to the same straight line.

4.9. Point M is the midpoint of the segment AB and point M' is the midpoint of the segment $A'B'$. Prove that the midpoints of the segments AA', BB' and MM' lie on the same straight line.

4.10. Prove that the midpoints of the sides of an arbitrary quadrilateral are the vertices of a parallelogram.

4.11. (a) Prove that in an arbitrary quadrilateral the midlines, when intersecting, are divided in half.

(b) Prove that in an arbitrary quadrilateral the line segment connecting the midpoints of the diagonals passes through the intersection point of the medians and is bisected at that point.

The solution of a number of problems on a mutual position of three points A, B, C, which do not lie on the same straight line, is based on the use of the formula

$$\overrightarrow{OM}=\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}), \tag{2}$$

where M is the centre of gravity of the triangle ABC, and O is an arbitrary point.

Example 4.4. Assume that $ABCDEF$ is an arbitrary hexagon and U, V, W, X, Y, Z are the midpoints of its sides. Prove that the centres of gravity of the triangles UWY and VXZ coincide (Fig. 13.7).

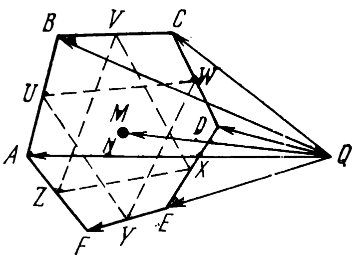

Fig. 13.7

Solution. Since the points U, V, W, X, Y and Z are the midpoints of a hexagon, it follows that

$$\overrightarrow{OU}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB}),\quad \overrightarrow{OV}=\frac{1}{2}(\overrightarrow{OB}+\overrightarrow{OC}),\quad \overrightarrow{OW}=\frac{1}{2}(\overrightarrow{OC}+\overrightarrow{OD}),$$

$$\overrightarrow{OX}=\frac{1}{2}(\overrightarrow{OD}+\overrightarrow{OE}),\quad \overrightarrow{OY}=\frac{1}{2}(\overrightarrow{OE}+\overrightarrow{OF}),\quad \overrightarrow{OZ}=\frac{1}{2}(\overrightarrow{OF}+\overrightarrow{OA}),$$

where O is an arbitrary point. Designating as M and N the centres of gravity of the triangles UWY and VXZ, we have, by formula (2),

$$\overrightarrow{OM}=\frac{1}{3}(\overrightarrow{OU}+\overrightarrow{OW}+\overrightarrow{OY})=\frac{1}{6}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}+\overrightarrow{OE}+\overrightarrow{OF}),$$

$$\overrightarrow{ON}=\frac{1}{3}(\overrightarrow{OV}+\overrightarrow{OX}+\overrightarrow{OZ})=\frac{1}{6}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}+\overrightarrow{OE}+\overrightarrow{OF}).$$

Thus, $\overrightarrow{OM}=\overrightarrow{ON}$, whence it follows that the point M coincides with the point N.

4.12. Given a triangle ABC. Prove that the equation $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=0$ holds if and only if O is the centre of gravity of the triangle ABC.

4.13. (a) Assume that M and N are the centres of gravity of the triangles ABC and DEF. Prove that

$$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=3\overrightarrow{MN}.$$

(b) Assume that A, B, C, D, E, F are arbitrary points of a plane. Prove that

$$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\overrightarrow{AE}+\overrightarrow{BF}+\overrightarrow{CD}.$$

4.14. Point M is the centre of gravity of the triangle ABC. Prove that

$$\vec{CA} + \vec{CB} = 3\vec{CM}.$$

4.15. A straight line l, which cuts the sides AC and BC at points P and Q, respectively, is drawn through the centre of gravity of the triangle ABC. Prove that

$$\frac{|AP|}{|PC|} + \frac{|BQ|}{|QC|} = 1.$$

4.16. The vertices A_1, B_1, C_1 of the triangle ABC belong to the sides BC, CA, and AB of the triangle ABC respectively, and the centres of gravity of the two triangles coincide. Prove that the points A_1, B_1, and C_1 divide the sides of the triangle ABC in equal ratios.

When solving problems connected with calculations of the ratios of the areas of some plane figures, use is often made of the following *property of the areas of triangles.* If the area of the triangle ABC is S and points M and N chosen on the sides AC and BC are such that

$$|CM| : |CA| = k_1,$$
$$|CN| : |CB| = k_2,$$

then the area of the triangle MCN is k_1k_2S.

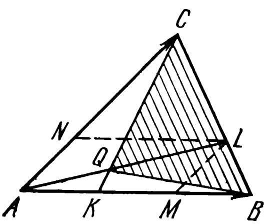

Fig. 13.8

Example 4.5. In the triangle ABC a point K taken on the side AB is such that $|AK| : |BK| = 1 : 2$, and a point L taken on the side BC is such that $|CL| : |BL| = 2 : 1$. Assume that Q is the point of intersection of the straight lines AL and CK. Find the area of the triangle ABC if the area of the triangle BQC is known to be equal to 1.

Solution. Assume $\vec{AB} = \mathbf{a}$, $\vec{AC} = \mathbf{b}$ (Fig. 13.8). Since $|BL|/|LC| = 1/2$, we obtain $S_{BQL} = 1/3$, $S_{LQC} = 2/3$ by virtue of the property of areas formulated above.

Let us find the ratio $|QL|/|AL|$. The straight line which passes through the point L parallel to the side AC divides the side AB in the ratio 2 : 1 (reckoning from the vertex A) and $\vec{AM} = \frac{2}{3}\mathbf{a}$. The straight line which passes through the point L parallel to the side AB divides the side AC in the ratio 1 : 2 (reckoning from the vertex A) and $\vec{AN} = \frac{1}{3}\mathbf{b}$. Therefore,

$$\vec{AL} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}.$$

Since the vectors $\vec{AQ}$ and $\vec{AL}$ are collinear (the points A, Q, L lie on the same straight line), we have

$$\vec{AQ} = \mu\vec{AL} = \frac{\mu}{3}(2\mathbf{a} + \mathbf{b}). \qquad (*)$$

Similarly, we can show for the point K that

$$\overrightarrow{CK}=\frac{2}{3}\overrightarrow{CA}+\frac{1}{3}\overrightarrow{CB}=\frac{1}{3}(\mathbf{a}-3\mathbf{b}),$$

$$\overrightarrow{CQ}=\lambda\overrightarrow{CK}=\frac{\lambda}{3}(\mathbf{a}-3\mathbf{b}).$$

But $\overrightarrow{AQ}=\overrightarrow{AC}+\overrightarrow{CQ}$, whence

$$\frac{\mu}{3}(2\mathbf{a}+\mathbf{b})=\mathbf{b}+\frac{\lambda}{3}(\mathbf{a}-3\mathbf{b}).$$

From the condition of uniqueness of a resolution of a vector into components with respect to two noncollinear vectors **a** and **b**, we get a system of equations $2\mu = \lambda$, $\mu = 3 - 3\lambda$, from which we find that $\mu = 3/7$.

We can now find the relation $\frac{|QL|}{|AL|}$

$$\frac{|QL|}{|AL|}=\frac{|AL|-|AQ|}{|AL|}=1-\frac{|AQ|}{|AL|},$$

and according to equation (*) we have $\frac{|QL|}{|AL|}=1-\mu=\frac{4}{7}$. Hence $\frac{S_{ABC}}{S_{QBC}}=\frac{1}{1-\mu}=\frac{7}{4}$, and since $S_{QBC}=1$, the required area of the triangle is equal to 7/4.

Answer. 7/4.

4.17. In the triangle ABC whose area is equal to 6 a point K taken on the side AB divides that side in the ratio $|AK| : |BK| = 2 : 3$, and a point L taken on the side AC divides AC in the ratio $|AL| : |LC| = 5 : 3$. The point Q of intersection of the straight lines CK and BL is at the distance 1.5 from the straight lines AB. Find the length of the side AB.

4.18. Given a triangle ABC. Points M and N are taken on the sides AB and BC respectively; $|AB| = 5|AM|$, $|BC| = 3|BN|$. The segments AN and CM meet at a point O. Find the ratio of the areas of the triangles OAC and ABC.

4.19. Point K divides the median AD of the triangle ABC in the ratio 3 : 1, reckoning from the vertex. In what ratio does the straight line BK divide the area of $\triangle ABC$?

4.20. A point taken on each median of the triangle divides the median in the ratio 1 : 3, reckoning from the vertex. Find the ratio of the area of the triangle with vertices at those points to that of the original triangle.

The solution of some problems presupposes the use of a vector **c** which is collinear with the bisector of the angle between the vectors **a** and **b**. It is convenient to represent the vector **c** in the following form:

$$\mathbf{c}=\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}. \quad (3)$$

Problems 4.21-4.24 can be solved with the use of formula (3).

4.21. In the triangle ABC find $\overrightarrow{AA_1}$, where AA_1 is the bisector of the interior angle A of the triangle.

4.22. In the triangle ABC the median BD meets the bisector AF at a point O. The ratio of the area of the triangle DOA to that of the triangle BOF is 3/8. Find $|AC| : |AB|$.

4.23. In the triangle ABC the bisector AA_1 divides the side BC in the ratio $|BD| : |CD| = 2 : 1$. In what ratio does the median CE divide this bisector?

4.24. The bisectors AD and BE of the triangle ABC meet at a point O. Find the ratio of the area of the triangle ABC to that of the quadrilateral $ODCE$ knowing that $|BC| = a$, $|AC| = b$, $|AB| = c$.

It is convenient to use the following *property of vectors* in solving certain problems. If three vectors **a**, **b**, and **c** are noncoplanar, then the equation

$$\alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c} = \mathbf{0} \qquad (4)$$

yields an equation $\alpha = \beta = \gamma = 0$.

The consequence of equation (4) is the uniqueness of the representation of any space vector as a linear combination of three noncoplanar vectors **a**, **b** and **c**, the equations

$$\mathbf{d} = \alpha_1 \mathbf{a} + \beta_1 \mathbf{b} + \gamma_1 \mathbf{c},$$

$$\mathbf{d} = \alpha_2 \mathbf{a} + \beta_2 \mathbf{b} + \gamma_2 \mathbf{c}$$

yields equations

$$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_2, \quad \gamma_1 = \gamma_2.$$

The *condition of coplanarity* of three vectors **a**, **b**, and **c** has the form

$$\alpha \mathbf{a} + \beta \mathbf{b} + \gamma \mathbf{c} = \mathbf{0},$$

where α, β and γ are certain real numbers at least one of which is nonzero (i.e. $\alpha^2 + \beta^2 + \gamma^2 > 0$).

Example 4.6. Given three noncoplanar vectors **a**, **b** and **c**. Prove that the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$, $\mathbf{c} - \mathbf{a}$ are coplanar.

Solution. To prove this statement, it is sufficient to find numbers α, β, γ satisfying the following conditions:

$$\alpha (\mathbf{a} + \mathbf{b}) + \beta (\mathbf{b} + \mathbf{c}) + \gamma (\mathbf{c} - \mathbf{a}) = \mathbf{0}, \qquad (*)$$

$$\alpha^2 + \beta^2 + \gamma^2 > 0. \qquad (**)$$

We reduce (*) to the form

$$\mathbf{a} (\alpha - \gamma) + \mathbf{b} (\alpha + \beta) + \mathbf{c} (\beta + \gamma) = \mathbf{0}. \qquad (***)$$

Since **a**, **b**, **c** are noncoplanar, it follows from equation (4) that α, β, γ must satisfy the system of equations

$$\alpha - \gamma = 0, \quad \alpha + \beta = 0, \quad \beta + \gamma = 0.$$

The triple of numbers $\alpha = 1$, $\beta = -1$, $\gamma = 1$ is one of the solutions of this system for which condition (**) is satisfied. Consequently, the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$, $\mathbf{c} - \mathbf{a}$ are coplanar.

4.25. Given three noncoplanar vectors **a**, **b** and **c**. Prove that the vectors $\mathbf{a} + 2\mathbf{b} - \mathbf{c}$, $3\mathbf{a} - \mathbf{b} + \mathbf{c}$ and $-\mathbf{a} + 5\mathbf{b} - 3\mathbf{c}$ are coplanar.

4.26. Given three nonzero vectors **a**, **b**, **c** each two of which are pairwise noncollinear. Find their sum if the vector $\mathbf{a} + \mathbf{b}$ is collinear with the vector **c**, and the vector $\mathbf{b} + \mathbf{c}$ is collinear with the vector **a**.

4.27. Given three noncoplanar vectors **a**, **b** and **c**. Find the numbers p and q for which the vectors $p\mathbf{a} + q\mathbf{b} + \mathbf{c}$ and $\mathbf{a} + p\mathbf{b} + q\mathbf{c}$ are collinear.

4.28. $ABCD$ is a parallelogram, O is its centre, Q is an arbitrary point of space. Express the vector $\overrightarrow{QO}$ in terms of the vectors $\overrightarrow{QA} = \mathbf{a}$, $\overrightarrow{CD} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{c}$.

The solution of problems 4.29-4.33 is based on the use of the following vector relation. If A, B, C, D are four points belonging to the same plane and O is an arbitrary point of space, then

$$\overrightarrow{OD} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB} + (1 - \alpha - \beta)\overrightarrow{OC}, \qquad (5)$$

where α and β are some numbers.

4.29. Given a parallelepiped $ABCDA_1B_1C_1D_1$. A plane cuts the straight lines AB, AD, AA_1, AC_1 at points B_0, D_0, A_0 and C_0 respectively. Prove that if $\overrightarrow{AC_0} = \lambda_1\overrightarrow{AC_1}$, $\overrightarrow{AB_0} = \lambda_2\overrightarrow{AB_1}$, $\overrightarrow{AD_0} = \lambda_3\overrightarrow{AD_1}$, $\overrightarrow{AA_0} = \lambda_4\overrightarrow{AA}$, then

$$\frac{1}{\lambda_1} = \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}.$$

4.30. Points K, L, M, N are taken on the sides OA_1, A_1A_2, A_2A_3, and A_3O, respectively, of a nonplane quadrilateral $OA_1A_2A_3$, and

$$\overrightarrow{OK} = \alpha\overrightarrow{KA}, \quad \overrightarrow{A_1L} = \beta\overrightarrow{LA_2}, \quad \overrightarrow{A_2M} = \gamma\overrightarrow{MA_3}, \quad \overrightarrow{A_3N} = \delta\overrightarrow{NO}.$$

Prove that for the four points K, L, M, N to belong to the same plane, it is necessary and sufficient that the equality $\alpha\beta\gamma\delta = 1$ be satisfied.

4.31. Given two triangles $A_1A_2A_3$ and $A_4A_5A_6$ which do not lie in the same plane. Prove that the vectors $\overrightarrow{MN}$, $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are coplanar if M, N, P, Q, R and S are the midpoints of the segments A_1A_2, A_4A_5, A_2A_3, A_5A_6, A_3A_4, A_6A_1.

4.32. Given two triangles ABC and $A_1B_1C_1$ which do not lie in the same plane; M and N are the midpoints of the sides AC and BC, and M_1 and N_1 are the midpoints of the sides A_1C_1 and B_1C_1. Prove that if $\overrightarrow{AB} = \overrightarrow{A_1B_1}$, then the vectors $\overrightarrow{MM_1}$, $\overrightarrow{NN_1}$, and $\overrightarrow{CC_1}$ are coplanar.

4.33. Given two skew lines m and n. Points P, Q, R are given on the straight line m and points P_1, Q_1, R_1 on the straight line n, with $\overrightarrow{PQ} = k\overrightarrow{PR}$, $\overrightarrow{P_1Q_1} = k\overrightarrow{P_1R_1}$. Prove that the straight lines PP_1, QQ_1, RR_1 are parallel to the same plane.

When solving problems connected with the ratio of the volumes of parts of a tetrahedron formed upon a section of the tetrahedron by a plane, use is often made of the following assertion: if the volume of the tetrahedron $ABCD$ is V and points M, N, P taken on its edges DA, DB, DC respectively are such that

$$|DM| = k_1|DA|,\ |DN| = k_2|DB|,\ |DP| = k_3|DC|,$$

then the volume of the tetrahedron $MNPD$ is $k_1k_2k_3V$.

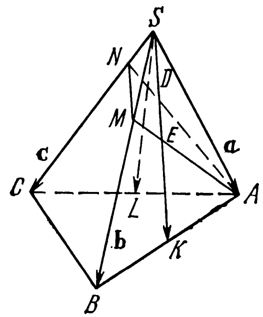

Fig. 13.9

Example 4.7. A plane passes through the vertex A of the base of a triangular pyramid $SABC$, bisects the median SK of the triangle SAB and cuts the median SL of the triangle SAC at a point D such that $2|SD| = |DL|$. In what ratio does the plane divide the volume of the pyramid?

Solution. We introduce the designations $\overrightarrow{SA} = \mathbf{a}$, $\overrightarrow{SB} = \mathbf{b}$, $\overrightarrow{SC} = \mathbf{c}$ (Fig. 13.9). It is evident that $k_1 = 1$. Assume that $\overrightarrow{SM} = k_2\mathbf{b}$, $\overrightarrow{SN} = k_3\mathbf{c}$, where M and N are the points of intersection of the plane of the section and the edges SB and SC respectively. Let us find k_2 and k_3. For that purpose we use the equations

$$\overrightarrow{SE} = \frac{1}{2}\overrightarrow{SK} = \frac{1}{4}(\mathbf{a}+\mathbf{b}), \quad \overrightarrow{SD} = \frac{1}{3}\overrightarrow{SL} = \frac{1}{6}(\mathbf{a}+\mathbf{c}).$$

Designating $\overrightarrow{SM} = k_2\mathbf{b}$, we can use equation (5) to represent the vector $\overrightarrow{SM}$ in the form

$$\overrightarrow{SM} = \alpha\mathbf{a} + \frac{\beta}{4}(\mathbf{a}+\mathbf{b}) + \frac{1}{6}(1-\alpha-\beta)(\mathbf{a}+\mathbf{c}).$$

Using the uniqueness of a resolution of a vector into components with respect to three noncoplanar vectors, we get a system of equations

$$0 = \frac{5}{6}\alpha + \frac{1}{12}\beta + \frac{1}{6}, \quad k_2 = \frac{1}{4}\beta, \quad 0 = \frac{1}{6}(1-\alpha-\beta),$$

from which we find $k_2 = 1/3$.

Similarly, from the equations

$$\overrightarrow{SN} = k_3\mathbf{c}, \quad \overrightarrow{SN} = \left(\frac{5}{6}\alpha + \frac{\beta}{12} + \frac{1}{6}\right)\mathbf{a} + \frac{\beta}{4}\mathbf{b} + \frac{1}{6}(1-\alpha-\beta)\mathbf{c}$$

we find $k_3 = 1/5$.

On the basis of the assertion formulated above, we obtain

$$V_{SAMN}=1\cdot\frac{1}{3}\cdot\frac{1}{5}V_{SABC},$$

and, consequently, the volume of the remaining part of the pyramid is equal to $\frac{14}{15}V_{SABC}$. Thus the required ratio of the volumes is 1:14.

4.34. Parallel sections ABC and $A_1B_1C_1$ are drawn in a trihedral angle with vertex S. Designating as V, V_1, V_2, V_3 the volumes of the tetrahedrons $SABC$, $SA_1B_1C_1$, SA_1BC, SAB_1C_1, respectively, show that

$$V_2=\sqrt[3]{V^2\cdot V_1} \quad \text{and} \quad V_2\cdot V_3=V\cdot V_1.$$

4.35. Given a regular quadrangular pyramid $SABCD$. A plane is drawn through the midpoints of the edges AB, AD and CS. In what ratio does the plane divide the volume of the pyramid?

4.36. The volume of the pyramid $ABCD$ is equal to 5. A plane which cuts the edge CD at a point M is drawn through the midpoints of the edges AD and BC. With this ratio, the lengths of the segment DM and of the edge MC are related as 2/3. Calculate the area of the section of the pyramid by the indicated plane if the distance between that area and the vertex A is equal to 1.

4.37. A plane cuts the lateral edges SA, SB and SC of the triangular pyramid $SABC$ at points K, L and M respectively. In what ratio does the plane divide the volume of the pyramid if it is known that $|SK|:|KA|=|SL|:|LB|=2$, and the median SN of the triangle SBC is bisected by that plane?

4.38. All the edges of the triangular pyramid $SABC$ are equal. A point M taken on the edge SA is such that $|SM|=|MA|$, a point N taken on the edge SB is such that $3|SN|=|SB|$. A plane drawn through the points M and N is parallel to the median AD of the base ABC. Find the ratio of the volume of the triangular pyramid which is cut off from the initial pyramid by the drawn plane to that of the pyramid $SABC$.

5. A Scalar Product of Vectors

A *scalar product* of two nonzero vectors is the product of the lengths of those vectors by the cosine of the angle between the vectors:

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\cdot|\mathbf{b}|\cos(\widehat{\mathbf{a},\mathbf{b}}). \tag{1}$$

The *necessary and sufficient condition of the perpendicularity* of two nonzero vectors is the equality of their scalar product to zero:

$$\mathbf{a}\cdot\mathbf{b}=0. \tag{2}$$

If $\varphi=(\widehat{\mathbf{a},\mathbf{b}})$, then

$$\mathbf{a}\cdot\mathbf{b}>0 \text{ for } 0\leqslant\varphi<\frac{\pi}{2};\ \mathbf{a}\cdot\mathbf{b}<0, \text{ for } \frac{\pi}{2}<\varphi\leqslant\pi. \tag{3}$$

The scalar product of a vector by itself is equal to the square of its length:

$$\mathbf{a}\cdot\mathbf{a} = \mathbf{a}^2 = |\mathbf{a}|^2. \qquad (4)$$

Properties of a scalar product:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} \quad \text{(commutative law);}$$
$$(\lambda\mathbf{a})\cdot\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b}) \quad \text{(associative law);}$$
$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} \quad \text{(distributive law).}$$

Formulas (1)-(4) make it possible to establish a relationship between vectors and elucidate the properties of vectors.

Example 5.1. It is known that the vectors $3\mathbf{a} - 5\mathbf{b}$ and $2\mathbf{a} + \mathbf{b}$ are mutually perpendicular and the vectors $\mathbf{a} + 4\mathbf{b}$ and $-\mathbf{a} + \mathbf{b}$ are also perpendicular to each other. Find the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

Solution. By the hypothesis

$$(3\mathbf{a} - 5\mathbf{b})\cdot(2\mathbf{a} + \mathbf{b}) = 0,$$
$$(\mathbf{a} + 4\mathbf{b})\cdot(-\mathbf{a} + \mathbf{b}) = 0.$$

Hence it follows that

$$6\mathbf{a}^2 - 7\mathbf{a}\cdot\mathbf{b} - 5\mathbf{b}^2 = 0, \qquad (*)$$
$$-\mathbf{a}^2 - 3\mathbf{a}\cdot\mathbf{b} + 4\mathbf{b}^2 = 0,$$

i.e. we have obtained two equations for three unknowns $\mathbf{a}^2$, $\mathbf{b}^2$ and $\mathbf{a}\cdot\mathbf{b}$. According to (1), the cosine of the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ can be calculated by the formula

$$\cos(\widehat{\mathbf{a}\cdot\mathbf{b}}) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}. \qquad (**)$$

From equations (*) we find

$$\mathbf{a}\cdot\mathbf{b} = \frac{19}{43}\mathbf{a}^2, \quad \mathbf{b}^2 = \frac{25}{43}\mathbf{a}^2. \qquad (***)$$

Squaring both sides of equation (**) and substituting the values of (***) into the resulting equation, we obtain

$$\cos^2(\widehat{\mathbf{a}\cdot\mathbf{b}}) = \frac{19^2}{25\cdot 43},$$

or

$$\cos(\widehat{\mathbf{a}\cdot\mathbf{b}}) = \frac{19}{5\sqrt{43}}, \quad \cos(\widehat{\mathbf{a}\cdot\mathbf{b}}) = -\frac{19}{5\sqrt{43}}.$$

Answer. $\arccos\frac{19}{5\sqrt{43}}$ or $\arccos\left(-\frac{19}{5\sqrt{43}}\right)$.

5.1. Given: $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $(\widehat{\mathbf{a}, \mathbf{b}}) = 2\pi/3$. Calculate: (a) $\mathbf{a}^2$; (b) $(\mathbf{a}+\mathbf{b})^2$; (c) $(3\mathbf{a} - 2\mathbf{b})(\mathbf{a} + 2\mathbf{b})$.

5.2. Knowing that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 1$, $|\mathbf{c}| = 4$ and $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, calculate $\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$.

5.3. Find the vectors $\mathbf{a}$ and $\mathbf{b}$ such that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$.

5.4. Prove that the vector $(\mathbf{ab})\,\mathbf{c} - (\mathbf{ac})\,\mathbf{b}$ is perpendicular to the vector $\mathbf{a}$.

5.5. Prove that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are arbitrary vectors and $\mathbf{a}$ is not perpendicular to $\mathbf{c}$, then there is a number k such that the vectors $\mathbf{a}$ and $\mathbf{b} + k\mathbf{c}$ are perpendicular to each other. Find k.

If the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the sides of ABC then the equation $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ yields an equation

$$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}\cdot\mathbf{b},$$

which is a vector notation of the cosine theorem. Problems 5.6-5.21 can be solved with the use of a vector notation of the cosine theorem.

5.6. A median CC_1 is drawn in the triangle ABC. Prove that if $|BC| > |AC|$, then the angle CC_1B is obtuse.

5.7. Prove that the angle C of the triangle ABC is acute, right or obtuse according as the median CC_1 drawn from the vertex C is larger than, equal to or smaller than $\frac{1}{2}|AB|$.

5.8. (a) Find the length of the median $|AD|$ of the triangle ABC knowing the lengths of the sides $|AC| = b$, $|AB| = c$ and the magnitude of the angle A.

(b) Find the length of the bisector $|AE|$ of the triangle knowing the lengths of the sides $|AC| = b$, $|AB| = c$ and the magnitude of the angle A.

5.9. Given the sides of the triangle ABC. Find:

(a) the length of the median $|AD| = m_a$;

(b) the length of the bisector $|AE| = l_a$.

5.10. In the triangle ABC the angle B is a right angle, the medians $|AD|$ and $|BE|$ are mutually perpendicular. Find the angle C.

5.11. In the triangle ABC points D and E chosen on the sides BC and AC respectively are such that $|BD| = |DC|$, $|AE| = 2|CE|$. Find $|BC| : |AB|$ if it is known that $AD \perp BE$ and $\angle ABC = 60°$.

5.12. In the quadrilateral $ABCD$ the angle A is equal to 120° and the diagonal AC is the bisector of that angle. It is known that $|AC| = \frac{1}{5}|AB| = \frac{1}{3}|AD|$. Find the cosine of the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{CD}$.

5.13. Prove that if the equality $a^2 + b^2 = 2c^2$ holds true for the triangle ABC, then $am_a + bm_b = 2cm_c$, where m_a, m_b, m_c are the lengths of the medians of the triangle, and a, b, c are the lengths of its sides.

5.14. In the triangle ABC a line segment A_1B_1 is drawn parallel to the side AB, the points A_1 and B_1 lying on the sides AC and BC respectively. Show that if $|AB_1| = |BA_1|$ then the triangle ABC is isosceles.

5.15. The medians AA_1 and BB_1 are drawn in the triangle ABC. Prove that if $\angle C + (\widehat{\overrightarrow{AA_1}, \overrightarrow{BB_1}}) = 180°$, then $|CA|^2 + |CB|^2 = 2|AB|^2$.

5.16. Prove that if G is the centre of gravity of the triangle ABC and O is some point of space, then

$$|OA|^2+|OB|^2+|OC|^2=3|OG|^2+|AG|^2+|BG|^2+|CG|^2$$

(*Leibniz's theorem*).

5.17. Prove that if O is the centre of the circle circumscribed about the triangle ABC and H is its orthocentre, then

(1) $\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$;
(2) $|OH|^2=9R^2-(a^2+b^2+c^2)$;
(3) $|AH|=2R|\cos A|$.

5.18. Prove that the centre O of a circumscribed circle, the centre of gravity C of a triangle and the orthocentre H of an arbitrary triangle belong to the same straight line (*Euler's line*), with $|OG|:|GH|=$ $=1:2$.

5.19. Prove that the distance from the centre O of a circle circumscribed about the triangle ABC to its centre of gravity G is defined by the formula

$$|OG|^2=R^2-\frac{1}{9}(a^2+b^2+c^2).$$

5.20. Prove that if Q is an arbitrary point, H is the orthocentre and O is the centre of a circle circumscribed about the triangle ABC, then

$$\overrightarrow{QO}=\frac{1}{2}(\overrightarrow{QA}+\overrightarrow{QB}+\overrightarrow{QC}-\overrightarrow{QH}).$$

5.21. A quadrilateral $ABCD$ is inscribed into a circle. Prove that if $|AB|^2+|CD|^2=4R^2$, where R is the radius of the circumscribed circle, then the diagonals of the quadrilateral are perpendicular.

A scalar product can be used to prove the validity of certain inequalities for the trigonometric functions of the angles of a triangle.

Example 5.2. Prove that the inequality

$$\cos 2A+\cos 2B+\cos 2C\geqslant -3/2$$

holds for every triangle ABC.

Solution. Assume that point O is the centre of the circle, with radius R, circumscribed about the triangle ABC. It is evident that $(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})^2\geqslant 0$. Removing the brackets, we obtain

$$\overrightarrow{OA}^2+2\overrightarrow{OA}\cdot\overrightarrow{OB}^2+\overrightarrow{OB}^2+2\overrightarrow{OB}\cdot\overrightarrow{OC}+2\overrightarrow{OC}\cdot\overrightarrow{OA}+\overrightarrow{OC}^2\geqslant 0.$$

Since the central angle formed by the radii OA and OB is double the angle C inscribed into the circle, we have

$$\overrightarrow{OA}\cdot\overrightarrow{OB}=R^2\cos 2C.$$

Similarly,

$$\overrightarrow{OC}\cdot\overrightarrow{OA}=R^2\cos 2B,\quad \overrightarrow{OB}\cdot\overrightarrow{OC}=R^2\cos 2A.$$

Since $\overrightarrow{OA}^2 = \overrightarrow{OB}^2 = \overrightarrow{OC}^2 = R^2$, the last inequality assumes the form

$$2R^2 (\cos 2A + \cos 2B + \cos 2C) + 3R^2 \geqslant 0,$$

or

$$\cos 2A + \cos 2B + \cos 2C \geqslant -3/2,$$

and that is what we wished to prove.

5.22. Prove that the inequality

$$\cos A + \cos B + \cos C \leqslant 3/2$$

holds for the angles of every triangle ABC.

5.23. Prove that the inequality

$$\sin^2 A + \sin^2 B + \sin^2 C \leqslant 9/4$$

holds for the angles of every triangle ABC.

5.24. Prove that the inequality

$$\cos 2A + \cos 2B - \cos 2C \leqslant 3/2$$

holds for the angles of every triangle ABC. Under what condition does this inequality turn into an equality?

5.25. Prove that the inequality

$$\cos \alpha + \cos \beta + \cos \gamma > -3/2$$

holds for any trihedral angle with plane angles α, β, γ.

Chapter 14

Combinatorics. The Binomial Theorem. Elements of the Theory of Probabilities

1. Arrangements. Combinations. Permutations

Assume that a set is given which consists of n different elements $\{a_1, \ldots, a_n\}$. We choose from it a set consisting of r elements, i.e. take a sample of size r. Samples can differ from one another either by the composition or by the order of the elements. If we assume that there are identical elements in a sample, then in some cases the size of the sample may exceed that of the initial set.

Telephone numbers are an example of these samples. Assume that a number consists of 12 digits and the telephone dial contains 10 digits. Then, when we dial a number, we are choosing 12 elements from a set consisting of 10 elements. Since the dial returns to its original position after a digit is dialled, there must be repeated digits in a telephone number. This means that a sample can contain one element repeated over and over again.

The number of different samples of size r with repeating elements, taken from an original set containing n different elements, is equal to n^r. If the elements in the sample are not repeated, then the size of the sample cannot exceed that of the original set. The number of different samples containing r nonrepeating elements taken from an original set of size n is

$$A_n^r = n(n-1)\ldots(n-r+1); \tag{1}$$

A_n^r also indicates the number of different ways in which n elements can be arranged in r positions and is, therefore, called the *number of permutations of n elements taken r at a time*. If $n = r$, then the samples only differ by the order in which their elements are taken. Samples of that kind are known as *permutations of n elements*. The number of different permutations is

$$P_n = n(n-1)\ldots 1 = n! \tag{2}$$

In some problems the order of the elements in the sample is of no importance, for example, a selection of 3 people to the presidium of a meeting consisting of 200 people, or buying 5 goods in a shop where there are 100 different items. In that case, samples of the same composition, i.e. samples whose elements coincide, are assumed to be indistinguishable. The number of samples of different compositions of size r taken from a set of size n is

$$\binom{n}{r} = \frac{A_n^r}{P_r} = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}; \tag{3}$$

$\binom{n}{r}$ is known as the *number of combinations of n elements taken r at a time.*

Example 1.1. The letters of the Morse code are sequences of dots and dashes. How many different letters can be formed by using 5 symbols?

Solution. The original set in this problem consists of two elements, a dot and a dash. Since five symbols are used, the sample contains five elements which can be repeated. Thus the number of different samples each of which represents a letter is equal to $2^5 = 32$.

1.1*. How many seven-digit telephone numbers are there?

1.2*. How many different telephone numbers are there if it is assumed that each number contains not more than seven digits (a telephone number may begin with a zero)?

1.3. Assume that the letters of a certain code are sequences of dots, dashes and spaces. How many different letters can be formed from 5 symbols?

1.4. In some country no two citizens have the same set of teeth (i.e. different ones are missing). What is the largest number of citizens the country can contain (the largest set of teeth is 32)?

1.5. Assume that $p_1, \ldots, p_m$ are different prime numbers. How many divisors are there for the number $q = p_1^{k_1} p_2^{k_2} \ldots, p_m^{k_m}$, where $k_1, \ldots, k_m$ are natural numbers (the divisors 1 and q are included)?

1.6. How many different seven-digit telephone numbers are there if the digits in the numbers are not repeated?

1.7*. How many different outcomes are there in an experiment consisting of n tosses of a coin? (The outcomes of two trials are assumed to be different if either the number or the order of the heads and tails in the trials are different.)

1.8. How many permutations are there of seven students divided into rows of three sitting side by side?

1.9. A collection of 30 volumes is on a book shelf. How many ways are there of arranging the series

(a) for volumes 1 and 2 to be side by side;

(b) for volumes 3 and 4 not to be side by side?

1.10*. How many different chords can you strike on ten keys of a piano if each chord contains from three to ten notes?

1.11*. A meeting of 40 people must choose a chairman, a secretary and 5 members of a committee. How many different committees can be formed?

If the number of different samples consisting of several heterogeneous groups of elements must be determined, it is convenient to assume that the elements of each group are chosen from a different original set, i.e. the number of initial sets is the same as the number of different groups whose elements are represented in the sample. Thus, for instance, let us assume that we must form a combined team of 24 sportsmen from eight regions, so that the team contains three sportsmen from each region. This sample contains 24 elements which are chosen from eight original sets, three elements being taken from each set.

Example 1.2. There are m white and n black balls in an urn. In how many ways can r balls be selected from the urn of which k balls will be white? (It is assumed that the balls of each colour are distinct, say, are numbered.)

Solution. There are $\binom{n}{k}$ ways of selecting k white from m white balls and $\binom{n}{r-m}$ ways of selecting the remaining $r-k$ black balls from the group of n balls. Every way of selecting the k white balls is associated with the $\binom{n}{r-k}$ different ways of selecting the black balls. Consequently, the total number of different samples is equal to the product $\binom{m}{k}\binom{n}{r-k}$.

1.12. We have ten roses and eight dahlias from which we must make a bouquet consisting of two roses and three dahlias. How many different bouquets can we make?

1.13. There are 36 cards in a pack of cards, four of which are aces. How many different hands of six cards are there containing two aces?

1.14. A team consists of two house-painters, three plasterers and one joiner. How many different teams can be formed from a staff of fifteen house-painters, ten plasterers and five joiners?

1.15. Six out of 48 numbers in a lottery are winning numbers. Those who guess all six numbers get the main prize. Smaller prizes are given to those who guess five, four, or even three of the six winning numbers. How many different cards with all the possible numbers can there be for which (a) five, (b) four, (c) three of the six numbers have been guessed, if six arbitrary numbers are crossed out in each card (cards with the same numbers crossed out are assumed to be identical)?

1.16. How many circles can be drawn through 10 points no four of which lie on the same circle and no three lie on the same straight line, if each circle passes through three points?

1.17. Ten cards have been taken from a pack of 52. In how many cases will there be at least one ace among the selected cards?

1.18. How many ways are there of choosing a hand of 6 cards containing an ace and a king of the same suit from a pack of 52 cards?

1.19. Ten men and six women participate in a tennis tournament. How many ways are there of forming four mixed pairs?

1.20*. How many different four-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contains one digit 1?

1.21. How many different four-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contains a digit 1? (The digits in the number cannot be repeated.)

2. Permutations and Combinations with a Given Number of Repetitions

We shall consider here samples whose elements are repeated a given number of times. Assume that a sample consists of m elements among which one element (let us assume it is the first one) is repeated n_1 times, another element (the second) is repeated n_2 times and so on, the kth element being repeated n_k times. It is evident that

$$n_1 + n_2 + \ldots + n_k = m.$$

A collection of integers $(n_1, \ldots, n_k)$ is called the *composition of a sample*. It defines the number of the groups of elements from which the sample is chosen and the number of like elements from each group present in it. Thus, for instance, a sample with the composition (1, 2, 4) consists of three groups of elements, with one element from the first group, two elements from the second group, and four repeated elements from the third group.

The number of different samples with the same composition is called the *number of permutations of m elements with a given number of repetitions* $n_1, \ldots, n_k$. It can be found from the formula

$$p_m(n_1, \ldots, n_k) = \frac{m!}{n! \ldots n_k!}. \quad (1)$$

Example 2.1. A train time-table must be compiled for various days of the week so that two trains a day depart for three days, one train a day for two days, and three trains a day for two days. How many different time-tables can be compiled?

Solution. The number of trains a day (the digits 1, 2, 3) are three groups of like elements from which a sample must be formed. In the time-table for a week the number 1 is repeated twice, the number 2 is repeated 3 times and the number 3 is repeated twice. The number of different time-tables is equal to

$$p(2, 3, 2) = \frac{7!}{2!\,3!\,2!} = 210.$$

The number of different compositions of a sample containing m elements from k groups of like elements is*

$$\overline{C}(k, m) = C(k+m-1, m) = \frac{(k+m-1)!}{m!\,(k-1)!}. \quad (2)$$

Example 2.2. R balls must be distributed among k boxes. How many ways are there of doing this? (Each box may hold all the balls.)

