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A Handbook of Methods of Approximate Fourier Transformation and Inversion of the Laplace Transformation

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МЕТОДЫ ПРИБЛИЖЕННОГО ПРЕОБРАЗОВАНИЯ ФУРЬЕ И ОБРАЩЕНИЯ ПРЕОБРАЗОВАНИЯ ЛАПЛАСА

СПРАВОЧНАЯ КНИГА

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A HANDBOOK OF METHODS OF APPROXIMATE FOURIER TRANSFORMATION AND INVERSION OF THE LAPLACE TRANSFORMATION

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by George Yankovsky

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Contents

List of Symbols

$f(t)$	original function
$F(p)$	image function under the Laplace transformation
$\varphi(p)$	complex Fourier transform of a function f
$\varphi_c(p)$	Fourier cosine transform of a function f
$\varphi_s(p)$	Fourier sine transform of a function f
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomial of degree n of the parameters α, β
$P_n^{*(\alpha,\beta)}(x)$	shifted Jacobi polynomial for the interval $[0, 1]$
$P_n^*(x)$	shifted Legendre polynomial for the interval $[0, 1]$
$T_n^*(x)$	shifted Chebyshev polynomial of the first kind for the interval $[0, 1]$
$U_n^*(x)$	shifted Chebyshev polynomial of the second kind for the interval $[0, 1]$
$L_n^{(\lambda)}(x)$	generalized Chebyshev-Laguerre polynomial of degree n
${}_1F_1(\alpha, \beta, z)$	confluent hypergeometric function
$\bigcup\limits_\xi M_\xi$	the sum of all sets M_ξ with respect to ξ
$\bigcap\limits_\xi M_\xi$	the intersection (common portion) of all sets M_ξ
$\mathrm{Var}\, f(x)$ $\langle a,b\rangle$	total variation of a function f on the interval $\langle a, b\rangle$ (the interval need not be indicated)
$g(t+0)$	$= \lim g(t+\delta)$ $(\delta > 0,\ \delta \to 0)$
$g(t-0)$	$= \lim g(t-\delta)$ $(\delta > 0,\ \delta \to 0)$
$\mathrm{res}\, f(a)$	residue of the function $f(z)$ at the pole $z = a$

Preface

The problem of approximate computation of inverse Laplace transforms and, in particular, numerical inversion arose out of the need to obtain a numerical solution where existing tables of functions and their transforms (images, or image functions) do not enable one to obtain the original function from the image function or require excessive computation.

Many methods of inversion have been constructed over the past two decades but they are still scattered among specialized journals and books and, as a rule, are not generally known. As far as the authors know, there are no books in the scientific literature that contain a systematic presentation of all these methods.[1]

The authors' purpose here has been to write a book with a full description of the present state of the inversion problem and to make the text useful for problem solving with the help of the Laplace transformation. The first aim was relatively easy to attain since the literature on the inversion problem is still slight and comparatively easy to survey.

[1] In a way, reference [13] is an exception, but its purpose was merely to unite all known auxiliary numerical tables connected with computing the Mellin integral. The mathematical theory of inversion of Laplace transforms is given only briefly, in the form of explanations to the tables.

As to how useful this text will actually be, only time will show, but the authors believe that a few remarks are in order. Most important undoubtedly is the remark that the problem has not been investigated with sufficient thoroughness and the results obtained are still small in number. For those readers who are not acquainted with the problem of inverting Laplace transforms, a few explanatory remarks should be made. What follows is a nonrigorous (but pictorial) presentation.

The inversion problem is merely the problem of finding the solution $f(x)$ of an integral equation of the first kind:

$$\int_0^\infty f(x) e^{-px} dx = F(p) \qquad (*)$$

where $F(p)$ is taken to be a known function of the complex argument p, the function being analytic in a certain half-plane of the form $\operatorname{Re} p > \gamma$ $(\gamma < \infty)$.

The equation $(*)$ is the Laplace transformation of the function $f(x)$ into $F(p)$. The kernel of the integral, e^{-px}, is an entire analytic function of the arguments x and p with smooth variation, and the operation of integration which performs the averaging of f with weight e^{-px} can appreciably smooth the peculiarities in the behaviour of the function f being transformed.

In the inversion problem, one uses the image function $F(p)$, which, due to its analyticity, varies very smoothly, to restore all possible irregularities in the behaviour of the original function $f(x)$. It is therefore to be expected that the apparatus used to solve the inversion problem cannot be simple and rough but must be complicated and sensitive even to small nuances in the behaviour of $F(p)$ in order to perform the delicate work of recovering the original function $f(x)$.

Note yet another property of the inversion problem—the instability of the original function f relative to small variations in the image F. This instability is evident from a glance at the transformation (*). Indeed, suppose f is the given original function and F is the corresponding image function. Subject f to any change, even a strong one, but over a very small interval. Call the new original function f_1. Such a change will hardly affect the integral and will alter the image function $F(p)$ only slightly, so that the new image function $F_1(p)$ will be close to $F(p)$.

We could have constructed a new original function f_1 by changing f on many intervals (instead of one) of the half axis $0 \leqslant x < \infty$, but in a way that the total sum of the lengths of these intervals is a small quantity, for it will still be possible to make the new image function $F_1(p)$ as close to $F(p)$ as desired so that to the close-lying image functions $F(p)$ and $F_1(p)$ there will correspond the original functions $f(x)$ and $f_1(x)$ that differ radically over many intervals of the half axis $0 \leqslant x < \infty$.

We can now give a more complete explanation as to why and in what respect the content of this text is to be regarded as insufficient and we can attempt to indicate the reasons why we believe that these defects will hardly be removed in the near future.

At the present time we stand at the beginning of the construction of a theory of approximate inversion of the Laplace transform. In most published papers we find either modifications of familiar methods of computation or newly devised computational schemes that have no counterparts in the past. Instances of the former kind are methods of computing the complex Mellin integral based on the idea of interpolation or the idea of Gauss of attaining the highest possible degree of accuracy, and others. Now the use of orthogonal polynomials for inversion appeared only in connection with the Laplace transformation and did not have closely related analogs in the past.

The construction of a computational method or any indication of the possibility of construction is only the first step towards constructing a theory of the method. The steps that follow include determining the conditions of convergence of the rule and the rate of convergence, finding estimates of a priori and a posteriori errors, elaborating ways of improving convergence if it is not sufficiently fast, and so on. Such investigations of the inversion problem are particularly important due to its instability.

As the reader will see from the contents of this book, few results of this kind have been obtained, and then only in the simplest cases.

The main reasons that hamper a rapid solution of the problems at hand are those two properties of the inversion problem that were mentioned above: the unavoidable complexity of the mathematical apparatus involved in inverting Laplace transforms and the instability of the inversion problem relative to variations in the functions F, which instability must give rise to some form of instability in any computational process of the solution of this problem.

The fact that the rules for approximate inversion of transforms have not been studied with sufficient thoroughness makes this handbook incomplete in many respects. It is hard to expect a rapid improvement in our knowledge in these questions in the immediate future, and so for some years to come this book will most likely remain incomplete.

So far we have spoken only of inverting Laplace transforms and have not mentioned harmonic analysis at all. This is because in this book harmonic analysis is touched on only to the extent that it is connected with the inversion problem; namely, we have taken from harmonic analysis only such problems involving computation of Fourier integrals as are closely related to inverting Laplace transforms; and computing Fourier integrals is for us one of several possible ways of solving the inversion problem. For this reason, all that has been said about inversion refers also to computing Fourier integrals.

This text is aimed at a broad category of readers engaged in the theory of the Laplace transformation or its scientific and technical applications. For this reason, the authors did not strive for particular brevity in presentation and attempted to make the text accessible to nonmathematicians as well. It is assumed the reader has a basic knowledge of analysis and the theory of functions of a complex variable as given in any extended college course of mathematics.

The authors

Part One

INVERSION OF THE LAPLACE TRANSFORMATION

Chapter 1

Introduction

1.1 *Basic concepts in the theory of the Laplace transformation*

The past few decades have seen particularly frequent and successful use of operational methods on the basis of the Laplace transformation applied in mathematics, mechanics and engineering. These methods have found extensive applications in the theory of thermal conductivity, in electric and radio engineering, in the study of nonstationary phenomena in electric networks, in problems of the dynamics of automatic control systems, in the theory of linear differential, integral, and difference equations, and in many other problems.

The operational method of problem solving may be divided into four stages:

(1) one passes from the desired original function $f(t)$ to the image function $F(p)$;

(2) operations are performed on $F(p)$ that correspond to operations on $f(t)$; an equation in $F(p)$ is then obtained which frequently turns out to be much simpler than the equation in the original function; for example, an ordinary differential equation is replaced by an algebraic equation, a partial differential equation is replaced by an ordinary differential equation, and so forth;

(3) the image equation thus obtained is solved for $F(p)$;

(4) from the image function $F(p)$ thus found one passes to the original function $f(t)$, which is the desired function.

In many cases, the most difficult stage is the fourth, i.e. finding the original function $f(t)$ from the image function $F(p)$, or computing the inverse Laplace transform. Extensive tables of correspondences between original functions and image functions are available so that one can find the original function from a given image function. However, these tables do not by far encompass all cases that may be encountered in practice and, what is more, it often happens that the original function is expressed in terms of very complicated functions that are hard to compute and are not always tabulated. Then it is either impossible or undesirable to find the exact original function. This necessitates the construction of approximate methods for computing inverse Laplace transforms that permit finding the original function in a broad class of cases. These methods are discussed in Part One of this text.

Let us now recall certain familiar facts concerning the theory of the Laplace transformation.

Given on the half axis $0 \leqslant t < \infty$ a function $f(t)$ that is integrable[1] with its absolute value on any finite interval $[a, b]$ $(0 \leqslant a < b < \infty)$.

---

[1] Here and henceforth we assume the Riemann integral.

Let us introduce the complex parameter $p = \sigma + i\tau$ and define the Laplace transform of the function f by

$$F(p) = \int_0^\infty f(t) e^{-pt} dt \qquad (1.1.1)$$

Here, the value of the improper integral over the half axis $[0, \infty)$ will be taken to mean the limit to which the integral over the finite interval $[0, B]$ tends as $B \to \infty$, so that

$$\int_0^\infty f(t) e^{-pt} dt = \lim_{B\to\infty} \int_0^B f(t) e^{-pt} dt \qquad (1.1.2)$$

We say that the *Laplace transformation is applicable to the function* f *for the value* p *of the parameter if for that value of* p *the integral* (1.1.1) *converges.*

It can be verified that if the transformation (1.1.1) is applicable to f for $p = p_0 = \sigma_0 + i\tau_0$, then it is applicable to f for any value $p = \sigma + i\tau$ for which $\mathrm{Re}(p - p_0) = \sigma - \sigma_0 > 0$. Indeed, consider the function $\varphi(t) = \int_0^t f(u)\, e^{-p_0 u}\, du$. If the integral (1.1.1) converges for $p = p_0$, then $\varphi(t)$ has a finite limit $\lim_{t\to\infty} \varphi(t)$ and consequently is bounded on the half axis $0 \leqslant t < \infty$:

$$|\varphi(t)| \leqslant Q < \infty$$

To prove the convergence of the integral (1.1.1), we take advantage of the Bolzano-Cauchy criterion. Take three arbitrary positive numbers a, b, c and assume $\mathrm{Re}(p - p_0) = \sigma - \sigma_0 \geqslant c$. In the computations given below we made use of integration by parts:

$$\left|\int_a^{a+b} f(t)\,e^{-pt}\,dt\right| = \left|\int_a^{a+b} e^{-(p-p_0)t}\,d\varphi(t)\right|$$

$$= \left|\varphi(t)\,e^{-(p-p_0)t}\Big|_a^{a+b} + (p-p_0)\int_a^{a+b} \varphi(t)\,e^{-(p-p_0)t}\,dt\right|$$

$$\leqslant e^{-ca}\cdot 2Q + |p-p_0|\,Q\int_a^{\infty} e^{-ct}\,dt = e^{-ca}\left(2+\frac{|p-p_0|}{c}\right)Q$$

The last term in this chain of relations does not depend on b and for any fixed p (as a increases) becomes less than any preassigned positive number. Therefore the Bolzano-Cauchy criterion holds true for the integral under the limit sign in (1.1.2), and the integral (1.1.1) converges.