Solution. We shall assume for convenience that we have k boxes in each of which the number of balls can vary from 0 to R. Then, supposing that each box is associated with a group of identical elements we have k different groups from which we form a sample containing R elements with repetitions. The different ways of distributing the balls correspond to the different compositions of the sample, i.e.

$$C(R+k-1, R) = \frac{(R+k-1)!}{R!\,(k-1)!}.$$

2.1. How many different combinations of letters can be formed from the letters in the word "Mississippi"?

* Formula (2) can be obtained by counting the number of permutations with repetitions from $m + k - 1$ elements, where m is the number of elements of the original sample and $k - 1$ is the number of boundaries separating the groups of like elements.

2.2*. How many different collections of cakes, eight cakes in each collection, can be formed from four kinds of cake?

2.3*. A lift with seven people stops at ten floors. From zero to seven people go out at each floor. How many ways are there for the lift to empty? (The ways only differ by the number of people leaving at each floor.)

2.4*. In a game of bridge a pack of 52 cards is distributed among four players, 13 cards to each player. How many ways are there of dealing the cards?

2.5*. How many ways of tossing 12 dice are there in which each of the values 2, 3, 4, 5, 6 occurs twice?

2.6*. There are m white and n black balls, with $m > n$. How many different ways are there in which all the balls are put in a row so that no black balls are side by side?

2.7*. We toss a coin and assume a head is a success and a tail is a failure. How many trials will lead to 52 successes out of 100 tosses of the coin? (An experiment is a series of 100 trials; two experiments are assumed to be different if the results of at least two tosses do not coincide.)

2.8*. Two variants of a test paper are distributed among 12 students. How many ways are there of seating of the students in two rows so that the students sitting side by side do not have identical papers and those sitting in the same column have the same paper?

2.9. Twenty books about mathematics and logic are on a bookshelf. Prove that the greatest number of permutations of a collection consisting of five books on mathematics and five books on logic is when there are ten books in each science.

3. The Binomial Theorem

The natural power of the sum of two quantities can be found from the formula

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{m} a^{n-m}b^m + \ldots + \binom{n}{n} b^n. \quad (1)$$

The right-hand side of the formula is known as the *expansion of power of a binomial*, and $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$ are called the *binomial coefficients*. The general form of the terms on the right-hand side of the formula is usually written*

$$T_k = \binom{n}{k} a^{n-k}b^k, \qquad k = 0, 1, 2, \ldots, n. \quad (2)$$

The total number of terms in the series is $n + 1$.

* Use is also often made of the formula

$$T_k = \binom{n}{k} a^{n-k}b^k = \frac{n(n-1)\ldots(n-k+1)}{k!} a^{n-k}b^k.$$

Example 3.1. Find the term in the expansion of $\left(x+\frac{1}{x^4}\right)^{10}$ which does not contain x (i.e. contains x raised to the power zero).

Solution. According to formula (2) for a general term,

$$T_k = \binom{10}{k} x^{10-k} \left(\frac{1}{x^4}\right)^k.$$

By the hypothesis, the number k must satisfy the equation

$$10 - k - 4k = 0. \qquad (*)$$

The only root of equation (*) is $k = 2$. Thus the second term of the expansion

$$T_2 = \binom{10}{2} x^8 \frac{1}{x^8} = \binom{10}{2} = 45$$

is the required term.

Answer. 45.

Example 3.2. Find the sixth term of the expansion of $(y^{1/2}+x^{1/3})^n$ if the binomial coefficient of the third term from the end is equal to 45.

Solution. Let us first find the power of the binomial. According to the hypothesis the number n must satisfy the equation

$$\binom{n}{n-2} = \binom{n}{2} = \frac{n(n-1)}{2} = 45,$$

whose roots are $n_1 = 10$, $n_2 = -9$. Since $n_2 = -9$ is not a natural number, the root is $n = 10$, and, consequently, the sixth term of the expansion can be represented as

$$T_6 = \binom{10}{6} (y^{1/2})^4 (x^{1/2})^6 = \binom{10}{4} y^2x^3 = 210y^2x^3.$$

Answer. $210y^2x^3$.

3.1. Find the sum of the binomial coefficients if the power of the binomial is 10.

3.2. Which term in the expansion of $(x + x^{-2})^{12}$ does not contain x?

3.3*. Find out which term in the expansion of the binomial $(\sqrt{x} + \sqrt[4]{x^{-3}})^n$ contains $x^{6.5}$ if the ninth term has the largest coefficient.

3.4*. Find out which term in the expansion of $\left(\frac{x}{a}+\frac{a}{x^2}\right)^8$ contains x^2.

3.5*. Prove that the sum of all the coefficients of the expansion of $(2y - x)^k$ is equal to 1 for any natural k.

3.6. The binomial coefficients of the second and ninth terms of the expansion of $(5x^{-3/2} - x^{1/3})^n$ are equal. Find out which term in the expansion does not contain x.

3.7**. Find the largest term in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^{100}$.

3.8. Which is the greatest term in the expansion of $\left(\frac{9}{10}+\frac{1}{10}\right)^{100}$?

3.9*. The sum of the binomial coefficients in an expansion is equal to 1024. Find the term of the expansion of $\left(x^2+\frac{1}{x}\right)^n$ which contains x raised to the eleventh power.

3.10*. Prove that if the power n of a binomial is an odd number, then the sum of the binomial coefficients of the even terms is equal to the sum of the binomial coefficients of the odd terms.

3.11. In the binomial expansion of

$$\left(a\sqrt[5]{\frac{a}{3}}-\frac{b}{\sqrt[7]{a^3}}\right)^n$$

find the term containing a^3 if the sum of the binomial coefficients of the odd terms is 2048.

3.12. Find the greatest term in the expansion of $(\sqrt{5}+\sqrt{2})^{20}$.

3.13. The third term in the expansion of $\left(2x+\frac{1}{x^2}\right)^m$ does not contain x. For which x's is that term equal to the second term in the expansion of $(1+x^3)^{30}$?

3.14*. For which positive values of x is the fourth term in the expansion of $(5+3x)^{10}$ the greatest?

3.15*. Find x for which the fiftieth term in the expansion of $(x+y)^{100}$ is the greatest if it is known that $x+y=1$, $x>0$, $y>0$.

3.16. Find x for which the kth term of the expansion of $(x+y)^n$ is the greatest if $x+y=1$ and $x>0$, $y>0$.

Example 3.3. In the binomial expansion of

$$(\sqrt[3]{3}+\sqrt{2})^5$$

find the terms which do not contain irrational expressions.

Solution. A general term in the expansion is

$$T_k=\binom{5}{k}(\sqrt[3]{3})^{5-k}(\sqrt{2})^k=\binom{5}{k}\cdot 3^{\frac{5-k}{3}}2^{\frac{k}{2}}.$$

The resulting expression is rational if $\frac{5-k}{3}$ and $\frac{k}{2}$ are integers. The number k must, evidently, be an even number smaller than 5. We make sure by direct verification that the only value it can assume is 2. Consequently, there is only one such term in the binomial expansion, that is

$$T_2=\binom{5}{2}\cdot 3\cdot 2=6\cdot\frac{5\cdot 4}{2}=60.$$

Answer. 60.

3.17. Find the terms which do not contain irrational expressions in the expansion of $(\sqrt[5]{3}+\sqrt[7]{2})^{24}$.

3.18*. How many rational terms are there in the expansion of $(\sqrt{2}+\sqrt[4]{3})^{100}$?

3.19.** In the binomial expansion of $\left(\sqrt{x}+\frac{1}{2\sqrt[4]{x}}\right)^n$ the first three coefficients form an arithmetic progression. Find all the rational terms in the expansion.

3.20*. Prove that

$$1-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\ldots+(-1)^n\binom{n}{n}=0.$$

3.21. By comparing the coefficients in x on both sides of the equation

$$(1+x)^m(1+x)^n=(1+x)^{m+n},$$

prove that

$$\binom{n}{k}\binom{m}{0}+\binom{n}{k-1}\binom{m}{1}+\ldots+\binom{n}{0}\binom{m}{k}=\binom{m+n}{k}.$$

3.22. Using the result of the preceding problem, find the sum of the squares of the binomial coefficients. Prove that the sum of the squares of the binomial coefficients is equal to $\binom{2n}{n}$.

3.23*. Prove that the equality

$$1-10\binom{2n}{1}+10^2\binom{2n}{2}-10^3\binom{2n}{3}+\ldots-10^{2n-1}\binom{2n}{1}+10^{2n}=(81)^n$$

holds true.

Some combinatorial formulas can be obtained by differentiating or integrating both sides of the expansion of $(1+x)^n$, which is valid for all x.

Example 3.4. Prove that the equality

$$n\binom{n}{0}+(n-1)\binom{n}{1}+\ldots+\binom{n}{n-1}=n\cdot 2^{n-1}$$

holds true.

Solution. By differentiating the binomial expansion for $(1+x)^n$, we have

$$\left(x^n+\binom{n}{1}x^{n-1}+\ldots+\binom{n}{n}\right)'$$

$$=nx^{n-1}+(n-1)\binom{n}{1}x^{n-2}+\ldots+\binom{n}{n-1}.$$

On the other hand, the equality

$$[(1+x)^n]'=n(1+x)^{n-1}$$

holds true. Substituting the value $x = 1$ into the identity

$$n(1+x)^{n-1} = nx^{n-1} + (n-1)\binom{n}{1}x^{n-2} + \ldots + \binom{n}{n-1},$$

we obtain the required equality

$$n \cdot 2^{n-1} = n\binom{n}{0} + (n-1)\binom{n}{1} + \ldots + \binom{n}{n-1}.$$

3.24. Prove that

$$n(n-1)\binom{n}{0} + (n-1)(n-2)\binom{n}{1} + \ldots + 2\binom{n}{n-2}.$$

3.25*. Prove that

$$\frac{C(n, 0)}{n+1} + \frac{C(n, 1)}{n} + \frac{C(n, 2)}{n-1} + \ldots + \frac{C(n, n)}{1} = \frac{2}{n+1}\left(2^n - \frac{1}{2}\right).$$

3.26*. Prove that

$$n\binom{n}{0} - (n-1)\binom{n}{1} + (n-2)\binom{n}{2} - (n-3)\binom{n}{3} + \ldots + (-1)^{n-1}\binom{n}{n-1} = 0.$$

3.27*. Prove that

$$\frac{C(n, 1)}{n} - \frac{C(n, 2)}{n-1} + \ldots - \frac{(-1)^n C(n, n)}{2} = \begin{cases} 0, & n = 2l, \\ \frac{2}{n+1}, & n = 2l+1. \end{cases}$$

3.28*. Simplify the expression $P_1 + 2P_2 + \ldots + nP_n$.
3.29*. Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \ldots + \binom{n-10}{m} = \binom{n+1}{m+1} - \binom{n-10}{m+1}.$$

3.30. Prove the inequality

$$\binom{2n+x}{n}\binom{2n-x}{n} \leqslant \binom{2n}{n}^2.$$

4. Calculating the Probability of an Event by Means of Combinatorial Formulas

Assume that several people are to take part in a lottery in which 10 tickets are drawn. Each ticket is labelled with the name of a participant and then all the tickets are thoroughly shuffled. A ticket is selected at random and the person whose name is on the ticket gets a prize. What is the probability that a certain person will get the prize. If the name of the person appears on only one ticket, he has one chance in ten. If it is written on two tickets, he has two chances in ten and so on.

The selection of any of the tickets labelled with the name of that participant is a favourable outcome. The number of favourable outcomes is obviously the number of tickets with his name. The chance a participant has of getting a prize is given by the ratio of the number of favourable outcomes to the total number of outcomes that are equally possible. To find that number, the number of favourable outcomes must be divided by the number of all the outcomes of the experiment.

When the experiment is carried out many times, the ratio of the number of outcomes in which the participant wins to all the outcomes of the experiment approaches the ratio of the number of the participant's tickets to all the tickets in the lottery. Therefore, the ratio of the number of the possible favourable outcomes to the number of all the possible outcomes of the experiment is naturally considered to be the probability of winning.

Let us now approach the concept of probability more formally. For that purpose we introduce the following definition. Any one of the equipossible outcomes of the experiment is called an *elementary event* (in the example given above the selection of one of the tickets is an elementary event). The set of all the equipossible outcomes is known as the *space of elementary events*, or the *sample space*, and each elementary event is known as a *point in that space* (in the example given above the space of elementary events consisted of ten points).

The collection of the elementary events combining all the outcomes in which event A takes place is called the *set of elementary events which are favourable for event A*. The *probability of event A* is the ratio of the number of elementary events which are favourable for it to the number of all the possible elementary events. If the number of outcomes favourable to event A is m and the number of all points constituting the sample space is n, then the probability $P(A)$ of the event A is

$$P(A) = m/n. \tag{1}$$

In the problems in which the number of all the possible elementary events is finite, the number of elementary events favourable to event A can be found directly.

Example 4.1. Fifteen of a class of 20 students are members of a mathematics society. What is the probability that a randomly chosen student is a member of the society?

Solution. Assume that event A is the randomly chosen student being a member of the society. Then the number of elementary events favourable to the event A is 15, while the number of all the elementary events in this case is 20. Consequently, the probability is

$$P(A) = 15/20 = 3/4.$$

Answer. 3/4.

Example 4.2. Two dice are tossed. Which event is more probable: a score of 11 or a score of 4?

Solution. We associate the outcome of the experiment with an ordered pair of numbers (x, y), where x is the number of points on the first die and y is the number of points on the second die. Thus the sample space consists of a set of pairs (x, y), where x and y assume the values from 1 to 6. The number of such pairs is 36. Two elementary events to which the pairs (6, 5) and (5, 6) correspond are favourable to event A, which is that the points on the two dice sum to 11. Three elementary events to which the pairs (1, 3), (3, 1), (2, 2) correspond are favourable to the event B (that the points on the two dice sum to 4).

The probabilities of events A and B are thus

$$P(A) = 2/36 = 1/18 \quad \text{and} \quad P(B) = 3/36 = 1/12$$

and consequently event B is more probable.

4.1. What is the probability that a leaf from a new calendar torn out at random corresponds to the first day of a month? (The year is assumed not to be a leap-year.)

4.2. What is the probability that a randomly chosen number from one to twelve is a divisor of 12? (Unity is assumed to be a divisor of any number.)

4.3*. What is the probability of a randomly chosen two-digit number being divisible by 3?

4.4. Find the probability that a randomly chosen term of the sequence $U_n = n^2 + 1$ $(n = 1, 2, \ldots, 10)$ is divisible by 5.

4.5. There are ten white and three red balls in an urn. What is the probability of drawing a red ball from the urn?

4.6.** A coin is thrown thrice. Which of the events is more probable: event A consisting in the tails occurring all three times or event B consisting in the tails occurring twice and the head's occurring once? Calculate the probability of these events.

4.7. Two dice are thrown. What is the probability of the score being 7?

4.8. Two dice are thrown. What is the probability of the score being even?

4.9. One standard article was lost when 100 articles 10 of which were substandard were transported. Find the probability of a randomly chosen article being up to standard.

4.10. Given the conditions of the preceding problem find the probability that an article selected at random will be substandard.

4.11*. There are three children in a family. What is the probability that all are boys? (The probability of a child being born a boy or a girl is assumed to be the same.)

In some cases it is convenient to use combinatorial formulas for calculations of the probability of an event.

Example 4.3. Find the probability that all the students in a group of 40 were born on different days of the year.

Solution. The outcomes of the experiment are different samples of an initial set of 365 taken 40 at a time. A sample may contain repeated elements (since any day can be the birthday of several people). Consequently, the sample space contains $(40)^{365}$ different samples. The samples which do not contain repeated elements correspond to the favourable events. There are A_{365}^{40} samples of this kind. Thus the required probability is $P(A) = A_{365}^{40}/40^{365}$.

Solve the following problems using the formulas for distributions, combinations and permutations.

4.12. There are n white and m red balls in an urn. What is the probability that two balls drawn at random are red?

4.13. As he was dialling a telephone number, a subscriber forgot the last three digits and, remembering only that they were different, dialled at random. What is the probability that he dialled the right number.

4.14. At the end of a day 60 melons remained in a shop, of which 50 were ripe. A customer chooses two melons. What is the probability that they were both ripe?

4.15. There are n white, m black, and k red balls in an urn. Three balls are drawn at random. What is the probability that they are all of different colours?

4.16. An examination paper includes questions on four topics taken from a curriculum of 45 topics. A student has not revised 15 of the topics. What is the probability that he gets a paper in which he can answer all the questions?

4.17. The integers 1 to 15 are written on individual cards. Two cards are selected at random. What is the probability that the digits written on the cards will sum to 10?

In the following problems it is convenient to use the formula for the number of permutations with a given number of repeatitions.

4.18*. What is the probability that a random arrangement of blocks with the letters $i, i, i, t, t, n, n, o, a, v$ in a row results in a word "invitation".

4.19. Seven students randomly occupy a row of seven seats. Find the probability of three definite students sitting side-by-side.

4.20. Four books on algebra and three books on geometry are put on a book-shelf at random. What is the probability that the books on each topic will be side-by-side?

4.21*. Find the probability that in a game of bridge (four players get 13 cards each) each player gets an ace.

4.22. When a coin was tossed 10 times, five heads and five tails were attained. What is the probability that all five heads came up on the first five tosses?

Example 4.4. Ten out of fifteen building workers are plasterers and five are house-painters. A team of five is chosen at random. What is the probability that the team will have three painters and two plasterers?

Solution. The sample space consists of all the samples of different compositions containing five elements from a set of size 15. The number of such samples is $\binom{15}{5}$. The favourable events are associated with samples containing three painters and two plasterers. The number of ways of choosing three painters out of five is $\binom{5}{3}$ and, independent of the preceding choice, two plasterers can be chosen in $\binom{10}{2}$ ways.

Consequently, the number of samples corresponding to a favourable event is the product $\binom{5}{3}\binom{10}{2}$. Thus the probability is defined by the expression

$$P(A)=\binom{5}{3}\binom{10}{2}\Big/\binom{15}{5}.$$

Answer. $\binom{5}{3}\binom{10}{2}\Big/\binom{15}{5}$.

In a general case the probability of obtaining a sample of size $k+r$, where k elements belong to a group of n elements and r elements belong to another group of m elements, is given by

$$P(A)=\binom{m}{r}\binom{n}{k}\Big/\binom{m+n}{r+k}. \qquad (2)$$

4.23. There are 15 articles in a box, five of which are coloured. Five articles are selected at random. Find the probability that four of them are coloured and one is not.

4.24. In a lot of N articles n articles are standard. A random choice of m articles is made. Find the probability that k articles out of the chosen ones are standard.

4.25. What is the probability of winning the main prize in a lottery in which six numbers must be guessed out of 48? What is the probability of guessing five, four or three numbers?

4.26. What is the probability of getting one ace, or an ace and a king when six cards are dealt from a pack of 52 cards?

4.27. There are six tickets to the theatre, four of which are for seats on the first row. What is the probability that of three tickets selected at random two will be in the first row?

4.28. Twenty teams take part in football matches. They are divided at random into two groups, each group consisting of 10 teams. What is the probability that the two strongest teams will be in the same group?

5. Problems in Calculating Probabilities by Geometrical Methods

There are some problems which cannot be solved by the direct calculation of elementary event, which requires that each event is equally possible and the number are finite. Let us consider an example. Assume that a power line connecting points A and B has been damaged by a storm. What is the probability that the damage occurred in the part of the line between points C and D, which themselves lie between A and B? In this case the set of elementary events is infinite since the break may occur at any point along the line between A and B. It is natural to assume that the probability of the break in any section of the line is proportional to the length of that section. Since the probability of the damage along the whole segment is unity (the break has occurred), the probability of the break in CD is

$$P(A) = |CD|/|AB|.$$

Let us assume that the outcomes of an experiment which is carried out an infinite number of times are distributed uniformly in a domain S. This means that the probability of an event E, that the outcome of the experiment lies in a certain part of domain S, is proportional to the magnitude of that part and does not depend on its position or shape.

Thus

$$P(E) = m(s)/m(S), \tag{1}$$

where $P(E)$ is the probability that a point chosen at random from the domain S is in the domain s, and $m(s)$ and $m(S)$ are the magnitudes of the respective domains.

Example 5.1. A telephone subscriber has ordered a call. He may get it at any time in the next hour. What is the probability of the call taking place in the last 20 minutes of the hour?

Solution. Assume that event E is the call taking place in the last 20 minutes of the hour. Let us represent the sample space as a line segment of length 60. The elementary events favourable to the event E lie in the last third of the segment. Consequently,

$$P(E) = 1/3.$$

5.1. A mine-field is laid so that the mines are placed every 100 m along a straight line. What is the probability that a ship 20 m wide will blow up if it passes through the mine-field at right angles to the line?

5.2. A circle of radius R surrounds a smaller circle of radius r. Find the probability that a point thrown at random into the larger circle will also fall in the smaller one. (The probability of the point falling in a circle is assumed to be proportional to the area of the circle and independent of its position.)

If a random event whose probability must be found reduces to deciding whether a point lies within a certain section of a plane figure, the boundaries of the figure and the section being the curves of known functions, then the calculation of the areas in expression (1) reduces to the calculation of definite integrals.

Example 5.2. Two real numbers, x and y, are selected at radom, given that $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Find the probability that $y^2 \leqslant x$.

Solution. Assume that a point with coordinates (x, y) corresponds to the pair of numbers x and y. The sample space is a square whose sides are unit segments of the coordinate axes. A figure whose set of points corresponds to the outcomes favourable to the event $y^2 \leqslant x$ is bounded by the graphs of the functions $y = 0$, $x = 1$ and $y^2 = x$. This area can be calculated by the formula

$$S = \int_0^1 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}.$$

Since the area of a unit square is unity, it follows that

$$P(A) = 2/3.$$

Answer. 2/3.

5.3. Two real numbers x and y are chosen at random and are such that $|x| \leqslant 3$, $|y| \leqslant 5$. What is the probability of the fraction x/y being positive?

5.4. Two real numbers x and y are chosen at random and are such that $|x| \leqslant 1$, $0 \leqslant y \leqslant 1$. What is the probability that $x^2 < y$?

5.5. Two real numbers x and y are chosen at random and are such that $|x| \leqslant 1$ and $|y| \leqslant 1$. What is the probability that $|x| < |y|$?

5.6*. Two positive numbers x and y each of which does not exceed two are taken at random. Find the probability that $xy \leqslant 1$, $y/x \leqslant 2$.

5.7*. Two positive numbers x and y are taken at random, neither of them exceeding unity. What is the probability of their sum not exceeding unity if the sum of their squares exceeds 1/4?

5.8*. A parabola touches the base of a square and passes through its upper vertices. What is the probability that a point thrown in the square at random will fall in the domain between the upper side of the square and the parabola?

5.9. A parabola touches a semicircle and passes through the endpoints of its diameter. What is the probability that a point thrown into the semicircle at random will fall in the domain bounded by the arc of the semicircle and the parabola?

5.10. Suppose a function $f(x)$ given on the segment $[0, 1]$ is such that $f'(x) > 0$ for $x \in [0, 1]$, with $f(0) = 0$, $f(1) = 1$. Prove that when a point is thrown into a square whose sides are the interval $[0, 1]$ of the x-axis and the interval $[0, 1]$ of the y-axis the greatest probability that it will fall in the domain bounded by the curves $y = f(x)$, $y = f(a)$, $y = 0$, $y = 1$ is attained for $a = 1/2$.

5.11*. A domain is bounded by the curves $x = 0$, $x = \pi/2$, $y = \sin x$, $y = \sin a$. A point is thrown into a rectangle whose sides are the interval $[0, \pi/2]$ of the x-axis and the interval $[0, 1]$ of the y-axis. For what value of a is the probability of the point falling into the domain is the least?

In order to solve some problems by geometrical methods, it is convenient to introduce a Cartesian system of coordinates.

Example 5.3. Two friends agreed to meet at a definite place between 11 a.m. and noon. The first to arrive waits for his friend for a quarter of an hour and then leaves. Find the probability of the meeting taking place if each of them arrives at an arbitrary moment between 11 and 12 a.m.

Solution. Since the arrival of each friend is accidental, we can choose a segment of unit length and associate the arrival of the first person with randomly chosen point of the segment and the arrival of the second person with another random segment of unit length. We lay off those segments on the coordinate axes, the first on the x-axis and the second on the y-axis. Thus the sample space is a square of unit area inscribed in the first quadrant of the coordinate plane. Every point with coordinates (x, y) is a pair representing the arrivals of the two friends. The elementary events (x, y) which are favourable to the friend's meeting must satisfy the condition

$$|x - y| \leqslant 1/4. \qquad (*)$$

A geometrical interpretation of the required event is associated with the intersection of the band $(*)$ and the unit square consisting of the points whose coordinates (x, y) satisfy the inequalities $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. The area of the figure where the band and square intersect is the probability of a meeting since the area of a unit square is unity.

Answer. 7/16.

5.12. At a random moment within a period of 20 minutes student A telephones student B, waits for 2 minutes and then puts down the receiver. During the same 20 minutes student B arrives home at a random moment, stays for 5 minutes and then leaves. What is the probability that the two will have a talk?

5.13*. Two points, B and C, are thrown at random onto a unit segment of the abscissa. Find the probability that the length of the segment BC will be smaller than the distance between the origin and the nearest point.

5.14. A person lives in a town B connected by a railway with towns A and C. The trains running between towns A and C stop at town B. A train leaves town B in each direction every hour. A person arrives at the station at a random moment and takes the first train to arrive at the station. How must the time-table be compiled for the probability that the train will leave for A to be 5 times the probability that the train will leave for town C?

5.15*. *Buffon's needle problem.* A plane is ruled with parallel straight lines at distance of $2a$ apart. A needle $2l$ long $(l < a)$ is thrown on the plane at random. Find the probability that the needle will hit any of the lines.

6. Calculating the Probabilities of Compound Events

Events are divided into certain, impossible and random events. It follows from experience that *certain* events always occur, *impossible* events never occur, *random* events may occur and may not occur. For example, a certain event is that a white ball is taken from an urn which contains only white balls, and an impossible event is that a white ball is taken from an urn containing only black balls. If there are both white and black balls in an urn, then the selection of a ball of a definite colour from the urn is a random event.

A certain event coincides with the whole sample space Ω and a random event A is a subset of that space. An impossible event ϕ does not contain any elementary events.

The *sum of two events* A and B is an event C being the occurrence of either the event A or the event B. The sum of two events is designated as

$$C = A + B. \qquad (1)$$

The following example elucidates the concept of the sum of two events. Assume that a boy has bought tickets for two lotteries, "Sprint" and "Start". Let us consider a random event C that the boy wins in one lottery at least. The occurrence of this event is connected with the occurrence of at least one of the following events: an event A—there are winning tickets of the "Sprint" lottery among the tickets the boy has bought; an event B—there are winning tickets of the "Start" lottery among those bought by the boy.

The *product of two events* A and B is an event C which is the occurrence of both these events. The product of two events is expressed as

$$C = A \cdot B. \tag{2}$$

Events A and B are said to be *mutually incompatible* if their product is an impossible event:

$$A \cdot B = \varnothing.$$

The following example elucidates the concept of the product of two events.

There are Mercedes and Jaguar cars among those involved in an accident. Some of the cars turned upside down. An event A that a randomly chosen car that did not turn upside down is a Jaguar is a product of two events: event B—the car did not turn upside down and event C—the car is Jaguar, i.e. $A = B \cdot C$.

The definition of the probability of a compound event A, which is a combination of simpler events $A_1, \ldots, A_k$ whose probabilities are known, is based on the formulas for the addition and multiplication of probabilities. The following examples elucidate the meaning of these formulas.

We conduct an experiment which consists in throwing two dice and calculating the probability of an event C being that the score does not exceed three. The space of elementary events occurring as a result of the experiment can be represented as ordered pairs of integers varying from 1 to 6. There are 36 pairs of this kind. Among these events, the events (1, 1), (1, 2), (2, 1) are favourable to the event C. Thus, according to the definition given in Sec. 4, we infer that the probability of the event C is

$$P(C) = 3/36 = 1/12.$$

Let us consider now the event C as a combination of more simple events. For that purpose we note that the event C occurs if an event A—the score is equal to two, or an event B—the score is equal to three, occurs. Thus the event C is the sum of the events A and B: $C = A + B$. From the original space of elementary events only the pair (1, 1) is favourable to the event A and the pairs (1, 2) and (2, 1) are favourable to the event B. Consequently, the probabilities of the events A and B are equal to

$$P(A) = 1/36, \quad P(B) = 1/18$$

respectively. Thus, in the given case, there holds an equality

$$P(C) = P(A) + P(B).$$

Note that in this example the events A and B are incompatible (the total cannot equal two and three at the same time).

Let us calculate the probability of an event C that a card drawn at random from a pack of 52 cards is either an ace or a heart. The sample space in this example consists of 52 elements. The elementary events which are favourable to the event C are either a card of hearts being drawn from the pack (there are 13 cards of the same suit in a pack) or an ace (there are 4 aces in a pack). Taking into account that one of the aces is also of hearts and, consequently, 16 elementary events prove to be favourable, we obtain

$$P(C) = 16/52 = 4/13.$$

Let us now represent C as a combination of more simple events: an event A—the card drawn at random is a heart, and an event B—the card drawn at random is an ace. Then, according to the definition of the sum of two events, $C = A + B$. The probabilities of the events A and B are equal, respectively, to

$$P(A) = 1/4, \quad P(B) = 1/13.$$

We shall also need the probability of the product of the events A and B, i.e. the event $D = A \cdot B$ is that the card drawn at random is the ace of hearts. The probability of the event D is evidently equal to

$$P(D) = 1/52.$$

It is easy to verify that in the given case there holds an equality

$$P(A + B) = P(A) + P(B) - P(D).$$

The following formula generalizes the examples we have considered:

$$P(A + B) = P(A) + P(B) - P(AB). \tag{3}$$

i.e. the probability of the sum of two events A and B is equal to the sum of the probabilities of those events minus the probability of their product.

When the events A and B are incompatible, formula (3) assumes the form

$$P(A + B) = P(A) + P(B). \tag{4}$$

Let us consider an experiment consisting in throwing two dice and calculating the probability of an event C consisting in the fact that the number of dots appearing on the top face of the first die exceeds 3 and that of the second die exceeds 4. The elementary events favourable to the event C are ordered pairs of numbers (4, 5), (4, 6), (5, 5), (5, 6), (6, 5), (6, 6). Thus

$$P(C) = 6/36 = 1/6.$$

Let us now represent the event C as a combination of more simple events: an event A that the score on the first die exceeds three, and an event B—the score on the second die exceeds four. Then, according to the definition of a product of events, the event C can be represented as a product of the events A and B: $C = A \cdot B$.

Let us calculate the probabilities of the events A and B. Note first of all that the sample spaces arising upon the throw of each die separately consist of six equally possible outcomes. The elementary events favourable to the event A consist in 4, 5 or 6 dots appearing on the top face of the first die. Consequently, $P(A) = 1/2$.

The elementary events favourable to the event B are that 5 or 6 dots appear on the top face of the second die. Consequently, $P(B) =$

1/3. It is easy to verify that in this case there holds a relation

$$P(C) = P(A) \cdot P(B). \qquad (5)$$

The events A and B for which relation (5) holds true are said to be *independent*. Thus formula (5) can be used to calculate the probability of the product of two events when they are independent. If condition (5) is not fulfilled for events A and B, then those events are said to be *dependent*. In that case we can speak of the so-called conditional probability of the occurrence of the event A provided that the event B did occur.

Assume that we must calculate the probability of an event A that the sum of the dots appearing on the top faces of two dice does not exceed four, if it is known that one dot appeared on the top face of one die (event B). Since the event B did occur, we can take it to be certain and consider a new sample space consisting of 11 elementary events which are favourable to the event B:
(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1). In this new sample space, 5 elementary events, (1, 1), (1, 2), (1, 3), (2, 1), (3, 1), are favourable to the event A. Thus the probability of the event A in this sample space is equal to 5/11. We call the quantity obtained a *conditional probability of the event A* provided that the event B took place and designate it as $P(A/B)$.

Let us now consider the original sample space arising from a throw of two dice and calculate the probability of the event $C = A\ B$ that the score of the dice does not exceed four and a unity appears on the top face of one of the dice. The elementary events favourable to the event C can be represented by the following pairs of numbers: (1, 1), (1, 2), (1, 3), (2, 1), (3, 1). Thus $P(C) = 5/36$. Eleven elementary events which are favourable to the event B have been considered above. Consequently, $P(B) = 11/36$. It is easy to verify the validity of the relation

$$P(C) = P(A/B)\, P(B).$$

The following formula for multiplication of probabilities generalizes the examples we have considered:

$$P(AB) = P(A)\, P(B/A) = P(B)\, P(A/B), \qquad (6)$$

i.e. the probability of the product of two events A and B is equal to the product of the probability of one of the events by the conditional probability of the other event calculated on the assumption that the first event did occur.

For the case of three events the formula generalizing formula (6) has the form

$$P(ABC) = P(A/BC)\, P(BC) = P(A/BC)\, P(B/C)\, P(C). \qquad (7)$$

Example 6.1. Two balls are drawn from an urn containing n white and m black balls. What is the probability of the balls being of different colours?

Solution. We represent the event C that the selected balls will be of different colours as $C = A + B$, where the event A is that the first ball is white and the second ball is black; and the event B is that the first ball is black and the second ball is white. Since the events A

and B are incompatible, we have, according to (4),

$$P(C) = P(A) + P(B). \qquad (*)$$

The probabilities of the events A and B can be calculated by formula (6). We represent the event A as $A = W, Bl$, where the letters W and Bl written in the indicated order mean that the first selected ball was white and the second black. Then

$$P(A) = (P(W) P(Bl/W).$$

The probability of the event W is the ratio of the number of white balls to that of all the balls in the run. The conditional probability of the fact that the second ball is black, provided that the first ball is white, is the ratio of the original number of black balls to all the balls in the urn minus one. Thus

$$P(A) = \frac{n}{n+m} \cdot \frac{m}{n+m-1}.$$

Similarly,

$$P(B) = \frac{m}{n+m} \cdot \frac{n}{n+m-1}.$$

Substituting the expressions obtained into formula (*), we get

$$P(C) = \frac{2nm}{(n+m)(n+m-1)}.$$

Answer. $\frac{2nm}{(n+m)(n+m-1)}$.

Solve the following problems using the formulas for multiplication and addition of probabilities.

6.1. There are n white and m black balls in an urn. Two balls are drawn. What is the probability of both balls being white; both balls being black?

6.2*. Solve problem 6.1 under the condition that the selected balls are replaced and their colour is recorded.

6.3. Four cards are drawn from a pack of 52 cards. What is the probability of all of them being of different suits?

6.4. A die is thrown several times. What is the probability of one dot appearing on the top face of the die on the third throw?

6.5. Twenty cars were driven to a service station. Five of them had a fault in the running gear, eight had a fault in the motor and ten had no faults. What is the probability that a car has faults both in the running gear and in the motor?

6.6. When preparing for an examination in mathematics a student must be ready with the answers to 20 questions in the fundamentals of mathematical analysis and 25 questions in geometry. However, he had only time enough to prepare answers to 15 questions in the analysis and 20 questions in geometry. An examination paper contains three questions, two of which are in analysis and one in geometry. What is the probability:

(a) that the student will pass the exam with an excellent mark (will answer all three questions); (b) with a good mark (will answer any two of the three questions)?

A *complement* of the random event A (or an *opposite event*) is an event C that during an experiment the event A did not occur. A complement to the event $\bar{A}$ is designated as A. The probabilities of the events A and $\bar{A}$ are related as

$$P(A) + P(\bar{A}) = 1. \qquad (8)$$

If the compound event A is represented as

$$A = A_1 + \ldots + A_k, \qquad (9)$$

where A_i are events whose probabilities are known, then it is sometimes convenient to calculate the probability $P(A)$ using the formula

$$\bar{A} = \bar{A}_1 \cdot \bar{A}_2 \ldots \bar{A}_k, \qquad (10)$$

which relates the complements of the events being considered. Thus, if A_i are independent, we obtain

$$P(A) = 1 - P(\bar{A}) = 1 - P(\bar{A}_1) \ldots P(\bar{A}_k) = 1 - [1 - P(A_1)] \ldots \ldots [1 - P(A_k)]. \qquad (11)$$

If all the events A_i are equally probable, formula (11) assumes a more simple form

$$P(A) = 1 - (1 - p)^k, \qquad (12)$$

where p is the probability of the event A_i.

Example 6.2. One bomb is enough to destroy a bridge. Find the probability of destruction if three bombs are dopped onto the bridge with the probabilities of hitting equal to 0.3, 0.4, 0.7 respectively.

Solution. Let us calculate the probability of the event $\bar{A}$ consisting in the destruction of the bridge. We designate as $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ the events consisting in the first, the second and the third bomb not hitting the bridge respectively. Then $\bar{A} = \bar{A}_1\bar{A}_2\bar{A}_3$. Since the independence of A_i yields the independence of $\bar{A}_i$, we have

$$P(\bar{A}) = P(\bar{A}_1) P(\bar{A}_2) P(\bar{A}_3) = 0.3 \cdot 0.4 \cdot 0.7 = 0.084.$$

Consequently, the probability of destroying the bridge is

$$P(A) = 1 - P(\bar{A}) = 0.916.$$

Answer. 0.916.

6.7. There are n white, m black and k red balls in an urn. Three balls are drawn at random. What is the probability of at least two balls being of the same colour?

6.8. There are 15 text-books on a book-shelf, 5 of which are bound. Three text-books are selected at random. What is the probability of at least one of them being bound?

6.9. There are n tickets in a lottery, l of which are winning tickets. A person buys k tickets. What is the probability of at least one of the tickets being a winning ticket?

6.10. In one radar surveillance cycle a target is detected with probability p. In each cycle the location of the target occurs independently of other cycles. What is the probability of the target being detected in n cycles?

6.11*. There is a target at which n shots are fired. Every shot hits the target with the probability p. How many shots must be fired for the probability of hitting the target to be not less than P?

The probability of the event A which can occur only upon the occurrence of one of several incompatible events $B_1, B_2, \ldots, B_n$ is equal to the sum of the products of the probabilities of each of those events by the conditional probability of the event A provided that the given event did occur:

$$P(A) = P(B_1) P(A/B_1) + P(B_2) P(A/B_2) + \ldots + P(B_n) P(A/B_n). \tag{13}$$

Equation (13) is called the *formula for total probability*.

Example 6.3. There are 6 Masters of Sport and 4 firstgrade sportsmen on the first team, and there are 6 first-grade sportsmen and 4 Masters of Sport on the second team. A combined team formed from the players of the two teams consists of 10 players, 6 sportsmen from the first team and 4 sportsmen from the second team. One sportsman is chosen at random from the combined team. What is the probability of his being a Master of Sport?

Solution. Assume that the event B_i ($i = 1, 2,$) is that the sportsman chosen at random is from the ith team. Then the probability of the events B_i are equal to $P(B_1) = 3/5$ and $P(B_2) = 2/5$ respectively. Assume that the event A consists in that the player chosen at random is a Master of Sport. Then, provided that the event B_i has occurred (i.e. it is known to which team the sportsman belongs), the conditional probabilities of the event A are equal to $P(A/B_1) = 3/5$ and $P(A/B_2) = 2/5$ respectively. Using the total probability formula, we obtain

$$P(A) = \frac{3}{5} \cdot \frac{3}{5} + \frac{2}{5} \cdot \frac{2}{5} = \frac{13}{25}.$$

Answer. 13/25.