Moreover, let p vary in a bounded closed region D lying inside the half-plane $\operatorname{Re} p > \sigma_0$. Clearly, for $p \in D$ there exist numbers c and M such that the following inequalities hold true: $\operatorname{Re}(p-p_0) = \sigma - \sigma_0 \geqslant c$ and $|p-p_0| \leqslant M$. From the inequalities thus obtained it follows that the Bolzano-Cauchy criterion will hold uniformly with respect to the parameter p in D. Since the integral $\int_0^b f(t)\,e^{-pt}\,dt$ is clearly an entire analytic function of p, it follows, from its uniform convergence in D to the integral (1.1.1), that $F(p)$ is an analytic function regular in D and since D is any interior region of the half-plane $\operatorname{Re} p > \sigma_0$, the function $F(p)$ is regular throughout this half-plane.

We now consider the set E of all real values $p = \sigma$ of the parameter p for which the Laplace transform (1.1.1) is applicable to the function f, and denote by γ the lower bound (the greatest lower bound of the values σ) of this set:

$$\gamma = \inf_E \sigma$$

The value of γ is determined on the basis of the following facts.

1. When γ has a finite value, we can say that the Laplace transform (1.1.1) is applicable to f throughout the open half-plane $\operatorname{Re} p > \gamma$, and $F(p)$ is a regular function of p at least in this half-plane and will not be applicable to f for a single value of p in the half-plane $\operatorname{Re} p < \gamma$.

2. When $\gamma = -\infty$, the Laplace transform (1.1.1) is applicable to f for any value of p and $F(p)$ is a regular function over the entire complex plane p.

3. When $\gamma = +\infty$, the Laplace transform (1.1.1) is not applicable to f for any value of p.

The number γ may be called the *boundary of the index of convergence*, and the straight line $\operatorname{Re} p = \sigma = \gamma$ the *boundary of the region of convergence* of the Laplace transform.

In this connection, one ordinarily refines the concept of the original function. The function f is called the *original function* if it has the following properties:

(1) f is defined on the axis $-\infty < t < \infty$ and is integrable with absolute value on every finite interval;

(2) the function f vanishes for $t < 0$;

(3) the Laplace transform is applicable to f for at least one value of p.

The following theorem holds true.

Theorem 1. *To every original function* f *there corresponds a number* γ $(-\infty \leqslant \gamma < \infty)$ *such that the transform* (1.1.1) *is applicable to* f *for any* p, *where* $\operatorname{Re} p = \sigma > \gamma$; *here*, $F(p)$ *is a regular function in the half-plane* $\operatorname{Re} p = \sigma > \gamma$. *The transform* (1.1.1) *is not applicable to* f *for any value of* p *for which* $\operatorname{Re} p < \gamma$.

The function F is called the *Laplace image function* of f.

It is sometimes difficult to find the value of γ. There is a simple procedure for estimating the upper bound of γ in many cases.

Theorem 2. *If there are two numbers* M $(0 \leqslant M \leqslant \infty)$ and α $(-\infty < \alpha < \infty)$ *such that for any* $t \geqslant 0$ *the inequality*

$$|f(t)| \leqslant Me^{\alpha t} \tag{1.1.3}$$

holds true, then $\alpha \geqslant \gamma$.

Proof. Let $\operatorname{Re} p = \sigma > \alpha$

$$\int_0^\infty |f(t)\, e^{-pt}|\, dt = \int_0^\infty |f(t)|\, e^{-\sigma t}\, dt = \int_0^\infty M e^{-(\sigma-\alpha)t}\, dt = \frac{M}{\sigma-\alpha} < \infty$$

Thus, for any real value $p = \sigma$ exceeding α, the integral (1.1.1) will be absolutely convergent, and since γ is the lower bound of such values of p, it must be true that $\gamma \leqslant \alpha$, which completes the proof.

The condition (1.1.3) holds for a broad class of functions, in particular for most functions encountered in applications; for this reason, the original functions f are frequently defined somewhat differently: namely, the first two properties are retained, while the third is replaced in the following fashion.

There exist two numbers M and α such that the inequality

$$|f(t)| \leqslant Me^{\alpha t}, \quad 0 \leqslant t < \infty$$

is valid.

Also note the variation of the image function $F(p)$ as the point p goes to infinity. A sufficient theorem for what follows is

Theorem 3. *If the integral* (1.1.1) *converges absolutely for the value* $p = p_0 = \sigma_0 + i\tau_0$, *then* $F(p)$ *tends to zero as* p *goes to infinity by any law as long as* p *remains in the half-plane* $\operatorname{Re} p = \sigma \geqslant \sigma_0$.

Proof. Let $p = \sigma + i\tau$ and $\sigma \geqslant \sigma_0$. By assumption, the integral (1.1.1) is absolutely convergent for $p_0 = \sigma_0 + i\tau_0$ and so it will converge for the value of p that is taken. Take a positive number A whose value will be determined below and split the integral (1.1.1) into two summands:

$$F(p)=\int_0^\infty f(t)\,e^{-pt}\,dt=\int_0^A f(t)\,e^{-pt}\,dt+\int_A^\infty f(t)\,e^{-pt}\,dt=I_1+I_2$$

First estimate the second integral:

$$|I_2|\leqslant\int_A^\infty |f(t)\,e^{-p_0t}|\,e^{-(\sigma-\sigma_0)t}\,dt\leqslant\int_A^\infty f(t)\,e^{-p_0t}|\,dt$$

If we have an arbitrary positive number ε, the number A can be chosen so that the last term of the inequalities is less than $\varepsilon/3$. Then we have $|I_2|<\varepsilon/3$ for any value of p for which $\operatorname{Re} p=\sigma\geqslant\sigma_0$. Therefore it suffices to be sure that I_1 tends to zero as $p\to\infty$.

The function f is taken to be absolutely integrable on any finite interval, say on $[0, A]$, and therefore there is a function $g(t)$, defined and continuously differentiable on $[0, A]$, for which the following inequality holds true:

$$\int_0^A |f(t)-g(t)|\,e^{-\sigma_0t}\,dt<\frac{\varepsilon}{2}$$

For the integral I_1 we have

$$I_1=\int_0^A [f(t)-g(t)]\,e^{-pt}\,dt+\int_0^A g(t)\,e^{-pt}\,dt=I_3+I_4$$

The integral I_3 may be estimated as follows:

$$|I_3|\leqslant\int_0^A |f(t)-g(t)|\,e^{-\sigma t}\,dt\leqslant\int_0^A |f(t)-g(t)|\,e^{-\sigma_0t}\,dt<\frac{\varepsilon}{3}$$

To estimate I_4, first perform the integration by parts:

$$I_4=-\frac{1}{p}\,g(t)\,e^{-pt}\Big|_0^A+\frac{1}{p}\int_0^A g'(t)\,e^{-pt}\,dt$$

Then

$$|I_4| \leqslant \frac{1}{|p|} [|g(0)| + |g(A)| e^{-\sigma_0 A}] + \frac{1}{|p|} \int_0^A |g'(t)| e^{-\sigma_0 t} \, dt = \frac{1}{|p|} M$$

$$M = |g(0)| + |g(A)| e^{-\sigma_0 A} + \int_0^A |g'(t)| e^{-\sigma_0 t} \, dt$$

If $|p| > 3M/\varepsilon$, then $|I_4| < \varepsilon/3$ and

$$|I_1| \leqslant |I_3| + |I_4| < \frac{2}{3} \varepsilon$$

Finally, if we take advantage of the estimate of I_2 obtained above, we can say that when $|p| > 3M/\varepsilon$ the inequality

$$|F(p)| \leqslant |I_1| + |I_2| < \frac{2}{3} \varepsilon + \frac{1}{3} \varepsilon = \varepsilon$$

will hold for $F(p)$, and since ε is an arbitrary positive quantity, it follows that $\lim_{p \to \infty} F(p) = 0$ $(\operatorname{Re} p = \sigma \geqslant \sigma_0)$.

1.2 *Complex integral for computing inverse Laplace transforms*

We now give one of a number of possible general methods for computing the inverse Laplace transform when such an inversion is given with the aid of a complex integral in which the integration is carried out along some straight line parallel to the imaginary axis of the complex plane.

To obtain the rule for inversion we will proceed from the Fourier double integral. Given on the real axis t $(-\infty < t < \infty)$ an arbitrary function $g(t)$. We say that the function g is representable by a *Fourier double complex integral* if for all values of t $(-\infty < t < \infty)$ the following equation[1] holds true:

[1] The conditions that are sufficient for representing a function by the Fourier integral are given in Chap. 7.

$$\frac{1}{2}[g(t+0)+g(t-0)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\tau t}\int_{-\infty}^{\infty}g(x)e^{-i\tau x}\,dx\,d\tau \quad (1.2.1)$$

This equation is called *Fourier's formula.*

The inner integral with respect to the variable x is assumed to be absolutely convergent for all τ, which is clearly equivalent to the absolute integrability of $g(t)$. The outer integral is regarded in the sense of the principal value, that is, as the limit of the integral taken along the interval $[-b, b]$ symmetric relative to the point $\tau = 0$ provided $b \to \infty$.

Let us establish a connection between the Fourier formula (1.2.1) and the inverse Laplace transform. Suppose that $f(t)$ is an arbitrary original function and that for a certain value c the product $f(t)e^{-ct}$ is an absolutely integrable function on the half axis $[0, \infty)$. Suppose $g(t) = f(t)e^{-ct}$. Note that for negative t the function f and, hence, $g(t)$ are equal to zero.

We assume that $g(t)$ can be represented as the Fourier integral (1.2.1). To simplify notation, we will assume that at the discontinuity points of $g(t)$ the following equation holds:

$$g(t)=\frac{1}{2}[g(t+0)+g(t-0)]$$

This condition only slightly alters the problem, since the set of discontinuity points of f and g is of measure zero and altering their values on such a set will not affect the magnitudes of the integrals. It clearly holds at all points of continuity of $g(t)$. The Fourier formula for the function g takes the form

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\tau t}\int_{0}^{\infty}g(x)e^{-i\tau x}\,dx\,d\tau$$

or, if we revert to f,

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(c+i\tau)t}\int_{0}^{\infty} f(x)\,e^{-(c+i\tau)x}\,dx\,d\tau \qquad (1.2.2)$$

Let us consider the Laplace transform of the function f:

$$F(p)=\int_{0}^{\infty} f(t)\,e^{-pt}\,dt$$

Due to the absolute integrability of $f(t)\,e^{-ct}$, on the straight line $p=c+i\tau$ $(-\infty<\tau<\infty)$ the transform converges for all τ and, hence, it will converge in the half-plane $\operatorname{Re} p=\sigma\geqslant c$. Besides, $F(p)$ will be a regular analytic function for $\operatorname{Re} p=\sigma>c$.

The value of the image function $F(p)$ is of interest when $\sigma=c$:

$$F(c+i\tau)=\int_{0}^{\infty} f(t)\,e^{-(c+i\tau)t}\,dt$$

The image function $F(p)$ stands under the sign of the inner integral in (1.2.2) so that for $f(t)$ the equation

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(c+i\tau)\,e^{(c+i\tau)t}\,d\tau$$

holds true. This is exactly the representation of the original function f in terms of its image function F; it is preferably written with the aid of the complex variable $p=\sigma+i\tau$. On the straight line $p=c+i\tau$ $(-\infty<\tau<\infty)$ we have $dp=i\,d\tau$, and the resulting equation becomes

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(p)\,e^{pt}\,dp \qquad (1.2.3)$$

This is called *Mellin's formula*.

The foregoing enables us to regard the following theorem as proven.

Theorem 4. *Let the original function* f (t) *be such that for any value* c *the function* g (t) = f (t) e^{-ct} *satisfies the conditions*:

(1) g (t) *is absolutely integrable on the half axis* [0, ∞);

(2) g (t) *is representable by the Fourier double integral.*

Then the representation (1.2.3) *of the original function* f *in terms of its image* F *is true*; *it is assumed here that the following equation holds at the points of discontinuity of* f:

$$f(t)=\frac{1}{2}[f(t+0)+f(t-0)].$$

Thus the problem of computing the inverse Laplace transform reduces to the problem of computing the contour integral (1.2.3) of a certain regular function. Since it is by far not always possible to find an exact expression of the integral (1.2.3) in terms of known functions, one can attempt to set up rules for obtaining it numerically. But this is a hard problem since, firstly, the contour of integration in the integral (1.2.3) is infinite and, secondly, the integrand e^{pt} oscillates on the line of integration, these oscillations being the stronger, the greater the value assumed by the parameter t.

On the other hand, the function $F(p)$ under the integral sign is not an arbitrary function but is an image function with all the properties that were mentioned above. This somewhat simplifies computing the integral (1.2.3) since the properties of the image function $F(p)$ may be taken into account beforehand when setting up rules for a numerical inversion of the Laplace transform.

Setting up rules for computing the Mellin integral (these take into account the properties of the image function) will be discussed in Chapters 4 to 6.

1.3 *Representing functions by the Laplace integral*

We list the conditions that are sufficient for a given function of a complex variable, $F(p)$, which is analytic in the half-plane $\operatorname{Re} p > \alpha$, to serve as the image function of a certain original function; that is, so that it can be represented by a convergent Laplace integral.

Let us first recall, without proof, certain facts from the theory of functions of a complex variable that we will need in the sequel.

Jordan's lemma. *If on a certain sequence of arcs of circles* C_{R_n}, $|p| = R_n$, $\operatorname{Re} p < c$ ($R_n \to \infty$, c *is fixed*), *the function* $F(p)$ *tends to zero uniformly with respect to* $\arg p$, *then for any positive* t

$$\lim_{n\to\infty} \int_{C_{R_n}} F(p)\, e^{pt}\, dp = 0 \tag{1.3.1}$$

If the same conditions hold on the sequence of arcs of circles C'_{R_n}, $|p| = R_n$, $\operatorname{Re} p > c$, *then for any negative* t

$$\lim_{n\to\infty} \int_{C'_{R_n}} F(p)\, e^{pt}\, dp = 0. \tag{1.3.2}$$

Theorem 5 (Cauchy's theorem). *If a function* $f(z)$ *is analytic in a simply connected domain* D, *then the integral of the function along any closed contour* C *lying in* D *is equal to zero*:

$$\int_C f(z)\, dz = 0.$$

Theorem 6 (residue theorem). *Let a single-valued function* $f(z)$ *be continuous on the boundary* C *of a domain* D *and be everywhere analytic inside this domain, except for a finite number of singular points* $a_1, a_2, \ldots, a_n$. *Then*

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{res} f(a_k) \tag{1.3.3}$$

where $\operatorname{res} f(a)$ *is the residue of the function* f *at the singular point* a.

The residue of the function $f(z)$ at a pole of order n may be found from the formula

$$\operatorname{res} f(a) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)] \quad (1.3.4)$$

For poles of the first order the formula (1.3.4) can be simplified to

$$\operatorname{res} f(a) = \lim_{z \to a} [(z-a) f(z)] \quad (1.3.5)$$

Here, if in the neighbourhood of the point a the function $f(z)$ is defined as the quotient of two analytic functions at this point,

$$f(z) = \frac{\varphi(z)}{\psi(z)}$$

and $\varphi(a) \neq 0$ while $\psi(z)$ has a zero of the first order at a (that is, $\psi(a) = 0$, $\psi'(a) \neq 0$), then (1.3.5) can be replaced by

$$\operatorname{res} f(a) = \lim_{z \to a} \frac{\varphi(z)}{\psi(z)} (z-a) = \lim_{z \to a} \frac{\varphi(z)}{\frac{\psi(z) - \psi(a)}{z-a}} = \frac{\varphi(a)}{\psi'(a)} \quad (1.3.6)$$

Now let us return to the proof of the theorem on the representability of $F(p)$ by the Laplace integral.

Theorem 7. *If the function* F (p) *is analytic in the half-plane* Re p $> \alpha$ *and tends to zero as* $|p| \to \infty$ *in any half-plane* Re p $\geqslant c > \alpha$ *uniformly with respect to* arg p, *and the integral*

$$\int_{c-i\infty}^{c+i\infty} F(p)\, dp \quad (1.3.7)$$

converges absolutely, then F (p) *is the image function of*

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(p)\, e^{pt}\, dp \quad (1.3.8)$$

which means it can be represented by the convergent Laplace integral

$$F(p) = \int_0^{\infty} e^{-pt} f(t)\, dt$$

for Re p > c; *here the Laplace integral converges absolutely.*

Proof. Take a number p_0 such that Re $p_0 > c$; then from (1.3.8) it follows that

$$\int_0^{\infty} e^{-p_0 t} f(t)\, dt = \frac{1}{2\pi i} \int_0^{\infty} e^{-p_0 t} \left(\int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp \right) dt \qquad (1.3.9)$$

Consider the inner integral. In it, $p = c + iy$, $dp = i\, dy$ and hence it can be rewritten as

$$\int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp = ie^{ct} \int_{-\infty}^{\infty} e^{iyt} F(c+iy)\, dy$$

Let us estimate the last integral:

$$\left| \int_{-\infty}^{\infty} e^{iyt} F(c+iy)\, dy \right| \leqslant \int_{-\infty}^{\infty} |F(c+iy)|\, dy \qquad (1.3.10)$$

By virtue of the conditions of the theorem, the integral (1.3.7) is absolutely convergent and so the integral on the left of (1.3.10) converges uniformly with respect to t and, hence, we can interchange the order of integration in (1.3.9):

$$\int_0^{\infty} e^{-p_0 t} f(t)\, dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(p)\, dp \int_0^{\infty} e^{(p-p_0)t}\, dt$$

$$= -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{F(p)}{p-p_0}\, dp \qquad (1.3.11)$$

The last equation is true since the inner integral converges due to the fact that Re $(p - p_0) < 0$ and $t > 0$.

Consider the arc C'_R: $|p| = R$, Re $p > c$. By the theorem, on this arc max $|F(p)| = \alpha_R \to 0$ as $R \to \infty$ and hence

$$\left| \int\limits_{C'_R} \frac{F(p)}{p-p_0}\, dp \right| \leqslant \frac{\alpha_R}{R-|p_0|}\, \pi R$$

and this integral too tends to zero as $R \to \infty$. From this it follows that the straight-line path of integration in (1.3.11) can be replaced by the closed contour $\widetilde{C}_R$ made up of the arc C'_R and the line segment $[c+ib,\ c-ib]$ traversed downwards. Then the formula (1.3.11) can be written as:

$$\int\limits_0^\infty e^{-p_0 t} f(t)\, dt = \frac{1}{2\pi i} \int\limits_{\widetilde{C}_R} \frac{F(p)}{p-p_0}\, dp \qquad (1.3.12)$$

We drop the minus sign since we have changed the direction along the straight line. We compute the integral in the right member of (1 3.12) by using the residue theorem. The function $\frac{F(p)}{p-p_0}$ inside the contour $\widetilde{C}_R$ has only one singular point—a pole of the first order at the point $p = p_0$. Its residue at this point can be computed from (1.3.6) and will be equal to $F(p_0)$. Then from (1.3.3) we find

$$\int\limits_0^\infty e^{-p_0 t} f(t)\, dt = F(p_0) \qquad (1.3.13)$$

And since p_0 is any point of the half-plane $\operatorname{Re} p > c$, it follows from (1.3.13) that $F(p)$ is a convergent Laplace integral for all p for which $\operatorname{Re} p > c$. Later on we will show that this integral is absolutely convergent as well.

We will now show that if the conditions of the theorem are fulfilled, then $f(t)$, which is represented by the integral (1.3.8), will have property (2) in the definition of an original function. Indeed, for $t < 0$, we have by the Jordan lemma

$$\lim_{R\to\infty} \int\limits_{C'_R} e^{pt} F(p)\, dp = 0$$

Hence, the straight-line path of integration in the formula (1.3.8) may be replaced by the contour $\widetilde{C}_R$ which was defined earlier. Then, for $t < 0$, we get, by the Cauchy theorem,

$$f(t) = \frac{1}{2\pi i} \int_{\widetilde{C}_R} e^{pt} F(p)\, dp = 0$$

since the integrand is analytic inside $\widetilde{C}_R$. Hence, property (2) is fulfilled for f. Besides, we will show that a condition of the form (1.1.3) is fulfilled for the function f when $\alpha = c$. Indeed, from (1.3.8) follows

$$|f(t)| = \left| \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp \right|$$

$$\leqslant \frac{1}{2\pi} e^{ct} \int_{-\infty}^{\infty} |F(c+iy)|\, dy = Me^{ct} \quad (1.3.14)$$

so that the inequality (1.1.3) holds.