6.12. An examination is carried out according to the following scheme: if a certain paper has been drawn, the examinator puts it aside, i.e. the other students cannot draw it. A student knows the answers to the questions on k papers, $k < n$. In what case is the probability of the student drawing the paper he knows greater, when he is the first to answer or the last?

6.13*. There are two balls in an urn whose colours are not known (each ball can be either white or black). We put a white ball in the urn. What is now the probability of drawing a white ball?

6.14. One gun is selected from five guns among which there are 3 sniper guns and 2 ordinary guns, and a shot is fired from that gun. Find the probability of hitting the target if the probability of hitting when firing the sniper gun is 0.95 and that using the ordinary gun is 0.7.

6.15. There are two urns. There are m white and n black balls in the first urn and k white and l black balls in the second urn. One ball is taken from the first urn and placed into the second. What is the probability now:

(a) of drawing a white ball from the first urn;

(b) of drawing a white ball from the second urn?

6.16. There are two lots of identical articles with different amounts of standard and defective articles: there are N articles in the first lot, n of which are defective, and M articles in the second lot, m of which are defective. K articles are selected from the first lot and L articles from the second and a new lot results. What is the probability of an article selected at random from the new lot being defective?

6.17*. Under the conditions of the preceding problem find the probability of at least one article out of three selected at random from the new lot being defective?

Answers and Hints

Chapter 1

Sec. 1. 1.1. 1, 3. **1.2.** 0. **1.3.** 3, $3\pm\sqrt{20}$. **1,4.** 0, -2, $\frac{-2\pm\sqrt{66}}{2}$. **1.5.** 2, $\frac{1}{2}$. **1.6.** 0, -2. **1.7.** 3, $\frac{2}{3}$. **1.8.** 2, -4. **1.9.** -2, 1. **1.10*.** $\varnothing$. *Hint.* Designate $z=x^2-5x+6$. **1.11*.** $\frac{-5\pm\sqrt{85}}{2}$, $\frac{-5\pm\sqrt{5}}{2}$. *Hint.* Designate $z=x^2+5x$. **1.12*.** $-\frac{3}{2}$, 0, $\frac{-3\pm\sqrt{65}}{4}$. *Hint.* Designate $z=2x^2+3x$. **1.13.** ± 3, ± 2. **1.14.** $\pm\sqrt[4]{\frac{5}{2}}$. **1.15.** $\frac{1}{2}(\sqrt[3]{2}+1)$, $\frac{1}{2}(-\sqrt{5}+1)$. **1.16*.** No solutions. *Hint.* Designate $y=\frac{1+x^2}{x}+1$. **1.17.** 0, 5. **1.18.** $-\frac{1}{2}$, $\frac{3}{4}$, $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{1}$. **1.19.** -1. **1.20.** 1, $-\frac{5}{2}$. **1.21*.** $-\frac{2}{3}$, $-\frac{1}{2}$, 3. *Hint.* Use the binomial formula. **1.22.** 1, $a+\sqrt{a}$, $a-\sqrt{a}$. **1.23.** 4, 3, 2, -5. **1.24.** $\frac{2a\pm\sqrt{26a^2\pm 2\sqrt{25a^4+4b^4}}}{2}$. **1.25.** -1, 12. **1.26.** $\frac{-5\pm\sqrt{21}}{6}$. **1.27.** -3, 4. **1.28.** -3. 2. **1.29.** $-5\pm\sqrt{85}$, $-5\pm\sqrt{5}$. **1.30*.** -1, 0. *Hint.* Designate $y=x^2+x$. **1.31.** $\frac{-5\pm\sqrt{21}}{2}$. **1.32.** -2, ± 1, $\frac{1}{2}$. **1.33.** 1, $-\frac{5}{3}$, $-\frac{3}{5}$. **1.34.** -2, $\frac{3}{2}$, $\frac{2}{3}$. **1.35.** $\frac{-4+\sqrt{6}\pm\sqrt{18-8\sqrt{6}}}{2}$, $\frac{-4-\sqrt{6}\pm\sqrt{18+8\sqrt{6}}}{2}$. **1.36.** $\frac{1}{2}$. **1.37.** -3, 5. **1.38.** 2.

1.39. 5. 1.40. $-\frac{1}{3}$. 1.41. $\frac{1}{4}$, 5. 1.42*. $a+b$, 0, $\frac{2ab}{a+b}$, $\frac{a^2+b^2}{a+b}$. *Hint*. Introduce the auxiliary unknown $z=\frac{a+b}{2}-x$. 1.43. 1. 1.44. −1. 1.45. $\frac{1}{3}$. 1.46. [1, 2]. 1.47. −8, 2. 1.48. −4, −2, 0, 2, 4. 1.49. 0. 1.50. $(-\infty, 2]\cup[3, \infty)$. 1.51. 0, ±1. 1.52. $1\frac{3}{4}$, $2\frac{1}{2}$, $3\frac{1}{4}$.

Sec. 2. 2.1. 8. 2.2. 5. 2.3. 8. 2.4. −1, 3. 2.5. No solutions. 2.6. $\frac{7\pm\sqrt{153}}{16}$. 2.7. 2. 2.8. No solutions. 2.9. −1, 2. 2.10. 3. 2.11. No solutions. 2.12. −5, 4. 2.13. $\frac{17}{16}$. 2.14. No solutions. 2.15. $\frac{7+\sqrt{41}}{2}$. 2.16. −61, 30. 2.17. 2. 2.18. 8, $8\pm\frac{12\sqrt{21}}{7}$. 2.19. −6, −5, $-\frac{11}{2}$. 2.20. −1. 2.21. −2. 2.22. 0. 2.23. 1, $-\frac{1}{3}$. 2.24. ±4. 2.25. −1. 2.26. $\frac{-Bp\pm\sqrt{B^2p^2+A[(p^2+c-c_1)^2-4p^2c]}}{2Ap}$. 2.27*. ±21. *Hint*. Rationalize the denominator. 2.28. $\frac{2}{7}$, 5. 2.29*. 3. *Hint*. Use the fact that $\sqrt{x-2}\sqrt{4-x}=\sqrt{6x-x^2-8}$. 2.30. −17, 23. 2.31. −7, 2. 2.32. $-\frac{1}{511}$, 2. 2.33. ±7. 2.34*. 5. *Hint*. Designate $y=\frac{\sqrt{x+4}+\sqrt{x-4}}{2}$. 2.35*. 1. *Hint*. Use the inequality $\frac{1}{a}+a\geqslant 2$ which is valid for $a>0$. 2.36. 1024. 2.37. 3, 5. 2.38. $\pm 2\sqrt{2}$. 2.39. −5, 2. 2.40. −2, 0. 2.41*. $\frac{66}{119}$. *Hint*. Note that the product of the terms on the left-hand side of the equation is 66. 2.42. $\frac{3-\sqrt{73}}{4}$, 0, $\frac{3}{2}$, $\frac{3+\sqrt{73}}{4}$. 2.43. 15. 2.44*. −3, 6. *Hint*. Designate $y=x^2-3x+7$. 2.45. $\frac{3}{4}$. 2.46. −3, 4. 2.47. 0. 2.48. 9. 2.49. [−1, 0]. 2.50. [0, 3]. 2.51. 2. 2.52. $\pm\frac{\sqrt{5}}{2}$. 2.53. No solutions. 2.54. 5. 2.55. [1, 2.5], 13. 2.56. [5, 10]. 2.57. 4. 2.58. ±2. 2.59. 0. 2.60. $\frac{1}{2}$, 5. 2.61. −1. 2.62*. −6, 1. *Hint*. Introduce the auxiliary unknowns $u=\sqrt{x-2}$, $v=\sqrt{x+7}$. **2.63***. ± 1. *Hint.* Factor out $\sqrt{x+1}$. **2.64.** No solutions. **2.65.** 0. **2.66.** 1. **2.67.** 0, $\frac{63a}{65}$. **2.68.** $\frac{(2+\sqrt{3})^n+1}{(2+\sqrt{3})^n-1}$, $\frac{(2-\sqrt{3})^n+1}{(2-\sqrt{3})^n-1}$. **2.69.** 1.

Sec. 3. 3.1. (1, 2, 3). **3.2.** (8, 4, 2). **3.3.** (1, −2, −1). **3.4.** (1, −3, −2). **3.5.** $(abc,\ ab+ac+bc, a+b+c)$. **3.6.** $\left(-\frac{ab}{(b-1)(1-a)}, \frac{b}{(a-1)(b-a)}, \frac{a}{(b-1)(b-a)}\right)$. **3.7.** For $a \neq 0$, $a \neq -3$ the system is determinate, for $a = -3$ the system is indeterminate, for $a = 0$ the system is inconsistent. **3.8.** For $a \neq 0$ the system is determinate, for $a = 0$ the system is indeterminate. **3.9.** For $a = 0$ the system is indeterminate, for $a = 2$ the system is inconsistent, for the remaining a it is determinate. **3.10.** For $a = 0$, $a = 1$ the system is inconsistent, for $a = -1$, $a = 2$ the system is indeterminate, for the remaining a it is determinate. **3.11.** For $a + b \neq 0$ the system is determinate, for $a + b = 0$ it is indeterminate. **3.12.** For $a = 0$, $b \neq 0$; $a = 0$, $b = 0$; $a \neq 0$, $b = 0$ the system is indeterminate, for the remaining values of the pair a, b the system is determinate. **3.13.** The system is indeterminate for $p=\frac{3b+14a}{16}$, $m=\frac{5b+2a}{64}$. **3.14.** They are consistent. **3.15.** $a = 1$, $b = -1$. **3.16.** (1, −1), (1, −2), (−1, −1), (−1, −2). **3.17.** $a=1$. **3.18.** $\left(0, 0, \frac{9}{4}\right)$, (2, −1, 1). **3.19.** $a = -4$. **3.20.** $a = 3$. **3.21.** (1, 4), (4, 1). **3.22.** (0.6, 0.3), (0.4, 0.5). **3.23.** (3, 2), (2, 3). **3.24.** (14, −11), (11, −14). **3.25.** (4, 2), (2, 4). **3.26.** (1, 4), (−1, 6). **3.27.** (1, 2). **3.28.** (4, 1), (1, 4). **3.29.** (2, 1), (−2, −1). **3.30.** (4, 1), (1, 4), $\left(\frac{-5\pm\sqrt{41}}{2}, \frac{-5\mp\sqrt{41}}{2}\right)$, where the signs are either both upper or both lower. **3.31.** $\left(\pm\sqrt{\frac{ab\pm\sqrt{a^2b^2-4ab}}{2b}}, \pm\sqrt{\frac{ab\mp\sqrt{a^2b^2-4ab}}{2b}}\right)$. **3.32.** (1, 2), (2, 1). **3.33.** (3, 1) (1, 3). **3.34.** (1, 2), (2, 1). **3.35.** (1, 1). **3.36.** (±3, ±2), (±2, ±3). **3.37.** (±3, ±1), (±1, ±3). **3.38.** (±3, ±2), (±2, ±3). **3.39*.** (2, 1), (1, 2). *Hint.* Pass to the unknowns $u = x + 1$, $v = y + 1$. **3.40*.** (±2, ±1), (±2, ±1). *Hint.* Represent the first equation as a quadratic equation with respect to the variable $z = (x + y)/(x - y)$. **3.41*.** (2, 3), $\left(-\frac{3}{4}, -4\right)$. *Hint.* Divide the first equation by the second. **3.42.** (5, 1), (1, 5), (3, 2), (2, 3). **3.43.** (2, 1) (−1, −2), $(1 \pm \sqrt{2},\ 1 \mp \sqrt{2})$. **3.44.** (−2, −4), $\left(\frac{5}{3}, \frac{10}{3}\right)$. **3.45.** (1, 4),

$-5, 4), (5, -4), (-1, -4)$. **3.46***. $(3, 5), (5, 3), (-3, -5), (-5, -3)$. *Hint.* Into the first equation represented in the form $x^4 + y^4 + 2x^2y^2 = 931 + x^2y^2$ substitute $(x^2 + y^2)$ whose expression is found from the first equation. **3.47***. $(2, 1), (1, 2), (-3, 0), (0, -3), (1, -2), (2, -1)$. *Hint.* Represent the first equation as $(x + y)^2 + (xy - 1)^2 = 10$. **3.48***. $(5, 2), (-2, -5)$. *Hint.* Factor $x^5 - y^5$. **3.49**. $(2, -1, -1), (-1, -1, 2), (-1, 2, -1)$. **3.50**. All permutations of the numbers $(1, 0, 0)$.

3.51*. $\left(2, -\frac{1}{2}, 4\right)$, $\left(-2, \frac{1}{2}, -4\right)$, $\left(-\sqrt{\frac{15}{2}}, 2\sqrt{\frac{15}{2}}, \frac{2}{5}\sqrt{\frac{2}{15}}\right)$ and $\left(\sqrt{\frac{15}{2}}, -2\sqrt{\frac{15}{2}}, -\frac{2}{5}\sqrt{\frac{2}{15}}\right)$.

Hint. Introduce new variables $u = xy$, $v = xz$, $w = yz$. Find the expressions for u and w in terms of v from the first and the second equation of the resulting system, the third equation, after the substitution, will be quadratic with respect to v.

3.52. $\{0, 0, 0\}$. **3.53**. $x = \frac{a(b^2+c^2)}{2bc}$, $y = \frac{b(a^2+c^2)}{2ac}$, $z = \frac{c(a^2+b^2)}{2ab}$.

3.54. $\left(\pm\sqrt{\frac{2abc(ab-bc+ca)}{(ab+bc-ca)(-ab+bc+ca)}}, \pm\sqrt{\frac{2abc(ab+bc-ac)}{(ab-bc+ac)(-ab+bc+ca)}}, \pm\sqrt{\frac{2abc(-ab+bc+ca)}{(ab-bc+ca)(ab+bc-ca)}}\right)$.

3.55. $(1, 3, 9), (9, 3, 1)$. **3.56**. $(0, 1, -1), (-1, 2, -1), (-1, 1, 0)$. **3.57**. $(3, -1, -1), (0, 2, -1), (0, -1, 2)$. **3.58** All permutations of 1, 2, 3. **3.59**. $(3, 6, 10)$ and $(6, 3, 10)$.

3.60. $\left(\pm\frac{a}{\sqrt{a+b+c}}, \pm\frac{b}{\sqrt{a+b+c}}, \pm\frac{c}{\sqrt{a+b+c}}\right)$.

3.61. $\left(\pm\frac{2}{\sqrt{(-a+b+c)(a+b-c)}}, \pm\frac{2}{\sqrt{(a-b+c)(-a+b+c)}}, \pm\frac{2}{\sqrt{(a-b+c)(a+b-c)}}\right)$ and $(0, 0, 0)$. One of the coordinates has a positive sign and the other have like signs. **3.62**. $(0, 0, 0)$, $(\pm 1, \pm 1, \pm 1)$, $(0, \pm\sqrt{2}, \pm\sqrt{2})$, $(\pm\sqrt{2}, 0, \pm\sqrt{2})$ and $(\pm\sqrt{2}, \pm\sqrt{2}, 0)$. **3.63**. $\left(1, 0, -\frac{1}{2}\right)$, $\left(-1, 0, \frac{1}{2}\right)$. **3.64**. $(\pm 1, \pm 2, \pm 5)$, one of the coordinates has a positive sign and the other have like signs. **3.65**. $(0, 0, 0)$. **3.66**. $(-5, -3, 0), (3, 1, -2)$. **3.67**. $(2, -1)$. **3.68**. $(2, 3)$, $\left(\frac{13}{3}, \frac{5}{3}\right)$. **3.69**. $\left(\frac{25}{3}, \frac{16}{3}\right)$. **3.70**. $(4, 4)$. **3.71**. $(2, 3), (-2, -3), (2, -3), (-2, 3)$. **3.72**. $(1, 1)$. **3.73**. $(25, 9)$, $\left(\frac{49}{4}, \frac{81}{4}\right)$. **3.74**. $(5, 4)$.

Chapter 2

Sec. 1. 1.1. 1. 1.2. $ab(a-b)^2$. 1.3. a^2+a+1. 1.4*. $\frac{1}{3}$. *Hint.* Use (4) to write all the logarithms to a certain common base 1.5. $(\log_2 x+1)^3$.
1.6. $\frac{1}{\log_a b-1}$. 1.7. 6. 1.8. 3. 1.9. $a(b+3)$. 1.10. $\frac{a+b}{1-b}$.
1.11. $a\cdot b\cdot c+1$. 1.12. $\log_8 a=\frac{1}{1-\log_8 c}$. 1.13*. $\frac{1}{\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1}}$. *Hint.* Pass to the logarithms to the base x. 1.18*. *Hint.* On the left-hand side of the expression pass to the logarithms to the base N. 1.19*. *Hint.* When simplifying the expression, take into account the fact that the root of an even degree is understood in an arithmetic sense. 1.20*. *Hint.* See the Hint to 1.18*. 1.21*. *Hint.* See the Hint to 1.18*.

Sec. 2. 2.1. 4. 2.2. 2. 2.3. $\frac{3}{5}$. 2.4. -3. 2.5. $-\frac{1}{2}$. 2.6. $-\frac{3}{2}$, 4. 2.7. $\log_3 2$. 2.8. $\frac{\pi}{2}(2k+1)$, $\frac{\pi}{4}+\pi k$, $k\in Z$. 2.9. -5, $\frac{93}{11}$. 2.10. $\frac{7}{5}$. 2.11. 81. 2.12. $\frac{5}{3}$. 2.13. $-\frac{5}{2}$, 3. 2.14. πn, $n\in Z$. 2.15. $-2+\sqrt{4-2\log_3 5}$. 2.16. 5. 2.17. 2, -2. 2.18. $\log_3(2+\sqrt{5})$, $\log_3\frac{\sqrt{5}-1}{2}$. 2.19. 3, $\log_6 8$. 2.20. 9, 81. 2.21. 1, $-\frac{1}{3}$. 2.22. 1, -1, 0. 2.23. $\frac{1}{2}$. 2.24. 0. 2.25. -1, 1. 2.26. 1, $\log_4 3$. 2.27. $\frac{4}{3}$, $\frac{\log_2 3+3}{3}$. 2.28. 0. 2.29. -2, 2. 2.30. -2, 2. 2.31. -2, 2. 2.32. 1. 2.33. 1, 3. 2.34. $-\frac{1}{2}$. 2.35. 2, 3, 4, 11. 2.36. $\frac{1}{3}$, 2, 4. 2.37. 1, $a^{1/\pi}$. 2.38. $\frac{1}{5}$, 25. 2.39. 10, 10^{-4}. 2.40*. 2. *Hint.* Divide both sides of the equation by 2^x and use the property of monotonicity of an exponential function. 2.41*. 3. *Hint.* See the Hint to 2.40*. 2.42*. 1. *Hint.* Compare the greatest value of the function appearing on the left-hand side of the equation and the least value on the right-hand side. 2.43*. 1. *Hint.* Find y_1 and y_2 which are the roots of the quadratic equation with respect to the variable $y=2^x$, solve the equations $y_1(x)=2^x$ and $y_2(x)=2^x$ using the property of monotonicity of the functions appearing in them. 2.44*. 3. *Hint.* See the hint to 2.43*. 2.45*. 1. *Hint.* Make a substitution $y=x-1$ and use the hint to 2.43*.

Sec. 3. 3.1. $-\frac{8}{3}$. 3.2. 3, 2. 3.3. 2. 3.4. 2. 3.5. -1, 7. 3.6. 1, 10^{-3}, 10^{-2}. 3.7*. $\frac{1}{81}$, 9. *Hint.* Take the logarithms of both sides of the equation to the base 3. **3.8*.** $10^{\sqrt{\log(13/3)}}$, $10^{-\sqrt{\log(13/3)}}$, $10^{\sqrt{\log(7/3)}}$, $10^{-\sqrt{\log(7/3)}}$. *Hint.* Introduce the unknown $y = x^{\log x}$. **3.9*.** 10^{-1}, $10^{(1+\sqrt{3})/2}$, $10^{(1-\sqrt{3})/2}$. *Hint.* Take the logarithms of both sides of the equation to the base 10. **3.10*.** 10, 0.1. *Hint.* See the Hint to 3.9*. **3.11.** 10. **3.12.** 10, 10^4. **3.13*.** —10, —1. *Hint.* Use the identity $\sqrt{x^2} = -x$ which is valid for $x < 0$. **3.14.** $\log_3 10$ $\log_3 28 - 3$. **3.15*.** $\frac{1}{3}$, $\frac{1}{15}$. *Hint.* Take the logarithms of both sides of the equation to the base 15. **3.16*.** 1, 0.1, 0.01. *Hint.* Having simplified the right-hand side of the equation, take its logarithms to the base 10. **3.17.** 3, $3 + \sqrt{2}$. **3.18.** 3. **3.19.** 7. **3.20.** 1, 3. **3.21.** —10. **3.22.** 2. **3.23.** 2, 3. **3.24.** 1. **3.25.** 1. **3.26.** 3. **3.27.** 3. **3.28.** 1. **3.29.** $\pm\frac{1}{2}$. **3.30.** 1, 3. **3.31.** 4, $\frac{\sqrt[3]{4}}{2}$. **3.32.** 1, 4, $\frac{1}{4\sqrt[5]{8}}$. **3.33.** 8, $\frac{1}{\sqrt[3]{4}}$. **3.34.** b^2+1, $b > 0$ and $b \neq 1$. **3.35.** $\frac{1}{2}$, $\frac{1}{8}$. **3.36*.** $\frac{1}{81}$, 3. *Hint.* Solve the quadratic equation with respect to the variable $y = \log_3 x$. When solving equations of the form $\log_3 x = y_1(x)$ and $\log_3 x = y_2(x)$, use the property of monotonicity of the functions entering into different sides of the equation. **3.37*.** 1. *Hint.* Compare the greatest value of the function appearing on the left-hand side of the equation with the least value appearing on the right-hand side. **3.38*.** 2. *Hint.* See the Hint to 3.37*. **3.39*.** $\frac{1}{4}$, 2. *Hint.* See the hint to 3.36*.

Sec. 4. 4.1. (5, 5). **4.2.** (4, 2). **4.3.** $\left(\frac{1}{2}, -\frac{3}{2}\right)$. **4.4.** (6, 6). **4.5.** (3, 9), (9, 3). **4.6*.** (1, 4). *Hint.* 4^x, $2^{(x-y)/2}$, 2^{3-y} are successive terms of a geometric progression. **4.7.** (2, 2). **4.8.** (2, 1). **4.9.** (1, 9), (16, 1). **4.10.** (—2, 7). **4.11.** (1, 1). **4.12.** (2, 4). **4.13.** (1, —1), (5, 3). **4.14.** (16, 3), $\left(\frac{1}{64}, -2\right)$. **4.15.** (27, 4), $\left(\frac{1}{81}, -3\right)$. **4.16.** (4, 1). **4.17.** (6, 2). **4.18.** (9, 16). **4.19*.** (4, 1), (—4, —1). *Hint.* The first equation is a quadratic equation with respect to $z = 2^{\sqrt{xy}}$ and the second is a quadratic equation with respect to $u = \frac{x+y}{x-y}$. **4.20*.** $(\sqrt{3}, 1)$, $(-\sqrt{3}, 1)$. Designate $z = x^2 + y$, $u = 2^{y-x^2}$ and use $6^{x^2-y} = 3^{x^2-y}/2^{y-x^2}$. **4.21.** (2, 2, 1).

Sec. 5. 5.1. 4. 5.2. 3. 5.3. 2. 5.4. 2. 5.5. 4. 5.6. $1+\sqrt{1+\log 2}$, $1-\sqrt{1+\log 2}$. 5.7. 1, 4. 5.8**. 2, $-\frac{1}{\log 5}$. *Solution.* Dividing both sides by $2^2 \cdot 5^2$, we get $5^{x-2} \cdot 2^{\frac{3x}{x+1}-2} = 1 \Leftrightarrow \left(5 \cdot 2^{\frac{1}{1+x}}\right)^{x-2} = 1 \Leftrightarrow 5 \cdot 2^{\frac{1}{x+1}} = 1$ or $x-2=0 \Leftrightarrow x = -\frac{1}{\log 5}$ or $x = 2$. 5.9. $\log\frac{3}{4}\Big/\log\frac{\sqrt{5}-1}{2}$. 5.10. $\frac{2}{3}$. 5.11*. $-1-\sqrt{\frac{1}{2}\log(1+\sqrt{11})}$, $-1+\sqrt{\frac{1}{2}\log(1+\sqrt{11})}$. *Hint.* The numbers 10^{3x^2+7x}, 10^{x^2+5x}, $10^{-(x+x^2)}$ are successive terms of a geometric progression. 5.12*. No solutions. *Hint.* Investigate the behaviour of the functions appearing on different sides of the equation. 5.13. $-\sqrt{2}$, $\sqrt{2}$. 5.14. 1, 8. 5.15. $-\frac{1}{2}$, $\frac{1}{2}$, $x > 3$. 5.16. $(\log_2 3 - 2)^{-1}$, $(1-\log_3 4)^{-1}$. 5.17. (5, 0.5). 5.18. 10, $10^{1/9}$. 5.19. $\left[\frac{1}{6}, \infty\right)$ 5.20. $\frac{1}{9}$. 5.21. 2, 7/3. 522. 1. 5.23. 8. 5.24. 1. 5.25. 5. 5.26. 2. 5.27*. (3, 9). *Hint.* Take the logarithms in the first equation to the base x. 5.28. $\left(\frac{2}{3}, \frac{27}{8}, \frac{32}{3}\right)$. 5.29. (4, 2), (—4, 2). 5.30*. (1, 1), $\left(\frac{16}{81}, \frac{4}{9}\right)$. *Hint.* Taking the logarithms in both equations, to the base 10, rationalize the system with respect to the unknowns $u = \frac{\log x}{\log y}$, $v = \sqrt[4]{x}+\sqrt{y}$. 5.31*. $\left(100, \frac{1}{100}\right)$, (100, 100), $\left(\frac{1}{100}, 100\right)$, $\left(\frac{1}{100}, \frac{1}{100}\right)$. *Hint.* Pass to decimal logarithms in the first equation. 5.32. (2, 6), (6.2). 5.33. (1, 1, 1), $(4, 2, \sqrt{2})$. 5.34. (a^4, a, a^7), $\left(\frac{1}{a^4}, a, \frac{1}{a^7}\right)$.

Chapter 3

Sec. 1. 1.1. $(-\infty, -2) \cup (2, +\infty)$. 1.2. $\left(-\frac{9}{2}, -2\right) \cup (3, +\infty)$. 1.3. $[-1, 3]$. 1.4. $(-\infty, -2) \cup (-1, 0]$. 1.5. $(-8, 1]$. 1.6. $(-\infty, -1) \cup (3, 7)$. 1.7. $[-1, 2]$. 1.8. $(-\infty, -2) \cup (1, 2) \cup (3, +\infty)$. 1.9. $(1, 2]$. 1.10. $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{3}{2}, 4\right)$. 1.11. $(-\infty,$

1) $\cup \left(\frac{3}{2}, 2\right) \cup (3, +\infty)$. **1.12.** $\left[-5, \frac{-9-\sqrt{61}}{8}\right)$. **1.13.** $\left(\frac{\sqrt{13}-5}{2}, 1\right]$. **1.14.** $(-\infty, -5] \cup \left[1, \frac{8+\sqrt{22}}{3}\right)$. **1.15.** $[-46, 3)$. **1.16.** $\left(-\frac{3+\sqrt{5}}{2}, 1\right]$. **1.17.** $[-6, 0) \cup (3, 4]$. **1.18.** $[-5, -1) \cup (1, +\infty)$. **1.19.** $[-1, +\infty)$. **1.20.** No solutions. **1.21.** $\left(-1, \frac{5-\sqrt{13}}{2}\right) \cup (2, +\infty)$. **1.22.** $\left(-\infty, \frac{\sqrt{37}-13}{6}\right]$. **1.23.** $x \in \mathbf{R}$. **1.24.** $(5, +\infty)$. **1.25.** $(-2, -1] \cup \left[-\frac{2}{3}, \frac{1}{3}\right)$. **1.26.** $x \in \mathbf{R}$. **1.27.** $\left[\frac{7}{6}, \frac{3}{2}\right]$. **1.28.** $[1-\sqrt{17}, \sqrt{5}-1]$. **1.29.** $x \in (-\infty, \sqrt{2}-1) \cup (\sqrt{2}+1, +\infty)$. **1.30.** $\left(-\frac{3+\sqrt{65}}{2}, 3\right)$. **1.31.** $(-\infty, -2-\sqrt{2}] \cup [1+\sqrt{3}, +\infty)$. **1.32.** $(-\infty, -1-\sqrt{3}] \cup [1-\sqrt{5}, +\infty)$. **1.33.** $[-1, 2) \cup (8, 5+\sqrt{18}]$.

Sec. 2. **2.1.** $(-\infty, \log_2(-1+\sqrt{7})$. **2.2.** $(-2, +\infty)$. **2.3.** $(-\infty, -1]$. **2.4.** $\left(-\infty, -\sqrt{2\log_2 \frac{\sqrt{5}+1}{2}}\right] \cup \left[\sqrt{2\log_3 \frac{\sqrt{5}+1}{2}}, +\infty\right)$. **2.5.** $[-\sqrt{\log_3 4}, \sqrt{\log_3 4}]$. **2.6.** $[-4, 2) \cup (0, +\infty)$. **2.7.** $[-10, +5]$. **2.8.** $[0, 1]$. **2.9.** $[\log_{13} 5, 1]$. **2.10.** $(-\sqrt{7}, -\sqrt{3}) \cup (\sqrt{3}, \sqrt{7})$. **2.11.** $[2, 18)$. **2.12.** $(-\log_2 9, +\infty)$. **2.13.** $(-\infty, \log_5 3)$. **2.14.** $\left(\frac{1}{2}\log_5 6, \log_6 5\right)$. **2.15.** $[-1, 3)$. **2.16.** $(2, +\infty)$. **2.17.** $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{5}{8}, +\infty\right)$. **2.18.** $[0, 4]$. **2.19.** $(-\infty, 0) \cup (\log_3 2, 1)$ **2.20.** $(-1/2+k, k] \cup [1/4+k, 1/2+k)$, $k \in \mathbf{Z}$.

Sec. 3. **3.1.** $\left(-\frac{3}{2}, -1\right)$. **3.2.** $(3, +\infty)$. **3.3.** $\left[-\frac{1}{2}, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$. **3.4.** $(-\infty, -1)$. **3.5.** $\left[\frac{1}{4}, 2\right]$. **3.6.** $(-\sqrt{3}, \sqrt{3})$. **3.7.** $\left(\frac{1}{2}, 4\right)$. **3.8.** $\left(-\frac{3}{2}, -\frac{23}{16}\right)$. **3.9.** $(4, 7)$. **3.10.** $(1, +\infty)$. **3.11.** $(3, 7)$. **3.12.** $(10^{-2}, 10)$. **3.13.** $[-2, 6) \cup \left[\frac{20}{3}, 7\right)$.

3.14. $\left(0, \frac{1}{3}\right] \cup [9, +\infty)$. **3.15.** $(1, 2] \cup [3, 4)$. **3.16.** $(-7, 1)$. **3.17.** $(-5, -4) \cup (-3, -1)$. **3.18.** $(0, 1] \cup [2, +\infty)$. **3.19.** $[1, +\infty)$. **3.20.** $(-\infty, \log_3 2)$. **3.21.** $\left(-\frac{3}{2}, -1\right) \cup \left(\frac{\sqrt[3]{10}-3}{2}, \frac{\sqrt[3]{100}-3}{2}\right)$. **3.22.** $(0, 2) \cup (4, +\infty)$. **3.23.** $(0, 1) \cup (2, +\infty)$. **3.24.** $\left(0, \frac{\sqrt{17}-3}{2}\right]$. **3.25.** $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ **3.26.** $(1, 3) \cup (3, 5)$. **3.27.** $\left[\frac{\sqrt{5}-3}{2}, 1\right)$. **3.28.** $(2^{-15}, 2^{-9}] \cup [2^9, +\infty)$. **3.29.** $(-2, -1) \cup [0, \infty)$. **3.30.** $(0, 1) \cup \left(\sqrt{6}-1, \frac{5}{2}\right)$. **3.31.** $(0, 1) \cup [4, 2^{1+\sqrt{3}}]$. **3.32.** $\left(-1, \frac{1}{2}\right] \cup \left[1, \frac{5}{2}\right)$. **3.33.** $[2^{-\sqrt{1/8}}, 1) \cup (1, \sqrt[3]{2})$. **3.34.** $\left(0, \frac{1}{2}\right] \cup (1, +\infty)$. **3.35.** $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right) \cup \left(\frac{\sqrt{3}}{2}, 1\right)$. **3.36.** $\left(\frac{1}{2}, 1\right) \cup \left(1, \frac{3}{2}\right) \cup \left(\frac{3}{2}, 2\right)$. **3.37.** $(-4, -3) \cup [1, +\infty)$. **3.38.** $(1, 2)$. **3.39.** $\left(0, \frac{3-\sqrt{5}}{2}\right] \cup \left[\frac{3+\sqrt{5}}{2}, +\infty\right)$. **3.40.** $\left(\frac{3+\sqrt{5}}{2}, 3\right]$. **3.41.** $(-\infty, -7) \cup (-5, -2] \cup [4, +\infty)$. **3.42.** $(-\infty, 0] \cup [\log_6 5, 1)$. **3.43.** $\left(2-\sqrt{2}, \frac{3}{4}\right] \cup \left[\frac{13}{4}, 2+\sqrt{2}\right)$. **3.44.** $\left(0, \frac{1}{10^{\sqrt{\lg 5}}}\right) \cup (10^{\sqrt{\log 5}}, +\infty)$. 3.45. $\left(\frac{1}{2}\log_2 7, \log_2 3\right]$. 3.46. $\left(-1, -\frac{2}{\sqrt{5}}\right) \cup \left(\frac{2}{\sqrt{5}}, 1\right)$. 3.47. $\left(\frac{1}{2}, \frac{3}{2}\right]$. 3.48. $(1000, +\infty)$. 3.49. $\left(\frac{1}{\sqrt[3]{3}}, 1\right) \cup (9, +\infty)$. 3.50. $\left(\frac{\sqrt{5}+1}{2}, 2\right)$. 3.51. $(1, 2)$. 3.52. $(0, 1) \cup (1, \sqrt[10]{10})$. 3.53. $(3, \pi) \cup \left(\pi, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 5\right)$. 3.54. $\left(2\pi k, \frac{\pi}{6}+2\pi k\right) \cup \left(2\pi k+\frac{5\pi}{6}, (2k+1)\pi\right)$, $k \in \mathbf{Z}$. 3.55. $\left(\frac{1}{2}, \frac{3}{4}\right)$. 3.56. $\left(\pi k+\frac{\pi}{4}, \pi k+\frac{\pi}{3}\right]$, $k \in \mathbf{Z}$. 3.57. $\left(2\pi k+\arcsin\frac{2}{3}, 2\pi k+\frac{\pi}{2}\right) \cup \left(2\pi k+\frac{\pi}{2}, (2k+1)\pi-\arcsin\frac{2}{3}\right)$, $k \in \mathbf{Z}$. 3.58. $\left(\frac{\pi}{2}, 2\right) \cup \left(-\frac{\pi}{6}, 1\right] \cup \left(2\pi k-\frac{\pi}{6}, \frac{\pi}{2}+2\pi k\right)$, $k \in \mathbf{Z}$, $k \neq 0$. **3.59*.** $x=1$, $x=2$.

Hint. Find the domain of definition of the unknown x and verify the validity (or nonvalidity) of the given inequality for every integer from the domain of definition. **3.60.** $x = -1$, $x = 0$, $x = 1$, $x = 2$. **3.61.** 6, 7, 8.

Sec. 4. **4.1.** $x_{1,2} = \pm\frac{1}{2}(-1+\sqrt{1-4a})$ for $a<0$, $x=0$ for $a=0$, for $a>0$ the equation has no solutions. **4.2.** $a>\frac{11}{9}$. **4.3.** $2\sqrt{2}\leqslant a<\frac{11}{3}$. **4.4.** $\frac{-7-\sqrt{45}}{2}\leqslant m\leqslant -4+2\sqrt{3}$. **4.5.** $m>1$. **4.6.** For no m. **4.7.** $\frac{1}{2}<a\leqslant 1$. **4.8.** $-3\leqslant a\leqslant 3$. **4.9.** $-\frac{1}{\sqrt{2}}<a<\frac{1}{\sqrt{2}}$. **4.10.** $p=-\frac{1}{2}$. **4.11.** $\alpha_1=6$, $\alpha_2=-\frac{3}{2}$. **4.12.** (0, 0), (1, −2). **4.13.** (a) For $a<0$ there are no solutions, for $a=0$ there are three solutions, for $0<a<1$ there are four solutions, for $a=1$ there are two solutions, for $a>1$ there are no solutions. (b) For $a<0$ there are no solutions, for $a=0$ there are two solutions, for $0<a<4$ there are four solutions, for $a=4$ there are three solutions, for $a>4$ there are two solutions. **4.14.** $\frac{1}{2}<a<1$. **4.15.** $a<-2$. **4.16.** For $|a|\leqslant\sqrt{2}$ the inequality has no solutions. For $|a|<\sqrt{2}$ the solution is $a-\sqrt{a^2-1}+1<x<a+\sqrt{a^2-1}-1$. **4.17.** For $a\leqslant 0$ and $a\geqslant 4$ the inequality has no solutions, for $0<a<2$ the solution is $-a\leqslant x\leqslant a$, for $a=2$ the solution is $-2<x<2$, for $2<a<4$ the solution is $-\frac{a}{2}\sqrt{a(4-a)}<x<\frac{a}{2}\sqrt{a(4-a)}$ **4.18.** $a\leqslant 50$. **4.19.** $k<\frac{1}{2}$, $k>\frac{3}{2}$. **4.20.** $-\frac{1}{2}<a<1$. **4.21.*** $-\frac{13}{4}<a<3$. *Hint*. Writing the inequality in the form $3-x^2>|x-a|$, construct and investigate the graphs of the functions appearing on the left-hand and right-hand sides of the inequalities. **4.22.** $\alpha=1$, $\alpha=\frac{7}{4}$. **4.23.** $\alpha=1$, $\alpha=0$, **4.24.** $\alpha=100$. **4.25.** $|h|\geqslant\sqrt{5}+2$, $|h|\leqslant\sqrt{5}-2$. **4.26.** $h>\frac{5}{2}$. **4.27.** $h=(2k+1)\pi$, $k\in\mathbf{Z}$. **4.28.** $m=-2$, $n=-4$. **4.29.** $x=\log_2 a$ for $0<a\leqslant 1$. For $a\leqslant 0$ and $a>1$ the equation has no solutions. **4.30.** $x=\pm\log_{12}(1+\sqrt{1-a}$ for $a\leqslant 1$. For $a>1$ there are no solutions. **4.31.** $0<a<\frac{1}{\sqrt[3]{36}}$. **4.32.** $a\leqslant-\sqrt{7}$, $a\geqslant\sqrt{7}$. **4.33.** For $a=0$ there are no solutions, $x<-2+\log_3 a$ for $a>0$, $x<\log_3(-a)$ for $a<0$. **4.34.** $\alpha\geqslant 2$. **4.35.** $\alpha\geqslant 1$.