Let us now return to the integral (1.3.13) and show that it converges absolutely. True enough, let $p_0 = x_0 + iy_0$, then by virtue of (1.3.14) and the fact that $x_0 > c$,

$$\int_0^{\infty} |e^{-p_0 t} f(t)|\, dt \leqslant M \int_0^{\infty} e^{-(x_0 - c)t}\, dt = \frac{M}{x_0 - c}$$

1.4 *Ill-conditioning of the problem of computing inverse Laplace transforms*

The problem of recovering the original function $f(t)$ from the operational image function $F(p)$ may be regarded as a problem in solving the first-order integral equation

$$\int_0^{\infty} e^{-pt} f(t)\, dt = F(p) \quad (1.4.1)$$

which belongs to the class of what are called *ill-conditioned problems*. These problems have two properties that greatly complicate their solution: they are not solvable for all values of the numerical or functional parameters defining their solution, and small variations in these parameters may be associated with a large change in the solution (the instability of such problems was mentioned in the preface and we are compelled to repeat some of the things said there). We now give a more detailed explanation of these properties for the problem of computing inverse Laplace transforms and for the associated integral equations.

For the sake of definiteness consider the case where the image function $F(p)$ is known on the real half axis $p > \alpha$ and the argument p is a real variable. The equation (1.4.1) does not have a solution for all functions $F(p)$ that are continuous or even smooth when $p > \alpha$. In particular, it is unsolvable if $F(p)$ is not an analytic function when $p > \alpha$. Now suppose that $F(p)$ is the image function of some function f and the equation (1.4.1) is, hence, solvable. Replace $f(t)$ by some other function $f_1(t)$ that differs from $f(t)$ by a perturbation of large magnitude on a rather small interval and is coincident with $f(t)$ on the remaining portion of the half axis $[0, \infty)$. The new original function f_1 will be associated with the image function $F_1(p)$ that differs but slightly from $F(p)$ for arbitrary $p > \alpha$. Hence, to a small change in the right-hand member of (1.4.1) there can correspond an arbitrarily large variation in the solution f in a uniform metric. It can be demonstrated that a similar situation will obtain in other metrics.

The ill-conditioning of the problem of solving equation (1.4.1), which is equivalent to computing the inverse Laplace transform, can complicate the numerical solution but does not make it impossible. To the solution of equation (1.4.1) we can apply the regularization methods developed by A. N. Tikhonov, V. K. Ivanov and others. We will not dwell on these methods here.

Chapter 2

Some Analytical Methods for Computing Inverse Laplace Transforms

2.1 *Finding the original function via the inversion formula*

It was shown in Chap. 1 that if $f(t)$ is the original function and $F(p)$ is the Laplace image function, then to compute the original function from the image we can take advantage of the complex integral

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(p)\, e^{pt}\, dp \qquad (2.1.1)$$

where c is the abscissa in the half-plane of the absolute convergence of the Laplace integral. True, it is rather difficult to utilize (2.1.1) for a direct computation of $f(t)$, since it requires a knowledge of the function $F(p)$ for complex values $p = c + iy$ $(-\infty < y < \infty)$ and the integral is improper with an oscillating kernel. But since (2.1.1) is an integral of an analytic function taken along a contour in the complex plane, it can be transformed by applying familiar methods of the theory of functions of a complex variable, for example, by changing the path integration, computing residues, and the like. In certain cases such transformations permit obtaining an expression of practical convenience for the original function, from which we can extract important properties of the function defined by the complex integral.

The methods of computing the original function with the aid of such transformations of the complex integral (2.1.1) will be considered in the following subsections.

2.1a. Expansion of the original function in series in terms of exponential functions. For an important class of image functions $F(p)$ we can obtain a series expansion of the original function whose terms correspond to the singular points of the image functions. Namely, the following theorem holds true.

Theorem 1. (1) *Let the function* F (p) *be meromorphic*; (2) *let the function* F (p) *be analytic in some half-plane* Re p > > α; (3) *there exists a system of circles*

$$C_n : |p| = R_n, \quad R_1 < R_2 < \ldots \quad (R_n \to \infty)$$

on which F (p) *tends to zero uniformly with respect to* arg p; (4) *for any* c > α *the integral* $\int_{c-i\infty}^{c+i\infty} F(p)\,dp$ *converges absolutely.*

Then for the original function of F (p) *we have*

$$f(t) = \sum_{p_k} \operatorname{res}_{p_k} F(p)\, e^{pt} \qquad (2.1.2)$$

where the residues are computed with respect to all poles of the function F (p) *and the summation is over groups of poles lying in the annular regions between adjacent circles* C_n.

Proof. Under the conditions of Theorem 1, Theorem 7 of Chap. 1 holds true; according to Theorem 7, Chap. 1, $F(p)$ is the image function of

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp \qquad (2.1.3)$$

Denote by C_n' the portion of the circle C_n to the left of the straight line $\operatorname{Re} p = c$, by $c \pm ib_n$, the points of intersection of this straight line with the circle C_n, and by

Γ_n, the closed contour that is made up of the section $[c - ib_n,\ c + ib_n]$ and the arc C'_n and is traversable counterclockwise. Since by the Jordan lemma, for $t > 0$,

$$\lim_{n\to\infty} \int_{C'_n} e^{pt} F(p)\, dp = 0$$

the integral in (2.1.3) can be replaced by the following integral:

$$f(t) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\Gamma_n} e^{pt} F(p)\, dp \qquad (2.1.4)$$

Now, applying the Cauchy residue theorem, we get

$$f(t) = \lim_{n\to\infty} \sum_{(\Gamma_n)} \operatorname{res}_{p_k} F(p)\, e^{pt}$$

where the residues are taken at all singular points of $F(p)$ lying inside Γ_n. The equation thus obtained proves the theorem.

2.1b. Special cases of expanding the original function in series in terms of exponential functions. We consider the case where $F(p)$ is a rational fractional function. We then have

Theorem 2. *If the function* $F(p) = \frac{A(p)}{B(p)}$ *is rational fractional and the degree of the polynomial* $A(p)$ *is less than that of the polynomial* $B(p)$, *then its original function is*

$$f(t) = \sum_{k=1}^{n} \frac{1}{(n_k - 1)!} \lim_{p\to p_k} \frac{d^{n_k-1}}{dp^{n_k-1}} \{F(p)(p - p_k)^{n_k} e^{pt}\} \qquad (2.1.5)$$

where p_k *are the poles of* $F(p)$, *and* n_k *are their multiplicities, and the sum is taken over all poles.*

Proof. First of all note that $F(p)$ is an image function. This follows from the theorem on the partial-fraction expansion of a fractional rational function, from the linearity of the Laplace transformation, and from the validity of the formula

$$t^n e^{p_0 t} \risingdotseq \frac{n!}{(p-p_0)^{n+1}}$$

where the image function is on the right side and the original function is on the left side. Hence

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}F(p)\,dp \qquad (2.1.6)$$

where $c > \max \operatorname{Re} p_k$, and p_k are the poles of $F(p)$.

As in the preceding theorem, the integral (2.1.6) can be replaced by the integral (2.1.4) because the Jordan lemma is applicable due to the fact that $F(p) \to 0$ as $p \to \infty$.

Applying the residue theorem to the integral (2.1.4) and the formula (1.3.4) to compute the residues at the poles, we arrive at formula (2.1.5).

In particular, if all the poles are simple, then (2.1.5) is simplified to

$$f(t)=\sum_{k=1}^{n}\frac{A(p_k)}{B'(p_k)}e^{p_k t} \qquad (2.1.7)$$

[we took advantage of (1.3.6) to compute the residues at the simple poles].

Remark. If the polynomial $B(p)$ has real coefficients, then to each complex root p there corresponds a complex conjugate root $\overline{p}$. If the polynomial $A(p)$ also has real coefficients, then

$$\frac{A(\overline{p})}{B'(\overline{p})}e^{\overline{p}t}=\overline{\frac{A(p)}{B'(p)}e^{pt}}$$

and, hence, the sum of the expressions $\frac{A(p)}{B'(p)}e^{pt}$, which sum is computed for the complex conjugate roots p_k and $\overline{p}_k$, will be equal to $2\,\mathrm{Re}\,\frac{A(p_k)}{B'(p_k)}\,e^{p_k t}$, and so formula (2.1.7) can in this case be represented as

$$f(t)=\sum \frac{A(p_k)}{B'(p_k)}\,e^{p_k t}+2\mathrm{Re}\sum \frac{A(p_k)}{B'(p_k)}\,e^{p_k t}$$

where the summation in the first term is carried over all real roots of $B(p)$ and in the second term over all complex roots with positive imaginary parts.

2.2 *Expanding the original function into power series*

Suppose that the image function $F(p)$ is analytic at the point at infinity. Then, as we know from operational calculus, $F(\infty)=0$. Expand the function $F(p)$ in a Laurent series about the point at infinity and show that its original function can be obtained by taking the sum of the originals of the terms of this expansion. Knowing that the function $t^{n-1}/(n-1)!$ is the original function of $1/p^n$, we state the following theorem.

Theorem 3. *If* F (p) *is analytic at the point at infinity and in that neighbourhood has a Laurent expansion*

$$F(p)=\sum_{k=1}^{\infty}\frac{c_k}{p^k} \tag{2.2.1}$$

then the original function of F (p) *is*

$$f(t)=\sum_{k=1}^{\infty}\frac{c_k}{(k-1)!}\,t^{k-1} \tag{2.2.2}$$

which is an entire function.

Proof. By the hypothesis of the theorem, the function $F(p)$ is analytic in the circle $|p|\geqslant R$. Set $p=1/q$ and $F(p)=F\left(\frac{1}{q}\right)=\Phi(q)$. The function $\Phi(q)=\sum\limits_{k=1}^{\infty}c_k q^k$

will be analytic in the circle $|q| \leqslant 1/R$, and on the basis of Cauchy's inequality[1] the following inequalities will be valid for its coefficients:

$$|c_k| \leqslant MR^k \quad (k = 1, 2, \ldots)$$

From the inequalities just obtained we get, for any t,

$$f(t) \leqslant \sum_{k=1}^{\infty} |c_k| \frac{|t|^{k-1}}{(k-1)!} \leqslant MR \sum_{k=0}^{\infty} \frac{(R|t|)^k}{k!} = MRe^{R|t|} \qquad (2.2.3)$$

From this it is clear that the series (2.2.2) converges over the whole plane t, that is, $f(t)$ is an entire function of the variable t.

From the inequality (2.2.3) it follows directly that for $t > 0$,

$$|f(t)| \leqslant Ce^{Rt}$$

Thus, for the function $f(t)$ an inequality of the form (1.1.3) holds and it is the original function.

Multiplying the series (2.2.2) by e^{-pt}, we get a series that converges uniformly for all values of t, which means it can be integrated termwise with respect to t from zero to infinity. Then

$$\int_0^{\infty} e^{-pt} f(t)\, dt = \int_0^{\infty} \sum_{k=1}^{\infty} c_k \frac{t^{k-1}}{(k-1)!} e^{-pt}\, dt$$

$$= \sum_{k=1}^{\infty} c_k \int_0^{\infty} \frac{t^{k-1}}{(k-1)!} e^{-pt}\, dt = \sum_{k=1}^{\infty} \frac{c_k}{p^k}$$

---

[1] Cauchy's inequality for coefficients: if a function $f(z)$ is analytic in a circle $D = |z - z_0| < R$ and is continuous on the boundary of that circle, then the inequality $|f^{(n)}(z_0)| \leqslant \frac{Mn!}{R^n}$, where M is the maximum modulus of $f(z)$ in the domain D, holds true.