4.36. $2<c\leqslant 3$. 4.37. $0<c\leqslant 8$. 4.38. $\frac{-1-\sqrt{17}}{2}<a<\frac{-1-\sqrt{5}}{2}$, $\frac{-1+\sqrt{5}}{2}<a<\frac{-1+\sqrt{17}}{2}$. 4.39. $a<-2$. 4.40. $-3<x<-1$. 4.41. $x>3$. 4.42. (1, 0). 4.43. $x=-\arcsin\frac{2\cos a}{\sqrt{1+4\cos^2 a}}+(-1)^n\arcsin\frac{2}{\sqrt{1+4\cos^2 a}}+\pi n$, k, $n\in \mathbf{Z}$, for $2\pi k-\frac{\pi}{6}\leqslant a\leqslant\frac{\pi}{6}+2\pi k$ and $\frac{5\pi}{6}+2\pi k\leqslant a\leqslant\frac{7\pi}{6}+2\pi k$. 4.44. $x=(-1)^k\times \arcsin 10^{a-\sqrt{2(a^2-1)}}+\pi k$ for $a<-\sqrt{2}$ or $a\geqslant\sqrt{2}$, $x=(-1)^k\times\arcsin 10^{a\pm\sqrt{2(a^2-1)}}+\pi k$ for $-\sqrt{2}\leqslant a\leqslant -1$, for $-1<a<\sqrt{2}$ the equation has no solutions. 4.45. $b<-2-\sqrt{8}$, $b>2$. 4.46. $-3\leqslant a\leqslant -2$, $x=\pm\frac{1}{2}\arccos 2a+5+\pi k$, $k\in\mathbf{Z}$. 4.47. $a_{1,2}=\pm 2$. 4.48. $x=\pi k+(-1)^k\arcsin\frac{b}{b-1}$ for $b<\frac{1}{2}$ $\left(b\neq\frac{1}{3},\ b\neq 0, b\neq -1\right)$. For $b\geqslant 1/2$ the equation has no solutions. 4.49. For $a\leqslant 0$ the equation has no solutions, $x=\pi k\pm\arcsin 2^{-\sqrt{\frac{\log_2 3}{2a}}}$, $k\in\mathbf{Z}$, for $a>0$. 4.50. $0<x<1$ for $a\geqslant 1$, $0<x<\arcsin a$, $1<x<\pi/2$ for $a\leqslant\sin 1$, $0<x<1$, $\arcsin a<x<\pi/2$ for $\sin 1<a<1$. 4.51. (0, 0), (1, 0). 4.52. $k=1$, $x=\tan\frac{\pi}{4}(1-\sqrt{7})$, $y=\cos\frac{\pi}{4}(1+\sqrt{7})$. 4.53. $a<-1$, $a=0$.

Chapter 4

Sec. 1. 1.33. Tan $\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ for $\alpha\in\left[0,\ \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}+\pi\right]$; $\cot\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ for $\alpha\in\left[\pi,\ \frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},\ 2\pi\right]$. 1.34. $2\tan^3\alpha$ for $\alpha\neq\frac{\pi}{2}+\pi n$.

Sec. 2. 2.1. -1. 2.2. $\frac{1}{64}$. 2.3. $\frac{\sqrt{6}-\sqrt{2}}{4}$. 2.4. $\frac{1}{4}$. 2.5. 1. 2.6. 4. 2.7. $\frac{3}{2}$. 2.8*. $\frac{\sqrt{5}-1}{4}$. *Hint.* Use the equation $\cos(3\cdot 18°)=\sin(2\cdot 18°)$. 2.9*. $\frac{1}{8}\left(\sqrt{3}\cdot\sqrt{10+2\sqrt{5}}-\sqrt{5}+1\right)$.

Hint. Use the equation $\sin 42° = \sin (60° - 18°)$. 2.10. 0.96. 2.11. 2. 2.12. $a^4 - 4a^2 + 2$. 2.13. $\frac{3n - n^3}{2}$. 2.14. $\frac{1-m}{1+m}$. 2.15. $\frac{a^2 - b^2}{a^2 + b^2}$. 2.16. 3. 2.17. $\frac{4a}{a^2 + b^2 + 2b}$. 2.18*. $-\sqrt{3}$. *Hint.* Considering the conditions as a system of trigonometric equations, find α and β as the roots of that system which belong to the indicated intervals. 2.19. $\frac{\cot \beta}{\cot \alpha} = \frac{p+q}{p-q}$. 2.20. $\tan \frac{\alpha}{2} = 2$, $\tan \frac{\alpha}{2} = -\frac{1}{3}$. 2.21. -2, $-\frac{1}{2}$, $\frac{1}{2}$, 2. 2.22. $4x^3 - 3x - m = 0$. 2.23. $\tan \frac{\alpha}{2} = m$. 2.24. $\frac{2}{a}$. 2.25. $\frac{\sqrt{15}}{8}$. 2.26. $\frac{\sqrt{3}}{2}$. 2.27. $\frac{77}{85}$. 2.28. $4\sqrt{5}$. 2.29*. $\pi - 2$. *Hint.* $\sin 2 = \sin (\pi - 2)$ and $\pi - 2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. 2.30. $\frac{1 + 2\cdot\sqrt{30}}{2\sqrt{2} - \sqrt{15}}$. 2.31. $\frac{\sqrt{2}}{2}$. 2.32. $\frac{4\sqrt{2} + \sqrt{5}}{9}$. 2.33. $\frac{3}{5} + \frac{\sqrt{13}}{13}$.

2.42**. Yes, it does. *Solution.* Designating $x = \cos y$, we transform the argument of the second term by the formula $\frac{1}{2}\cos y + \frac{\sqrt{3}}{2}\sin y = \cos\left(\frac{\pi}{3} - y\right)$. For the variable y the equation assumes the form $\arccos(\cos y) + \arccos\left[\cos\left(\frac{\pi}{3} - y\right)\right] = \frac{\pi}{3}$ or, after simplification, the form $y + \left|y - \frac{\pi}{3}\right| = \frac{\pi}{3}$. For $y \in \left[0, \frac{\pi}{3}\right]$ the given equation turns into an identity.

2.43*. Yes, it does. *Hint.* See the Hint to problem 2.42. 2.44. $\frac{1}{2}\left[n \cos \alpha - \frac{\cos [(n+2)\alpha] \cos n\alpha}{\sin \alpha}\right]$. 2.45. $\frac{\cos [(n+2)\alpha] \sin n\alpha}{\sin \alpha}$. 2.46. $\frac{1}{2^n} \cot \frac{\alpha}{2^n} - 2\cot 2\alpha$. 2.47. $\frac{1}{2}$. 2.48. $\frac{n}{2}$. 2.49. $\frac{1}{2}$. 2.50. 0. 2.51. $\tan \frac{(n+1)\alpha}{2}$.

Sec. 3. Unless otherwise specified, the letters n, l, m, k encountered in the answers assume integral values. 3.1. $-\frac{\pi}{2} + 2\pi n$, $(-1)^n \frac{\pi}{6} + \pi n$. 3.2. πn. 3.3. $\pm\frac{\pi}{3} + \pi n$. 3.4. $\frac{\pi}{2} + k\pi$, $\pm\frac{\pi}{3} + 2k\pi$. 3.5.* $\left[\frac{\pi}{6} + \pi n, \frac{\pi}{3} + \pi n\right] \cup \left[\frac{2\pi}{3} + \pi n, \frac{5\pi}{6} + \pi n\right]$ *Hint.* Simplify the expression isolating perfect squares in the radicands.

3.6. $\arctan \frac{1+\sqrt{5}}{4}+\pi n$, $-\arctan \frac{\sqrt{5}-1}{4}+\pi n$. 3.7. $\arctan \frac{1}{2}+k\pi$, $\arctan \frac{7}{2}+k\pi$. 3.8. $\frac{\pi}{2}+2k\pi$, $\pi(2k+1)$. 3.9. $\pm \frac{\pi}{3}+k\pi$. 3.10. $-\frac{\pi}{4}+\pi k$, $\pm \arctan \frac{\sqrt{2}}{2}+\pi k$, $\frac{2\pi k}{5}$, $\frac{\pi}{11}(2k+1)$. 3.11. $\frac{\pi}{2}+2k\pi$, $(-1)^n \frac{\pi}{6}+\pi n$. 3.12. $\arctan \frac{1}{2}+k\pi$. 3.13. $\frac{\pi}{4}\times(4k+1)$. 3.14. $\pi(2k+1)$, $\frac{\pi}{2}(4k-1)$. 3.15. $\frac{\pi}{12}(6k\pm 1)$. 3.16. $\frac{\pi k}{2}$.

3.17*. $x=\frac{k\pi}{2}\pm\frac{1}{2}\arcsin\left(2\sqrt{\frac{a-1}{2a-3}}\right)$ for $a\in\left[\frac{1}{2}, 1\right]$, for the other values of a there are no solutions. *Hint.* The values of a for which the equation has solutions can be found from the condition $|\sin x|\leqslant 1$. 3.18. $\frac{\pi}{4}+k\pi$, $\frac{\pi}{12}+\frac{k\pi}{7}$. 3.19. $-\frac{\pi}{108}+\frac{\pi k}{9}$, $\frac{7\pi}{24}+\frac{\pi k}{2}$. 3.20. $-\frac{\pi}{20}-\frac{2\pi n}{5}$, $\frac{3\pi}{100}+\frac{2\pi n}{25}$. 3.21. $\frac{21}{16}\pi$, $\frac{11\pi}{8}$. 3.22. $\frac{\pi}{12}+\pi n$, $\frac{5\pi}{36}+\frac{\pi n}{3}$. 3.23. No solutions. 3.24. $\frac{\pi}{2}+2\pi n$. 3.25. $\arctan \frac{3}{2}+\pi k$, $\pi k-\arctan \frac{1}{2}$. 3.26. πk, $\pm \arctan\sqrt{2}+\pi k$, $\pm\frac{\pi}{3}+\pi k$. 3.27. $\frac{\pi}{2}(4k+1)$. 3.28. $\frac{\pi}{4}+2k\pi$. 3.29. $2k\pi$, $\frac{\pi}{2}+2k\pi$. 3.30. $\frac{3\pi}{4}+\pi n$, $(-1)^n\frac{\pi}{4}-\frac{\pi}{4}+\pi n$. 3.31*. $x=\frac{\pi}{4}+2k\pi\pm\arccos\frac{a+2}{a\sqrt{2}}$ for $a\in(-\infty, -1)\cup(-1, -2(\sqrt{2}-1)\cup[2(\sqrt{2}+1), +\infty)$. *Hint.* Represent it as an equation with respect to the variable $t=\sin x+\cos x$. The values of a for which the equation has solutions can be found from the condition $|t|\leqslant\sqrt{2}$.

3.32. $\frac{k\pi}{3}$. 3.33. $x=k\pi\pm\frac{1}{2}\arccos\frac{\sqrt{1+16a^2}-1}{4a}$ for $a\neq 0$, $x=k\pi\pm\frac{\pi}{4}$ for $a=0$. 3.34. $\frac{2k+1}{8}\pi$. 3.35. $\frac{\pi}{4}+\frac{\pi n}{2}$. 3.36. πn. 3.37. $\frac{\pi}{16}(2k+1)$, $\frac{\pi}{3}(3k\pm 1)$. 3.38. $\frac{\pi k}{2}$, $\frac{\pi k}{9}$. 3.39. $\frac{\pi}{4}(2k+1)$, $\frac{\pi}{5}(2k+1)$, $\frac{\pi}{7}(2k+1)$. 3.40. $\frac{\pi}{8}(2k+1)$. 3.41. $\frac{\pi}{4}+\frac{\pi n}{2}$, $\frac{\pi}{2}+\pi n$. 3.42. $\pm\frac{\pi}{6}+\frac{\pi k}{2}$. 3.43. $\arctan 3+\pi k$, $\arctan\frac{1}{4}+\pi k$.

3.44. $\frac{\pi}{4}(8k+1)$. 3.45. $\frac{\pi}{4}+k\pi$, $\frac{\pi}{12}+2k\pi$, $\frac{5\pi}{12}+2k\pi$. 3.46. $2k\pi$, $\frac{\pi}{2}+2k\pi$. 3.47. $\frac{\pi}{2}+k\pi$, $-\frac{\pi}{4}+k\pi$, $2k\pi$. 3.48. $\frac{\pi}{4}+\frac{k\pi}{2}$, $\frac{\pi}{2}+k\pi$. 3.49. $(-1)^n \arcsin\frac{\sqrt{3}-1}{2}+\pi n$. 3.50. $\frac{\pi k}{3}$. 3.51. $\frac{3\pi}{4}+\pi n$, $\frac{3}{16}\pi+\frac{\pi n}{4}$. 3.52. $\frac{\pi n}{8}$, $\frac{\pi}{8}(2n+1)$. 3.53. $-\frac{\pi}{4}+k\pi$, $\pm\frac{2\pi}{3}+2k\pi$. 3.54. $k\pi$, $(-1)^{k+1}\frac{\pi}{6}+k\pi$, $\pm\frac{\pi}{3}+2k\pi$. 3.55. $-\frac{\pi}{4}+k\pi$, $\pm\frac{\pi}{3}+\pi$. 3.56. $\frac{\pi}{10}(2k+1)$, $\frac{\pi}{6}(2k+1)$. 3.57. $\frac{\pi k}{2}-1$, $\frac{\pi}{10}(2k+1)-1$. 3.58. $\frac{\pi k}{4}$. 3.59. $\frac{\pi n}{2}$, $\frac{\pi}{8}(2n+1)$. 3.60. $\frac{2}{5}\pi n$, $\pi(2n+1)$, $\frac{\pi}{2}(2n+1)$. 3.61. $\frac{\pi}{8}(2n+1)$, $\frac{\pi n}{3}$. 3.62. $\frac{\pi n}{2}$, $\frac{\pi}{8}(2n+1)$. 3.63. $\frac{\pi n}{4}+(-1)^n\frac{\pi}{24}$. 3.64. $\frac{\pi n}{3}$. 3.65. $\frac{\pi}{4}(4n-1)$, πn. 3.66. $\frac{\pi}{4}(2n+1)$, $2\pi n$, $\frac{\pi}{2}(4n+1)$. 3.67. $\frac{\pi}{2}(2n+1)$, $\pi n\pm\frac{1}{2}\arccos(\sin^2\alpha)$. 3.68. $\frac{\pi}{2}(2k+1)$, $(-1)^k\frac{\pi}{6}+\pi k$, $\frac{\pi}{4}(2k+1)$, $\frac{\pi}{14}(2k+1)$. 3.69. $\frac{\pi k}{8}$. 3.70. $\frac{\pi}{6}\times(2k+1)$, $(-1)^{k+1}\frac{\pi}{12}+\frac{\pi k}{3}$. 3.71. $\pm\frac{\pi}{3}+\pi n$. 3.72. $-\frac{\pi}{4}+\pi n$, $2\pi n$, $-\frac{\pi}{2}+2\pi n$, $\frac{\pi}{4}(2n+1)$. 3.73. $-\frac{\pi}{4}+\pi n$, 3.74. $\frac{\pi}{4}+\frac{\pi n}{2}$, $\frac{\pi}{16}+\frac{\pi n}{8}$. 3.75. $\frac{\pi}{2}+\pi n$, $(-1)^n\frac{\pi}{24}+\frac{\pi n}{4}$. 3.76. $\frac{\pi}{4}+\frac{\pi n}{2}$, $\pm\frac{\pi}{3}+\pi n$. 3.77. πn, $\frac{\pi}{4}+\frac{\pi n}{2}$, $\pm\frac{1}{2}\arccos\left(-\frac{3}{4}\right)+\pi n$. 3.78. $-\frac{\pi}{4}+\pi n$, $(-1)^n\frac{\pi}{12}+\frac{\pi n}{2}$. 3.79. πn, $\frac{\pi}{20}+\frac{\pi n}{10}$. 3.80. $\pm\frac{\pi}{4}+2\pi n$, $\pm\frac{\pi}{6}+\pi n$. 3.81. $\frac{\pi}{2}+\pi n$, $\frac{\pi}{12}+\frac{\pi n}{6}$. 3.82. $\frac{\pi}{8}+\frac{\pi n}{4}$. 3.83. $\frac{\pi}{4}+\frac{\pi n}{2}$. 3.84. $\pm\frac{\pi}{3}+\pi n$. 3.85. $\pm\frac{\pi}{6}+\pi n$. 3.86*. $\pm\arccos\left(-\frac{\sqrt{2}}{4}\right)+2\pi n$. *Hint.* Introduce the variable $t=x+\frac{\pi}{4}$ and set up an equation for $y=\sin t+\cos t$. 3.87. $\frac{\pi n}{3}$.

3.88. $(-1)^{n+1}\frac{\pi}{6}+\pi n$. 3.89. $\frac{\pi}{2}+\pi n$. 3.90. $\frac{\pi n}{5}$, $-\frac{\pi}{2}+2\pi n$. 3.91. $(-1)^n\frac{\pi}{30}+\frac{\pi n}{5}$, $\frac{\pi}{16}+\frac{\pi n}{4}$, $-\frac{\pi}{4}+\pi n$. 3.92. $\frac{\pi}{4}+\frac{\pi n}{2}$. 3.93. $(-1)^n\frac{\pi}{12}+\frac{\pi n}{2}$, $\frac{\pi}{14}+\frac{2\pi n}{7}$, $\frac{\pi}{6}+\frac{2\pi n}{3}$. 3.94. πn, $-\frac{\pi}{4}+\pi n$. 3.95. $\pm\frac{1}{2}\arccos(\sqrt{3}-1)+\pi k$. 3.96. $\frac{\pi k}{2}$, $\frac{\pi}{8}+\frac{\pi k}{4}$. 3.97. $\frac{\pi}{2}+\frac{2\pi n}{3}$, $n\neq 3l$. 3.98*. $\left(\frac{\pi}{2}+2\pi k,\ 2\right)$. *Hint.* Estimate the left-hand and right-hand sides of the equation. 3.99. $-\frac{\pi}{4}+\pi k$, $-\frac{\pi}{3}+2\pi k$, $-\frac{\pi}{6}+2\pi k$. 3.100. $\frac{\pi}{4}+k\pi$. 3.101. $\frac{\pi}{4}+\frac{\pi k}{2}$, $(-1)^k\frac{\pi}{30}+\frac{\pi k}{5}$. 3.102. $\frac{\pi k}{8}$, $\frac{2\pi k}{9}$. 3.103. $\frac{3\pi}{8}+\frac{\pi k}{2}$, $-\frac{\pi}{4}+\pi k$. 3.104. $-\frac{\pi}{4}+\pi k$, $(-1)^n\frac{\pi}{12}+\frac{\pi k}{2}$. 3.105. $\frac{\pi}{4}+\frac{\pi n}{2}$. 3.106. $-\frac{\pi}{4}+\pi k$, $\frac{3}{8}\pi+\frac{\pi k}{2}$. 3.107. $-\frac{\pi}{2}+2\pi k$, $(-1)^k\frac{\pi}{6}+\pi k$. 3.108. $(-1)^k\frac{\pi}{6}+\pi k$, $\frac{\pi}{2}+\pi k$, 3.109. $-\frac{\pi}{12}+\frac{\pi}{2}k$, $\frac{\pi}{6}+\pi k$, 3.110. $-\frac{\pi}{4}+\pi k$, πk. 3.111. $\frac{\pi}{4}+\frac{\pi}{2}k$, πk. 3.112*. $\left(\arccos\frac{5}{13}+2\pi n,\ 2\right)$. *Hint.* Estimate the left-hand and right-hand sides of the equation. 3.113. $\frac{1}{2}+\frac{7}{4}\pi$. 3.114. $-\frac{3}{4}\pi$, $-\frac{\pi}{2}$, $-\frac{\pi}{4}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$. 3.115. $\frac{\pi}{2}$, π, 2π, $\frac{5\pi}{2}$. 3.116. $\pm\frac{\pi}{2}$, $\frac{2\pi}{11}k$, $k=0, \pm 1, \pm 2, \pm 3$. 3.117*. $\frac{-5\pm\sqrt{25+2\pi(2k+1)}}{2}$, $k=-2, -1, 0, 1, 2, 3, \ldots$ *Hint.* The values of k satisfying the equation must be obtained from the condition of the existence of real roots.

3.118*. $\frac{(4m+1)\pi\pm\sqrt{(4m+1)^2\pi^2-240}}{12}$, $\frac{-(4n+1)\pi\pm\sqrt{(4n+1)^2\pi^2+240}}{12}$, n is any number, m is any number except for ± 1 and 0. *Hint.* See the Hint to problem 3.117*. 3.119*. $\frac{1+(4k+1)\pi\pm\sqrt{1+2(4k+1)\pi}}{2}$, k is any integral non-negative number. *Hint.* See the Hint to problem 3.117*.

3.120**. *Solution.* The initial equation can be written in the form

$$\cos\left(\frac{\pi}{4}+\frac{\cos x+\sin x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\cos x-\sin x}{2}\right)=0.$$

Solving the equation $\cos\left(\frac{\pi}{4}+\frac{\cos x+\sin x}{2}\right)=0$ by the method of an auxiliary angle, we have $\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)=\frac{\pi}{2}(4k+1)$. Estimating the left-hand and right-hand sides of the last equation, we infer that it has no solutions for all $k\in \mathbb{Z}$. We can prove by analogy that the equation $\cos\left(\frac{\pi}{4}+\frac{\cos x-\sin x}{2}\right)=0$ has no solutions either.

3.121*. $2k\pi\pm\varphi$, $\frac{\pi}{2}(4k\pm 1)\mp\varphi$, $\varphi=\frac{1}{2}\arcsin\frac{9}{16}$, the signs are either both upper or both lower. *Hint.* See the solution of 3.120**. 3.122*. $\frac{1}{2}\operatorname{arccot}\left(k+\frac{1}{4}\right)+\pi n$, $\frac{(-1)^n}{2}\arcsin\frac{4}{4l+1}+\pi n$, $l\neq -1$, $l\neq 0$. *Hint.* See the solution of 3.120**. 3.123. $-\frac{5\pi}{4}+\frac{3}{2}$, $-\pi+1$, $-\frac{3\pi}{4}+\frac{3}{2}$, $-\frac{\pi}{4}+\frac{3}{2}$, 1, $\frac{\pi}{4}+\frac{3}{2}$.

3.124. $-\frac{\pi}{6}+2\pi n$. 3.125. (*a*) 0, $\frac{\pi}{6}$, π, (*b*) πn, $\frac{\pi}{6}+2\pi n$.

3.126. $\arctan\sqrt{\frac{7}{2}}+\pi n$.

3.127**. $\frac{3\pi}{4}+2\pi n$. *Solution.* The equation is equivalent to the equation $\sin x=\frac{\sqrt{2}}{2}$ whose roots are $x=\frac{\pi}{4}+2\pi n$ and $x=\frac{3\pi}{4}+2\pi n$, 2π is the least common multiple of the periods of the equation and the inequality and, consequently, the values $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ must be verified. Substituting $\frac{\pi}{4}$ into the inequality, we obtain a numerical inequality $2^{\frac{\cos\frac{7\pi}{4}}{\cos 3+\sin 3}}>2^{\cos\frac{\pi}{2}}$. Since $2^{\cos\frac{\pi}{2}}=1$, $\cos\frac{7\pi}{4}>0$, $\cos 3+\sin 3=\sqrt{2}\cos\left(\frac{\pi}{4}-3\right)<0$, the numerical inequality obtained is not valid. We make sure by analogy that $\frac{3\pi}{4}$ satisfies the inequality.

3.128*. $\frac{3\pi}{8}+\pi k$. *Hint.* See the solution of 3.127**. 3.129*. $\frac{5\pi}{8}+\pi k$. *Hint.* See the Hint to 3.128*. 3.130*. $-\frac{3\pi}{4}+2\pi k$. *Hint.* See the Hint to 3.128*. 3.131. $\frac{\pi}{2}(4k+1)$. 3.132. $\left(\frac{\pi}{2}\times(4k+1),\ \frac{\pi}{2}(4n+1)\right)$, $\left(\frac{\pi}{2}(4k-1),\ \frac{\pi}{2}(4n-1)\right)$. 3.133. $\frac{\pi}{4}\times(4k+1)$.

3.134**. $\left(\frac{\pi}{3}(6m+1), \frac{\pi}{3}(6k+1)\right)$, $\left(\frac{\pi}{3}(6m-1), \frac{\pi}{3}(6k-1)\right)$.

Solution. With respect to the variables $u=\tan\frac{x}{2}$ and $v=\tan\frac{y}{2}$, the initial equation assumes the form

$$\frac{1-u^2}{1+u^2}+\frac{1-v^2}{1+v^2}-\frac{(1-u^2)(1-v^2)}{(1+u^2)(1+v^2)}+\frac{4uv}{(1+u^2)(1+v^2)}=\frac{3}{2}$$

or, after simplification, the form $u^2+v^2-8uv+9u^2v^2+1=0$, the last equation is equivalent to the equation $(u-v)^2+(3uv-1)^2=0$. Returning to the initial unknowns, we have

$$\tan\frac{x}{2}=\frac{\sqrt{3}}{3}, \qquad \tan\frac{x}{2}=-\frac{\sqrt{3}}{3},$$

or

$$\tan\frac{y}{2}=\frac{\sqrt{3}}{3} \qquad \tan\frac{y}{2}=-\frac{\sqrt{3}}{3}.$$

3.135. $\left(\frac{\pi}{2}(4k+1), \frac{\pi}{2}(4l+1)\right)$. 3.136. $\left(\frac{\pi}{2}(4k-1), \frac{\pi}{2}\times(4l-1), \frac{\pi}{2}(4m-1)\right)$. 3.137. $\frac{\pi}{4}(8k+1)$. 3.138. $\left(\frac{\pi}{2}+m\pi, \frac{\pi}{2}+l\pi\right)$.

Sec. 4. 4.1. $\left(\frac{\pi n}{2}-\pi k-\frac{\pi}{4}, \frac{\pi n}{2}+\pi k+\frac{\pi}{4}\right)$. 4.2. $\left(\pi m \pm \frac{\varphi+\psi}{2}, \pi n \pm \frac{\varphi-\psi}{2}\right)$, $\left(\pi m+\frac{\pi}{2}\pm\frac{\varphi-\psi}{2}, \pi n+\frac{\pi}{2}\pm\frac{\varphi+\psi}{2}\right)$, in each solution the signs are either both upper or both lower, $\psi=\arcsin 0.535$, $\psi=\arcsin 0.185$. 4.3 $\left(\pi k+\pi n+\frac{\pi}{3}, \frac{\pi k}{2}-\pi n+\frac{\pi}{3}\right)$, $\left(\pi k+\pi n-\frac{\pi}{3}, \pi k-\pi n-\frac{\pi}{3}\right)$. 4.4. $\left(\frac{7\pi}{24}+\pi k+\pi n, \frac{\pi}{24}+\pi k-\pi n\right)$, $\left(\frac{\pi}{24}+\pi k+\pi n, \frac{7\pi}{24}+\pi k-\pi n\right)$, $\left(-\frac{\pi}{24}+\pi k+\right.$

$\pi n,\ -\frac{7\pi}{24}+\pi k-\pi n\Big)$, $\Big(-\frac{7\pi}{24}+\pi k-\pi n,\ -\frac{\pi}{24}+\pi k+\pi n\Big)$.
4.5. $\Big(\frac{\pi}{2}+2\pi l,\ \frac{\pi}{6}+2\pi p\Big)$, $\Big(-\frac{\pi}{6}+2\pi l,\ -\frac{\pi}{2}+2\pi p\Big)$.
4.6. $(2\pi m,\ 2\pi n)$, $\Big(\frac{2\pi}{3}(3m\pm 1),\ \frac{2\pi}{3}(3n\mp 1)\Big)$, $\Big(\pi m+\pi\pm\varphi,\ \pi+\frac{\pi}{6}\pm\varphi\Big)$, $\Big(\pi-\frac{\pi}{6}\pm\varphi,\ \pi-\frac{\pi}{6}\mp\varphi\Big)$, $\varphi=\arccos\frac{\sqrt{3}-\sqrt{11}}{4}$.
4.7. $\Big(2\pi k+\frac{\pi}{4},\ \frac{4\pi l}{3}+\frac{\pi}{6}\Big)$, $\Big(-\frac{\pi}{4}+2\pi k,\ \frac{4\pi l}{3}-\frac{\pi}{2}\Big)$, $\Big(-\frac{3\pi}{4}+2\pi k,\ \frac{4\pi l}{3}-\frac{\pi}{6}\Big)$, $\Big(\frac{3\pi}{4}+2\pi k,\ \frac{4\pi l}{3}-\frac{\pi}{6}\Big)$.
4.8. $(2\pi k\pm\varphi,\ 2\pi l\pm\psi)$, $(2\pi k+\pi\pm\varphi,\ 2\pi l+\pi\mp\psi)$, $\varphi=\arctan\frac{\sqrt[4]{500}}{5}$, $\psi=\arcsin\frac{\sqrt[4]{500}}{5}$ and either both upper signs or both lower signs are taken. **4.9.** $(2\pi n,\ 2\pi k+\pi)$. **4.10.** $(\pi k,\ 2\pi n)$, $\Big(\arccos\frac{1}{7}+2\pi k,\ -\frac{2\pi}{3}+2\pi n\Big)$, $\Big(-\arccos\frac{1}{7}+2\pi k,\ \frac{2\pi}{3}+2\pi n\Big)$.
4.11. $\Big((-1)^n\Big(-\frac{\pi}{4}+\pi k\Big)+\pi n,\ -\frac{\pi}{4}+\pi k\Big)$, $\Big((-1)^n\arctan\frac{1}{2}+(-1)^n\pi k+\pi n,\ \arctan\frac{1}{2}+\pi k\Big)$. **4.12.** $\Big(\frac{\pi}{2}+\pi k,\ \pi n\Big)$, $\Big(\frac{3\pi}{4}+2k\pi+\pi n,\ -\frac{\pi}{4}+\pi n\Big)$, $\Big(\frac{\pi}{2}-\arctan 2+2k\pi+n\pi,\ -\arctan 2+\pi n\Big)$. **4.13.** $\Big((-1)^{n+1}\frac{\pi}{8}-\frac{n\pi}{2},\ -\frac{1}{5}\arctan\frac{\sqrt{2}}{2}+\frac{\pi k}{5}\Big)$. **4.14.** $\Big((-1)^k\frac{\pi}{9}+\frac{\pi k}{3},\ \Big(-\frac{1}{7}\Big)\arctan\frac{\sqrt{3}}{2}+\frac{\pi n}{7}\Big)$.
4.15. $\Big(\pm\frac{3\pi}{4}+2\pi n,\ (-1)^m\frac{\pi}{6}+\pi m\Big)$. **4.16.** $\Big((-1)^n\frac{\pi}{3}+\pi n,\ \pm\frac{\pi}{4}+2\pi m\Big)$. **4.17.** $\Big(\pm\frac{\pi}{6}+2\pi n,\ (-1)^k\frac{\pi}{6}+\pi k\Big)$.
4.18. $\Big(\frac{\pi}{3}+\pi k,\ \sqrt{2}\Big)$. **4.19.** $\Big(\pm\frac{\pi}{6}+2k\pi,\ \frac{\pi}{18}+\frac{\pi n}{3}\Big)$.
4.20. $\Big(\frac{\pi}{4}(2n+1),\ \pi(2p+1)\Big)$. **4.21.** $\Big(\frac{(-1)^k}{2}\arcsin\frac{2}{5}+\frac{(-1)^n}{2}\arcsin\frac{4}{5}+(k+n)\frac{\pi}{2},\ \frac{(-1)^k}{2}\arcsin\frac{2}{5}-\frac{(-1)^n}{2}\arcsin\frac{4}{5}+(k-n)\frac{\pi}{2}\Big)$. **4.22.** $\Big(\frac{5\pi}{6},\ \frac{7\pi}{6}\Big)$, $\Big(\frac{\pi}{6},\ \frac{11\pi}{6}\Big)$, $\Big(\frac{7\pi}{6},\ \frac{7\pi}{6}\Big)$, $\Big(\frac{11\pi}{6},\ \frac{11\pi}{6}\Big)$. **4.23.** $\Big(\frac{\pi}{4}(2k+1),\ \frac{\pi}{2}+l\pi,\ \frac{\pi}{2}+m\pi\Big)$.

4.24. $\left(\varphi+\pi n,\ \frac{\pi}{3}-\varphi-\pi n\right)$, where $\varphi=\arctan\frac{-7\pm\sqrt{193}}{8\sqrt{3}}$. 4.25. $\left(\frac{\pi}{6}(6k+(-1)^k),\ \frac{\pi}{3}(6l+3\pm 1)\right)$. 4.26*. $\left(\frac{\pi}{4}+\pi k,\ \arctan 2+\pi l,\ \arctan 3+\pi m\right)$, $\left(\pi k+\arctan 2,\ \frac{\pi}{4}+\pi l,\ \pi m+\arctan 3\right)$, where $k+l+m=0$. *Hint.* Calculating the tangents of both sides of the first equation, rationalize the system with respect to $\tan x$, $\tan y$, $\tan z$. 4.27*. $\left(\arctan 2\sqrt{5}\pm\pi k,\ \arctan\sqrt{5}\pm\pi l,\ \arctan\frac{\sqrt{5}}{3}\pm\pi m\right)$, if we take the upper signs, then we have $k+l+m=0$ and if we take the lower signs, then we have $k+l+m=2$. *Hint.* See the Hint to 4.26*. 4.28. $\left(\frac{7\pi}{36}+k\pi,\ \frac{5\pi}{36}+k\pi\right)$, $\left(\pi n-\frac{11\pi}{36},\ \pi n-\frac{13\pi}{26}\right)$. 4.29. $\left(\frac{5\pi}{24}-\frac{k\pi}{2},\ \frac{\pi}{24}-\frac{k\pi}{2}\right)$. 4.30. $\left(-\frac{\pi}{6}+\frac{(-1)^k}{2}\arcsin\frac{2-3\sqrt{3}}{6}+\frac{k\pi}{2},\ \frac{\pi}{6}+\frac{(-1)^k}{2}\times\arcsin\frac{2-3\sqrt{3}}{6}+\frac{k\pi}{2}\right)$. 4.31. $\left(\frac{1}{2},-\frac{1}{2}\right)$, $\left(-\frac{1}{2},\frac{1}{2}\right)$, (1, 0), (−1, 0), (0, 1), (0, −1). 4.32. $a=2\pi l$, $\left(\pm\frac{\pi}{3}+k\pi+l\pi,\ \mp\frac{\pi}{3}-k\pi+\pi l\right)$, $a=\pi(2l+1)$, $\left(\pm\frac{\pi}{6}+\frac{\pi(2l+1)}{2}+k\pi,\ \frac{\pi(2l+1)}{2}-k\pi\mp\frac{\pi}{6}\right)$, $a\neq\pi m$: there are no solutions.

Sec. 5. 5.1. $-3\tan 1$. 5.2. $-\frac{3}{2}$. 5.3. $-\frac{1}{2}$, $\frac{\sqrt{3}}{2}$ 5.4. $\sqrt{3}$. 5.5*. $\cos a$ for $a\in(0,\pi]$, $\cos\frac{1}{2}a$ for $a\in[-2\pi,0)$, for $a\in(-\infty,2\pi)\cup(\pi,+\infty)$ there are no solutions. *Hint.* When solving the quadratic equation for the variable $z=\arccos x$, take into account that $0\leqslant\arccos x\leqslant\pi$. 5.6. $\sqrt{2}$. 5.7. $-\frac{3}{5}$. 5.8. $\frac{1}{2}$. 5.9. 1. 5.10. 0, 1. 5.11. No solutions. 5.12. −2. 5.13. 0, $\frac{\sqrt{2}}{2}$, $-\frac{\sqrt{2}}{2}$. 5.14. 0, −1, 1. 5.15. $\frac{1}{2}$. 5.16. $\frac{\sqrt{3}}{2}$. 5.17. 0, $\frac{1}{2}$, $-\frac{1}{2}$. 5.18. $-\frac{1}{2}$, 0, $\frac{1}{2}$. 5.19. $\frac{1}{2}$, 1. 5.20. [0, 1]. 5.21. [−1, 0]. 5.22. (0,1].

5.23. $\left[\frac{\sqrt{2}}{2},\ 1\right]$. 5.24. $[0,\ 1]$. 5.25. $[-1,\ 1]$. 5.26. $[0,\ \infty)$. 5.27. $(-1,\ 1)$. 5.28. $(0,\ 1)$. 5.29*. $x=2a\sqrt{1-a^2}$ for $a\in\left[-\frac{\sqrt{2}}{2};\ \frac{\sqrt{2}}{2}\right]$, for the other values of a there are no solutions. *Hint*. The domain of the permissible values of a can be found from the condition $|\,2\arcsin a\,|\leqslant\frac{\pi}{2}$. 5.30*. $x=\sqrt{1-4a^2}$ for $a\in\left[0,\ \frac{1}{2}\right]$, for the other values of a there are no solutions. *Hint*. See the hint to 5.29*.

Sec. 6. 6.1. $\left(-\frac{\pi}{6}+2\pi n,\ \frac{7\pi}{6}+2\pi n\right)$. 6.2. $\left(\arctan 2+\pi n,\ \frac{\pi}{2}+\pi n\right)$. 6.3. $(\pi n,\ \operatorname{arccot}(-3)+\pi n)$. 6.4. $\left[1-\frac{2}{3}\pi+2\pi n,\ 1-\frac{\pi}{3}+2\pi n\right]$. 6.5. $\left[0,\ \sqrt{\frac{\pi}{6}}\right]\cup\left[\sqrt{\frac{7\pi}{6}+2\pi k},\ \sqrt{\frac{13\pi}{6}+2\pi k}\right]\cup\left[-\sqrt{\frac{\pi}{6}},\ 0\right]\cup\left[-\sqrt{\frac{13\pi}{6}+2\pi k},\ -\sqrt{\frac{7\pi}{6}+2\pi k}\right]$, $k=0,\ 1,\ 2,\ \ldots$ 6.6. $\left(\frac{5\pi}{4},\ \frac{5\pi}{4}+2\pi n\right)$. 6.7*. No solutions. *Hint*. Use the inequality $|\sin x|\leqslant 1$. 6.8. $\left[-\frac{\pi}{2}+2\pi n,\ \frac{\pi}{2}+2\pi n\right]$. 6.9. $\left(\frac{\pi}{4}+\pi n,\ \frac{3\pi}{4}+\pi n\right)\cup\left(\frac{3\pi}{4}+\pi n,\ \pi(n+1)\right)$. 6.10. $\left(-\frac{\pi}{2}+\pi n,\ -\arctan 2+\pi n\right)\cup\left(-\frac{\pi}{4}+\pi n,\ \frac{\pi}{4}+\pi n\right)$. 6.11. $\left(-\frac{1}{2}\arcsin\frac{2}{3}+\pi n,\ \pi n\right)\cup\left(\frac{\pi}{2}+\pi n,\ \frac{\pi}{2}+\frac{1}{2}\arcsin\frac{2}{3}\right)$. 6.12. $\left(\frac{\pi}{6}(12k+1),\ \frac{\pi}{6}(12k+3)\right)\cup\left(\frac{\pi}{2}(4k+2),\ \frac{\pi}{2}(4k+3)\right)$. 6.13. $\left(\frac{2\pi}{3}+2\pi k,\ \frac{5\pi}{3}+2\pi k\right)$. 6.14. $\left(\frac{7\pi}{6}+2\pi k,\ \frac{11\pi}{6}+2\pi k\right)$. 6.15. $(\pi+2\pi k,\ \pi+\varphi+2\pi k]\cup[2\pi k-\varphi,\ 2\pi k)$, $\varphi=\arcsin\frac{1}{3}$. 6.16. $\left(\frac{\pi}{6}+2\pi k,\ \frac{\pi}{2}+2\pi k\right)\cup\left(\frac{\pi}{2}+2\pi k,\ \frac{5\pi}{6}+2\pi k\right)$. 6.17. $\left[-\frac{5\pi}{6}+2\pi k,\ -\frac{\pi}{6}+2\pi k\right]$ and $\frac{\pi}{2}+2k\pi$. 6.18. $\left(\frac{\pi}{6}+\frac{2\pi k}{3},\ \frac{7\pi}{18}+\frac{2\pi n}{3}\right)\cup\left(\frac{\pi}{2}+\frac{2\pi n}{3},\ \frac{11\pi}{18}+\right.$

$\left.\frac{2\pi n}{3}\right)$. 6.19. $\left(\pi n, \frac{\pi}{6}+\pi n\right) \cup \left(\frac{\pi}{2}+\pi n, \frac{5\pi}{6}+\pi n\right)$.
6.20. $\left(\frac{\pi n}{2}+\frac{\pi}{8}, \frac{\pi n}{2}+\frac{\pi}{4}\right) \cup \left(\frac{\pi n}{2}+\frac{\pi}{4}, \frac{\pi n}{2}+\frac{3\pi}{8}\right)$.
6.21. $\left(\frac{\pi n}{2}+\frac{5\pi}{24}, \frac{\pi}{2}(n+1)+\frac{\pi}{24}\right)$. 6.22. $\left(\frac{\pi}{6}+2\pi k, \frac{5\pi}{6}+2\pi k\right)$ and $-\frac{\pi}{2}+2\pi k$. 6.23. $\left(-\frac{\pi}{4}+\pi n, \pi n\right) \cup \left(\frac{\pi}{4}+\pi n, \frac{\pi}{2}+\pi n\right)$. 6.24. $\left(\frac{3\pi}{4}+2\pi n, \frac{9\pi}{4}+2\pi n\right) \cup (-\pi+2\pi n, 2\pi n)$.

Sec. 7. 7.1. $[-1, 1]$. 7.2. $[-1, 1]$. 7.3. $\left[\frac{1}{4}, 1\right]$. 7.4. $\left[-1, \frac{\sqrt{3}}{2}\right)$. 7.5. $\left(-\frac{\sqrt{3}}{3}, \infty\right)$. 7.6. $(-\infty, \cot 2)$. 7.7. $(-\infty, \tan 1)$. 7.8. $\varnothing$. 7.9. $(-\infty, \infty)$. 7.10. $\left[-1, \cos\frac{1}{2}\right]$. 7.11. $\left(-1, \frac{1}{\sqrt{2}}\right)$. 7.12. $[-1, 0)$. 7.13. $(1, \infty)$. 7.14. $\left[0, \frac{1}{2}\right)$. 7.15. $\left(\frac{\sqrt{2}}{2}, 1\right) \cup \left(-1, -\frac{\sqrt{2}}{2}\right)$.

Sec. 8. 8.8*. Introduce an auxiliary angle and use the inequality $|\sin x| \leqslant 1$.

8.9**. *Solution.* Let us represent the expression we have to estimate in the form

$$\frac{a}{2}(1-\cos 2x)+\frac{b}{2}\sin 2x+\frac{c}{2}(1+\cos 2x)$$
$$\Leftrightarrow \frac{a+c}{2}+\frac{1}{2}[(c-a)\cos 2x+b\sin 2x].$$

Then we must use the inequality proved in 8.8*.

8.10*. *Hint.* Represent the left-hand side as a function of half the argument. 8.11*. *Hint.* Apply the technique used in 8.2. 8.12*. *Hint.* See the Hint to 8.11*.

8.14**. *Solution.* The initial inequality is equivalent to the inequality

$$\frac{\cos\alpha+\cos\beta+\cos\gamma}{2} \leqslant \frac{3}{4}.$$

Since $\cos\beta+\cos\gamma=2\cos\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}$ we can take into account the hypothesis and use the reduction formula to obtain

$$\cos\alpha+\cos\beta+\cos\gamma \leqslant \cos\alpha+2\sin\frac{\alpha}{2}.$$

Since $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$, the problem reduces to finding the greatest value of the function $1 - 2\sin^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}$. Isolating a perfect square, we get $1 + \frac{1}{2} - 2\left(\sin\frac{\alpha}{2} - \frac{1}{2}\right)^2 \leqslant \frac{3}{2}$. Thus $\cos\alpha + \cos\beta + \cos\gamma \leqslant \frac{3}{2}$, whence follows the validity of the initial inequality.

8.20*. *Hint.* Estimate the difference $\cos x - \left(1 - \frac{x^2}{2}\right)$ using the representation $1 - \cos x = 2\sin^2\frac{x}{2}$. 8.21*. *Hint.* Use the result of the preceding problem and of Example 8.3. 8.22*. *Hint.* See the Hint to 8.20*.

Chapter 5

Sec. 1. 1.1. $\cos 0 + i\sin 0$. 1.2. $3(\cos\pi + i\sin\pi)$. 1.3. $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$. 1.4. $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. 1.5. $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$. 1.6. $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$. 1.7. $5\left[\cos\left(-\arccos\frac{3}{5}\right) + i\sin\left(-\arccos\frac{3}{5}\right)\right]$. 1.8. $5\left[\cos\left(\pi + \arccos\frac{3}{5}\right) + i\sin\left(\pi + \arccos\frac{3}{5}\right)\right]$. 1.9. $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. 1.10. $\cos\left(\frac{3\pi}{2} + \alpha\right) + i\sin\left(\frac{3\pi}{2} + \alpha\right)$. 1.11. $-\frac{1}{2^{50}}$. 1.12. $\pm(1+i)$. 1.13. $\pm 2(1-i)$. 1.14. $\sqrt{5}\left[\cos\left(\pi k + \frac{\arcsin(-4/5)}{2}\right) + i\sin\left(\pi k + \frac{\arccos(-4/5)}{2}\right)\right]$ $(k=0,\ 1)$. 1.15. $\pm\frac{1 \pm i}{\sqrt{2}}$. 1.16. $\pm\frac{1}{2}[\sqrt{3} + 1 - (\sqrt{3} - 1)\,i]$, $\pm[\sqrt{3} - 1 + (\sqrt{3} + 1)\,i]$. 1.17. $\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$, $\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}$ $\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}$, $\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}$.

1.18. $\sqrt[7]{5}\left[\cos\dfrac{2\pi k + \arccos\frac{3}{5}}{7} + i\sin\dfrac{2\pi k + \arccos\frac{3}{5}}{7}\right]$.

1.19. $\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$ $(k=0,\ 1,\ 2)$.

Sec. 2. 2.1. A circle of unit radius with centre at the origin. 2.2. A positive semiaxis Ox which includes the point O. 2.3. A ray emanating from the origin (without the point O) which makes an angle of $\pi/3$ with the Ox axis. **2.4.** The interior part of the circle bounded by concentric circles of radii 1 and 4 with centre at the origin. **2.5.** The set of all exterior points of a circle of unit radius with centre at the point (1/2, 0). **2.6.** The set of all points which lie in the interior of a circle of radius $\sqrt{99}$ with centre at the origin. **2.7.** The Oy axis. **2.8.** The straight line $y = 2x + 3/2$. **2.9.** A half-plane lying above the straight line $y = -1/2$. **2.10.** The set of all points which lie in the interior of a circle bounded by concentric circles of radii 1 and 4 (including the circle) with centre at the point (0, —1). **2.11.** The straight line $y = -x$. **2.12.** The set of all points of the rectangle $|q| \leqslant 1, |p| \leqslant 2$. **2.13.** (1) The straight lines defined by the equation $ay + bx = 0$. (2) The straight lines defined by the equation $y = -b$. **2.14.** The set of all points which lie outside the circle with centre at the point (1, 0) and radius 10. **2.15.** The Ox axis and points with coordinates which satisfy the conditions $x = -\frac{1}{2}, \; -\frac{\sqrt{3}}{2} < y < \frac{\sqrt{3}}{2}$.

Sec. 3. 3.1. $1-i, \frac{4-2i}{5}$. **3.2.** $0, -1, \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$. **3.3.** $2-\frac{3i}{2}$. **3.4.** $0, \cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}$ $(k=0, 1, 2, 3, 4)$. **3.5.** (2, 1), $\left(\frac{3}{2}, \frac{1}{2}\right)$. **3.6.** (6, 1). **3.8.** -1. **3.9.** $z_1=1, z_2=i, z_3=-i$. **3.10.** (a) $(i, i), (-i, -i)$, (b) $(i, i), (-i, -i)$. **3.11.** $|a+bi| = 1$, $a+bi \neq -1$. **3.12.** All real and all purely imaginary numbers. **3.13.** $z = -1 - i$ for $a = 1$, $z = \frac{-a^2 \pm \sqrt{2-a^2}}{a^2-1} - i$ for $1 < a \leqslant \sqrt{2}$. For $a > \sqrt{2}$ the equation has no solutions. **3.14*.** $a > 2$. *Hint.* Investigate the mutual positions of the circles $|z+\sqrt{2}| = a^2 - 3a + 2$ and $|z + i\sqrt{2}| = a^2$. **3.15.** $-\frac{21}{10} < a < -\frac{5}{6}$. **3.16.** $z = 2 - \frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{2}}{2} - 2\right)$.

Chapter 6

Sec. 1. 1.2. The sequence increases monotonically. **1.3*.** The sequence increases monotonically beginning with the second term. *Hint.* Compare the ratio y_{n+1}/y_n and unity. **1.4.** $c = 0, d \neq 0, a/d > 0$ or $c \neq 0, d/c > -1, ad > bc$. **1.5.** $y_1 = 0$ is the least term, the greatest term does not exist. **1.6.** $y_3 = 4$ is the greatest term, the least term does not exist. **1.7*.** $x_8 = 24$ is the least term, the greatest term does not exist. *Hint.* Find the extremal points for the function $f(x) =$

$2x + \frac{512}{x^2}$. **1.8.** (a) $n \geqslant 31$, (b) $n \geqslant 301$. **1.9*.** None. *Hint.* Solve the inequality $2 < |x^2 - 5x + 6| < 6$ in integral positive numbers. **1.10*.** Beginning with $n = 3$. *Hint.* Consider the interval of monotonicity of the function $f(x) = x^2 - 5x + 6$. **1.11*.** For integers belonging to the interval $\left[1, \left[-\frac{1}{\ln q}\right]\right]$ the sequence increases monotonically, for integers belonging to the interval $\left[\left[-\frac{1}{\ln q}\right] + 1, \infty\right)$ the sequence decreases monotonically. If $-1/\ln q$ is an integer, the sequence decreases monotonically beginning with the term numbered $-1/\ln q$. *Hint.* Consider the intervals of monotonicity of the function $f(x) = xq^x$. **1.12.** It is $x_n \in [1/2, 2]$. **1.13.** It is $x_n \in [1, 4/3]$. **1.14*.** It is $x_n \in [0, 1/2]$. *Hint.* Verify the fact that the sequence $f(x) = x^2 - 2x + 3$ increases on the interval $[1, \infty]$. **1.15*.** It is $x_n \in [0, 1]$. *Hint.* Reduce the expression in parentheses to a common denominator.

Sec. 2. 2.7*. *Hint.* Take some neighbourhood of the point a and compare the values of the terms of the sequence, which have not got into that neighbourhood, and the endpoints of the neighbourhood. **2.8*.** *Hint.* Take some neighbourhood of the point a which does not include q. **2.9*.** No. *Hint.* Take the neighbourhoods of the points p and q which do not intersect. **2.11*.** No, it does not. *Hint.* Consider even and odd values of n. **2.12*.** $\lim\limits_{n\to\infty} x_n = 0$. *Hint.* Use the inequality $|\sin x| \leqslant 1$. **2.13.** (a) $\lim\limits_{n\to\infty} x_n = 1$; (b) the sequence has no limit.

Sec. 3. 3.1. $-1/2$. **3.2.** 1. **3.3.** 0. **3.4.** 0. **3.5.** -1. **3.6*.** 7/15. *Hint.* Divide the numerator and the denominator by 3^{n+1} and use formula (3.4). **3.7*.** 0. *Hint.* Verify the fact that for any n the power base is smaller than 3/4. **3.8*.** 0. *Hint.* See the Hint to 3.7*. **3.9*.** 0. *Hint.* Use the inequality $|\sin n!| \leqslant 1$. **3.10.** 0. **3.11.** $-5/2$. **3.12.** 1/2. **3.13*.** 0. *Hint.* Multiply and divide by $n^2 - n\sqrt[3]{1-n^3} + \sqrt[3]{(1-n^3)^2}$. **3.14*.** $\lim\limits_{n\to\infty} x_n = 2$. *Hint.* Find a recurrence expression for the sequence (x_n^2), $n \in \mathbb{N}$, and use formula (3.5). **3.15*.** $\lim\limits_{n\to\infty} x_n = a$. *Hint.* Reduce the recurrence relation to the form $x_{n+1} = x_n + (x_n - a)^2$ and use formula (3.5). **3.16.** $\lim\limits_{n\to\infty} x_n = \sqrt{a}$.

Sec. 4. 4.1. 119/3. **4.2*.** $(a_1 = 2, d = -3)$, $(a_1 = -10, d = -3)$. *Hint.* Use formula (4.4). **4.3*.** (1/3, 2/3, 1). *Hint.* Use formula (4.4). **4.4.** $a_n = p + q - n$. **4.5*.** *Hint.* Use the fact that if the numbers u, v, w are three successive terms of an arithmetic progression, then $v - u = w - v$. **4.6*.** *Hint.* Use the identity $\frac{1}{ab} = \frac{1}{b-a}\left[\frac{1}{a} - \frac{1}{b}\right]$.

4.7. 29. **4.8.** 9. **4.9.** 7. **4.10*.** *Hint.* Consider the sum of the terms, which are equidistant from the ends, among $a_1, \ldots, a_{m+n}$. **4.11*.** 82 350. *Hint.* An even number divisible by 3, by 6. **4.12.** $d = 2a_1$, $a_1 \neq 0$ or $d = 0$, $a_1 \neq 0$. **4.13*.** 1275. *Hint.* Use the formula $x^2 - y^2 = (x - y)(x + y)$. **4.14*.** 1064. *Hint.* Use the Hint to 4.10*. **4.15.** 98. **4.16*.** $a_n = 8n - 4$. *Hint.* Use formula (4.6). **4.17*.** *Hint.* Express the sums $a_1 + a_{3n}$, $a_1 + a_{2n}$, $a_1 + a_n$ in terms of S_{3n}, S_{2n}, S_n respectively and use the relation $a_{3n} + a_n = 2a_{2n}$. **4.18*.** *Hint.* Use the hypothesis to obtain a relationship between a_1 and d and use that relationship when proving the assertion. **4.19*.** For $a \geqslant 12$. *Hint.* Consider the set of values of the function $f(x) = 25^x + 25^{-x} + 5^{1+x} + 5^{1-x}$. **4.20.** $x = \log_2 5$. **4.22*.** *Hint.* Show that $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ is not a rational number. **4.23*.** No, they cannot. *Hint.* See the hint to 4.22*. **4.24*.** Yes, it can. *Hint.* If the lengths of the sides are a, b, c and d respectively, then the necessary and sufficient condition for the possibility of inscribing a circle into a quadrilateral consists in the fact that $a + c = b + d$.

Sec. 5. 5.2. $b_1 = 5$, $b_5 = 405$. **5.3.** (7, −14, 28, −56). **5.4.** $S_4 = 40$. **5.5.** (1, 3, 9). **5.6.** (1, 3, 9). **5.7.** (3, 6, 12); $\left(\frac{3}{2}(9+\sqrt{65}), -6, \frac{3}{2}(9-\sqrt{65})\right)$. **5.8.** (2, 4, 8, 16); (16, 8, 4, 2). **5.9.** (2, 4, 8) or (8, 4, 2). **5.10.** (1, 5, 25); (25, 5, 1). **5.11.** $q = 2$. **5.12.** $\left(\frac{S_n}{S'_n}\right)^{n/2}$. **5.13.** $\frac{1}{(2+\sqrt{3})^{m-1}}, \frac{1}{(2+\sqrt{3})^{m-2}}, \frac{1}{(2+\sqrt{3})^{m-3}}, \ldots$ **5.14.** $\frac{u_1^2(q^{2n}-1)}{q^2-1}$. **5.17*.** $S_n = 2n + \frac{x^{2n+2} - x^{4n+2} - x^{2n} + 1}{(1-x)\cdot x^{2n}}$. *Hint.* Square the expressions in the parentheses and sum up the resulting geometric progressions. **5.18*.** $S_n = 4 - \frac{1}{2^{n-2}} - \frac{n}{2^{n-1}}$. *Hint.* Consider the difference $S_n - \frac{S_n}{2}$. **5.19*.** $S_n = \frac{nx^{n+1}}{(x-1)} - \frac{x^{n+1}-x}{(x-1)^2}$. *Hint.* Multiply both sides of the equation by x and subtract xS_n from S_n. **5.20.** $b_4 = \frac{3}{16}$; $q = \frac{1}{4}$. **5.21*.** $\frac{3}{5}$. *Hint.* Use the method proposed in Example 5.2. **5.22.** $\left(6, 3, \frac{3}{2}, \ldots\right)$. **5.23.** $\left(6, -\frac{1}{2}\right)$. **5.24.** $\frac{S^2}{2S-1}$. **5.25*.** $x > 0$, $x \neq \pm a$, $S = \frac{(a+x)^3}{4(a-x)ax}$. *Hint.* To find the common ratio q of the progression, divide b_2 by b_1. With respect to x solve the inequality of the form $|q(x)| < 1$. **5.26.** $p = \frac{8a}{2-\sqrt{2}}$, $S = 2a^2$. **5.27.** The condition has the form $a^{p-m}b^{m-k}c^{k-p} = 1$. **5.28.** The numbers 11, 12, 13 cannot be terms of the same geometric progression.

Sec. 6. 6.1. (4, 8, 16); $\left(\frac{4}{25}, -\frac{16}{25}, \frac{64}{25}\right)$. **6.2.** (3, 6, 12); (27, 18, 12). **6.3*.** 931. *Hint.* Use the representation of a number in the decimal notation, i.e. represent the required number in the form $a \cdot 10^2 + b \cdot 10 + c$, where a is the number of hundreds, b is the number of tens and c is the number of unities. **6.4.** (32,16, 8, 0); (2, 6, 18, 30). **6.5.** (2, 10, 18, . . .); (2, 6, 18, . . .). **6.6.** $b_7 = 27$. **6.7.** (24, 27, 30, ..., 54) and (24, 24, 24, . . .). **6.8.** $q = 3/2$, $q = 1$.

Sec. 7. 7.1. Yes, $x_n \in \left[0, 1\frac{1}{2}\right]$. **7.2.** No, it is not. **7.3*.** Yes, it is, $x_n \in [-8, 11]$. *Hint.* See the hint to 1.4. **7.4*.** Nine terms. *Hint.* Use the formula for the sum of a geometric progression. **7.7*.** *Hint.* Use the property of the sides of a triangle. **7.8.** $\lim_{n\to\infty} S_n = a^2/2$. **7.9.** 1/3. **7.10.** 0.

7.11*. 630, 135, 765. *Hint.* Assume that x is the number of hundreds, y is the number of tens and z is the number of unities. Then the first condition of the problem leads to an equation $x \cdot 100 + y \cdot 10 + z = 45p$, where p is an integer. Since, x, y, z are successive terms of an arithmetic progression, it follows that $2y = x + z$. Using the conditions of the problem, we can form a system of two equations in four unknowns:

$$x \cdot 100 + y \cdot 10 + z = 45p, \quad 2y = x + z,$$

which must be solved in the set of nonnegative integers.

7.12.** *Solution.* According to (4.3), using the conditions of the problem, we obtain

$$\frac{a_1 + a_n}{2} = np, \quad \frac{a_1 + a_k}{2} = kp.$$

Eliminating a_1 in the system, we get an equation $\frac{a_n - a_k}{2} = p(n-k)$. Using (4.2), we have $d(n-k) = 2p(n-k)$, and, consequently, $d = 2p$, $a_1 = p$. Using again formula (4.3), we obtain the required expression:

$$S_p = \frac{2p + 2p(p-1)}{2} p = p^3.$$

7.13*. *Hint.* Multiply and divide each term by the expression which is an adjoint of the denominator. **7.15*.** (8, 4, 2, 1, 1/2, 1/4). *Hint.* It follows from the first five equations that the numbers x, y, z, s and t form a geometric progression. **7.16*.** *Hint.* Represent each number entering into the expression being proved as the sum of terms of the corresponding geometric progression. **7.17*.** $A = 2$, $B = 32$. *Hint.* Use the Vieta theorem. **7.18.** 0. **7.19.** 0.5. **7.20*.** $3\sqrt{3}/2$. *Hint.* Under the limit sign is the sum of n terms of a geometric progression with a common ratio $q = 1/3$. **7.21.** 1.5. **7.22.** 1/3. **7.23.** 1.25. **7.24*.** 1. *Hint.* Use the relation $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. **7.25*.** $\frac{1}{da_1}$. *Hint.* See the Hint to 7.24*. **7.26.** 1/4.

Chapter 7

Sec. 1. 1.8*. *Hint.* Consider the sequences

$$x_n^{(1)} = \frac{2}{\pi(1+4n)}, \quad x_n^{(2)} = \frac{2}{\pi(3+4n)}.$$

Sec. 2. 2.1. 1. **2.2.** 0. **2.3.** 1. **2.4.** 2. **2.5.** 0. **2.6.** 0. **2.7.** 6. **2.8.** $3x^2$. **2.9.** ∞. **2.10.** $\frac{1}{a}$ for $a \neq 0$, for $a = 0$ there is no limit. **2.11.** (a) $-\frac{3}{4}\sqrt{2}$; (b) $\frac{21}{4}\sqrt{\frac{2}{3}}$. **2.12.** (a) $-1/3$; (b) 1. **2.13*.** $-3/2$. *Hint.* Isolate the factor $(x-1)^2$ in the numerator and the denominator. **2.14.** 4/3. **2.15.** 3/2. **2.16.** $\sqrt{2}/2$. **2.17.** 0. **2.18.** 3/2. **2.19.** 4. **2.20.** $-2/3$. **2.21.** 1.5. **2.22.** 1.5. **2.23*.** -1.75. *Hint.* Pass to the variable $y = -x$. **2.24.** 2. **2.25.** 8. **2.26.** 3/4. **2.27.** n/m. **2.28*.** $\cos a$. *Hint.* Use the formula $\sin x - \sin a = 2 \sin\frac{x-a}{2} \cos\frac{x+a}{2}$. **2.29*.** π. *Hint.* Designate $\pi/n = x$. **2.30*.** π. *Hint.* Designate $x + 2 = y$ and use the formula $\tan \pi(y+2) = \tan \pi y$. **2.31.** $-1/\sqrt{2}$. **2.32*.** -24. *Hint.* Reduce the expression appearing in the numerator to the form $\frac{\tan x \sin(x-\pi/3) \sin(x+\pi/3)}{\cos^2 x \cos^2(\pi/3)}$ and use the relation $\cos\left(x+\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}-x\right)$. **2.33*.** $\frac{\sqrt{3}}{3}$. *Hint.* Reduce the expression appearing in the denominator to the form $4 \sin\frac{\pi/3-x}{2} \sin\frac{\pi/3+x}{2}$. **2.34.** 0. **2.35*.** 1. *Hint.* Add and subtract a unity in the numerator.

Sec. 3. 3.9*. *Hint.* Use the formula $\cos(x+\Delta x) - \cos x = -2 \sin\frac{\Delta x}{2} \sin\left(x+\frac{\Delta x}{2}\right)$. **3.10*.** *Hint.* Use the inequality $\ln(1+x) \leqslant x$. **3.11*.** *Hint.* Use the result of the preceding problem. **3.13.** The function $y = \tan x$ is discontinued at points $x = \pi/2 + \pi n$, $n \in Z$. **3.14.** $\tilde{f}(0) = 1$. **3.15.** $\tilde{f}(0) = 2$. **3.16.** $\tilde{f}(0) = 1$. **3.17.** $\tilde{f}(81) = 1/6$. **3.18.** $A = 1$. **3.19.** $A = 0$. **3.20.** $A = 3/2$. **3.21.** $A = 1/2$. **3.22.** $A = 2/m^2$. **3.23*.** $A = 2/\pi$. *Hint.* When calculating the limit $\lim\limits_{x\to 1} f(x)$, introduce the designation $z = 1 - x$. **3.24.** $b/a = \pi/2$.

3.25. $a = 1$. 3.26. $2a - b = 0$. 3.27. $3a - b = 0$. 3.28. $a + b + 1 = 0$. 3.29. $b = 4$. 3.30. $a = 3/2 + 2n$. 3.31. $a = 3/4$.

Sec. 4. 4.1. -1. 4.2. 3/4. 4.3. Yes. ($2 > -7/22 + 1/4$). 4.4. Yes. ($1/2 + \sqrt{2} > -3 + \log 5$). 4.5. 3/7. 4.6. 2/3. 4.7. ∞. 4.8. ∞. 4.9. $1/\cos^2 a$. 4.10. $-\cot a$. 4.11. $\cos^3 a$. 4.12. $\sqrt{2}/4$. 4.13. $\sqrt{2}/2$. 4.14. 1/2. 4.15. 1/2. 4.16. $2\sqrt{2}$. 4.17. ∞. 4.18. 1/2. 4.19. ∞. 4.20. $\sqrt{3}/3$. 4.21. 2. 4.22. 0. 4.23. 2. 4.24. 2. 4.25. $\sin 2a$. 4.26. 1/2. 4.27. 1. 4.28. 1/2. 4.29. $\sin 2a/\cos^4 a$. 4.30. $-2\sin 2a$. 4.31. 3. 4.32. $3\sqrt{2}$. 4.33. $-\sqrt{2}$. 4.34. 0. 4.35. $2a/\pi$. 4.36. a/π. 4.37. 2. 4.38. 1/2. 4.39. a. 4.40. 3/2. 4.41. $\tilde{f}(3) = 2$. 4.42. $\tilde{f}(0) = -1/8$. 4.43. $\tilde{f}(0) = -4$. 4.44. $A=3/4$. 4.45. $A=1/2$.

Chapter 8

Sec. 1. 1.1. $f'(x) = -\frac{1}{x^2}$. 1.2. $f'(x) = -\sin x$. 1.3*. $f'(x) = e^x$. *Hint*. Use the equation $\lim\limits_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1$. 1.4*. $f'(x) = \frac{1}{x}$. *Hint*. Use the equation $\lim\limits_{\Delta x \to 0} \frac{\ln(1+\Delta x)}{\Delta x} = 1$. 1.5*. $f'(x) = nx^{n-1}$. *Hint*. Use the binomial formula. 1.6. $f'(x) = 0$. 1.10*. *Hint*. See problem 7.1.6*.

1.12. $\frac{2}{\sqrt{x}(2-\sqrt{x})^2}$. 1.13. $\frac{-2x}{\sqrt{1-x^4}\,(1+x^2)}$. 1.14. $\frac{\cos\sqrt{x}}{4\sqrt{x}\,\sin\sqrt{x}}$.

1.15. $\frac{e^{\sqrt{\ln(ax^2+bx+c)}}\,(2ax+b)}{2\sqrt{\ln(ax^2+bx+c)}\,(ax^2+bx+c)}$. 1.16. $\frac{1}{1+x^2}$.

1.17. $\frac{2abmnx^{n-1}\,(a+bx^n)^{m-1}}{(a-bx^n)^{m+1}}$. 1.18. $\sin^3 x \cos^2 x$.

1.19. $\frac{1}{\sin^4 x \cos^4 x}$. 1.20. $-\frac{1}{x^2}\tan\frac{x-1}{x}$. 1.21. $f'(x) = 1$.

1.22. $f'(x) = 0$. 1.23. $f'(x) = -2x$. 1.24. $f'(x) = \frac{1-m}{m}x^{(1-2m)/m} + 3\frac{1-n}{n}x^{(1-2n)/n}$. 1.25. $f'(x) = \frac{x}{\sqrt{1+x^2}}$. 1.26. $f'(x) = \frac{1}{\sqrt{2}} \times (8-t^3)^{-2/3}t^2$. 1.27. $f'(x) = \frac{8}{x^2}$. 1.28. $f'(x) = \frac{1}{6}(x^2-1)^{-5/6}2x$.

1.29. $f'(x) = -\frac{x}{2(x^2-1)^{5/4}}$. 1.30. $f'(x) = \sqrt{2}$. 1.31. $f'(x) =$

$\frac{1}{2}\sqrt{\frac{a}{x}}$. **1.32*.** $f'(x)=\begin{cases}-1, & x\in[1,2),\\ 1, & x\in(2,\infty).\end{cases}$ *Hint.* To simplify the form of the function $f(x)$, designate $\sqrt{x-1}=t$. **1.33.** $f'(x)=\begin{cases}2, & x\in(0,1),\\ 2/3, & x\in(1,\infty).\end{cases}$ **1.34.** $f'(x)=\begin{cases}-1/2, & x\in(-\infty,0),\\ 1/2, & x\in(0,\infty).\end{cases}$

1.35*. $f'(x)=\begin{cases}0, & x\in[2,4),\\ 1/\sqrt{x-2}, & x\in(4,\infty).\end{cases}$ *Hint.* Designate $\sqrt{2x-4}=t$.

Sec. 2. 2.1. $f(x)$ increases for $x\in(-\infty,-2)\cup(-1,+\infty)$; $f(x)$ decreases for $x\in(-2,-3/2)\cup(-3/2,-1)$. **2.2.** $f(x)$ decreases for $x\in(0,1)\cup(1,e)$, $f(x)$ increases for $x\in(e,\infty)$. **2.3.** $f(x)$ increases for $x\in\mathbf{R}$. **2.4.** $f(x)$ increases for $x\in(2,3)$, $f(x)$ decreases for $x\in(3,\infty)$. **2.5.** $f(x)$ decreases for $x\in(-\infty,0)\cup(1,\infty)$, $f(x)$ increases for $x\in(0,1)$. **2.6.** $f(x)$ decreases for $x\in(-\infty,0)\cup(0,\infty)$. **2.7*.** $a\in(-\infty,-2,-\sqrt{5})\cup(\sqrt{5},+\infty)$. *Hint.* The hypothesis is equivalent to the condition $f'(x)>0$ for $x\in\mathbf{R}$. Find the values of the parameter a for which the auxiliary function $g(t)$, which results from $f'(x)$ upon the substitution $t=\cos x$, is positive on the interval $[-1,1]$. **2.8*.** $a\in(6,\infty)$. *Hint.* Having calculated the value of the derivative, find the domain of variation of a for which it is positive. Use the inequality $|\cos\alpha x|\leqslant\cos 0$ for any α. **2.9.** $x_{max}=-2$, $y_{max}=8$; $x_{min}=2$, $y_{min}=0$.

2.10. $x_{max}=\frac{\pi}{3}+k\pi$, $y_{max}=\frac{\pi}{3}+k\pi+\frac{\sqrt{3}}{2}$; $x_{min}=-\frac{\pi}{3}+k\pi$, $y_{min}=-\frac{\pi}{3}+k\pi-\frac{\sqrt{3}}{2}$, $k\in Z$. **2.11.** $x_{min}=-\frac{1}{2}$, $y_{min}=-\frac{1}{2}e^{-3/4}$; $x_{max}=1$, $y_{max}=1$. **2.12.** $x_{max}=3$, $y_{max}=\frac{1}{3}$; $x_{min}=-3$, $y_{min}=-\frac{1}{3}$. **2.13.** $x_{max}=-2$, $y_{max}=25$; $x_{min}=1$, $y_{min}=-2$. **2.14.** $x_{min}=e$, $y_{min}=e$. **2.15.** $x_{max}=-1$, $y_{max}=17$; $x_{min}=3$, $y_{min}=-47$. **2.16.** $x_{max}=0$, $y_{max}=-2$, $x_{min}=2$, $y_{min}=2$.

Sec. 3. 3.1. $\max\limits_{x\in[-1,1]} f(x)=3$; $\min\limits_{x\in[-1,1]} f(x)=1$.

3.2. $\max\limits_{x\in[-2,1]} f(x)=17$; $\min\limits_{x\in[-2,1]} f(x)=0$. **3.3.** $\max\limits_{x\in[0,\pi]} f(x)=\frac{3\sqrt{3}}{8}$; $\min\limits_{x\in[0,\pi]} f(x)=0$. **3.4.** $\min\limits_{x\in[0,\pi/2]} f(x)=0.5$; $\max\limits_{x\in[0,\pi/2]} f(x)=\frac{3}{4}$.

3.5. $\max\limits_{x\in[-\pi/2,\pi/2]} f(x)=\frac{\pi}{4}$; $\min\limits_{x\in[-\pi/2,\pi/2]} f(x)=-\frac{\pi}{4}$,

3.6. $\max\limits_{x \in [\pi, 3\pi/2]} f(x) = 0$; $\min\limits_{x \in [\pi, 3\pi/2]} f(x) = -1$. 3.7*. $\max\limits_{x \in \mathbf{R}} f(x) = \frac{4}{8-\sqrt{2}}$, $\min\limits_{x \in \mathbf{R}} f(x) = \frac{4}{8+\sqrt{2}}$. *Hint*. Pass to the variable $y = \sin x + \cos x$. 3.8. $\max\limits_{x \in [\pi/6, \pi/3]} f(x) = 4\frac{\sqrt{3}}{3}$, $\min\limits_{x \in [\pi/6, \pi/3]} f(x) = 2$.
3.9*. $\min\limits_{x \in [0, \pi]} f(x) = 3$. *Hint*. Pass to the variable $y = \cos x$.
3.10. $\max\limits_{x \in [-2, 0]} f(x) = f(0) = 1$; $\min\limits_{x \in [-2, 0]} f(x) = f(-1) = 0$.
3.11. (a) $\max\limits_{x \in [0, 2]} f(x) = f(2) = 4$, $\min\limits_{x \in [0, 2]} f(x) = 2$, (b) $\max\limits_{x \in [-2, 0]} f(x) = f(-2) = 4$, $\min\limits_{x \in [-2, 0]} f(x) = 2$. 3.12. $\max\limits_{x \in \mathbf{R}} f(x) = 2$, $\min\limits_{x \in \mathbf{R}} f(x) = -2$. 3.13. $\max\limits_{x \in [1/2, 4]} f(x) = 21 + 3 \ln 2$, $\min\limits_{x \in [1/2, 4]} f(x) = 0$.
3.14. $x_{\min} = \frac{1}{3}$, $\max\limits_{x \in [0, 3]} f(x) = 105$. 3.15. $\max\limits_{x \in [0, 10]} f(x) = f(5) = 5$, $\min\limits_{x \in [0, 10]} f(x) = f(0) = f(10) = 0$. 3.16*. $\max\limits_{x \in [0, 3]} f(x) = f(3) = 4\sqrt{6}$, $\min\limits_{x \in [0, 3]} f(x) = f(1) = 0$. *Hint*. Pass to the variable $y = (x-1)^2$ and use the fact that $g(u) = \sqrt{u}$ is a monotonically increasing function.

Sec. 4. 4.1. $[0, \infty)$. 4.2. $[3, 3/\cos^2 1.5]$. 4.3. An empty set. 4.4*. $[-1/3, 1]$. *Hint*. We can find the maximum and minimum values of the initial function, but there is also another way consisting in considering the values of y for which the equation $y(x^2 - 3x + 3) = x - 1$ has real solutions with respect to x. 4.5. (a) $y \in [0, 1/2]$, (b) $y \in [-1/2, 1/2]$. 4.6*. *Hint*. Consider the inequality relating the expressions which are inverse of the left-hand and right-hand sides of the initial inequality. 4.7*. *Hint*. Represent $f(x)$ in the form $f(x) = 2 \sin x - 2 \sin^3 x$ and, using the substitution $t = \sin x$, reduce the problem to the proof of the validity of the inequality $\min\limits_{t \in [-1, 1]} g(t) > -7/9$, where $g(t) = 2t - 2t^3$. 4.8*. *Hint*. See the Hint to 4.7*. 4.9*. *Hint*. See the Hint to 4.7*. 4.11*. *Hint*. See the Hint to 4.4*.
4.12*. $a \in \left(-\infty, \frac{-3-\sqrt{5}}{16}\right)$. *Hint*. Using the second equation, obtain an inequality with a parameter with respect to one unknown. Then find the least value of the function for each a and indicate the set of all values of a for which that value is smaller than 4. 4.13*. 44. *Hint*. Use the equation $a_9 + a_3 = a_1 + a_{11}$. 4.14*. $a = -4/3$, $a = -8/3$. *Hint*. Use the property of a geometric progression: $a_n^2 = a_{n-1}a_{n+1}$. 4.15. 2/3. 4.16. $\sqrt{3} - 1$. 4.17*. $a = 9$. *Hint*. Find min

of the function $y = \frac{4}{\sin x} + \frac{1}{1-\sin x}$ on the interval $(0, \pi/2)$, see also the hint to 4.7*. **4.18***. *Hint.* Use the relation between the arithmetic mean and the geometric mean of two numbers. **4.19***. *Hint.* Represent z in the form $z = (x+y+1)^2 + (x-2)^2 - 3$. **4.20***. $a=1$. *Hint.* Use the Vieta theorem and represent the sum of the squares of the roots of the equation as a function of a. **4.21***. *Hint.* Find the greatest and the least value of the function $\frac{x}{1+x^2}$ for $x \in \mathbf{R}$, see also the Hint to 4.4*. **4.25***. *Hint.* See the Hint to 4.4*. **4.26***. *Hint.* Use the representation $\sin^6 x + \cos^6 x = 1 - \frac{3}{4}\sin^2 2x$.