That is,

$$F(p) = \sum_{k=1}^{\infty} \frac{c_k}{p^k}$$

which completes the proof.

2.3 *Expanding the original function into generalized power series*

Theorem 3 can be extended to generalized power series (see [6]). Here we confine ourselves to the simplest case.

Theorem 4. *Let* F (p) → 0 *as* p → ∞, Re p < c (c *a positive number*), *and, in a finite* p-*plane, let it have no singularities except the coordinate origin* p = 0, *which is a branch point of the power type.*

Then from the expansion of F (p) *into a generalized power series of the form*

$$F(p) = p^{\alpha} \sum_{k=0}^{\infty} c_k p^{k\beta} \tag{2.3.1}$$

where β *is a positive number, it follows that the original function of* F (p) *is the series*

$$f(t) = \frac{1}{t^{\alpha+1}} \sum_{k=0}^{\infty} \frac{c_k}{\Gamma(-\alpha-k\beta)} \frac{1}{t^{k\beta}} \tag{2.3.2}$$

in which all terms with integral nonnegative $\alpha+k\beta$ *are crossed out.*

Proof. Consider a closed contour $C^*_{R,r}$ made up of the segment $[c - ib, c + ib]$, the arc C'_R of the circle $|p| = R$, $\operatorname{Re} p < c$, $\operatorname{Im} p < 0$, the arc C''_R of the same circle defined by the inequalities $\operatorname{Re} p < c$ and $\operatorname{Im} p > 0$, a two-sided cut along a segment of the real axis $-R < \operatorname{Re} p < -r$, and the circle C_r: $|p| = r$.

Since the function $e^{pt} F(p)$ is analytic inside the contour $C^*_{R,r}$, the integral of this function along the contour $C^*_{R,r}$ is equal to zero and, hence, the integral along the segment $[c - ib, c + ib]$ may be replaced by an integral along the remaining portion of the contour. Besides, by the Jordan lemma, the integral of $e^{pt} F(p)$ along $C'_R + C''_R$ for $t > 0$ will tend to zero as $R \to \infty$, and for this reason the inversion formula can be written thus:

$$f(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int\limits_{C^*_{R,r}} F(p)\, e^{pt}\, dp = \frac{1}{2\pi i} \int\limits_{C^*_r} F(p)\, e^{pt}\, dp \qquad (2.3.3)$$

where C^*_r is the contour made up of the two-sided cut $-\infty < \operatorname{Re} p < -r$ and the circle $|p| = r$ with the point $p = -r$ deleted.

Substitute the expression (2.3.1) for $F(p)$ into formula (2.3.3) and integrate term-by-term[1] to get

$$f(t) = \sum_{k=0}^{\infty} c_k \left(\frac{1}{2\pi i} \int\limits_{C^*_r} p^{\alpha+k\beta} e^{pt}\, dp \right)$$

Introduce a new variable of integration by setting $z = pt$; since $t > 0$, the shape of the contour of integration will not change and we get

$$f(t) = \sum_{k=0}^{\infty} c_k \frac{1}{2\pi i} \frac{1}{t^{\alpha+k\beta+1}} \int\limits_{C^*_{rt}} z^{\alpha+k\beta} e^z\, dz \qquad (2.3.4)$$

---

[1] Termwise integration of the series along an infinite straight line requires supplementary conditions. It suffices, for instance, to require convergence of the integral, along the cut $(-\infty, -r]$, of the sum of the series $|e^z| \sum\limits_{k=0}^{\infty} |c_k|\, |p|^{k\beta}$ (see, for example, [23]).

As we know from complex variable theory, the integral representation of the function $1/\Gamma(x)$ is of the form

$$\frac{1}{\Gamma(x)}=\frac{1}{2\pi i}\int_{C^*} e^z z^{-x}\,dz$$

where C^* is a contour of the form C^*_{rt}, and the function vanishes at the points $x = 0, -1, -2, \ldots$.

Therefore we can write formula (2.3.4) as

$$f(t)=\sum_{k=0}^{\infty} c_k \frac{1}{t^{\alpha+k\beta+1}}\frac{1}{\Gamma(-\alpha-k\beta)}=\frac{1}{t^{\alpha+1}}\sum_{k=0}^{\infty}\frac{c_k}{\Gamma(-\alpha-k\beta)}\frac{1}{t^{k\beta}}$$

where terms with integral nonnegative $\alpha+k\beta$ have to be crossed out. This completes the proof of the theorem.

Chapter 3

Methods of Numerical Inversion of Laplace Transforms Based on the Use of Special Expansions

3.1 *Computing inverse Laplace transforms by polynomials orthogonal on a finite interval*

In this chapter we construct some methods, which for the most part permit only a fundamental possibility of carrying out inversion of Laplace transforms with the aid of series. The coëfficients of such series can be found from infinite systems of equations with triangular matrices solvable sequentially or by other equivalent methods.

These systems are unstable with respect to the growth of errors. The problems of estimating errors and determining the conditions of convergence of actual computational processes have not yet been investigated.

3.1a. Statement of the problem. The problem of computing the inverse Laplace transform can be solved by methods based on a series expansion of the original function in terms of orthogonal functions, in particular, in terms of Chebyshev, Legendre and Jacobi polynomials. This problem, which in final form reduces to the problem of moments on a finite interval, has been studied by many mathematicians.

Let us consider the statement of this problem in the form given in the works of V. M. Amerbaev and in the book [3] of V. A. Ditkin and A. P. Prudnikov.

Suppose we know the Laplace transform $F(p)$ of the function $\beta(t) f(t)$:

$$F(p) = \int_0^\infty e^{-pt}\beta(t) f(t)\, dt \qquad (3.1.1)$$

where $f(t)$ is the desired function and $\beta(t)$ is nonnegative and absolutely integrable on $[0, \infty)$. Suppose that $f(t)$ is integrable on any finite interval $[0, T]$ and belongs to the class $L_2(\beta(t), 0, \infty)$:

$$\int_0^\infty \beta(t)\,|f(t)|^2\,dt < \infty \tag{3.1.2}$$

It is required to construct the function $f(t)$ from the image function $F(p)$ of the function $\beta(t) f(t)$.

In the integral (3.1.1) we introduce the change of variable $x = e^{-t}$; then (3.1.1) reduces to the form

$$F(p) = \int_0^1 x^p \omega(x)\,\varphi(x)\,dx \tag{3.1.3}$$

where

$$\varphi(x) = f(-\ln x), \quad \omega(x) = \frac{\beta(-\ln x)}{x}$$

By virtue of the conditions that are imposed on the functions $f(t)$ and $\beta(t)$, the integral (3.1.3) converges everywhere in the half-plane $\operatorname{Re} p \geqslant 0$, and so to the variable p can be assigned the values 0, 1, 2, . . . and we can obtain the "weighted moments" of the function $\varphi(x)$:

$$\mu_k = F(k) = \int_0^1 x^k \omega(x)\,\varphi(x)\,dx \tag{3.1.4}$$

Then the problem at hand can be formulated thus: find a function $\varphi(x)$ from its "weighted moments" μ_k or, what is the same thing, find the function $f(t)$ from the values of the image function of $\beta(t) f(t)$ at the integral points $p = k$ ($k = 0, 1, 2, \ldots$). In a particular case, this problem can be simplified, and from the first $n + 1$ "weighted moments" we seek a polynomial

$$q_n(x) = \sum_{k=0}^{n} c_k x^k$$

such that its "weighted moments" coincide with the specified moments of the function $\varphi(x)$, that is, so that the equations

$$F(k) = \int_0^1 x^k \omega(x) q_n(x)\, dx = \mu_k \quad (0 \leqslant k \leqslant n) \qquad (3.1.5)$$

hold.

If such a polynomial exists, then the image functions of $\beta(t) q_n(e^{-t})$ and $\beta(t) f(t)$ coincide at the points $p = k$ ($k = 0, 1, \ldots, n$) and $q_n(e^{-t})$ may be considered a certain approximation to $f(t)$.

Remark. If $e_1, e_2, \ldots, e_n$ constitute a system of vectors in Euclidean space, then the *Gram determinant* (or Gramian) for this system is

$$\begin{vmatrix} (e_1 e_1) & (e_1 e_2) & \ldots & (e_1 e_n) \\ (e_2 e_1) & (e_2 e_2) & \ldots & (e_2 e_n) \\ \cdot & \cdot & \cdot & \cdot \\ (e_n e_1) & (e_n e_2) & \ldots & (e_n e_n) \end{vmatrix}$$

where $(e_k e_j)$ is the scalar product of the vectors e_k and e_j.

For a system of functions $f_i(x)$ ($i = 1, 2, \ldots, n$) for which the scalar product is defined as $\int_a^b \omega(x) f_k(x) f_j(x)\, dx$, the *Gram determinant* is

$$\begin{vmatrix} \int_a^b \omega f_1^2\,dx & \int_a^b \omega f_1 f_2\,dx & \ldots & \int_a^b \omega f_1 f_n\,dx \\ \int_a^b \omega f_2 f_1\,dx & \int_a^b \omega f_2^2\,dx & \ldots & \int_a^b \omega f_2 f_n\,dx \\ \cdot & \cdot & \cdot & \cdot \\ \int_a^b \omega f_n f_1\,dx & \int_a^b \omega f_n f_2\,dx & \ldots & \int_a^b \omega f_n^2\,dx \end{vmatrix}$$

where ω and f_i are functions of x.

The Gram determinant is always zero or greater than zero. It is zero if and only if the vectors $e_1, e_2, \ldots, e_n$ or the functions $f_1, f_2, \ldots, f_n$ are linearly dependent.

Now we show that the conditions (3.1.5) uniquely determine the polynomial $q_n(x)$. Indeed, the equations (3.1.5) constitute a system of $n+1$ linear algebraic equations in the $n+1$ unknowns $c_0, c_1, \ldots, c_n$, which are the coefficients of the polynomial $q_n(x)$. The determinant of this system is the Gramian of the functions $1, x, x^2, \ldots, x^n$, and since they are linearly independent, the determinant is nonzero, whence it follows that the system (3.1.5) has a solution which is unique. This means the polynomial $q_n(x)$ exists and the conditions (3.1.5) define it uniquely.

Note yet another property of the polynomial $q_n(x)$. In the class of polynomials of degree not exceeding n the polynomial $q_n(x)$ defined by the conditions (3.1.5) makes the following functional an absolute minimum:

$$J(c_0, c_1, \ldots, c_n) = \int_0^1 \omega(x)\left[\varphi(x) - \sum_{k=0}^{n} c_k x^k\right]^2 dx \quad (3.1.6)$$

True enough, write the system obtained from the minimality conditions of the functional (3.1.6):

$$\frac{\partial J}{\partial c_k} = -2\int_0^1 \omega(x)\left[\varphi(x) - \sum_{i=0}^{n} c_i x^i\right] x^k\, dx = 0$$

or

$$\int_0^1 \omega(x)\, x^k q_n(x)\, dx = \int_0^1 \omega(x)\, x^k \varphi(x)\, dx \quad (k = 0, 1, \ldots, n)$$

The latter system coincides with (3.1.5) and so the functional (3.1.6) has a stationary value on the polynomial $q_n(x)$. We now show that $q_n(x)$ makes the functional (3.1.6) an absolute minimum in the class of polynomials of degree not exceeding n.