Sec. 5. 5.1. $18 = 9 + 9$. **5.2.** $36 = 6 \cdot 6$. **5.3.** $40 + 80 + 60 = 180$. **5.4.** $\frac{a}{2} + \frac{a}{2} = a$. **5.5***. $p = -2$, $q = 0$, the distance is equal to 1. *Hint.* The distance from the vertex of the parabola to the Ox axis is the ordinate of the vertex. **5.6.** $\frac{4}{\sqrt{3}}$. **5.7.** The coordinates of the vertices of the rectangle which lie on the parabola are $\left(\frac{2}{3}a, \pm 2\sqrt{\frac{pa}{3}}\right)$. **5.8.** The altitude of the cone which has the greatest surface constitutes 4/3 of the radius of the ball. **5.9.** The diameter of the base and the altitude of the cylinder are equal to $2/\sqrt{3}$. **5.10***. The area of the circle circumscribed about the isosceles right triangle with legs $\sqrt{2S}$ is the least. *Hint.* Use the fact that the hypotenuse of a right triangle is the diameter of a circumscribed circle. **5.11***. $\varphi = \pi/3$. *Hint.* Use the fact that the triangle ABD is a right triangle and represent the lateral side and the smaller base in terms of the diameter of the circumscribed circle. **5.12***. The triangle one of whose base angles is equal to $\pi/2 - \alpha/2$ has the greatest perimeter. *Hint.* Use the sine theorem. **5.13.** 30 sq units. **5.14.** $2a$. **5.15.** 5. **5.16.** $\alpha = 2 \arcsin \frac{\sqrt{3}}{3}$, $H = \frac{8R}{3\sqrt{3}}$. **5.17***. $\alpha = \pi/3$. *Hint.* Use the formula $r = S/p$, where S is the area and p is half the perimeter of the triangle. **5.18.** $\alpha = \arccos \frac{V_r}{V_b}$. **5.19.** $\alpha = \pi/4$. **5.20***. $h = (l^{2/3} - d^{2/3})^{3/2}$. *Hint.* The relation between the arguments of the function whose greatest value we have to find can be found from the similitude of right triangles whose hypotenuses are the exterior and interior parts of the rigid bar with respect to the tower. **5.21.** The length is 30 cm and the width is 20 cm. **5.22.** (a) $x = y = \frac{d}{\sqrt{2}}$, (b) $x = \frac{d}{\sqrt{3}}$, $y = d\sqrt{\frac{2}{3}}$. **5.23.** $h = \frac{r}{\sqrt{2}}$. **5.24.** The side of the area adjacent to the wall must be twice as large as the other side. **5.25.** $AM = a\sqrt[3]{p}\,(\sqrt[3]{p} + \sqrt[3]{q})^{-1}$. **5.26***. To the point of the segment AB which is at the distance of 1 km from B. *Hint.* The time in which the point B can be attained, as a function of the coordinate of the point at which the boat must land, must be represented as the sum of two terms one of which is the time of sailing and the other is the time of travelling along the bank. 5.27*. $T(t_0)=\frac{2}{27}\frac{m_0^3g^2}{k^2}\left(\frac{g\cdot\text{cm}^2}{\text{s}^2}\right)$ in time $t_0=\frac{2m_0}{3k}$ (s). *Hint.* The kinetic energy T at time moment t is $T(t)=\frac{m(t)V^2(t)}{2}$, where $m(t)$ is the mass of the rain drop at the moment t, and $V(t)$ is the speed attained by the moment t. 5.28*. 20 km/h, 720 rubles. *Hint.* Use the fact that the cost of a unity way consists of two quantities, the first of which is proportional to the cube of the speed and the other is inversely proportional to the first degree of the speed. 5.29. $x(p)=\min\left(\frac{100}{\sqrt{3}}, a\right)$, where $x(p)$ is the distance from the railway station to the point P. 5.30. $\frac{23}{410}$ h. 5.31*. $1\frac{27}{43}$ h. *Hint.* At the time moment t the distance between the train and the car is the third side of a triangle whose other two sides are the distance travelled by the train and the distance remaining for the car to travel, respectively. 5.32*. In $\frac{a}{2v}$ h. *Hint.* See the Hint to 5.31*. 5.33. $y=\frac{2}{3}$ h. 5.34. The diamond was broken in two. 5.35. The resistance must be the same and equal to $\frac{R}{2}$. 5.36*. $\alpha=\max\left\{\arccos\frac{1}{k}, \arctan\frac{h}{d}\right\}$. *Hint.* The time of travelling of the messenger, as a function of the coordinate of the point of landing, consists of the times of travelling by water and along the bank. 5.37*. $\frac{\sin\alpha}{\sin\beta}=\frac{V_1}{V_2}$, where α is the angle of incidence and β is the angle of refraction of the ray. *Hint.* Express the path the ray traversed in each medium in terms of the coordinate of the point of incidence at the interface between the media. Find the ratio of the paths traversed in each medium to their projections onto the interface at which the time of traversing the whole way between points A and B is the minimal. 5.38*. $\alpha=\beta$ where α is the incidence angle and β is the refraction angle. *Hint.* See the hint to 5.37*. 5.39. $I_1=\frac{IR_2}{R_1+R_2}$, $I_2=\frac{IR_1}{R_1+R_2}$, i.e. the currents must be branched in inverse proportion to the resistances through which the currents must be passed. 5.40. 6000 rubles. 5.41*. $n=8$. The cost is $2.8\sqrt{2}$ mln rubles approx. *Hint.* If $f(x)$ is a function which expresses the dependence of the cost on the constructed living area, then we must seek the least value of the function $F(n)=nf\dfrac{40\,000}{n}$, where n is the number of constructed houses. **5.42*.** $4\sqrt{2}$ m. *Hint.* Find the distance from which the tangent of the angle of view is the largest (the tangent is a monotone function of its argument). **5.43*.** $(\sqrt{4b^2-3a^2}-b)\cdot 3^{-1/2}$. *Hint.* Find the distance for which the tangent of the angle formed by the point of the stop of the bus and two opposite sides of the facade of the palace is the largest. Express that angle as the difference between the angles at which the far and the nearest (with respect to the highway) ends of the facade of the palace can be seen. **5.44*.** $\varphi=\arctan K$, $F=\dfrac{KP}{\sqrt{1+K^2}}$. *Hint.* Use the fact that the sum of the forces in the plane of the movement must be equal to zero. **5.45*.** $\alpha=\beta$. *Hint.* See the Hint to the preceding problem. **5.46.** $\sqrt{2ap/K}$. **5.47.** $\rho=2.4$ $(x_0=5/3,\ y_0=5/9)$. **5.48*.** $(1/2,\ 7/4)$. *Hint.* The problem reduces to seeking the point C of the parabola which is at the largest distance from the straight line BD. **5.49*.** The smallest perimeter of the triangle AMB is $\sqrt{10}+2\sqrt{5}+\sqrt{34}$. The position of point M for which the smallest perimeter can be attained is $M(0, 0)$. *Hint.* Consider the point which is symmetric with respect to the point A about the straight line $y=x$. **5.50*.** The point M must bisect the segment of the straight line which is included between the sides of the angle. *Hint.* Investigate the variation of the area of the triangle upon a change in the slope of the straight line passing through M. **5.51.** The two remaining vertices result from the intersection of the sides of the angle by the straight line connecting the points which are symmetric images of the point M with respect to the sides of the angle. **5.52.** $2R\sin\dfrac{2\pi}{9}$. **5.53.** $\dfrac{a}{4b}\sqrt{3b^2-a^2}$. **5.54.** The length of the side of the base is 2 cm, the volume is 4 cm^3. **5.55.** The sides of the rectangle which has the largest area is $R\sqrt{2}/2$. **5.56.** $5\pi/9$. **5.57*.** $S_{\max}=R^2\tan\dfrac{\alpha}{2}$. *Hint.* Consider two cases: in the first case two vertices of the required rectangle lie on one of the radii which form the sector, and in the second case one vertex lies on each radius and two vertices lie on the arc of the sector. In the second case the sector must be divided into two similar sectors, and then the problem reduces to the first case considered for each half separately.

Sec. 6. 6.1. $(\sqrt{2},\ 2-\sqrt{2})$ and $(-\sqrt{2},\ 2+\sqrt{2})$. **6.2.** $y=2$. **6.3.** $y=-\sqrt{3}x+\dfrac{\pi\sqrt{3}+3}{2}$. **6.4.** $\arctan 9$, $y=9x-23\dfrac{1}{4}$. **6.5*.** $(1/2,\ -15/32)$. *Hint.* The coordinates of the point of tangency can be obtained from the equation $f'(x)=k$, where k is the slope of the tangent line. **6.6.** $(0,\ 2)$. **6.7.** 2. **6.8.** $y=8x+4$. **6.9.** $x_0=$

$+\arcsin\frac{1}{4\sqrt{2}}+\frac{\pi}{4}$. **6.10.** $\pi n, \frac{\pi}{8}+\frac{\pi n}{4}$, $n \in \mathbf{Z}$. **6.11.** (8, 0), (0, 0). **6.12*.** (3, —15), 21/2, 21. *Hint.* The hypothesis makes it possible to find at once the angle formed by the required tangent and the positive direction of the Ox axis. **6.13*.** $a = 1$. *Hint.* The condition of intersection of a straight line and a parabola is equivalent to the existence of two real roots of the respective quadratic equation, the half-sum of the abscissas of those roots must be equal to 2 by the hypothesis. **6.14.** $y = -3x - 4$. **6.15.** $y = x + 4$, $y = -x + 4$. **6.17.** (2, 8/3), (3, 7/2). **6.18.** $3\pi/4$.

6.19.** *Solution.* We differentiate each equation considering y to be a function of x. We have $y + y'x = 0$ and $2x - 2yy' = 0$. Finding the expression for y' in each equation, we have $y' = -y/x$, $y' = x/y$, respectively. Consequently, at any point M (x_0, y_0) which is a point of intersection of curves, the product of the slopes of the tangents is equal to -1.

6.20*. *Hint.* Show that the product of the slopes at the points of intersection of the curves of different families is equal to -1.

6.21. $\left(\sqrt{\frac{c}{a}}, b\sqrt{\frac{c}{a}}+2c\right)$, $\left(-\sqrt{\frac{c}{a}}, 2c-b\sqrt{\frac{c}{a}}\right)$ for $ac > 0$, (0, 0) for $c=0$; for $ac < 0$ there is no solution. **6.22** $(a+\sqrt{a^2-(5a+b-6)};\ 2a^2+\sqrt{a^2-(5a+b-6)}\,(2a-5)-10a-b)$, $(a-\sqrt{a^2-(5a+b-6)};\ 2a^2-\sqrt{a^2-(5a+b-6)}\,(2a-5)-10a-b)$, if $a^2-(5a+b-6) > 0$; $(a, 2a^2-10a-b)$, if $a^2-(5a+b-6)=0$; now if $a^2-(5a+b-6) < 0$, then there is no solution. **6.23.** $y = -\frac{1}{x_0^2}x+\frac{2}{x_0}+1$, where $x_0 = \frac{2}{b-1}$, if $a=0$, $b \neq 1$: $x_0 = \frac{a}{2}$ for $a \neq 0$, $b=1$; $x_0 = \frac{1}{b-1}$ for $a \neq 0$, $b = 1+\frac{1}{a}$; $x_0 = \frac{-1\pm\sqrt{1+a(1-b)}}{1-b}$ for $a > 0$, $b \neq 1$, $b < 1+\frac{1}{a}$ and for $a < 0$, $b \neq 1$, $b > 1+\frac{1}{a}$. In the other cases ($a=0$, $b=1$; $a>0$, $b>1+1/a$; $a < 0$, $b < 1 + 1/a$) there is no solution. **6.24*.** $y = -x + 5/2$. *Hint.* The condition of intersection of two curves $y = f_1(x)$ and $y = f_2(x)$ is equivalent to the consistency of the system of equations $y = f_1(x)$ and $y = f_2(x)$ whose solutions are the coordinates of the intersection points. **6.25.** (1/8, 1/16). **6.26*.** $\varphi = \pi/3$. *Hint.* The equired angle is the angle between the tangents to the circle which are drawn through the point (8, 0). **6.27.** (—0.4, 8.8) if the point M moved along the circle counterclockwise; (6, 4) if the point M moved in the inverse direction. **6.28.** $p = 2bk$. **6.29*.** The straight line $y = -1/2$. *Hint.* The locus of points from which the parabola can be seen at right angles is the set of intersection points of the tangents to the parabola which form a right angle. **6.30.** arctan (—4/3). **6.31*.** The circle $x^2 + y^2 = a^2 + b^2$. *Hint.* See the hint to 6.29*.

Sec. 7. **7.1*.** $-1/5$ m/s (the minus sign signifies that $y(t)$ decreases). *Hint.* If $y(t)$ is the law of variation of the way traversed by the upper end, and $x(t)$ is that of the lower end, then it must be taken into account that $x^2 + y^2 = 25$. **7.2.** $v(t) = -\frac{hl}{vt^2}$; the minus sign signifies that the shadow decreases. **7.3.** It decreases with the speed of 0.4. **7.4*.** 15 cm/s. *Hint.* The moment of the meeting can be found from the condition $x_1(t) = x_2(t)$.

7.5.** $2v\left|\sin\left(\frac{v_4}{2R}t\right)\right|$. *Solution.* We introduce a system of coordinates such that the wheel would roll along the Ox axis and the Oy axis would pass through the centre of the wheel for $t=0$. Then, by virtue of the law of independence of motions, we have the following laws of the variation of the abscissa and the ordinate of the nail:

$$x(t) = vt - R\sin\left(\frac{v}{R}t\right), \quad y(t) = R - R\cos\left(\frac{v}{R}t\right).$$

Thus,

$$x'(t) = v - \frac{v}{R}R\cos\left(\frac{v}{R}t\right) = v\left[1 - \cos\left(\frac{v}{R}t\right)\right],$$

$$y'(t) = \frac{R}{R}v\sin\left(\frac{v}{R}t\right) = v\sin\left(\frac{v}{R}t\right).$$

The speed of the nail at the time moment t is

$$v = \sqrt{(x')^2 + (y')^2} = v\sqrt{1 - 2\cos\left(\frac{v}{R}t\right) + 1}$$

$$= 2v\left|\sin\left(\frac{v}{2R}t\right)\right|.$$

7.6*. The velocity is equal to zero. *Hint.* When the point moves along the circle, the abscissa varies according to the law $x = R\cos\omega t$. **7.7*.** $h = \frac{v^2\sin^2\alpha}{2g}$. *Hint.* The variation of the height of a body proceeds according to the law $h(t) = v\sin\alpha t - \frac{gt^2}{2}$, the vertical component of the velocity at the point of the maximum height is equal to zero. **7.8.** 12 rad/s. The wheel will stop in 2 s. **7.9*.** $v(t) = \frac{2t^3 - 6t^2 + 12t}{\sqrt{t^4 - 4t^3 + 12t^2}}$. *Hint.* Express the distance between the bodies at the moment t as the third side of a triangle whose other two sides are $S_1(t)$ and $S_2(t)$. **7.10.** 40 km/h.

7.11.** At the time moment $t = \frac{t_1 + t_2}{2}$ the object must move with the speed $v = v_0 + at_2$. The law of motion of the object has the form

$$S(t)=\begin{cases} v_0t+at^2/2, & t\leqslant t_1,\\ v_0t+at_1t-at_1^2/2, & t_1<t\leqslant (t_1+t_2)/2,\\ v_0t+at_2t-at_2^2/2, & t>(t_1+t_2)/2.\end{cases}$$

Solution. The object is separated from the rocket at time moment t_1 and moves uniformly with the velocity attained by the rocket by the moment of the object separation. At a certain moment t the object increases its velocity instantaneously to some value v and again moves uniformly until it encounters the rocket at time moment t_2, their velocities being equal at that moment. Consequently, the law of motion of the object without the rocket is a polygonal line whose segments are tangents to the parabola $S(t)=v_0t+(at^2)/2$ at points t_1 and t_2. The abscissa t of the point of intersection of those tangents is the required moment of time. The velocity of the object at time moment t coincides with that at moment t_2 and can be found as the derivative of the function $S(t)$ at the moment t_2. Since $S'(t)=v_0+at$, the tangents to the parabola $S(t)$ at points t_1 and t_2 have the form

$$S(t)=S(t_1)+(v_0+at_1)(t-t_1), \qquad (*)$$
$$S(t)=S(t_2)+(v_0+at_2)(t-t_2).$$

Considering (*) to be a system of equations with respect to the pair of unknowns (S, t), we find that $t=(t_1+t_2)/2$ and, consequently, the law of motion of the object can be represented in the form given in the answer.

7.12*. $t_0=t_1-\sqrt{t_1^2-2S_1}$. *Hint.* The law of motion of the rocket, after switching off the engines, can be written as an equation of a tangent to the curve which is the graph of the law of motion.

Chapter 9

Sec. 1. 1.1*. $\arcsin\frac{x}{\sqrt{2}}-\ln(x+\sqrt{x^2+2})+c$. *Hint.* Divide the numerator by the denominator term-by-term and use formulas (16) and (17). **1.2.** $\frac{3}{5}x^{5/3}-\frac{12}{7}x^{7/6}+c$. **1.3*.** $-\frac{2}{3}(1-x)^{3/2}+\frac{2}{5}(1-x)^{5/2}+c$. *Hint.* Reduce $f(x)$ to the form $f(x)=\sqrt{1-x}-(1-x)\sqrt{1-x}$. To represent $f(x)$ in such a form, it is convenient to introduce a variable $t=1-x$, then $f(x)=g(t(x))$, where $g(t)=(1-t)\sqrt{t}=\sqrt{t}-t\sqrt{t}$. **1.4*.** $-8\left(1-\frac{x}{2}\right)^{1/2}+\frac{8}{3}\left(1-\frac{x}{2}\right)^{3/2}$. *Hint.* See the hint to problem 1.3*. **1.5*.** $-\frac{1}{4}(2x-1)^{-1}+\frac{1}{8}\times$ $x-1)^{-2}+c$. *Hint.* See the Hint to 1.3*. **1.6*.** $\frac{(1+x)^{5/2}}{5}-$

$\frac{(1+x)^{3/2}}{3}+c$. *Hint*. Represent $f(x)$ in the form $f(x)=\frac{\sqrt{(1+x)^3}}{2}-\frac{\sqrt{1+x}}{2}$. 1.7. $\frac{x^2}{2}-x+c$. 1.8. $t+\ln|t|+c$. 1.9. $\frac{12x^{5/6}}{5}+c$. 1.10. $-\frac{2}{3}x^{3/2}+4x^{-1/2}+c$. 1.11. $2x^{1/2}-\frac{2}{3}x^{3/2}+c$. 1.12. $-x+c$. 1.13. $x-\frac{x^3}{3}+c$. 1.14. $x+x^2+c$. 1.15. $mx^{1/m}+3nx^{1/n}+c$. 1.16. x^2-x+c. 1.17. $\frac{2(1+x)^{3/2}}{3}+c$. 1.18. $2x-8\ln|x|+c$. 1.19. $x+2\ln|x-2|+c$. 1.20. $\sqrt[6]{2x}+c$. 1.21*. $-\frac{1}{12}\cos 12x-\frac{1}{10}\cos 10x-\frac{1}{9}\cos 9x-\frac{1}{11}\cos 11x+c$. *Hint*. Reduce the integrand to the form $\sin 12x+\sin 10x+\sin 9x+\sin 11x$ and use rule (4) and formula (4) in the table of antiderivatives. 1.22. $-\frac{1}{4}\cos 4x+\frac{1}{5}\cos 5x+\frac{1}{6}\cos 6x-\frac{1}{7}\cos 7x+c$. 1.23. $\sin\alpha-\cos\alpha+\frac{1}{3}\sin 3\alpha-\frac{1}{3}\cos 3\alpha+c$. 1.24. $\frac{1}{4}x+\frac{1}{32}\sin 8x+c$. 1.25. $-\left(x+\frac{1}{\pi}\cos 4x\right)+c$. 1.26. $-\sqrt{2}\cos\frac{x}{2}+c$. 1.27. $-\frac{1}{2}\cos 2x+c$. 1.28. $-2\cos x+c$. 1.29. $-\frac{1}{8}\cos 8x+c$. 1.30. $-\frac{1}{8}\cos 4x+c$. 1.31. $\frac{1}{2}x-\frac{\sin x}{2}+c$. 1.32. $\frac{1}{4}\sin 2\alpha-\frac{1}{2}\alpha+c$. 1.33. $\tan x-x+c$. 1.34. $-\cot x-x+c$.

Sec. 2. 2.1. $y=x^3+1$. 2.2. $y=x^2$. 2.3. $y=5x-1$. 2.4. $y=\frac{x^2}{2}-\frac{1}{2}$. 2.5. $y=\frac{x^3}{3}+\frac{7}{3}$ and $y=\frac{x^3}{3}+1$. 2.6. $y=3\ln x+1$. 2.7. $S(t)=2-0.25\cos 2t$ (m). 2.8. $p\geqslant 2$, the pedestrians will once meet at $p=2$.

Sec. 3. 3.1. 8. 3.2. 1/2. 3.3. 0. 3.4. 8. 3.5*. $-2\frac{2}{3}$. *Hint*. Make a change $t=2-\frac{x}{2}$. 3.6*. $-1\frac{73}{135}$. *Hint*. See the hint to 3.5*. 3.7*. $\frac{3^{3/2}-2^{3/2}-1}{3}$. *Hint*. Rationalize the denominator. 3.8. $\frac{45}{4}$. 3.9. $\frac{46}{15}$. 3.10. $\frac{\pi-2}{4}$. 3.11. $\ln 2-\frac{1}{2}$. 3.12. $-\frac{1}{2}$. 3.13. $\frac{1}{8}$.

3.14. $\sqrt{2}$. 3.15. $\frac{1}{2}\left(\tan\frac{\pi}{3}-\tan\frac{\pi}{6}\right)$. 3.16. 2. 3.17. 1. 3.18. $2-\frac{\sqrt{2}}{2}$. 3.19. $2\sqrt{2}$. 3.20. $2\sqrt{2}+\frac{4}{3}(3^{3/2}-2^{3/2})$. 3.21. $\ln 2.5+2.5$. 3.22. $4\ln\frac{4}{3}$. 3.23. $2\sqrt{2}$. 3.24. $2\sqrt{2}-1$.

Sec. 4. 4.1. $\min\limits_{x\in[0,\,\pi/2]} F(x)=F(0)=0$, $\max\limits_{x\in[0,\,\pi/2]} F(x)=F\left(\frac{\pi}{2}\right)=1$. 4.2. $\min\limits_{x\in[-1,\,3]} F(x)=F(2.5)=-6.25$, $\max\limits_{x\in[-1,\,3]} F(x)=F(-1)=6$. 4.3. $\min\limits_{x\in[0,\,4]} F(x)=F(0)=0$, $\max\limits_{x\in[0,\,4]} F(x)=F(4)=\frac{16}{3}$. 4.4. $\max\limits_{x\in[-1/2,\,1/2]} F(x)=F\frac{1}{2}=-\frac{3}{8}$, $\min\limits_{x\in[-1/2,\,1/2]} F(x)=F\left(-\frac{1}{2}\right)=-\frac{5}{8}$. 4.5. $y=2-x$; $y=x-3$. 4.6. The curves coincide, $x\in\mathbf{R}$. 4.7. $\left(\frac{6}{5},\frac{36}{25}\right)$. 4.8. $\frac{x^3}{3}-\frac{5x^2}{2}+6x$. 4.9. $y=x-\frac{1}{4}$; $y=x+\frac{1}{4}$. 4.10. $S(t)=\frac{2t^3}{3}$. 4.11*. $A(x)=25x^2+100x$. *Hint.* The law of variation of the force F as a function of the path traversed x can be represented by the formula $F(x)=ax+b$, where the parameters a and b can be found from the hypothesis. The work done by the variable force is its antiderivative which vanishes for $x=0$. 4.12. $S(t)=\frac{2.5t^2}{2}-t$. 4.13. $S(t)=\frac{2}{3}t^2+\frac{10}{3}t$.

Sec. 5. 5.1. $(-\infty,\ -2)\cup(1/2,\ 3)$. 5.2. $(2,\ 3)$. 5.3. $[4,\ +\infty)$. 5.4. $A=-2/\pi$, $B=2$. 5.5. $a=1$. 5.6. $\sqrt{2\pi}$, $\frac{-1+\sqrt{8\pi+1}}{2}$. 5.7. 6 s. 5.10. Yes. 5.11. $A=7$, $B=-6$, $C=3$. 5.12. $\frac{\pi}{2}$, $\frac{7\pi}{6}$, $\frac{3\pi}{2}$, $\frac{11\pi}{6}$. 5.13. 1/2, 2. 5.14. $-\pi$, $-\frac{\pi}{3}$, 0.

Sec. 6. 6.1. $\frac{4}{3}$. 6.2. 9. 6.3. $\frac{343}{3}$. 6.4. $\frac{1}{3}+\ln 2$. 6.5. $3\times\left(1-\frac{1}{4\ln 2}\right)$. 6.6. $4\frac{1}{2}$. 6.7. $12-5\ln 5$. 6.8. $\frac{15}{4}-\ln 2$. 6.9. $\frac{8}{3}$. 6.10. 1. 6.11. $4-\frac{3}{\ln 2}$. 6.12. $7\ln 3$. 6.13. $\frac{9}{2}$. 6.14. $6\frac{1}{3}-\frac{3}{4\ln 2}$. 6.15. $\ln 2$. 6.16. $\frac{19}{24}$. 6.17. $\frac{1}{2}\ln 2$. 6.18. $\frac{8}{3}$. 6.19. $15-16\ln 2$.

6.20. $3\frac{8}{15}$. **6.21.** $\frac{\pi}{2}-\frac{1}{3}$. **6.22.** $9-8\ln 2$. **6.23*.** 8/9. *Hint.* To calculate the area of this figure, it is more convenient to use formula (6.3). **6.24*.** $\frac{3}{8}\pi r^2$. *Hint.* The domain of integration must be divided into two domains, the coordinates of the point of division can be found as a solution of the system $\begin{cases} x^2+y^2=r^2, \\ x-y=0, \\ y\geqslant 0. \end{cases}$ **6.25*.** $1-\pi/4$. *Hint.* $\max(x, y)=1$ are points which are two adjacent sides of a unit square inscribed into the angle of the first quadrant. **6.26.** 1/3. **6.27.** $\pi/2-1$. **6.28.** 1. **6.29*.** πab. *Hint.* Express y in terms of x for $y>0$ and $x>0$, when calculating the definite integral $4\int_0^a b\sqrt{1-x^2/a^2}\,dx$ use formula (9.3.4). **6.30*.** π. *Hint.* Isolate a perfect square with respect to the variable x.

6.31.** 4.3. *Solution.* The equation of the tangent to the curve $y=2x^2$ at the point with abscissa 2 has the form $y-8=8(x-2)$ since $y(2)=8$, $y'(2)=8$. The point of intersection of the tangent and the abscissa axis can be found from the equation $8x-8=0\Leftrightarrow$ $\Leftrightarrow x=1$. The domain of integration must be divided into two intervals [0, 1] and [1, 2]. On the interval [0, 1] we must calculate the area of the figure included between $y=2x^2$ and $y=0$ (the abscissa axis) and on the interval [1, 2], the figure included between $y=2x^2$ and $y=8x-8$. Thus

$$S=\int_0^1 2x^2\,dx+\int_1^2(2x^2-8x+8)\,dx=\frac{2x^3}{3}\Big|_0^1+\left(\frac{2x^3}{3}-\frac{8x^2}{2}+8x\right)\Big|_1^2$$

$$=\frac{2}{3}+\left(\frac{16}{3}-\frac{32}{2}+16-\frac{2}{3}+\frac{8}{2}-8\right)=\frac{4}{3}.$$

6.32. 9. **6.33.** $\ln 2-\frac{5}{8}$. **6.34*.** $2\frac{1}{4}$. *Hint.* The figure must be divided into two curvilinear trapezoids; the abscissa of the point of division is the abscissa of the point of intersection of the tangents. **6.35.** $\frac{45}{4}$.

6.36.** The parabola partitions the square into two parts whose areas are related as 1 : 2. *Solution.* We choose a system of coordinates such that the vertex of the parabola coincides with the point (0, 0) and the Oy axis is the axis of symmetry. Then the equation of the parabola assumes the form $y=ax^2$. The parameter a is chosen in the following way: we designate the length of the side of the square, the midpoint of whose base coincides with the origin, as l; then the point $(l/2, l)$ is the right upper vertex of the square which lies on the parabola, i.e. $l = a(l/2)^2$. From this equation we find that $a = 4/l$. The area S of the square is l^2 and the area cut off by the parabola can be found by the formula

$$S_n = 2\int_0^{l/2} \frac{4}{l} x^2\,dx = \frac{8}{l}\frac{x^3}{3}\Big|_0^{l/2} = \frac{l^2}{3}.$$

Thus the parabola partitions the square into two parts whose areas are related as 1:2.

6.37*. $\frac{S_n}{S} = \frac{3\pi - 8}{3\pi}$, where S is the area cut off by the parabola from the semicircle. *Hint.* Choose a system of coordinates such that the vertex of the parabola coincides with the origin and the Oy axis is the axis of symmetry of the parabola. Then the equation of the parabola has the form $y = ax^2$ and the equation of the circle has the form $(y - R)^2 + x^2 = R^2$. The relation between a and R can be established from the hypothesis (see the solution of problem 6.36**).

6.38.** $\frac{2}{3}$. *Hint.* The parameter a in the equation of the parabola can be found from the condition $f'(-5) = \tan(\pi - \arctan 20)$. **6.39*.** $a = 8$, $a = \frac{2}{5}(6 - \sqrt{21})$. *Hint.* Consider two cases: $a > 2$ and $a < 2$. In the second case take into account that in the passage through the point $x = 1$ the sign of the difference $\frac{1}{x} - \frac{1}{2x-1}$ changes. **6.40.** $\frac{9}{4}\left(1 - \frac{1}{\sqrt[3]{4}}\right)$. **6.41*.** $S = b$ for $a\sqrt{8/3b - 1}$; the problem has a solution for $b \in (0, 8/3)$. *Hint.* To express a as a function of b, we must solve for a an equation whose right-hand side is equal to b and the left-hand side is the area of the figure indicated in the hypothesis. The value of b for which the problem has a solution can be found from the condition $a(b) > 0$, where $a(b)$ is the required function. **6.42*.** $-\pi/6$, $\pi/3$. *Hint.* Take into account that the required value of a may be both larger than $\pi/6$ and smaller than $\pi/6$ (in the second case the area can be found by the formula

$S = \int_a^{\pi/6} |\sin 2x|\,dx$). **6.43*.** $a \in \left(0, \frac{4}{3}\right)$; $b = \frac{16}{9a^2} - 1$. *Hint.* See the Hint to 6.37*. **6.44*.** $y = \left[x - \arcsin\left(1 - \frac{\sqrt{2}}{4}\right)\right]\frac{4}{\sqrt{2}(4 - \sqrt{2})} + \frac{\sqrt{2}}{4}\sqrt{8\sqrt{2} - 2}$. *Hint.* When solving the problem, use the formula $\sqrt{1 + \cos 2x} = \sqrt{2}\,|\cos x|$.

Sec. 7. 7.1. 3/4. 7.2. $a = \sqrt[4]{3}$. 7.3. $a = 1$. 7.4*. $S(-1) = 125/6$, $\min_{k \in \mathbf{R}} S(k) = S(2) = 32/3$. *Hint.* The limits of integration can be found as the roots $x_1(k)$ and $x_2(k)$ of the equation $x^2 + 2x - 3 = kx + 1$. It should be borne in mind that the inequality $y_2(x) \leqslant y_1(x)$ is always satisfied on the interval $[x_1(k), x_2(k)]$.

7.5. $\min_{x_0 \in [1/2, 1]} S(x_0) = S\left(\frac{4}{5}\right) = \sqrt[3]{\frac{5}{4}} \cdot \frac{48}{25}$. 7.6*. (3/2, 13/4).

Hint. Use the equation $S = h\left(\frac{a+b}{2}\right)$, where $h = 1$, $a = f_{x_0}(1)$, $b = f_{x_0}(2)$, $f_{x_0}(x)$, is the equation of a tangent to $y = x^2 + 1$ at a point with the abscissa $x_0 \in [1, 2]$.

7.7*. $a = -1$. *Hint.* The function $f(x) = x^3 + 3x^2 + x + a$ is monotonic between two successive extrema. The further proof of the problem can be based on the following lemma.

Lemma. The area of the figure bounded by the straight lines $x = c$, $x = b$, $b > c$, by the graph of the differentiable monotone function $f(x)$ and the straight line $y = f(a)$, where $a \in [c, b]$, attains its least value in the case when $y = f\left(\frac{b+c}{2}\right)$.

Proof. For the sake of simplicity, let us prove this result in the case when $c = 0$, $b = 1$ and $f(b) = 1$. For a fixed value of a the area can be represented as the following function:

$$S(a) = \int_0^a [f(a) - f(x)\, dx + \int_a^1 [f(x) - f(a)]\, dx$$

$$= [f(a)\, x - F(x)] \Big|_0^a + F(x) \Big|_a^1 - f(a)\, x \Big|_a^1$$

$$= f(a)\, a - F(a) + F(0) + F(1) - F(a) - f(a) + f(a)\, a, \quad (*)$$

where $F(x)$ is a certain antiderivative of $f(x)$. Collecting terms in $(*)$, we obtain

$$S(a) = f(a)(2a - 1) - 2F(a) + F(0) + F(1). \qquad (**)$$

Differentiating $S(a)$ with respect to a with due account of the fact that $F'(a) = f(a)$, we get an equation for finding critical points:

$$S'(a) = f'(a)(2a - 1) + 2f(a) - 2f(a) = 0. \qquad (***)$$

Since $f(a)$ is monotonic by the hypothesis, it follows that $f'(a) \neq 0$ on the interval [0, 1] and, consequently, equation $(***)$ has a single root $a = 1/2$. For $a > 1/2$ we have $S'(a) > 0$ and $S(a)$ increases, and for $a < 1/2$ we have $S'(a) < 0$ and $S(a)$ decreases. Consequently, $S(a)$ attains its minimum value for $a = 1/2$.

7.8*. For $a = 1$ the area assumes the greatest value and for $a = 1/2$ it assumes its least value. *Hint.* Use the lemma indicated in the Hint to problem 7.7*. 7.9*. $a = 2/3$. *Hint.* Use the lemma in the Hint to problem 7.7*. 7.10*. For $a = 1/2$ the area has the least value and for $a \to 0$ it has its greatest value. *Hint.* See the lemma in the Hint to problem 7.7*. 7.11*. $a = 0$. *Hint.* See the lemma in the Hint to problem 7.7*.

Sec. 8. 8.1. 2π. 8.2*. $\frac{3}{10}\pi$. *Hint.* Consider the difference between the volumes of the bodies resulting from the rotation of the curves $y=\sqrt{x}$ and $y=x^2$. 8.3. $\frac{\pi}{4}\left[\frac{e^{2b}-e^{2a}}{2}+\frac{e^{-2a}-e^{-2b}}{2}+2(b-a)\right]$. 8.4*. $\frac{8}{3}\pi$. *Hint.* Pass to the functions $x_1(y)=1+\sqrt{1-y}$, $x_2(y)=1-\sqrt{1-y}$ and consider the volume of the required body as the difference of the volumes of two bodies resulting from the rotation of the figures bounded by the curves $x_1(y)$ and $x_2(y)$ about the Oy axis. 8.5*. $\pi^2/4$. *Hint.* The required volume is equal to that of the body resulting from the rotation of the curves $y = \sin x$, $x=\pi/2$ about the Ox axis. 8.6*. $\frac{3\pi}{2}$. *Hint.* See the Hint to 8.5*.

Sec. 9. 9.1. $a = 18$. 9.2*. 288. *Hint.* At the moments of the beginning of motion and of the stop the speed of the body is equal to zero. 9.3. 216 m. 9.4. $S(t)=\begin{cases} t^2 & \text{if } 0\leqslant t<3, \\ 6t+9 & \text{if } t\geqslant 3.\end{cases}$ 9.5*. 14/15 (J). *Hint.* $F(x) = k/x^2$, where k can be found from the hypothesis. 9.6*. 33.75 (J). *Hint.* $F(x) = kx$, where k can be found from the hypothesis.

Chapter 10

Sec. 1. 1.1. 32.5 km/h. 1.2. $\frac{17}{3}$. 1.3. 60 km/h. 1.4. 8 m/s. 1.5. 3×4 km. 1.6. 12 km/h, 10.5 km/h. 1.7. 6 h and 2 h. 1.8. $\frac{S}{4t}(3-\sqrt{5})$, $\frac{S}{4t}(\sqrt{5}-1)$. 1.9. 30 km/h. 1.10. 20 km/h, 60 km/h. 1.11. 56 km. 1.12. $v_r=1$ km/h. 1.13. $\frac{3S-v+\sqrt{9S^2+2Sv+v^2}}{2}$ km/h. 1.14. 14 h. 1.15. 60 km/h. 1.16. 48 km/h. 1.17*. 50 km and 150 km. *Hint.* Introduce the unknowns $\omega_1 = 1/V_1$ and $\omega_2 = 1/V_2$, where V_1 and V_2 are the speeds of the motor-cyclist and the cyclist, respectively. 1.18. 100 km/h. 1.19. 100 km/h. 1.20. $9+\sqrt{11}$ km/h. 1.21. The speed of the ships is 15 km/h and that of the river flow is 3 km/h. 1.22. 6 km/h, 21 km/h, 45 km. 1.23. 63 km/h. 1.24. The speed of the cyclist is 20 km/h and that of the car is 80 km/h. 1.25. The speeds of the pedestrian, the cyclist and the horseman are 6 km/h, 9 km/h and 12 km/h, respectively. The distance is 42 km. 1.26. 30 km/h, 20 km/h and 30 km. 1.27. 480 km. 1.28. 2 min. 1.29. 15 km. 1.30. 3 km/h, 45 km/h. 1.31. 3 km/h, 45 km/h. 1.32. 6 km/h. 1.33. $\frac{13}{3u+v}$. 1.34*. 50 km/h. *Hint.* Take into account the fact that the speed of the train proceeding from opposite direction with respect to the observer who is on the other train is equal to the sum of the speeds of the trains relative to a stationary observer. **1.35.** 108 km/h. **1.36.** 28 km, 20 km/h. **1.37.** 11 h 55 min. **1.38.** 7 km/h. **1.39.** The passenger train, 21 h, the goods train, 28 h. **1.40.** 15 h and 12 h. **1.41.** 6 h and 4 h. **1.42.** The speed of the first car is 9/8 times as high as that of the second. **1.43.** 16 h. **1.44.** The speed of the motorcyclist is 4 times as high as that of the cyclist. **1.45.** 20/3 h and 10/3 h. **1.46.** In 4 h. **1.47.** 1:2, 1:3. **1.48*.** 20 km/h and 40 km/h. *Hint.* The condition of proportionality of the speeds and times signifies that $v_1/t_1 = v_2/t_2$, where t_1 and t_2 are the times of movement with the speeds v_1 and v_2. **1.49.** The length of the circumference of the front wheel is 2 m and that of the rear wheel is 3 m. **1.50.** 90 km/h, 75 km/h, 60 km/h. **1.51.** 117 km, 24 km/h, 22.5 km/h.