Let $P_n(x)$ be an arbitrary polynomial of degree not exceeding n, and $P_n(x) \not\equiv q_n(x)$. Represent $P_n(x)$ in the form

$$P_n(x) = q_n(x) + \varepsilon_n(x)$$

Then

$$\int_0^1 \omega(x)\,[\varphi(x) - P_n(x)]^2\, dx = \int_0^1 \omega(x)\,[\varphi(x) - q_n(x)]^2\, dx$$

$$-2\int_0^1 \omega(x)\,[\varphi(x) - q_n(x)]\,\varepsilon_n(x)\, dx$$

$$+\int_0^1 \omega(x)\,\varepsilon_n^2(x)\, dx \qquad (3.1.7)$$

The second term on the right side of (3.1.7) is zero due to (3.1.5). Since $P_n(x)$ does not coincide identically with $q_n(x)$

and hence $\varepsilon_n(x)$ is not identically zero, the last term in (3.1.7) is positive and the following inequality will hold:

$$\int_0^1 \omega(x)[\varphi(x)-P_n(x)]^2\,dx > \int_0^1 \omega(x)[\varphi(x)-q_n(x)]^2\,dx$$

This completes the proof.

Note how the results just found are connected with series in terms of orthogonal polynomials. Denote by $p_n(x)$ $(n = 0, 1, 2, \ldots)$ a system of polynomials orthonormal on $[0, 1]$ with respect to the weight $\omega(x)$, and consider the corresponding generalized Fourier series for $\varphi(x)$:

$$\varphi(x) \sim \sum_{k=0}^{\infty} c_k p_k(x), \quad c_k = \int_0^1 \omega(x)\varphi(x)p_k(x)\,dx$$

We take a finite sum of n terms of the series

$$S_n(x) = \sum_{k=1}^{n} c_k p_k(x)$$

This is a polynomial of degree not exceeding n and it can be regarded as a certain approximation to the function φ.

It is easy to indicate an extremal property of such an approximation. Take arbitrary polynomials $P_n(x)$ of degree n and, among them, find one which has the least mean square deviation from φ. We already know that $q_n(x)$ is such a polynomial; now show that $q_n(x)$ coincides with $S_n(x)$. The polynomial P_n can be expanded in terms of the polynomials p_k $(k = 0, 1, \ldots, n)$:

$$P_n(x) = \sum_{k=0}^{n} A_k p_k(x)$$

Compute the mean square deviation of P_n from φ:

$$\delta_n^2 = \int_0^1 \omega(x)\,[\varphi(x) - P_n(x)]^2\,dx$$

$$= \int_0^1 \omega(x)\left[\varphi(x) - \sum_{k=0}^{n} A_k p_k(x)\right]^2 dx$$

$$= \int_0^1 \omega\varphi^2\,dx - 2\int_0^1 \omega\varphi \sum_{k=0}^{n} A_k p_k(x)\,dx + \int_0^1 \omega\left[\sum_{k=0}^{n} A_k p_k(x)\right]^2 dx$$

$$= \int_0^1 \omega\varphi^2\,dx - 2\sum_{k=0}^{n} A_k c_k + \sum_{k=0}^{n} A_k^2$$

$$= \int_0^1 \omega\varphi^2\,dx - \sum_{k=0}^{n} c_k^2 + \sum_{k=0}^{n} (A_k - c_k)^2$$

It is only the last sum, whose terms are all nonnegative, that depends on the choice of the polynomial $P_n(x)$ in the equation obtained; therefore, δ_n^2 attains a minimum if and only if $A_k = c_k$ $(k = 0, 1, \ldots, n)$. This means that the polynomial P_n that makes the value δ_n^2 a minimum, i.e. $q_n(x)$, must coincide with $S_n(x)$:

$$q_n(x) \equiv S_n(x) \qquad (3.1.8)$$

From (3.1.8) it follows, for one thing, that

$$\lim_{n\to\infty} q_n(x) = \lim_{n\to\infty} S_n(x) \sum_{k=0}^{\infty} c_k p_k(x)$$

and the convergence $q_n(x) \to \varphi(x)$ $(n \to \infty)$ is equivalent to the possibility of expanding the function $\varphi(x)$ in a series in terms of the orthogonal polynomials $p_n(x)$:

$$\varphi(x) = \sum_{k=0}^{n} c_k p_k(x)$$

The conditions for the possibility of such an expansion are in many cases known and, using them, we can obtain conditions under which the original function $f(t)$ can be found as the limit of the sequence of approximations $\beta(t)\, q_n(e^{-t})$ $(n = 1, 2, \ldots)$. Now consider certain special cases of the weight function $\omega(x)$.

3.1b. Inversion of Laplace transforms with the aid of Jacobi polynomials. Let the weight function be of the form

$$\omega(x) = x^{\alpha}(1-x)^{\beta}, \quad \alpha > -1, \quad \beta > -1 \qquad (3.1.9)$$

Construct the polynomial $q_n(x)$. But first consider the so-called *shifted Jacobi polynomials* $P_n^{*(\alpha,\beta)}(x)$. They differ from the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in that the interval of their definition is reduced to the interval $[0, 1]$ instead of the usual $[-1, 1]$, that is,

$$P_n^{*(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$$

Such polynomials depend on two parameters, which we denoted by α, β, and for any values of these parameters can be defined by

$$P_n^{*(\alpha,\beta)}(x) = \frac{(-1)^n}{n!} x^{-\alpha}(1-x)^{-\beta} \frac{d^n}{dx^n}\left(x^{\alpha+n}(1-x)^{\beta+n}\right) \qquad (3.1.10)$$

which is often called *Rodrigues' formula* for $P_n^{*(\alpha,\beta)}$. The polynomials $P_n^{*(\alpha,\beta)}(x)$ form an orthogonal system on $[0, 1]$ with respect to the weight $x^{\alpha}(1-x)^{\beta}$, and for them the following equations hold true:

$$\int_0^1 x^{\alpha}(1-x)^{\beta} P_n^{*(\alpha,\beta)}(x)\, P_m^{*(\alpha,\beta)}(x)\, dx = 0, \quad n \neq m \qquad (3.1.11)$$

$$r_n = \int_0^1 x^{\alpha}(1-x)^{\beta} [P_n^{*(\alpha,\beta)}(x)]^2\, dx$$

$$= \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\,(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)} \qquad (3.1.12)$$

Here, the orthonormal polynomials, which in Subsection 3.1a were denoted by $p_k(x)$, are

$$p_k^{*(\alpha,\beta)}(x)=\frac{1}{\sqrt{r_k}}P_k^{*(\alpha,\beta)}(x)$$

The coefficients c_k of the expansion of $\varphi(x)$ in the polynomials $p_k(x)$ have the values

$$c_k=\int_0^1 \omega(x)\,\varphi(x)\,p_k(x)\,dx$$

$$=\frac{1}{\sqrt{r_k}}\int_0^1 \omega(x)\,\varphi(x)\,P_k^{*(\alpha,\beta)}(x)\,dx=\frac{a_k}{\sqrt{r_k}}$$

For this reason, the polynomial $q_n(x)$, which coincides with the partial sum $S_n(x)$ of the generalized Fourier series, is of the form

$$q_n(x)=S_n(x)=\sum_{k=0}^{n}\frac{a_k}{r_k}P_k^{*(\alpha,\beta)}(x) \qquad (3.1.13)$$

It is possible to construct a simple expression for a_k in terms of the coefficients of the polynomials $P_k^{*(\alpha,\beta)}$ and the quantity $F(i)$. Let

$$P_k^{*(\alpha,\beta)}(x)=\sum_{i=0}^{k}\alpha_i^{(k)}x^i \qquad (3.1.14)$$

then

$$a_k=\int_0^1 \omega(x)\,\varphi(x)\,P_k^{*(\alpha,\beta)}(x)\,dx$$

$$=\sum_{i=0}^{k}\alpha_i^{(k)}\int_0^1 \omega(x)\,x^i\varphi(x)\,dx$$

$$=\sum_{i=0}^{k}\alpha_i^{(k)}\mu_i=\sum_{i=0}^{k}\alpha_i^{(k)}F(i) \qquad (3.1.15)$$

Using this formula, we can compute the coefficients a_k of the expansion (3.1.13) since the numbers α_i^k are known and the values of $F(i)$ are given.

Thus the problem of finding the polynomial $q_n(x)$ that approximates the function $\varphi(x)$ is solved.

In Subsection 3.1a we pointed out that the convergence of the sequence of approximations $q_n(x)$ to $\varphi(x)$ is equivalent to the possibility of expanding $\varphi(x)$ in a series in terms of the shifted Jacobi polynomials

$$f(t) = \varphi(x) = \sum_{k=0}^{\infty} \frac{a_k}{r_k} P_k^{*(\alpha,\beta)}(x)$$
$$= \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} q_n(x) \quad (3.1.16)$$

We now state a theorem (see [21]) that provides conditions sufficient for the possibility of such an expansion. We state it for ordinary Jacobi polynomials $P_n^{(\alpha,\beta)}$ considered on the interval $[-1, 1]$ but it can readily be extended to the expansion (3.1.16) in terms of shifted Jacobi polynomials.

First we give a theorem that yields a relationship between partial sums of Fourier series and of series in terms of Jacobi polynomials.

Theorem 1. *Given on an interval* $[-1, 1]$ *a measurable function* g (x); *let the integrals*

$$\left.\begin{aligned} &\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\,|g(x)|\,dx < \infty \\ &\int_{-1}^{1} (1-x)^{\alpha/2-1/4}(1+x)^{\beta/2-1/4}\,|g(x)|\,dx < \infty \end{aligned}\right\} \quad (3.1.17)$$

have finite values. If $S_n(x)$ *denotes the* n*th partial sum of the series in terms of Jacobi polynomials for the function* g (x) *and* $\sigma_n(\cos\theta)$ *denotes the* n*th partial sum of the Fourier cosine* *series of the function*

$$G(\theta) = (1-\cos\theta)^{\alpha/2+1/4}(1+\cos\theta)^{\beta/2+1/4} g(\cos\theta) \quad (3.1.18)$$

then in the interval $-1 < x < 1$ *the relation*

$$\lim_{n\to\infty} [S_n(x) - (1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}\sigma_n(x)] = 0 \quad (3.1.19)$$

is valid. It holds uniformly with respect to x *over any interval of the form* $-1+\varepsilon \leqslant x \leqslant 1-\varepsilon$, *where* $0 < \varepsilon < 1$.

From this theorem it follows, in particular, that if for some value θ $(0 < \theta < \pi)$ the Fourier series for the function $G(\theta)$ converges to $\frac{1}{2}[G(\theta+0) + G(\theta-0)]$, then for the corresponding value $x = \cos\theta$ the partial sum $S_n(x)$ will tend to $\frac{1}{2}[g(x+0) + g(x-0)]$.

As regards convergence of the Fourier series, the following theorem (see [24], Chap. 2) is sufficient in many cases.

Theorem 2. *Let* $G(\theta)$ *be a* 2π*-periodic function integrable with absolute value on* $[-\pi, \pi]$ *and let* l *be an arbitrary interval on the* x *axis. If* $G(\theta)$ *is of bounded variation on* l, *then the Fourier series for* $G(\theta)$ *converges to the value* $\frac{1}{2}[G(\theta+0) + G(\theta-0)]$ *at any point* θ *inside* l. *If* $G(\theta)$ *is also continuous on* l, *then the Fourier series converges to* $G(\theta)$ *uniformly with respect to* θ *on every subinterval within* l.

The function $G(\theta)$ defined by (3.1.18) is 2π-periodic and even. The associated Fourier series is a series in cosines of multiple arcs. Its convergence can be determined by the theorem that was just formulated. We assume that the interval l lies inside $[-\pi, \pi]$. Since the factor on the right of (3.1.18) in front of $g(\cos\theta)$ is continuous on l and assumes values not less than a positive number, the continuity of $G(\theta)$ on l is equivalent to the continuity on l of $g(\cos\theta)$. Besides, this factor is clearly of bounded variation on l

and for this reason the bounded nature of the variations of $G(\theta)$ and $g(\cos\theta)$ is the same.

Taking advantage of the two theorems given above, we now state the following theorem with respect to the possibility of expanding the function $\varphi(x)$ in a series (3.1.16) in terms of the shifted Jacobi polynomials, or, what is the same, with respect to the convergence of a sequence of approximations $q_n(x)$ to $\varphi(x)$.

Theorem 3. *Suppose the following conditions hold true for the function* $\varphi(x)$, $0 \leqslant x \leqslant 1$:

(1) *the integrals*

$$\int_0^1 x^{\alpha}(1-x)^{\beta}\,|\varphi(x)|\,dx$$

$$\int_0^1 x^{\alpha/2-1/4}(1-x)^{\beta/2-1/4}\,|\varphi(x)|\,dx$$

have finite values;

(2) $\varphi(x)$ *is of bounded variation on the interval* $l = [c, d]$ *lying inside* $[0, 1]$.

Then for any point x *lying inside* l *the equations*

$$\lim_{n\to\infty} q_n(x) = \sum_{k=0}^{\infty} \frac{a_k}{r_k} P_k^{*(\alpha,\beta)}(x) = \frac{1}{2}[\varphi(x+0)+\varphi(x-0)] \quad (3.1.20)$$

are valid. If $\varphi(x)$ *is also continuous on* l, *then on any interval of the form* $c+\delta \leqslant x \leqslant d-\delta$ $(0<\delta<\frac{1}{2}(d-c))$ *the following equation is valid uniformly with respect to* x:

$$\lim_{n\to\infty} q_n(x) = \sum_{k=0}^{\infty} \frac{a_k}{r_k} P_k^{*(\alpha,\beta)}(x) = \varphi(x) \quad (3.1.21)$$

Now, finally, let us return to the original problem of finding the function $f(t)$ from the image function $F(p)$ defined by the equation (3.1.1). To do this, make the change of variable $x = e^{-t}$ in an equation of the form (3.1.3) with weight function $\omega(x) = x^{\alpha}(1-x)^{\beta}$:

$$F(p) = \int_0^{\infty} e^{-pt} e^{-(\alpha+1)t} (1-e^{-t})^{\beta} f(t)\, dt$$

The same change must be made in the integrals taking part in condition (1) of Theorem 3. The calculations are simple and will not be carried out here.

Theorem 3 enables us to say that the assertion stated below is true for the problem that interests us.

Suppose that the quantities r_k *and* a_k *are computed from the values, at integer points* i (i = 0, 1, 2, . . .), *of the image function* F (p) *and from the polynomials* $P_k^{*(\alpha, \beta)}$ *via the rules* (3.1.15) *and* (3.1.12); *and also suppose the functions (assumed to be approximations to* f (t))

$$q_n(e^{-t}) = \sum_{k=0}^{n} \frac{a_k}{r_k} P_k^{*(\alpha, \beta)}(e^{-t})$$

have been formed.

If (1) *the integrals*

$$\int_0^{\infty} e^{-(\alpha+1)t} (1-e^{-t})^{\beta} |f(t)|\, dt$$

$$\int_0^{\infty} e^{-\frac{1}{2}\left(\alpha+\frac{3}{2}\right)t} (1-e^{-t})^{\frac{1}{2}\beta-\frac{1}{4}} |f(t)|\, dt$$

have finite values, (2) *the function* f (t) *is of bounded variation on the interval* [c, d] $(0 < c < d < \infty)$, *then for any* t, c < t < d, *it is true that*

$$\lim_{n\to\infty} q_n(e^{-t}) = \sum_{k=0}^{\infty} \frac{a_k}{r_k} P_k^{*(\alpha,\beta)}(e^{-t})$$

$$= \frac{1}{2}[f(t+0)+f(t-0)]$$

If, besides, the function f *is continuous on* [c, d], *then for any* δ $\left(0 < \delta < \frac{1}{2}(d - c)\right)$ *the following convergence occurs uniformly with respect to* t *on the interval* c + δ ⩽ t ⩽ ⩽ d − δ:

$$\lim_{n\to\infty} q_n(e^{-t}) = f(t).$$

In the next three subsections we consider special cases of Jacobi polynomials, Legendre polynomials, and Chebyshev polynomials of the first and second kinds, all of which are of special interest computationally. In these cases the computations can be somewhat more complete.

3.1c. Inversion of Laplace transforms with the aid of shifted Legendre polynomials. We consider the special case of the weight function (3.1.9) when $\alpha = \beta = 0$:

$$\omega(x) = 1 \quad \text{or} \quad \beta(t) = e^{-t}$$

The shifted Legendre polynomials $P_k^*(x)$ will be polynomials orthogonal on the interval [0, 1] with weight $\omega(x) = 1$. They are given by the formula (3.1.10) for $\alpha = 0$, $\beta = 0$ or by the formula

$$P_k^*(x) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(n+1)(n+2)\ldots(n+k)}{k!} x^k$$

$$= (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(n+k)!}{n!k!} x^k$$

The quantity r_n here is equal to

$$r_n = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\,(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)}$$

$$= \frac{\Gamma(n+1)\,\Gamma(n+1)}{n!\,(2n+1)\,\Gamma(n+1)} = \frac{1}{2n+1}$$

and the expansion of $f(t)$ in terms of shifted Legendre polynomials is of the form

$$f(t) = \sum_{k=0}^{\infty} (2k+1)\, a_k P_k^*(e^{-t}) \tag{3.1.22}$$

The quantities a_k are computed from (3.1.15), in which $\alpha_i^{(k)}$ are coefficients of the shifted Legendre polynomial $P_k^*(x)$.

3.1d. Inversion of Laplace transforms with the aid of shifted Chebyshev polynomials of the first kind. Now set $\alpha = \beta = -1/2$. The weight function is of the form $\omega(x) = x^{-1/2}(1-x)^{-1/2}$ and $\beta(t) = e^{-t/2}(1-e^{-t})^{-1/2}$. The shifted Chebyshev polynomials of the first kind $T_n^*(x)$ are an orthogonal system on $[0, 1]$ with respect to the weight $x^{-1/2}(1-x)^{-1/2}$. The Jacobi polynomials $P_n^{*(-1/2,\,-1/2)}(x)$ differ from $T_n^*(x)$ solely by a numerical factor, namely

$$P_n^{*(-1/2,\,-1/2)}(x) = C_n T_n^*(x)$$

where

$$C_n = \frac{\Gamma(2n)}{2^{2n-1}\,\Gamma(n)\,\Gamma(n+1)}$$

The polynomials $T_n^*(x)$ are of the form

$$T_n^*(x) = (-1)^n \sum_{k=0}^{n} (-1)^n \binom{n}{k} 2^k \frac{n(n+1)\,\ldots\,(n-k-1)}{(2k-1)!!}\, x^k$$

The values of r_n are computed from the formulas

$$r_n = \int_0^1 x^{-1/2}(1-x)^{-1/2}\,[T_n^*(x)]^2\,dx = \frac{\pi}{2} \qquad (n \neq 0, \quad r_0 = \pi)$$

and the expansion of $f(t)$ in terms of the shifted Chebyshev polynomials of the first kind is of the form

$$f(t)=\frac{1}{\pi}\left[a_0+2\sum_{k=1}^{\infty}a_kT_k^*(e^{-t})\right] \tag{3.1.23}$$

The coefficients a_k $(k=0, 1, \ldots)$ are computed from the formula (3.1.15) in which $\alpha_i^{(k)}$ are the coefficients of the shifted Chebyshev polynomial of the first kind $T_k^*(x)$.

Computationally, it is more convenient to make use of the trigonometric notation of the polynomials $T_n^*(x)$; namely,

$$T_n^*(x)=\cos[n\arccos(2x-1)] \tag{3.1.24}$$

Making the change of variable $2x-1=\cos\theta$ $(0\leqslant\theta\leqslant\pi)$ and taking into account that $x=e^{-t}$, $t=-2\ln\cos\frac{\theta}{2}$, we can rewrite the expansion (3.1.23) as

$$f\left(-2\ln\cos\frac{\theta}{2}\right)=\frac{1}{\pi}\left[a_0+2\sum_{k=1}^{\infty}a_k\cos k\theta\right] \tag{3.1.25}$$

3.1e. Inversion of Laplace transforms with the aid of shifted Chebyshev polynomials of the second kind. For $\alpha=\beta=1/2$ the weight function $\omega(x)$ and, respectively, $\beta(t)$ are of the form

$$\omega(x)=x^{1/2}(1-x)^{1/2},\quad \beta(t)=e^{-3t/2}(1-e^{-t})^{1/2}$$

The system of shifted Chebyshev polynomials of the second kind $U_n^*(x)$, which differ from the shifted Jacobi polynomials $P_n^{*(1/2,\,1/2)}$ only by a constant factor, $P_n^{*(1/2,\,1/2)}(x)=C_nU_n^*(x)$, where $C_n=\frac{(2n+1)!}{2^{2n}n!\,(n+1)!}$, is an orthogonal system of polynomials with respect to the weight function $\omega(x)$ on the interval $[0, 1]$. The polynomials $U_n^*(x)$ can be computed from the formula

$$U_n^*(x) = \frac{(-1)^n 2^n (2n+1)!!}{(n+2)(n+3)\dots(2n+1)}$$

$$\times \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^k (n+2)\dots(n+k+1)}{(2k+1)!!} x^k$$

For the polynomials $U_n^*(x)$ the quantity r_n is computed from the formula

$$r_n = \int_0^1 x^{1/2}(1-x)^{1/2}[U_n^*(x)]^2\,dx = \frac{\pi}{8}$$

and the expansion of $f(t)$ in terms of these polynomials is of the form

$$f(t) = \frac{8}{\pi} \sum_{k=0}^{\infty} a_k U_k^*(e^{-t}) \tag{3.1.26}$$

The coefficients a_k are computed from the formula (3.1.15), where $\alpha_i^{(k)}$ are the coefficients of the shifted Chebyshev polynomial of the second kind.