1.52. $\frac{13}{4}t$, $\frac{11}{5}t$. **1.53.** $4 \leqslant v \leqslant \frac{8+\sqrt{61}}{3}$. **1.54.** With the speed higher than $\frac{S+vt+\sqrt{(S-vt)^2+4tvl}}{2t}$. **1.55.** It will be enough. **1.56.** $2 \leqslant v < 6$. **1.57.** $5 < v < 10$. **1.58.** The village is farther away from the highway than the school is from the river. **1.59.** 4 s and 6 s. **1.60.** 1/80, 1/90. **1.61.** 4 and 6. **1.62.** $\frac{\pi R}{T}\left(\sqrt{1+\frac{4T}{t}} \pm 1\right)$.

1.63. $\frac{-a+\sqrt{a^2+240at}}{120t}S$. **1.64*.** $1\frac{1}{11}$ a. m. *Hint.* Use the fact that $\omega_{min}/\omega_h = 12$, where ω_{min} and ω_h are the angular velocities of the motion of the minutes hand and the hours hand respectively. **1.65.** By half a minute. **1.66.** 11 m approx. **1.67.** 9 km/h. **1.68.** 3 min. **1.69.** 1/4 h. **1.70.** 21 km. **1.71.** In 7 s after the first body begins to fall. **1.72.** 16 s. **1.73.** 60°. **1.74.** 10 s. **1.75.** $v_0 = 20$ m/s. **1.76.** In 5 s 0.5 m short of the boundary of the field. **1.77.** 20 m. **1.78.** 20 km/h. **1.79.** The second car was the first to stop. $a_2 = -8$ m/s². **1.80.** 2 s. **1.81*.** $a_1 : a_2 = 7/9$. *Hint.* Bear in mind that the times of the accelerations of the two trains are different. **1.82*.** $S_1 = \frac{2}{5}S$, $S_2 = \frac{1}{2}S$. *Hint.* See the Hint to 1.81*.

Sec. 2. 2.1. 24 m³ a day. **2.2.** 45 h. **2.3.** 132 min, 110 min. **2.4.** 6 min and 10 min. **2.5.** 6 min, 8 min, 12 min. **2.6.** $T + \sqrt{T(T-t)}$, $T - t + \sqrt{T(T-t)}$, $\sqrt{T(T-t)}$ $(T > t)$. **2.7.** $t_1 = 3$ h, $t_2 = 6$ h, $t_3 = 2$ h. **2.8.** 400 parts. **2.9.** In 14 days. **2.10.** In 10 days. **2.11.** Tractor of make A does 12 hectares, tractor of make B, 16 hectares. **2.12*.** 4 times. *Hint.* The condition in the form of an inequality is used to choose the unique value of the required unknown out of the two values obtained. **2.13.** 50 h. **2.14.** 9 days. **2.15.** 10 h and 8 h. **2.16.** 9 km a month. **2.17.** $6\frac{2}{3}$ h and $5\frac{1}{3}$ h. **2.18.** 5/2 m³. **2.19.** 3 m³/h. **2.20.** 60%. **2.21.** In 14 and 11 days. **2.22.** 4 h and 6 h. **2.23.**

The first has to read 20 pages a day and the second, 35 pages a day. **2.24.** 12 h. **2.25.** $c = 9\frac{11}{16}$. **2.26.** 600 m³. **2.27*.** 20 m³. *Hint.* Verify the solutions obtained by substitution into all the equations of the system. **2.28*.** In 40 h. *Hint.* Use the formula for the sum of an arithmetic progression. **2.29*.** (5/4) V. *Hint.* See the Hint to 2.28*. **2.30.** Assume that T_i ($i = 1, 2, 3$) is the time needed for the ith pump to pump out the water from its reservoir. Then $T_1 > T_3 > T_2$, and

$$T_1 : T_3 : T_2 = (1+\sqrt{\alpha}) : 1 : [\alpha(1+\sqrt{\alpha})], \quad \text{if} \quad \alpha < \frac{3-\sqrt{5}}{2},$$

$$T_1 : T_3 : T_2 = \frac{1-\alpha}{\alpha} : 1 : \frac{\alpha}{1-\alpha}, \quad \text{if} \ \frac{3-\sqrt{5}}{2} \leqslant \alpha < \frac{1}{2}.$$

2.31*. $t\,\frac{n-1}{n}$. *Hint.* Using the hypothesis, first show that all the pipes began operating before the reservoir was half full.

Sec. 3. 3.1. Approximately in 23 years. **3.2.** Approximately in 55 years. **3.3.** 12 kopecks, 80 kopecks. **3.4.** 5%. **3.5.** 10%. **3.6.** 10%. **3.7.** 200 rubles, 3-per-cent. **3.8.** 42.3%. **3.9.** 726. **3.10.** 50%. **3.11.** In the number of years which is larger than

$$\log\left(\frac{3Np-100M}{Np-100M}\right)\Big/\log\left(1+\frac{p}{100}\right).$$

3.12. More than in $\log\left(\frac{2Np-100n}{Np-100n}\right)\Big/\log\left(1+\frac{p}{100}\right)$ h.

Sec. 4. 4.1. Eleven 4.0 grades, seven 3.0 grades, ten 2.0 grades, and two 1.0 grades. **4.2.** Two 2.0 grades and seven 3.0 grades. **4.3.** Nine 5-storey houses and eight 8-storey houses. **4.4.** 10 Moskvich cars and 19 Volga cars. **4.5.** 33. **4.6*.** 25 boxes of the second kind and 4 boxes of the third kind. *Hint.* First find the cost of transporting one article in a box of each kind. **4.7*.** The first worked for 3 days and the second for 2 days. *Hint.* Find the number of days each excavator could operate. **4.8*.** 45.20. *Hint.* Bear in mind that the given condition is associated with two systems of equations, one of which is inconsistent. **4.9.** 13 min. **4.10.** 20 rafts. **4.11.** 15 or 95. **4.12.** 48. **4.13.** 32. **4.14*.** 5. *Hint.* By adding a certain digit to a number on the right we pass to a new number in which the number of unities is equal to the digit added and the number of tens, to the initial number. **4.15.** 6464. **4.16*.** 285 714. *Hint.* See the hint to 4.14*. **4.17.** 32. **4.18*.** 45 or 54. *Hint.* To obtain the sum of all even two-digit numbers, use the formula for the sum of the terms of an arithmetic progression with the difference $d = 2$ and $a_1 = 10$. **4.20.** 21 and 10. **4.21.** 31 and 41. **4.22*.** $A = 42$, $B = 35$. *Hint.* Use the formula $n = m \cdot p + k$, where n is the dividend, m is the divisor, p is the quotient and k is the remainder. **4.23*.** $N = 37$. *Hint.* See the hint to 4.22*. **4.24*.** $\frac{3}{10}, \frac{4}{17}, \frac{5}{26}$. *Hint.* The problem reduces to solving a system of quadratic inequalities in the set of natural numbers.

Sec. 5. 5.1. 1.5. kg. 5.2. $\frac{4}{5}r-24$, $32-\frac{4}{5}r$, $\frac{125}{4}\leqslant r\leqslant\frac{135}{4}$. 5.3. 7 kg. 5.4. 60 kg. 5.5. $\frac{2n-m+\sqrt{m^2+4n^2}}{2}$, $\frac{2n+m+\sqrt{m^2+4n^2}}{2}$. 5.6*. $\frac{mn}{m+n}$. *Hint.* Introduce x, which is the weight of the cut-off piece, as an unknown; c_1 and c_2 are the concentrations of copper in the first and the second piece respectively. 5.7. 5% and 11%. 5.8*. In the volume of 4 cm³. *Hint.* Use the formula $m=\rho V$ relating the mass, the density and the volume. 5.9. 12%, 24%, 48%. 5.10*. 29%. *Hint.* Introduce the concentrations c_1, c_2, c_3, c_4 as unknowns. The hypothesis produces a system of three equations for four unknowns c_1, c_2, c_3, c_4. When investigating the system, take into account that we seek the combination of the unknowns $\frac{2c_2+c_4}{3}$. 5.11. 13/4 times. 5.12. 5 g and 20 g. 5.13. 14 kg, 7 kg, 16 kg. 5.14. The first pipe feeds the liquid two times quicker than the second. 5.15. 50%. 5.16. 12.5 g. 5.17. 170 kg. 5.18. 40%, $43\frac{1}{3}$%. 5.19. 2 litres. 5.20. 10 litres and 90 litres. 5.21. 10 litres. 5.22. 1/6. 5.23. If $p=q$, then any number of pannings retain the percentage of gold; in that case the problem has a solution if $r\leqslant k$, the number of pannings being arbitrary. If $q<p$, then the number of pannings n is defined by the inequality $n\geqslant\log\frac{r(100-k)}{k(100-r)}\Big/\log\frac{100-q}{100-p}$. Now if $p<q$, then $n\leqslant\log\frac{r(100-k)}{k(100-r)}\Big/\log\frac{100-q}{100-p}$.

Chapter 11

Sec. 1. 1.1. 20 cm. 1.2. $\frac{2bc\cos(\alpha/2)}{b+c}$. 1.3. 75. 1.4. $m(m\cos\beta\pm\sqrt{c^2-m^2\sin^2\beta})\sin\beta$. 1.5. No. 1.6. $\frac{a^2}{8}\sin 2\alpha$. 1.7. $\frac{c}{2}\tan\alpha\times(r\cos\alpha-c)$. 1.8. $\frac{1}{2}cr\frac{\sin^2\alpha}{\cos\alpha}$. 1.9. $-2S\cos^2\alpha\cos 2\alpha$. 1.10. 288 cm². 1.11. $\sqrt{3}-1$. 1.12. $\frac{a^2\tan^2\alpha\sin 2\alpha}{2(1+\cos\alpha)}$. 1.13. 4. 1.14. $\frac{b^2\sin\alpha(5\sin\beta+3\cos\beta\tan\alpha)}{16\sin(\alpha+\beta)}$. 1.15. $\frac{1}{2}$. 1.16. $\frac{c^2}{2}\times\frac{\cos^2\beta\sin 2\alpha}{\cos(\alpha-\beta)\cos(\alpha+\beta)}$. 1.17*. $\frac{\sqrt{15}}{2}$. *Hint.* Complete the triangle to obtain a parallelogram and use Hero's formula. 1.18. $\frac{\sqrt{3}}{4}$. 1.19. $\frac{9}{2}\tan\alpha\cot\beta+\frac{7}{2}$. 1.20. $\left(\frac{2+\sqrt{3}}{\sqrt{6}}\right)$. 1.21. $\sqrt{3}$. 1.22. $4\sqrt{13}$. 1.23. 75. 1.24. $\frac{2}{5}l^2$. 1.25. $|MF|=\frac{a\sqrt{17}}{12}$. 1.26. 3, 5, 7. 1.27. $1+\sqrt{2}$. 1.28. $\frac{\pi}{4}\pm\arccos\frac{1+2\sqrt{2}}{4}$, $\frac{\pi}{4}\mp\arccos\frac{1+2\sqrt{2}}{4}$. 1.29. $|BC|=\sqrt{b(b+c)}$. 1.30. $\frac{3\sqrt{2}}{4}$. 1.31. $\frac{2}{3}$ cm². 1.32. $\frac{2-\sqrt{3}}{\sqrt{5}}$. 1.33. $\frac{|NC|}{|AC|}=\frac{3}{4}$. 1.34. $\frac{|AN|}{|NC|}=\frac{1}{9}$ or $=9$. 1.35. $\frac{1}{2}\arccos\frac{1}{4}+\frac{\pi}{2}=\pi-\arcsin\sqrt{\frac{5}{8}}$. 1.36. $4(1-\alpha)$. 1.37. $\frac{23}{90}$. 1.38. $\frac{25}{16}$. 1.39. $\frac{1}{12}$. 1.40. $\frac{\pi}{6}$. 1.41. $\frac{\beta(1+2\alpha+\alpha\beta)}{(1+\alpha)(1+\beta)(1+\alpha+\alpha\beta)}$.

Sec. 2. 2.1. $s=16\sqrt{3}$, $l=8$. 2.2. 256 cm³. 2.3. $\sqrt{3}$ m. 2.4. 2 cm. 2.5. 6 m. 2.6. 9.6 cm². 2.7. $\frac{a^2}{2}\sin 2\alpha$. 2.8. $\frac{4}{(\sqrt{3}+1)^2}$. 2.9. $\frac{\sqrt{10}}{2}$. 2.10. $\frac{1}{2}\sqrt{ab+\frac{(a-b)^2}{4\cos^2\alpha}}$. 2.11. l. 2.12. Side CD. 2.13. $\frac{12}{5}$. 2.14. $1+\frac{(1+3\tan\alpha)^2}{2\tan\alpha}-\frac{7}{8}\sqrt{a^2-16}$. 2.15. $\frac{1}{2}(a-b)^2\sin\alpha$. 2.16. $4h^2\cot\alpha-2h\sqrt{a^2-4h^2}$. 2.17. $\frac{7}{8}$. 2.18. 2. 2.19. $|AB|=|BC|=2$, $|AD|=\sqrt{3}$, $|DC|=1$, $S=\frac{3}{2}\sqrt{3}$. 2.20. $\frac{4}{5}$. 2.21. $\frac{15\sqrt{3}}{2}$ cm². 2.22. $2\sqrt{2}$ cm. 2.23. $|MK|=\frac{a\sqrt{13}}{6}$. 2.24. $\frac{|AM|}{|MD|}=\frac{2}{3}$. 2.25. $\frac{\sqrt{m^2+n^2}}{2}$. 2.26. $\frac{3}{2}S$. 2.27. $S=\frac{a+b}{4}\sqrt{(a+b)(3b-a)}$. 2.28. $\frac{4}{5}S$. 2.29. $\sqrt{35}$. 2.30. $\frac{5}{12}$. 2.31. $\frac{11}{12}$. 2.32. $\frac{37}{72}$. 2.33. 1. 2.34. $\frac{1}{2}\sqrt{\frac{\sqrt{17}-1}{2}}$. 2.35. $|KM|=2\sqrt{Q}\sqrt[4]{3}$, $|LN|=\sqrt[4]{\frac{Q}{\sqrt{3}}}$.

Sec. 3. 3.1. $\frac{9\sqrt{7}}{16}$. 3.2. $2\sqrt{Rr}$. 3.3. 8. 3.4. $\frac{a^2}{3}(3+\pi-3\sqrt{3})$. 3.5. $1-\sqrt{3}+\frac{2}{3}\pi$. 3.6. $\frac{150}{7}$ cm². 3.7. $\frac{5}{6}R^2(2\sqrt{3}+5\pi)$.

3.8. $\frac{1}{2}R^2\cot\alpha-\frac{1}{2}R^2\left(\frac{\pi}{2}-\alpha\right)$. 3.9. 8. 3.10. 7. 3.11. $S=\frac{\pi(4R^2-l^2)^2}{64R^2}$. 3.12. $r=\frac{d^2-(R_1-R_2)^2}{4(\sqrt{R_1}+\sqrt{R_2})^2}$. 3.13. 8 cm. 3.14. $\frac{a\sqrt{3}+r-\sqrt{4r^2+2ar\sqrt{3}}}{3}$. 3.15. $\frac{2}{3}(3-\sqrt{5})$. 3.16. $\left(\frac{3}{5}h, \frac{12}{5}h\right)$. 3.17. $3\sqrt{13}$. 3.18. $\frac{8}{5}$ cm². 3.19. $\frac{32}{\pi m^4}(\arccos m-m\times\sqrt{1-m^2})$. 3.20. $\frac{2\sin^2\alpha}{\alpha(1+\sin 2\alpha+\sin^2\alpha)}$. 3.21. The area of the square is larger than that of the circle. 3.22. $|AB|=\frac{6}{\sqrt{5}}$.

Sec. 4. 4.1. $\frac{\pi c}{1+\sqrt{2}}$. 4.2. $\frac{b\,|\cos(3\alpha/2)|}{2\sin\alpha\cos(\alpha/2)}$. 4.3. $r=\frac{R\sin(\alpha/2)}{1+\sin(\alpha/2)}$. 4.4. $\frac{abc}{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}}$.

4.5. $\frac{a^2(3\sqrt{3}-\pi)}{24}$. 4.6. $\frac{b+c-2\sqrt{bc}\cos\alpha}{2\sin^2\alpha}$. 4.7. $\sqrt{3}$, $2\sqrt{3}$ (or $2\sqrt{3}$, $\sqrt{3}$). 4.8. $\frac{5\sqrt{13}}{12}$. 4.9. $2R^2\frac{\sin^3(\alpha+\beta)\sin\beta}{\sin\alpha}$. 4.10. $\sqrt{15+6\sqrt{3}}$. 4.11. $R=c\frac{\sin(\alpha/2)\cos(\beta/2)}{\cos(\alpha/2+\beta/2)}$. 4.12. $S=\frac{1}{2}\frac{R^2(a+R)^3}{2(a-R)(a^2+R^2)}$. 4.13. $\frac{5\pi-6\sqrt{3}}{72}a^2$. 4.14. $\frac{a}{2}\times\frac{1-\sin(\alpha/2)}{1+\sin(\alpha/2)}\tan\frac{\alpha}{2}$. 4.15. $a\sqrt{\frac{13-4\sqrt{7}}{2}}$. 4.17. $S=\frac{7-4\sqrt{3}}{4}a^2\left(\frac{5}{6}\pi-\sqrt{3}\right)$. 4.18. $\frac{ab\sin\alpha}{a+b}\,\frac{1-\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}}$. 4.19. $R=2$.

4.20. $\frac{11}{10}$. 4.21. $r_1=\frac{\sqrt{6}}{6}(14-\sqrt{70})$, $r_2=\frac{\sqrt{6}}{8}(21-\sqrt{105})$. 4.22. $\frac{R\sqrt{3}(\sqrt{7}+5)}{\sqrt{7}}$. 4.23. 4 : 3. 4.24. $150+\frac{250}{\sqrt{3}}$. 4.25. $\frac{3\sqrt{15}}{2}$. 4.26. $\frac{5/4-\cos\beta}{2\sin\beta}\,\frac{b\sin\alpha}{\sin(\alpha+\beta)}$. 4.27. $\frac{\sqrt{7}}{4}$. 4.28. $\frac{125}{4}(3+\sqrt{3})$. 4.29. $\frac{50}{3}(6-\sqrt{3})$. 4.30. $128(3+2\sqrt{2}):49$. 4.31. $\sqrt{\frac{2}{4-\pi}}$.

4.32. $|AC| = \sqrt{10}$, $|AB| = 3\sqrt{2}$. 4.33. $\frac{4R^2 \sin\alpha \cos^4\alpha}{\cos 3\alpha}$. 4.34. $\frac{a^2\sqrt{3}}{26}$. 4.35. 22. 4.36. $S = \frac{1}{2} l(l-n)\sin\beta\left(1+\frac{l}{2n}\sin\beta\tan\frac{\beta}{2}\right)$.
4.37. $\frac{3\sqrt{3}}{5\pi-3}$. 4.38. $R_1 = \frac{\sin C}{\sin(B+C)} \frac{3-2\sqrt{2}\cos B}{4\sin B}$, $R_2 = \frac{\sin C}{\sin(B+C)} \frac{3+2\sqrt{2}\cos B}{4\sin B}$.
4.39. $\tan\frac{\alpha}{2}\sin 2\alpha$. 4.40. $\frac{1}{\sin\alpha+\cos\alpha-1}$. The ratio is the least for $\alpha = 45°$. 4.41. $\frac{\sin(\alpha/2)\sin 2\alpha}{\sin(3\alpha/4)\sin(7\alpha/4)}$. 4.42. $\arctan\frac{1}{\cos\alpha}$.
4.43. $\frac{|AE|}{|DE|} = \frac{\cos^2((\beta-\gamma)/2)}{\cos^2((\beta+\gamma)/2)}$, where $\gamma = \angle ACB$, $\beta = \angle ABC$.
4.44. $\frac{\sqrt{3}-1}{\sqrt{6}}$. 4.45. 25π. 4.46. 5 cm. 4.47. $\arccos\frac{3}{5}$, $\frac{\pi}{2} - \arccos\frac{3}{5}$ or $\arccos\frac{4}{5}$, $\frac{\pi}{2} - \arccos\frac{4}{5}$. 4.48. 14 cm. 4.49. 3, 4, 5.
4.50. $\frac{1}{2}$ dm. 4.51. $\sqrt{91}$ cm. 4.52*. $2\sqrt{5}$. *Hint.* Introduce the acute angle of the triangle α as an unknown and set up an equation to find α by means of the theorem on a tangent and a secant.
4.53. 240 cm². 4.54. $\frac{\pi}{18}$, $\frac{7\pi}{18}$. 4.55. $\frac{\pi}{12}$, $\frac{7\pi}{12}$. 4.56. $\frac{\pi}{6}$.
4.57. $\frac{\pi}{4} - \arccos\left(\frac{4}{\sqrt{6}} - \frac{1}{\sqrt{2}}\right)$, $\frac{\pi}{4} + \arccos\left(\frac{4}{\sqrt{6}} - \frac{1}{\sqrt{2}}\right)$.
4.58. $\frac{2}{3}$. 4.59. $\frac{S_\Delta}{S_0} = \frac{3\sqrt{3}(\sqrt{13}-1)}{32\pi}$. 4.60. $\frac{m\sqrt{k^2+2k\cos A+1}}{2(k+1)\cos(A/2)}$.
4.61. $\arccos\sqrt{2(1-S)}$. 4.62. $\frac{3\pm 2\sqrt{2}\sin(\alpha/2)}{2\cos^2(\alpha/2)} a$. 4.63. $R = \sqrt{2}$ cm.
4.64*. $\sqrt{\frac{3}{2}}$, $\sqrt{\frac{2}{3}}$. *Hint.* Introduce the distance from the point D to the point of tangency of the circle and the straight line AC as an unknown. 4.65. The triangle is equilateral, the length of the side is 8. 4.66. $|AB| = 10$, $|BC| = 6$, $|AC| = 12$.

Sec. 5. 5.1. $h^2\sqrt{3}$. 5.2. $R = \frac{a}{2|\cos\beta|}$. 5.3. $\frac{9\sqrt{3}r^2}{4}$.
5.4. $r\sqrt{7}$. 5.5. $\frac{75}{2}$. 5.6. 2. 5.7. $|O_1O_2| = \frac{3}{5}$. 5.8. 12.5 cm.

5.9. $\frac{2\pi}{3}$. 5.10. $2(\sqrt{6}-\sqrt{2})$. 5.11. $\frac{103\sqrt{17}}{130}$. 5.12. $\frac{4+3\sqrt{3}}{4}$. 5.13. $\left(\frac{2r}{\sin\gamma}-m\right)\left(r-\frac{r^2}{m}\tan\frac{\gamma}{2}\right)$. 5.14. $\frac{S_{tr}}{S_{cir}}=-\frac{4R^2\sin^3\alpha\cos\alpha}{\pi}$. 5.15. $\frac{\pi}{4}$, $\frac{3\pi}{4}$. 5.16. $2\pi^{-1}(\operatorname{cosec}\alpha+\operatorname{cosec}\beta)$. 5.17. $\frac{9}{2}r^2$. 5.18. 3 cm, 8 cm. 5.19. $\frac{4R^3}{S}$. 5.20. 8. 5.21. $12\sqrt{15}$. 5.22. 14.4. 5.23. 3. 5.24. 10 : 11. 5.25. The trapezoid is isosceles; 75° and 105°. 5.26. 210. 5.27. $\frac{9}{16}$. 5.28. $\left.\begin{matrix}|MN|\\|ND|\end{matrix}\right\}=a+l\sin^2\frac{\alpha}{2}\pm\sqrt{a^2+2al\sin^2\frac{\alpha}{2}-l^2\cos^2\frac{\alpha}{2}\sin^2\frac{\alpha}{2}}$. 5.29. $\frac{7}{\sqrt{85}}$, $\frac{34}{5}$. 5.30. $|AC|=\frac{2}{\sqrt{3}}$. 5.31. $\frac{5}{\pi}$.

Chapter 12

Sec. 1. 1.1. $\arccos\frac{1}{\sqrt{3}}$. 1.2. $d^3\sqrt{3}$. 1.3. $2a^3\sin\frac{\alpha}{2}\times\sqrt{\sin\frac{\alpha}{2}\sin\frac{3\alpha}{2}}$. 1.4. $\frac{b^3\sqrt{\cos 2\alpha}}{\sin\alpha}$. 1.5. $\varphi=\arccos(\sin\alpha\sin\beta)$. 1.6. 576 cm². 1.7. $\sqrt{2}d^2\sin 2\varphi\cos(45°-\alpha)$. 1.8. $a^2b\sin\alpha\sin\beta$. 1.10. $\frac{\sqrt{3}}{12}l^3\cos^2\alpha\sin\alpha$. 1.11. $\frac{6\sqrt{3}\sin(\alpha+30°)r^2}{\cos\alpha}$. 1.12. $\sin\alpha=\frac{\sqrt{6}}{3}$, $V=\frac{a^3}{6}$. 1.13. $\frac{a\sqrt{4b^2-a^2}}{2(a+2b)}\tan\alpha$. 1.14. $\frac{\sqrt{3}}{4}H^3\left(3\tan^2\frac{\alpha}{2}-1\right)$. 1.15. $V=\frac{a^3\cos^2(\alpha/2)\cot(\alpha/2)}{3(3-4\sin^2(\alpha/2))}$. 1.16. $\frac{a}{\sin\varphi}$. 1.17. $\frac{c^2\sin\alpha\cos\alpha}{2\cos\beta}$. 1.18. $V=\frac{1}{12}\sqrt{\frac{1}{2}(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}$. 1.19. $\frac{1}{3}l^3\sin\frac{\beta}{2}\sqrt{\cos^2\frac{\beta}{2}-\cos^2\alpha}$. 1.20. $\sin\frac{\theta}{2}=\frac{\sqrt{1+3\cos^2\alpha}}{2}$. 1.21. $\pi-2\arcsin\frac{1}{2\sin(\alpha/2)}$. 1.22. $\frac{b-a}{2\sqrt{3}}\tan\alpha$. 1.23. $a\sqrt{\sqrt{2}-1}$. 1.24. $\frac{2}{3}$. 1.25. $\frac{1}{12\sqrt{2}}\sqrt{(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}$.

1.26. $\frac{\sqrt{2}}{6}a^3\sqrt{\frac{\cot^2(\alpha/2)}{1-\cot^2(\alpha/2)}}$. 1.27. $-\frac{2}{3}l^3\frac{\cos(\beta/2)\cos\beta}{\sin^3(\beta/2)}$.
1.28. $\frac{d^2\tan(\alpha/2)}{2\cos\beta}$. 1.29. $V=\frac{4}{3}h^3\frac{\sin^2(\alpha/2)}{\cos\alpha}$. 1.30. $Q=\frac{\sqrt{3V}}{H}\sqrt{4H^3+3V}$. 1.31. $2\arcsin\left(\sqrt{2}\frac{\alpha}{2}\right)$.
1.32. $\operatorname{arccot}\sqrt{\frac{\tan^2\alpha-1}{2}}$. 1.33. $\frac{l^3\cot\alpha\cot\beta}{3[1/\sin^2\alpha+\cot^2\beta]^{3/2}}$.
1.34. $6\sqrt{2}-\sqrt{6}+4$. 1.35. $V=\frac{d^2\sqrt{3l^2-d^2}}{6}$, $S_{\text{lat}}=\sqrt{12l^2-d^2}\times\frac{d}{2}$. 1.36. $2\arcsin\frac{\sqrt{3}-1}{2}$. 1.37. $2\arccos\frac{\sqrt{3}}{2\sin(\varphi/2)}$.
1.38. $\frac{a^3(5+\sqrt{5})}{24}$. 1.39. $\arctan\left(\tan\alpha\cos\frac{\pi}{n}\right)$. 1.40. $\theta=2\arcsin\left(\frac{\cos(\pi/n)}{\cos(\alpha/2)}\right)$. 1.41. $\frac{(a^3-b^3)\sqrt{3}}{6}$.

Sec. 2. 2.1. $\frac{3a^2\sqrt{3}}{4}$. 2.2. $\arctan\frac{\sqrt{5}}{2}$. 2.3. $\sqrt{6}$. 2.4. $\frac{a\sqrt{3}}{3}$.
2.5. P coincides with the point C. 2.6. $\frac{3\sqrt{3}}{4}$. 2.7. $\frac{1}{3}\sqrt{\frac{1121}{170}}$.
2.8. $\sqrt{\frac{5}{2}}$. 2.9. $\frac{144\sqrt{3}}{5}$. 2.10. $\frac{H^2\sqrt{3}\cot\alpha}{\sin\alpha}$.
2.11. $\frac{(a^2-b^2)(a-b)}{8}\tan^2\alpha\tan\beta$. 2.12. $\operatorname{arccot}(\cos\alpha)$.
2.13. $\frac{S\sqrt{S}\sqrt[4]{6}}{2}$. 2.14. $\arccos\left(\tan\frac{\alpha}{2}\right)$. 2.15. $\frac{a^3\sqrt{2\cos\alpha}}{2\sin(\alpha/2)}$.
2.16. $\frac{c^3}{32}$. 2.17. $\frac{4}{\sqrt{3}}\text{ m}^2$. 2.18. $\frac{a^2\sin 2\alpha}{2\cos\varphi}$. 2.19. $S\sin\varphi\sqrt{S\sqrt{3}\cos\varphi}$.
2.20. $\frac{b}{8}\sqrt{15b^2+4l^2}$. 2.21. 3. 2.22. $\frac{a^2\tan\alpha}{8\sin\beta\tan\beta}$.
2.23. $\frac{(3m+2n)a^3\tan\alpha}{8n}:\frac{a^3\tan\alpha}{8}$, $S=\frac{a^2\sqrt{3}}{4\cos\alpha}$. 2.24. $\frac{3\sqrt{3}}{8}$.
2.25. $\frac{1}{2}(c+h)\cdot S$. 2.26. 7 : 17. 2.27. $\frac{7}{16}\sqrt{(a^2+b^2)c^2+4a^2b^2}$.
2.28. $\varphi=\arcsin\frac{1}{3}$. 2.29. $S=\frac{a^2\sqrt{3}}{16}\cos\alpha\sqrt{4+21\cos^2\alpha}$.

2.30. $\frac{a^2\sqrt{3\sin^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\sqrt{4\sin^2\frac{\alpha}{2}-1}+1}}{4\left(\sin\frac{\alpha}{2}+\sqrt{4\sin^2\frac{\alpha}{2}-1}\right)}$. 2.31. $S=$

$a^2 \sin^2 \frac{\alpha}{2} \sqrt{3-4\sin^2\frac{\alpha}{2}}$. 2.32. $\frac{2}{9}ab$. 2.33. $\frac{\sqrt{3}}{2(4\tan^2\alpha+1)\cos\alpha} l^2$.

2.34. $\frac{2\sqrt{11}}{49}a^2$. 2.35. $2a^2\cos^2\frac{\alpha}{2}\sqrt{-2\cos 2\alpha}$. 2.36. $S=4\sqrt{3}m^2$.

2.37. $\frac{a^3}{128}$. 2.38. $\frac{3\sqrt{2}}{4}a^2$. 2.39. $\sqrt{3}\frac{\sqrt[3]{v^2\cot^2\alpha}}{\cos\alpha}$. 2.40. $\frac{4ab}{9}\sin\frac{\alpha}{2}$.

2.41. 3 : 4. 2.42. $\frac{1}{6}$. 2.43. $\frac{6\sin\frac{\alpha}{2}\left(1-\frac{4}{3}\sin^2\frac{\alpha}{2}\right)}{\left(2\sin\frac{\alpha}{2}+\sqrt{1-\frac{4}{3}\sin^2\frac{\alpha}{2}}\right)^3}$.

2.44. $\frac{69}{100}$. 2.45. At the distance not large than $\frac{2}{3}|SD|$ from the point S. 2.46. 8 : 37. 2.47. $\frac{5\sqrt{2}ab}{16}$. 2.48. $\frac{1}{2}l^2\cos\alpha$. 2.49. $32\sqrt{3}$.

2.50. $a^2\left(1+2\sqrt{1+\frac{1}{\sqrt{2}}}\right)$. 2.51. $\frac{3}{5}$. 2.52. 1 : 1. 2.53. $\frac{25}{16}S$.

Sec. 3. 3.1. $\frac{d^3\cos^2\alpha\sin\alpha}{4\pi}$. 3.2. πs_2, $\frac{s_2\sqrt{\pi s_1}}{2}$. 3.3. $\frac{\pi r^3\sqrt{15}}{3}$.

3.4. $\frac{\pi S\sqrt{15}}{3}$. 3.5. $\frac{Sr}{3}$. 3.6. $\frac{\pi S\sqrt{S}}{3\sqrt[4]{3}}$. 3.7. $\frac{2}{3}\cdot\frac{\pi^2r^3}{\pi^2-1}$.

3.8. $\frac{\sqrt{5}}{5}$. 3.9. $\frac{2\pi-3\sqrt{3}}{10\pi+3\sqrt{3}}$. 3.10. $\frac{2\pi}{3}h^3$. 3.11. $\frac{\sin\alpha}{4\pi\cos\beta\cos^2\frac{\beta}{2}}$.

3.12. $\frac{l^2\cos\beta}{\cos^2\alpha}\sqrt{\sin(\beta+\alpha)\sin(\beta-\alpha)}$.

3.13. $\frac{\pi\sqrt{b^2-a^2}\,(b^2\cot^2\alpha-a^2\cot^2\beta)}{24(\cot^2\alpha-\cot^2\beta)^{3/2}}$. 3.14. $\frac{1}{4}(2S_1+2S_2+\pi d^2)$.

3.15. $\frac{2\sqrt{2}-1}{2\sqrt{2}+1}R$, $\frac{2\sqrt{2}+1}{2\sqrt{2}-1}R$. 3.16. $r\left(1+2\tan^2\frac{\alpha}{2}\pm\tan\frac{\alpha}{2}\sqrt{3+4\tan^2\frac{\alpha}{2}}\right)$. 3.17. $r\left(1+\frac{\sqrt{6}}{2}\right)$. 3.18. 4. 3.19. $r_1=\frac{c\sin\beta}{2\sin\alpha}$, $r_2=\frac{c\sin\alpha}{2\sin\beta}$, $r_3=\frac{c\sin\alpha\sin\beta}{2\sin^2(\alpha+\beta)}$. 3.20. $\frac{\rho^2-(R-r)^2}{4(\sqrt{R}+\sqrt{r})^2}$.

Sec. 4. 4.1. $\frac{\sqrt{H^2+2a^2}}{2}$. 4.2. 5 : 1. 4.3. $12R^2\sqrt{3}$. 4.4. $\frac{2+\sqrt{3}}{4}$.

4.5. $\frac{1}{4}a(\sqrt{3}-1)^2$. 4.6. $\frac{2}{3}R^3\sqrt{\frac{2}{3}}$. 4.7. $\frac{3-\sqrt{5}}{2}$, $\frac{3+\sqrt{5}}{2}$.

4.8. $\frac{1}{8} a \sqrt{41}$. 4.9. $\frac{8}{27} R^3 \sqrt{3}$. 4.10. $R=4$ cm. 4.11. $R=$
$\frac{a}{2\sqrt{3}} \sqrt{\frac{\sqrt{3}\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\sqrt{3}\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}} = \frac{a}{2\sqrt{3}} \sqrt{\frac{\sin\left(60^\circ-\frac{\alpha}{2}\right)}{\sin\left(60^\circ+\frac{\alpha}{2}\right)}}$.

4.12. $\frac{a\cos\frac{\alpha}{2}}{2\sqrt{\sin\left(\frac{\pi}{3}+\frac{\alpha}{2}\right)\sin\left(\frac{\pi}{3}-\frac{\alpha}{2}\right)}}$. 4.13. $\frac{a}{\sqrt{2}}$. 4.14. $\frac{a\sqrt{6}}{8}$.

4.15. $\frac{a\sqrt{3}}{3}(\sqrt{4\cot^2\alpha+1}-2\cot\alpha)$. 4.16. $\frac{2\pi\sin^2\alpha}{3\sqrt{3}(3+\cos^2\alpha)}$.

4.17. $b\left(1+\sqrt{\frac{2}{3}}\right)$, $b\left(1-\sqrt{\frac{2}{3}}\right)$. 4.18. $\frac{a}{2\sqrt{2}}$.

4.19. $\frac{a(2b-a)}{2\sqrt{3b^2-a^2}}$. 4.20. $\frac{4}{9}\pi\frac{\cot\frac{\beta}{2}-\frac{1}{\sqrt{3}}}{\left(\cot\frac{\beta}{2}+\frac{1}{\sqrt{3}}\right)^2}$. 4.21. $\frac{a(\sqrt{3}-1)}{4\sqrt{2}}$.

4.22. $\varphi_1=\frac{\pi}{3}$, $\varphi_2=2\arctan\frac{\sqrt{6}}{3}$. 4.23. $\frac{3\sqrt{3}\sin^2\alpha\cdot\cos^4\alpha}{2\pi}$.

4.24. $\frac{4\pi\sin^2\alpha\cos\alpha}{3\sqrt{3}(1+\cos\alpha)^3}$. 4.25. $\frac{4R\sqrt{3}\tan\alpha}{4+\tan^2\alpha}$. 4.26. $\frac{a}{2(1+\sqrt{6})}$.

4.27. $\frac{\sqrt{3}+1}{2\sqrt{2}+\sqrt{3}+1}$. 4.28. 4. 4.29 $\frac{\sqrt{219}b^2}{36}$. 4.30. $\frac{1}{2}\times$
$a^2(4b^2-a^2)^{1/2}(4a^2b^2-a^4-b^4)^{-1/2}$. 4.31. $\frac{b\cos\alpha\sin(\alpha/2)}{\cos(\alpha/2)+1}$.

4.32. $\frac{Hr}{r+\sqrt{r^2+4H^2}}$. 4.33. $\frac{\sqrt{2}}{2}$. 4.34. $\frac{a}{8}$. 4.35. $S=$
$3\sqrt{15}(\sqrt{5}+1)^2$, $\alpha=2\arcsin\sqrt{\frac{2}{5}}$. 4.36. $\frac{a}{6}$. 4.37. $\frac{2\pi a^2}{\sin^2 2\alpha}$.

4.38. $4R^2\sin 2\alpha$. 4.39. $4R^2\cos\alpha(\sin\alpha+\sqrt{-\cos 2\alpha})$. 4.40. $\frac{\sqrt{21}}{6}a$.

4.41. $\frac{1}{6}rR(R\pm\sqrt{R^2-r^2})$. 4.42. $\frac{4}{3}R^3\frac{(1+\cos\varphi)^3}{\cos\varphi\sin^2\varphi\sin\alpha}$.

4.43. $S=2\sqrt{2}R^2\cos\alpha\left(\sin\alpha+\sqrt{\sin^2\alpha+\frac{1}{2}\cos^2\alpha}\right)$.

4.44. $\frac{Q}{4}\sec^4\frac{\alpha}{2}$. 4.45. $\frac{a\sin\alpha\cos\alpha}{\sqrt{1+\cos^2\alpha}+\cos\alpha}$. 4.46. $\frac{ab}{\sqrt{2a^2-b^2}}$.

4.47. $\frac{\sqrt{3}(2-\sqrt{3})}{4}a^2$. 4.48. $\frac{c}{\sqrt{2}(\cos(\alpha/2)+\cos(\beta/2))}$.

4.49. $\frac{a\sqrt{3}}{4(1+\sqrt{7})}$. 4.50. $V=4(\sqrt{10}+1)^3$, $\alpha=2\arcsin\sqrt{\frac{11}{20}}$.

4.51. $\frac{a^3(2b-a)}{3\sqrt{2}\sqrt{2b^2-a^2}}$. 4.52. $\frac{21R^3}{16}$. 4.53. $S=\frac{8\sqrt{3}R^2}{\sin^2\alpha}$, $V=\frac{4\sqrt{3}}{3}R^3\frac{4-\sin^2\alpha}{\sin^2\alpha}$. 4.54. $\frac{32\sqrt{21}}{147}$. 4.55. $\frac{\left(6\sin^2\frac{\pi}{2n}+\frac{1}{2}n\sin\frac{2\pi}{n}\right)^3}{\pi n^2\sin^2\frac{2\pi}{n}}$.

4.56. $\frac{4}{3}\pi l^3\frac{\sin^3\alpha\cos^3\alpha}{(1+\cos\alpha)^3}$. 4.57. $V=\frac{2}{3}\pi R^3\frac{4-\sin^2\alpha}{\sin^2\alpha}$, $S=\frac{4\pi R^2}{\sin^2\alpha}$. 4.58. $\pi-4\arctan\frac{1}{2}$. 4.59. $\frac{\pi r^3\cot^3(\pi/4-\alpha/2)}{3\cos^2\alpha\sin\alpha}$.

4.60. $S\sin\alpha\sin 2\alpha\cos^2\frac{\alpha}{2}$. 4.62. $4\tan^3\frac{\alpha}{2}\cot\alpha$. 4.63. 2.

4.64. $\frac{27}{16\sin^2\alpha}$. 4.65. $\arcsin\sqrt{\frac{s}{S}}$. 4.66. $\frac{R}{2}\sqrt[3]{3\cot\frac{\alpha}{2}\operatorname{cosec}^2\frac{\alpha}{4}}$.

4.67. $\frac{5}{\sqrt{3}}r$. 4.68. $\frac{r(3+2\sqrt{2}+\sqrt{3})}{\sqrt{3}}$. 4.69. $\sin\frac{\alpha}{2}=\frac{1}{\sqrt{3}}$.

4.70. $\frac{\pi r^2(r+\sqrt{r^2+(d-r)^2})^3}{3(d-r)^2}$. 4.71. $\sqrt{\frac{R}{r}}\left(\sqrt{R+r}-\sqrt{R}\right)^2$.

4.72. $\frac{R(h_1+h_2)}{\sqrt{R^2+h_1^2}+\sqrt{R^2+h_2^2}}\;\frac{\sqrt{R^2+h_1^2}-R}{\sqrt{R^2+h_1^2}+R}$. 4.73. $\frac{9}{16}$. 4.74. $\frac{48}{125}\pi R^3$.

4.75. $R\left[r+R\cot\left(\frac{\pi}{4}+\frac{\alpha}{4}\right)\right]$. 4.76. $\frac{4\pi}{9}$. 4.77. $R=\frac{2r}{\sqrt{3}}\tan\left(\frac{\pi}{4}-\frac{\alpha}{4}\right)$.

Chapter 13

Sec. 1. **1.1.** (a) $(-6, -2, 4)$, (b) $(18, -5, 19)$. **1.2.** (a) $(-30, 21)$, (b) $(0, 0)$, (c) $\left(\frac{11}{2}, \frac{15}{2}\right)$ (d) $(25, -10)$. **1.3.** $\alpha=2$, $\beta=3$, $\gamma=5$. **1.4.** (a) $\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, (b) $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. **1.5.** $\left(\frac{7}{10}, \frac{3}{20}, \frac{3}{20}\right)$. **1.6.** (a) $\mathbf{c}=\mathbf{a}-\mathbf{b}$, (b) $\mathbf{c}=2\mathbf{a}-3\mathbf{b}$, (c) $\mathbf{c}=-\frac{3}{2}\mathbf{a}$. **1.7.** (a) $\overrightarrow{PQ}=(-3, 5, -3)$, (b) $\overrightarrow{PQ}=\left(-\frac{11}{10}, \frac{4}{3}, -\frac{1}{6}\right)$.

1.8. $\left(0, \frac{5}{2}\right)$. **1.9.** (a) $(-2, 1)$, (b) $(0, 2)$, (c) $(0, 2)$. **1.10.** $M_1(7, 0)$ and $M_2(-1, 0)$. **1.11.** $M(0, 1, 0)$. **1.12.** $M(-1, 0, 0)$. **1.13*.** (a) $\left(\frac{5}{3}, 1\right)$, (b) $\left(\frac{1}{3}, 4\right)$, (c) $\left(\frac{2}{3}, \frac{14}{3}\right)$. *Hint.* If the vertices of the triangle ABC are defined by their coordinates $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then the coordinates of the centre of gravity G of the triangle can be found from the equations $x=\frac{1}{3}(x_1+x_2+x_3)$, $y=\frac{1}{3}(y_1+y_2+y_3)$, $z=\frac{1}{3}(z_1+z_2+z_3)$. **1.14.** (a) $k=-\frac{9}{14}$, (b) $k=-\frac{5}{16}$, (c) $k=-3$. **1.15.** (a) Yes, $\mathbf{a}=\frac{3}{2}\mathbf{b}$, (b) Yes, $\mathbf{c}=-\frac{4}{3}\mathbf{d}$. **1.16.** $X=-\frac{5}{3}$, $Y=\frac{6}{5}$. **1.18.** $(4, 0)$, $(5, 2)$. **1.19.** $(-1, 2, 4)$, $(8, -4, -2)$. **1.20.** $\left(\frac{11}{7}, \frac{10}{7}, \frac{18}{7}\right)$. **1.21.** $\alpha=-1$, $\beta=4$. **1.22.** (a) 22, (b) -200, (c) 41, (d) $\sqrt{105}$. **1.23.** $\mathbf{e}_1=\left(-\frac{3}{5}, \frac{4}{5}\right)$, $\mathbf{e}_2=\left(\frac{3}{5}, -\frac{4}{5}\right)$. **1.24*.** $\mathbf{x}=\left(\frac{21}{65}, \frac{77}{65}\right)$. *Hint.* $\mathbf{x}=\mathbf{e}_1+\mathbf{e}_2=\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}$. **1.25.** -13. **1.26.** (a) $|\mathbf{a}|=\sqrt{3}$, (b) $|\mathbf{b}|=\sqrt{14}$. **1.27.** $(\sqrt{3}, \sqrt{3}, \sqrt{3})$ or $(-\sqrt{3}, -\sqrt{3}, -\sqrt{3})$. **1.28.** $6\sqrt{2}$. **1.29.** $(6, -2, 4)$ and $(-6, 2, -4)$. **1.30*.** $\frac{\sqrt{85}}{2}$. *Hint.* $\overrightarrow{AM}=\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC})=\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}$. **1.31.** (a) $\arccos\frac{6}{7}$, (b) $\arccos\left(-\frac{4}{3\sqrt{5}}\right)$ (c) $\arccos\frac{3}{7}$, (d) $\arccos\left(-\frac{2}{\sqrt{14}}\right)$, (e) $\arccos\frac{2}{7}$, (f) $\arccos\left(-\frac{2}{3\sqrt{5}}\right)$. **1.32*.** $\arccos\frac{2}{\sqrt{13}}$, $\arccos\left(-\frac{2}{\sqrt{29}}\right)$, $\arccos\left(-\frac{5}{\sqrt{26}}\right)$, 135°. *Hint.* $\mathbf{i}=(1, 0)$, $\mathbf{j}=(0, 1)$. **1.33.** $\frac{1}{11}$. **1.34*.** (a) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$, (b) $0, -\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}$, (c) $-1, 0, 0$, (d) $0, \frac{3}{5}, \frac{4}{5}$. *Hint.* $\mathbf{i}=(1, 0, 0)$, $\mathbf{j}=(0, 1, 0)$, $\mathbf{k}=(0, 0, 1)$. **1.35.** $\mathbf{p}=(-6, 8)$. **1.36.** $\mathbf{b}=(-24, -32, 30)$. **1.37.** 90°, $\sqrt{10}$. **1.38*.** $\mathbf{c}_1=(1, 0, 1)$ or $\mathbf{c}_2=\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}\right)$. *Hint.* Designating the coordinates of the vector $c=(X, Y, Z)$, derive a system of equations $\mathbf{ca}=1$, $\mathbf{cb}=1$, $\mathbf{c}^2=\mathbf{a}^2=\mathbf{b}^2=2$ and, solving it, get the answer. **1.39.** $\cos\alpha=\cos\beta=\cos\gamma=\frac{1}{\sqrt{3}}$. **1.40.** $\varphi=\frac{3\pi}{4}$. **1.41.** $Z=4$. **1.42.** $X=0$, $Y=2$. **1.43.** $\mathbf{c}=(-3, 3, 3)$, **1.44.** $\mathbf{c}=\left(\frac{4}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$. **1.45*.** $\mathbf{a}=(2, -2, -2)$. *Hint.* Using the length of the vector **a** and its perpendicularity to the vector **d**, set up two equations with respect to X, Y, Z: $X-Y+2Z=0$, $X^2+Y^2+Z^2=12$. Noting that $|\mathbf{b}|=|\mathbf{c}|$ we find that $2XY+YZ-XZ=0$, whence, solving three equations for X, Y, Z we get the answer. **1.46.** $|BD|=2\sqrt{6}$. **1.47.** $|AC|=5$, $\left(\frac{5}{2}, 1, 1\right)$. **1.48.** $\arccos\frac{63}{\sqrt{6441}}$. **1.49.** $\frac{43}{25\sqrt{13}}$. **1.50.** $|AA_1|=\sqrt{\frac{31}{2}}$, $|BB_1|=\frac{\sqrt{53}}{2}$, $|OG|=\frac{\sqrt{182}}{3}$, $\arccos\frac{14}{15}$. **1.51.** $(2+\sqrt{3}, 2+\sqrt{3})$ or $(2-\sqrt{3}, 2-\sqrt{3})$. **1.52.** $C_1(3, 6)$, $D_1(5, 3)$ or $C_2(-3, 2)$, $D_2(-1, -1)$. **1.53.** $A\left(\frac{1+7\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$, $C\left(\frac{1-7\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$. **1.54.** $|AA_1|=\frac{3}{4}\sqrt{10}$. **1.55.** $D(20, 23, 6)$. **1.56.** $A=4$. **1.57.** $A=7$. **1.58.** $|AB|=5$, $|BC|=5\sqrt{2}$, $|AC|=5$, $\angle A=90°$, $\angle B=\angle C=45°$. **1.59.** Acute. **1.60.** $\varphi=45°$. **1.61*.** $\overrightarrow{AH}=(2, 1)$. *Hint.* Take into account that $\overrightarrow{AH}=(X, Y)\perp\overrightarrow{BC}$ and $\overrightarrow{BH}=\overrightarrow{AH}-\overrightarrow{AB}\perp\overrightarrow{AC}$. **1.62.** $|OA_1|=\sqrt{\frac{3}{2}}$. **1.63.** $|AG|=\frac{\sqrt{51}}{3}$. **1.64*.** $A_1'(0, -2, 0)$ and $A_1''(2, 2, 2)$. *Hint.* Knowing the volume of the prism, we find its altitude $H=|AA_1|=\sqrt{6}$ and, designating the coordinates of the vertex $A_1(x_1, y_1, z_1)$, we relate the coordinates of the vector $\overrightarrow{AA_1}=(x-1, y, z-1)$ and its length. We get the other equation from the condition $\overrightarrow{AA_1}\perp\overrightarrow{AC}$. **1.65.** 18. **1.66.** 26.

Sec. 2. 2.1. (a) $x-y+1=0$, (b) $x-1=0$, (c) $y-2=0$. **2.2*.** $3x-2y-12=0$, $3x-8y+24=0$. *Hint.* Use the intercept equation of a straight line (4).

2.3. (a) $3x-2y-5=0$, (b) $x-5y-\frac{7}{6}=0$. **2.4.** $AB: 4x+y-6=0$,

$CD: x-4y-2=0,\ h=\dfrac{19}{\sqrt{17}},\ \cos\varphi=\dfrac{19}{\sqrt{17\cdot 58}},\ l_1: \dfrac{x-1}{\sqrt{26}+5\sqrt{17}}=$
$\dfrac{y-2}{-4\sqrt{26}-\sqrt{17}},\ l_2: (\sqrt{26}+5\sqrt{17})(x-1)+(-4\sqrt{26}-17)(y-2)=0.$

2.5. $y = 2x - 6$, $y = -2x + 6$. **2.6.** $x - 5y + 3 = 0$ or $5x + y - 11 = 0$. **2.7*.** C_1 (5, 10) and C_2 (3, 0). *Hint*. The area of the triangle ABC can be found from the formula

$$S=\frac{1}{2}|\mathbf{a}|\,|\mathbf{b}|\sqrt{1-\left(\frac{\mathbf{ab}}{|\mathbf{a}|\,|\mathbf{b}|}\right)^2}=\frac{1}{2}\sqrt{(|\mathbf{a}|\,|\mathbf{b}|)^2-(\mathbf{ab})^2}.$$

2.8. D (9, 0), **2.9.** $(x-1)^2+(y-1)^2=1$. **2.10.** $y=\dfrac{\sqrt{15}}{2}$ and $y=-\dfrac{\sqrt{15}}{2}$. **2.11*.** B (12, 5), C (—5, 12), D (—12, —5). *Hint*. The point C is symmetric with respect to the point A about the origin.

2.12.** $\left(x-\frac{1}{2}\right)^2+(y-\sqrt{2})^2=\frac{9}{4}$; $\left(x-\frac{1}{2}\right)^2+(y+\sqrt{2})^2=\frac{9}{4}$.

Solution. The centre of the required circle lies on the straight line which passes through the point (1/2,0) at right angles to the Ox axis

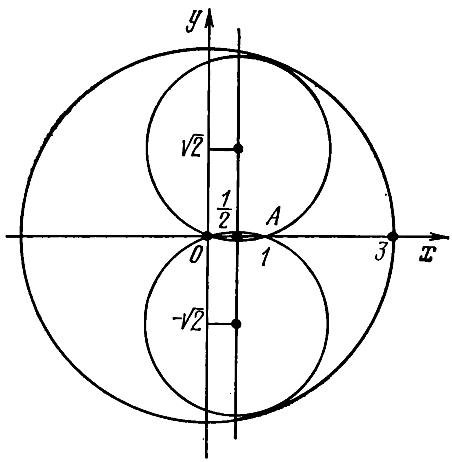

Fig. A.1

(Fig. A.1). The diameter of the required circle is equal to the radius of the given circle. Writing the equation of the required circle in the form $\left(x-\frac{1}{2}\right)^2+(y-y_0)^2=\frac{9}{4}$ and requiring that the circle in question should pass through the point A (1, 0), we find y_0.

2.13. $(x-1)^2+(y-1)^2=1$, $(x-5)^2+(y-5)^2=25$. **2.14.** $2x-y-2z=0$. **2.15.** (a) $3x+4y+6z-29=0$, (b) $2x-2y-z+9=0$, (c) $x-y+4z+11=0$. **2.16.** (a) $\arccos\frac{5}{6}$, (b) $\arccos\frac{5}{14}$, (c) $\arccos\frac{5}{11}$. **2.17*.** $\frac{\sqrt{6}}{9}$. *Hint.* Find the cosine of the angle between the vector **n** of the plane and the vector $\overrightarrow{AB}$. Using the definition of the angle between a straight line and a plane, find the sine of that angle. **2.18.** (a) $\arcsin\frac{18}{35}$, (b) $\arcsin\frac{23}{15\sqrt{10}}$. **2.19.** 10. **2.20.** (a) $\frac{3}{2}$, (b) 0, (c) 4. **2.21.** 3. **2.22.** $6x+2y+3z\pm 42=0$. **2.23.** $(-1, 0, 2)$. **2.24.** (a) $(0, 0, -2)$, (b) $(2, 3, 1)$. **2.25*.** 3. *Hint.* The vector $\mathbf{n}=(2, 2, -1)$ is parallel to the straight line which passes through the centre of the sphere at right angles to the given plane. The distance from the centre of the sphere to the plane is equal to 5. **2.26.** $(4, -3, 0)$ and $\left(\frac{4}{21}, \frac{97}{21}, \frac{40}{21}\right)$. **2.27.** $\left(x-\frac{3}{2}\right)^2+(y-1)^2+(z-1)^2=\frac{m^2}{2}-\frac{33}{4}$; for $m^2>\frac{33}{2}$ it is a sphere; for $m^2=\frac{33}{2}$ it is a point; for $m^2<\frac{33}{2}$ it is an empty set.

Sec. 3. **3.1.** $\arccos\frac{3}{\sqrt{14}}$. **3.2.** $\pi-\arccos\left(-\frac{1}{5\sqrt{13}}\right)$. **3.3.** $\arccos\frac{3}{\sqrt{10}}$. **3.4.** $\arccos\frac{1}{\sqrt{5}}$. **3.5*.** $\frac{1}{3}$. *Hint.* Choose a system of coordinates Oxy such that the axes Ox and Oy would pass through the legs BC and BA respectively. **3.7*.** $2ax+2by=a^2+b^2$, where a, b are the lengths of the legs. *Hint.* We choose a rectangular system of coordinates such that the Ox axis coincides with the leg CA and the Oy axis with the leg CB. **3.14.** $3a^2$, where a is the length of the side of the square. 3.15. $4a^2$, where a is the length of the side of the square. **3.16.** $\frac{6}{\sqrt{170}}$. **3.17.** $\frac{3}{\sqrt{170}}$. **3.18.** $\frac{1}{3}\sqrt{\frac{1121}{170}}$. **3.19.** $\sqrt{\frac{551}{850}}$. **3.20.** $\frac{1}{4}\sqrt{\frac{1373}{85}}$. **3.21.** $\arccos\frac{\sqrt{7}}{3}$.

3.22.** $\frac{7a^2\sqrt{17}}{24}$. *Solution.* Introducing the system of coordinates $Oxyz$ as shown in Fig. A.2, we find the coordinates of the points: $A\,(a, 0, 0)$, $E\left(0, \frac{a}{2}, a\right)$, $F\left(\frac{a}{2}, a, a\right)$. The equation of the plane which passes through those three points has the form $x+y+\frac{3}{2}z+a=0$. The cosine of the angle between the plane of the lower base and the given plane is $\cos\varphi = \frac{3}{\sqrt{17}}$. The area of the projection of the pentagon resulting from the section of the cube by the secant plane onto the plane of the lower base of the cube is $s = a^2 - \frac{a^2}{8} = \frac{7a^2}{8}$ and, consequently, the area of the pentagon itself is $S = \frac{s}{\cos\varphi} = \frac{7\sqrt{17}a^2}{24}$.

3.23.** $|A_1K| = |A_1L| = \frac{\sqrt{5}}{2}a$, $|KL| = \frac{\sqrt{6}}{2}a$, $\frac{7}{41}$. *Solution.* It follows from the hypothesis that $K\left(\frac{a}{2}, 0, 0\right)$, $L\left(0, a, \frac{a}{2}\right)$ (Fig. A.3). Then $|A_1L|^2 = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$, $|A_1L| = \frac{a\sqrt{5}}{2}$; $|A_1K|^2 = \frac{a^2}{4} + a^2$, $|A_1K| = \frac{a\sqrt{5}}{2}$. Let us consider two triangular pyramids: $NAKA_1$ and $NDML$. In the second pyramid we designate the unknown lengths of the legs ND and DM as x and y respectively.

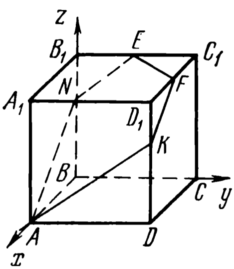

Fig. A.2

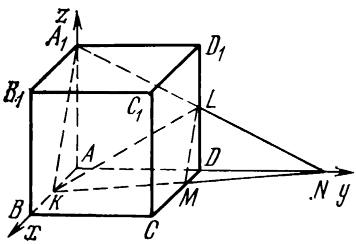

Fig. A.3

It follows from the similitude of the triangles AA_1N and DLN that $x = a$, and from the similitude of the triangles AKN and DMN it follows that

$$y = \frac{a}{4}; \quad |KL|^2 = \frac{a^2}{4} + a^2 + \frac{a^2}{4} = \frac{3a^2}{2}, \quad |KL| = \frac{a\sqrt{6}}{2}. \quad \text{Then}$$

$$V_{NAKA_1} = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{a}{2}a\cdot 2a = \frac{a^3}{6}; \qquad V_{NDML} = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{a}{2}\cdot\frac{a}{4}\cdot a = \frac{a^3}{48}.$$

From this we find the volume of one of the parts of the cube into which it is devided by the secant plane, $V_1 = V_{NAKA_1} - V_{NDML} = \frac{7a^3}{48}$. Then the volume of the second part of the cube is $V_2 = \frac{41}{48}a^3$, whence $V_1 : V_2 = 7 : 41$.

3.24. $\frac{3a^2\sqrt{3}}{4}$. 3.25. (a) $\frac{a^2}{4}$; (b) $\frac{7}{32}a^2$. 3.26. $8a^2$, where a is the length of the side of the cube. 3.27. $\frac{\pi}{2}, \frac{2}{\sqrt{3}}$. 3.28. $\frac{\pi}{4}, \frac{1}{\sqrt{3}}$. 3.29. $\frac{\pi}{3}, \frac{1}{\sqrt{3}}$. 3.30. (a) $\frac{a\sqrt{5}}{3}$, (b) $\frac{a\sqrt{5}}{5}$. 3.31. $\frac{a\sqrt{2}}{2}$.

Sec. 4. 4.4. $\overrightarrow{DC_1}=\frac{3}{4}\mathbf{a}+\frac{1}{4}\mathbf{b}$, $\overrightarrow{DC_2}=\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$, $\overrightarrow{DC_3}=\frac{1}{4}\mathbf{a}+\frac{3}{4}\mathbf{b}$. 4.5. $\overrightarrow{MC}=\frac{2}{3}\overrightarrow{MA}+\frac{1}{3}\overrightarrow{MB}$. 4.6. $\overrightarrow{MC}=\frac{1}{k+1}\overrightarrow{MA}+\frac{k}{k+1}\overrightarrow{MB}$. 4.17. $|AB|=4$. 4.18. $\frac{2}{11}$. 4.19. 1.5. 4.20. $\frac{25}{64}$. 4.21. $\overrightarrow{AA_1}=\frac{c\mathbf{b}+b\mathbf{c}}{b+c}$, where $\overrightarrow{AC}=\mathbf{b}$, $AB=\mathbf{c}$ and $|AC|=b$, $|AB|=c$. 4.22. $\frac{1}{2}$. 4.23. $\frac{1}{3}$. 4.24. $\frac{(a+b)(b+c)(a+b+c)}{ab(a+b+2c)}$. 4.26. 0. 4.27. $p=q=1$. 4.28. $\overrightarrow{QO}=\mathbf{a}-\frac{1}{2}\mathbf{b}+\frac{1}{2}\mathbf{c}$. 4.35. 1 : 8. 4.36. 3. 4.37. $\frac{8}{37}$. 4.38. $\frac{1}{6}$.

Sec. 5. 5.1. (a) 9, (b) 13, (c) -61. 5.2. -13. 5.3. The vectors **a** and **b** must be mutually perpendicular. 5.5. $k=-\frac{\mathbf{ab}}{\mathbf{ac}}$. 5.8. (a) $m_a=\frac{1}{2}\sqrt{b^2+2bc\cos A+c^2}$, (b) $l_a=\frac{2bc\cos(A/2)}{b+c}$. 5.9. (a) $m_a=\frac{1}{2}\sqrt{-a^2+2b^2+2c^2}$, (b) $l_a=\frac{2\sqrt{bcp(p-a)}}{b+c}$. 5.10. $\arctan\frac{\sqrt{2}}{2}$. 5.11. $\frac{3+\sqrt{73}}{8}$. 5.12. $\frac{2}{\sqrt{7}}$. 5.24. $\angle A=\angle B=30°$, $\angle C=120°$.

Chapter 14

Sec. 1. 1.1*. 10^7. *Hint.* From the initial set (0, 1, 2, . . ., 9) we take samples with replacements, which contain seven elements each. 1.2*. $\frac{10(10^7-1)}{9}$. *Hint.* Find the sum of the numbers which represent the quantity of different samples including one, two, and so on, to seven elements of the initial set. 1.3. 243. 1.4. 2^{32}. 1.5. The number of divisors q is equal to the product $(k_1+1)(k_2+1)\ldots(k_m+1)$. 1.6. A_{10}^7. 1.7*. 2^n. *Hint.* The initial set consists of two elements (H, T) and the samples with replacements consist of n elements. 1.8. 720. 1.9. (a) $2\cdot 29!$, (b) $28\cdot 29!$. 1.10*. 968. *Hint.* We must find the sum of the numbers of different chords containing three, four, and so on, to ten sounds. One chord, consisting of k sounds, is a sample of k

elements from the initial set which contains 10 elements; the order of the elements in the sample is inessential. **1.11***. $40 \cdot 39 \cdot C_{38}^{5}$. *Hint.* The chairman and the secretary form a sample without replacements consisting of two elements of the initial set which consists of 40 elements, five members of the commission form a sample without replacements of a certain composition taken from the initial set consisting of 38 members. **1.12.** $\binom{8}{5}\binom{10}{2}$. **1.13.** $\binom{32}{4}\binom{4}{2}$. **1.14.** $\binom{5}{4}\binom{15}{2}\binom{10}{3}$. **1.15.** (a) $42\binom{6}{5}$ different cards; (b) $\binom{42}{2}\binom{6}{4}$ different cards; (c) $\binom{42}{3}\binom{6}{3}$ different cards. **1.16.** 120. **1.17***. $\binom{52}{10}-\binom{48}{10}$. *Hint.* The required number is equal to the difference between the total number of the ways to draw 10 cards out of the pack of 52 and the number of the ways to draw 10 cards out of 48 so that there is no ace among the selected cards. **1.18.** $4\cdot\binom{44}{4}$. **1.19.** $\binom{10}{4}\binom{6}{4}$. **1.20***. 1225. *Hint.* Take into account that the digit notation of the number cannot begin with a zero. **1.21.** 750.

Sec. 2. 2.1. 2520. **2.2***. 3465. *Hint.* A sample with a given number of replacements of volume 8 is taken out of four groups of homogeneous elements. **2.3***. $\binom{16}{7}=\binom{16}{9}$. *Hint.* A sample with a given number of replacements of volume 7 is taken out of 10 groups of like elements. **2.4***. $\frac{52!}{(13!)^4}$. *Hint.* We seek the number of different samples of composition (13, 13, 13, 13). **2.5***. $\frac{12!}{2^6}$. *Hint.* Six different groups of homogeneous elements must form a sample with a specified number of replacements which contains 12 elements and has a composition (2, 2, 2, 2, 2, 2). **2.6***. $\binom{m+1}{n}$. *Hint.* We must consider a sample with a specified number of replacements which has a composition $(m+1, n)$ where $m+1$ is the number of gaps between m white balls and n is the number of black balls. The number of different arrangements is equal to the number of different samples of composition $(m+1, n)$. **2.7***. $\frac{100!}{48!\,52!}$. *Hint.* We seek the number of different samples of composition (n_1+n_2), where $n_1=52$ is number of successes and $n_1+n_2=100$. **2.8***. $2\,(6!)^2$. *Hint.* The number of permutations in the left-hand seats in the row must be multiplied by the number of permutations of the right-hand seats. Take into account that the right-hand and left-hand seats can be interchanged. **2.9***. *Hint.* Use the inequality $\binom{2n+k}{n}\binom{2n-k}{n}\leqslant\binom{2n}{n}^2$, which can be proved in a direct way.

Sec. 3. 3.1*. 1024. *Hint.* Expand the expression $(1+1)^{10}$ by the binomial formula. **3.2.** $k=4$. **3.3*.** $T_2=\binom{18}{2}x^{6.5}=153x^{6.5}$. *Hint.* Use the fact that the middle term has the greatest coefficient. **3.4.** $28x^2a^{-4}$. **3.5*.** *Hint.* Use the hint to 3.1*. **3.6.** -1375. **3.7**.** $\binom{100}{50}\left(\frac{1}{2}\right)^{100}$. *Solution.* Let us consider the relation between T_{k+1} and T_k. Since

$$T_{k+1}=\binom{100}{k+1}\left(\frac{1}{2}\right)^{100}, \quad T_k=\binom{100}{k}\left(\frac{1}{2}\right)^{100},$$

$$\frac{T_{k+1}}{T_k}=\frac{100!\,k!\,(100-k+1)!}{(k+1)!\,(100-k)!\cdot 100!}=\frac{100-k+1}{k+1}.$$

If $\frac{100-k+1}{k+1}>1$ then $T_{k+1}>T_k$ and if $\frac{100-k+1}{k+1}<1$, then $T_{k+1}<T_k$. We find that for $k<50$ the terms of T_k increase and for $k>50$ they decrease. Consequently the greatest term is $T_{50}=\binom{100}{50}\left(\frac{1}{2}\right)^{100}$.

3.8. The tenth term is the greatest term of the expansion. **3.9*.** $T_3=\binom{10}{3}x^{11}$. *Hint.* The binomial power can be obtained by the use of the Hint to 3.1*. **3.10*.** *Hint.* Use the expansion of $(1-1)^n$. **3.11*.** $-264a^3b^7$. *Hint.* Use the result of problem 3.10*. **3.12.** $314\,925\cdot 10^6$. **3.13.** $x=2$. **3.14*.** $5/8<x<20/21$. *Hint.* See the solution of problem 3.7**. **3.15*.** 1/2. *Hint.* Using the hypothesis, represent the 50th term of the expansion as a function of the argument x and solve the problem on seeking the greatest value of the function obtained on the interval [0, 1]. **3.16.** $x=\frac{n-k}{n}$. **3.17.** $\binom{24}{14}\cdot 3^2\cdot 2^2$. **3.18.** 26. **3.19**.** The first, the fifth and the ninth term of the expansion are rational. *Solution.* Since the coefficients $1, \frac{n}{2}, \frac{n(n-1)}{8}$ form an arithmetic progression, we can form an equation

$$\frac{n(n-1)}{8}+1=n,$$

whose roots are $n=8$ and $n=1$ respectively; $n=1$ is an extraneous root. For $n=8$ the general term of the expansion has the form

$$T_k=\binom{8}{k}x^{\frac{k}{2}}\left(\frac{1}{2}\right)^{8-k}x^{\frac{8-k}{4}}=2^{k-8}\binom{8}{k}x^{\frac{k+8}{4}}.$$

This term is rational if $k+8$ a multiple of 4, where $0\leqslant k\leqslant 8$. This condition is fulfilled for $k=0, 4, 8$. Consequently, the terms T_0, T_4, T_8 are rational.

3.20*. *Hint.* Use the binomial expansion for $(1-1)^n$. **3.23*.** *Hint.* Consider the binomial expansion for $(10-1)^{2n}$. **3.25*.** *Hint.* Find the increment of the antiderivative of the function $(1+x)^n$ on the interval [0, 1] directly and by writing the expression for $(1+x)^n$ by the binomial formula. **3.26*.** *Hint.* Find the derivative of the function

$(x-1)^n$ at the point $x = 1$. 3.27*. *Hint.* Compare the increments of the antiderivative of the function $(x-1)^n$ on the interval [0, 1] found directly and by expanding $(x-1)^n$ by the binomial formula. 3.28*. $(n+1)!-1$. *Hint.* Add $P_1 + P_2 + \ldots + P_n$ to the required expression and then subtract $P_1 + P_2 + \ldots + P_n$. 3.29*. *Hint.* Use the identity $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$.

Sec. 4. 4.1. 12/365. 4.2. 5/12. 4.3*. 1/3. *Hint.* The number of all two-digit numbers is 90. The number of two-digit numbers divisible by three can be found from the equation $99 = 12 + 3(n-1)$. 4.4. 0.4. 4.5. 3/13.

4.6**. $P(A) = 1/8$, $P(B) = 3/8$. *Solution.* The sample space consists of samples with replacements formed from the letters T and H. It contains $2^3 = 8$ elements. Only one sample (T, T, T) is favourable to the event A and three samples (T, T, H), (T, H, T), (H, T, T) are favourable to the event B. Thus, $P(A) = 1/8$, $P(B) = 3/8$.

4.7. 1/6. 4.8. 1/2. 4.9. 89/99. 4.10. 10/99. 4.11*. 1/8. *Hint.* See the solution of problem 4.6**. 4.12. $\binom{n}{2}\Big/\binom{n+m}{2}$. 4.13. 1/720. 4.14. 245/354. 4.15. $n\cdot m\cdot k/C(n+m+k,\ 3)$. 4.16. $\binom{30}{4}\Big/\binom{45}{4}$. 4.17. $4\binom{15}{2}$. 4.18*. 1/15120. *Hint.* The sample space consists of all permutations, with a given number of replacements, which have the composition 3, 2, 2, 1, 1, 1. Only one permutation of this kind is favourable. 4.19. $5\cdot3!4!/7!$. 4.20. $2\cdot4!3!/7!$. 4.21*. $24\cdot48!13^4/52!$. *Hint.* The sample space consists of all samples which have the composition (13, 13, 13, 13). Samples of the composition (12, 12, 12, 12), to each of which an ace is added, are considered to be favourable.

4.22. $\dfrac{5!\,5!}{10!}$. 4.23. $\dfrac{50}{C(15,\ 5)}$. 4.24. $\dfrac{C(n,\ k)\,C(N-n,\ m-k)}{C(N,\ m)}$.

4.25. $\dfrac{1}{C(48,\ 6)}$; $\dfrac{C(6,\ 5)\,C(42,\ 1)}{C(48,\ 6)}$; $\dfrac{C(6,\ 4)\,C(42,\ 2)}{C(48,\ 6)}$; $\dfrac{C(6,\ 3)\,C(42,\ 3)}{C(48,\ 6)}$.

4.26. $\dfrac{C(48,\ 5)\,C(4,\ 1)}{C(52,\ 6)}$; $\dfrac{C(44,\ 4)\,C(4,\ 1)\,C(4,\ 1)}{C(52,\ 6)}$.

4.27. $\dfrac{C(4,\ 2)\,C(2,\ 1)}{C(6,\ 3)}=0.6$. 4.28. $\dfrac{2\cdot C(18,\ 8)}{C(20,\ 10)}$.

Sec. 5. 5.1. 0.2. 5.2. $\dfrac{r^2}{R^2}$. 5.3. $\dfrac{1}{2}$. 5.4. $\dfrac{2}{3}$. 5.5. $\dfrac{1}{2}$. 5.6*. $\dfrac{1+3\ln 2}{8}$. *Hint.* Consider the ratio of the total area of the figures bounded by the curves $y=\dfrac{1}{x}$, $y=2x$, $x=2$, $y=0$ to the area of a square with side 2. 5.7*. $\dfrac{1}{2}-\dfrac{\pi}{16}$. *Hint.* See the hint to 5.6*. 5.8*. $\dfrac{2}{3}$. *Hint.*

Find the coefficients of the equation of the parabola $y = ax^2 + bx + c$ from the condition of its passage through the three indicated points, having chosen a requisite system of coordinates. **5.9.** $\frac{3\pi - 8}{\pi}$. **5.11*.** $\frac{1}{2}$. *Hint.* Use the statement made in problem 5.10. **5.12.** Approx. 0.314. **5.13*.** $\frac{1}{2}$. *Hint.* If we designate the distance from the point B to the origin as x, and that from the point C as y, then the sample space will be represented as a unit square inscribed into the first quadrant of the coordinate plane. The elementary events favourable to the event whose probability we must find are represented as points whose coordinates satisfy the inequality $|y - x| \leqslant \min(x, y)$. **5.14.** The trains going in the direction AC must arrive 10 minutes after the departure of the train going in the direction CA. **5.15*.** $\frac{2l}{\pi a}$. *Hint.* Introduce the system of coordinates Oxy, where x is the angle which the needle forms with the parallel, and y is the distance from the centre of the needle to the nearest parallel. In this case the sample space is associated with a rectangle with sides a and $\pi/2$, and the elementary events favourable to the condition of intersection of the parallel straight lines by the needle are associated with the points whose coordinates satisfy the inequality $l \sin x < y$.

Sec. 6. **6.1.** $\frac{n(n-1)}{(n+m)(n+m-1)}$, $\frac{2nm}{(n+m)(n+m-1)}$. **6.2*.** $\frac{n^2}{(n+m)^2}$, $\frac{m^2}{(n+m)^2}$. *Hint.* If the balls are replaced, then the events connected with the colour of the successively drawn balls are independent. **6.3.** $\frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49} \approx 0.1055$. **6.4.** $\frac{25}{216}$. **6.5.** $\frac{3}{5}$. **6.6.** (a) $\frac{20}{25} \cdot \frac{15}{20} \cdot \frac{14}{19} = \frac{42}{95}$, (b) $\frac{81}{190}$. **6.7.** $1 - \frac{6nmk}{(n+m+k)(n+m+k-1)(n+m+k-2)}$. **6.8.** $\frac{67}{91}$. **6.9.** $1 - \frac{(n-l)(n-l-1)\dots(n-l-k+1)}{n(n-1)(n-2)\dots(n-k+1)}$. **6.10.** $1-(1-p)^n$. **6.11*.** $n > \ln(1-P)/\ln(1-p)$, where n is the number of shots fired. *Hint.* The number of shots can be found from the condition stating that in a series of n shots the probability of hitting the target (at least once) is not smaller than P. **6.12.** *Hint.* The probability of passing the examination does not depend on whether the student is the first or the last to take the exam. **6.13*.** 2/3. *Hint.* Consider the following hypothesis: A—there were two white balls in the urn, B—there were two black balls in the urn, C—the balls in the urn were of different colours. The probabilities of the hypothesis are considered to be equal. **6.14.** 0.85.

6.15. (a) $\dfrac{(m-1)m+nm}{(m+n-1)(m+n)}$, (b) $\dfrac{(k+1)m+kn}{(k+l+1)(m+n)}$.

6.16*. $\dfrac{KnM+LmN}{(k+L)MN}$. *Hint*. See the hint to 6.13*. 6.17*. $[N(N-1)\times (N-2)(k+L)(k+L-1)(k+L-2)-k(k-1)(k-2)\times (N-n)(N-n-1)(N-n-2)-k(k-1)L(N-n)\times(N-n-1)(M-m)-k(L-1)(L-2)(N-n)(M-m)\times(M-m-1)-L(L-1)(L-2)(M-m)(M-m-1)\times(M-m-2)]/N(N-1)(N-2)(k+L)(k+L-1)\times(k+L-2)$. *Hint*. Consider the following hypotheses: H_0—all the three articles are from the first lot, H_1—two articles are from the first lot and one is from the second, H_2—one article is from the first lot and two from the second, H_3—all the three articles are from the second lot.

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