It is also more convenient, for the polynomials $U_n^*(x)$, to make use of the trigonometric notation:

$$U_n^*(x) = \frac{\sin[(n+1)\arccos(2x-1)]}{2\sqrt{x(1-x)}} \tag{3.1.27}$$

The expansion (3.1.26) then takes the form

$$f\left(-2\ln\cos\frac{\theta}{2}\right) = \frac{8}{\pi\sin\theta} \sum_{k=0}^{\infty} a_k \sin(k+1)\theta \tag{3.1.28}$$

where we make the same change of variable as in the preceding case.

3.1f. An alternative way of computing the a_k. Let us return to the series expansion of $f(t)$ in terms of the shifted Jacobi polynomials, that is, to (3.1.16), and let us see what expansion of the function $F(p)$, which is the Laplace transform of the function $\beta(t) f(t)$, corresponds to the expansion (3.1.16). To do this, expand the function x^p ($\operatorname{Re} p > 0$) in a series in terms of shifted Jacobi polynomials:

$$x^p = \sum_{k=0}^{\infty} \frac{b_k}{r_k} P_k^{*(\alpha, \beta)}(x)$$

where

$$b_k = \int_0^1 x^{\alpha} (1-x)^{\beta} x^p P_k^{*(\alpha, \beta)}(x)\, dx$$

To compute b_k we make use of (3.1.10) for the polynomials $P_k^{*(\alpha, \beta)}(x)$ and get

$$b_k = \int_0^1 x^{\alpha} (1-x)^{\beta} x^{p} \frac{(-1)^k}{k!} x^{-\alpha} (1-x)^{-\beta} \times \frac{d^k}{dx^k} [x^{\alpha+k} (1-x)^{\beta+k}]\, dx$$

$$= \frac{(-1)^k}{k!} \int_0^1 x^p \frac{d^k}{dx^k} [x^{\alpha+k} (1-x)^{\beta+k}]\, dx$$

Now, integrating by parts k times, we get

$$b_k = \frac{p(p-1) \ldots (p-k+1)}{k!} \int_0^1 x^{p+\alpha} (1-x)^{\beta+k}\, dx$$

$$= \frac{\Gamma(p+\alpha+1)\, \Gamma(k+\beta+1)}{\Gamma(p+\alpha+\beta+k+2)} \binom{p}{k}$$

Thus the expansion of x^p in terms of the polynomials $P_n^{*(\alpha,\beta)}(x)$ is of the form

$$x^p = \sum_{k=0}^{\infty} \frac{\Gamma(p+\alpha+1)\,\Gamma(k+\beta+1)}{r_k \Gamma(k+p+\alpha+\beta+2)} \binom{p}{k} P_k^{*(\alpha,\beta)}(x) \quad (3.1.29)$$

where r_k is found from the formula (3.1.12).

Remark. Denote by $L^2_{p(x)}$ the set of functions f that are defined on the interval $[a, b]$ and that are square integrable there with respect to the weight $p(x)$. Let the orthonormalizable system of functions $\varphi_k(x)$ ($k = 1, 2, \ldots$) be closed in the set $L^2_{p(x)}$, that is, such that for $f \in L^2_{p(x)}$ the Parseval equation holds:

$$\sum_{k=1}^{\infty} f_k^2 = \int_a^b p(x) f^2(x)\,dx, \quad f_k = \int_a^b p(x) f(x) \varphi_k(x)\,dx \quad (*)$$

Then for any two functions $f(x)$ and $g(x)$ belonging to $L^2_{p(x)}$, we have the *generalized Parseval equation*

$$\sum_{k=1}^{\infty} f_k g_k = \int_a^b p(x) f(x) g(x)\,dx \quad (**)$$

To prove this, consider the function $f + g$. Since it lies in $L^2_{p(x)}$, the following equation is, by $(*)$, valid:

$$\int_a^b p(x)[f(x) + g(x)]^2\,dx = \sum_{k=1}^{\infty} (f_k + g_k)^2$$

$$= \sum_{k=1}^{\infty} f_k^2 + 2\sum_{k=1}^{\infty} f_k g_k + \sum_{k=1}^{\infty} g_k^2$$

On the other hand,

$$\int_a^b p(x)[f(x)+g(x)]^2\,dx$$

$$= \int_a^b p(x) f^2(x)\,dx + 2\int_a^b p(x) f(x) g(x)\,dx + \int_a^b p(x) g^2(x)\,dx$$

$$= \sum_{k=1}^{\infty} f_k^2 + 2\int_a^b p(x) f(x) g(x)\,dx + \sum_{k=1}^{\infty} g_k^2$$

A comparison of the right members of the last two equations immediately yields (**).

Now let us take advantage of the generalized Parseval equation and apply it to the expansions (3.1.16) and (3.1.29):

$$F(p) = \int_0^1 x^{\alpha}(1-x)^{\beta}\varphi(x)\,x^p\,dx$$

$$= \sum_{k=0}^{\infty}\frac{a_k b_k}{r_k} = \sum_{k=0}^{\infty}\frac{\Gamma(p+\alpha+1)\,\Gamma(k+\beta+1)}{r_k\Gamma(p+\alpha+\beta+k+2)}\binom{p}{k}a_k$$

$$= \sum_{k=0}^{\infty}\frac{k!\,(2k+\alpha+\beta+1)\,\Gamma(k+\alpha+\beta+1)\,\Gamma(p+\alpha+1)\,\Gamma(k+\beta+1)}{\Gamma(k+\alpha+1)\,\Gamma(k+\beta+1)\,\Gamma(p+\alpha+\beta+k+2)}$$

$$\times\binom{p}{k}a_k = \Gamma(p+\alpha+1)\sum_{k=0}^{\infty}(2k+\alpha+\beta+1)$$

$$\times\frac{k!\,\Gamma(k+\alpha+\beta+1)\,p(p-1)\ldots(p-k+1)}{\Gamma(k+\alpha+1)\,\Gamma(p+\alpha+\beta+k+2)\,k!}a_k = \Gamma(p+\alpha+1)$$

$$\times\sum_{k=0}^{\infty}(2k+\alpha+\beta+1)\frac{\Gamma(k+\alpha+\beta+1)\,p(p-1)\ldots(p-k+1)}{\Gamma(k+\alpha+1)\,\Gamma(p+\alpha+\beta+k+2)}a_k \tag{3.1.30}$$

Remark. The *Schwarz-Bunyakovsky* (*Bunyakovsky-Cauchy*) *inequality* for convergent infinite series is

$$\sum_{n=1}^{\infty} (a_n b_n)^2 \leqslant \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2$$

where a_n and b_n are coefficients of the series.

An analog of this inequality for convergent integrals is the inequality

$$\int_a^b f(x)\, g(x)\, dx \leqslant \left(\int_a^b [f(x)]^2\, dx \right)^{1/2} \left(\int_a^b [g(x)]^2 \right)^{1/2}$$

Taking advantage of the Schwarz-Bunyakovsky inequality, we can show that the last series converges absolutely and uniformly in the half-plane $\operatorname{Re} p \geqslant 0$.

For instance if p is put equal to a positive integer n, the expansion (3.1.30) takes the form

$$F(n) = n!\,\Gamma(n+\alpha+1) \times \sum_{k=0}^{n} \frac{(2k+\alpha+\beta+1)\,\Gamma(k+\alpha+\beta+1)}{(n-k)!\,\Gamma(k+\alpha+1)\,\Gamma(n+\alpha+\beta+k+2)} a_k \qquad (3.1.31)$$

Setting $n = 0, 1, 2, \ldots$ in (3.1.31), we get an infinite triangular system of equations in the coefficients a_k. (We assume $0! = 1$.)

To make the coefficient of a_n in the nth equation of the system equal to 1, rewrite (3.1.31) as

$$\frac{\Gamma(2n+\alpha+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)} F(n) = \Gamma(n+\alpha+1) \sum_{k=0}^{n} \frac{(2k+\alpha+\beta+1)\,\Gamma(2n+\alpha+\beta+1)}{(n-k)!\,\Gamma(n+\alpha+\beta+1)} \times \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\,\Gamma(n+\alpha+\beta+k+2)} a_k \qquad (3.1.32)$$

The system (3.1.32) can be used to find the coefficients a_k of the expansion (3.1.16).

In the case of an expansion in terms of shifted Legendre polynomials, put $\alpha = \beta = 0$ in (3.1.32), then the coefficients a_k of the expansion (3.1.22) can be computed from the system

$$\frac{\Gamma(2n+1)}{n!\,\Gamma(n+1)} F(n) = \sum_{k=0}^{n} \frac{(2k+1)\,\Gamma(2n+1)}{(n-k)!\,\Gamma(n+k+2)} a_k$$

$$(n = 0,\ 1,\ 2,\ \ldots)$$

or

$$\binom{2n}{n} F(n) = \sum_{k=0}^{n} \frac{2k+1}{2n+1} \binom{2n+1}{n-k} a_k \qquad (3.1.33)$$

For expansions (3.1.25) and (3.1.28) in terms of shifted Chebyshev polynomials of the first and second kinds, we cannot obtain a system for finding a_k directly from (3.1.32) because these polynomials differ from the polynomials $P_k^{*(-1/2,\,-1/2)}(x)$ and $P_k^{*\,(1/2,\,1/2)}(x)$ by a constant factor. But if we carry out similar computations, we can obtain triangular systems of equations to determine the coefficients a_k of the expansions (3.1.25) and (3.1.28). For the expansion (3.1.25) the system has the form

$$2^{2n-1} F(n) = \frac{1}{2} \binom{2n}{n} a_0 + \sum_{k=1}^{n} \binom{2n}{n-k} a_k \ (n = 0,\ 1,\ 2,\ \ldots) \qquad (3.1.34)$$

For the expansion (3.1.28) it has the form

$$2^{2n} F(n) = \sum_{k=0}^{n} \frac{k+1}{n+1} \binom{2n+2}{n-k} a_k \quad (n = 0,\ 1,\ 2,\ \ldots) \quad (3.1.35)$$

3.1g. A remark on reducing the half-plane of regularity of the image function to the form Re $p \geqslant 0$ ($p \neq \infty$). In all preceding subsections we considered the problem of recovering the function $f(t)$ if we know the Laplace transform $F(p)$ of the function $f(t)$ with weight factor $\beta(t)$, that is, the function $\beta(t) f(t)$, and the abscissa of absolute convergence is zero. But in practice, for the most part, we know the Laplace transform $F(p)$ of $f(t)$ with a certain abscissa of absolute convergence γ_a that is not necessarily equal to zero; namely, we know that the transform is considered under the conditions

$$F(p)=\int_0^\infty e^{-pt} f(t)\,dt, \qquad \operatorname{Re} p \geqslant \gamma_0 > \gamma_a \tag{3.1.36}$$

Here we do as follows. Using the similarity and shift theorems for the Laplace transform, we can rewrite (3.1.36) as

$$hF(\gamma_0+ph)=\int_0^\infty e^{-\gamma_0 t/h} f\left(\frac{t}{h}\right) e^{-pt}\,dt \tag{3.1.37}$$

for any $h > 0$. The abscissa of absolute convergence of the integral (3.1.37) will be equal to zero.

Now, in order to obtain an expansion of the function $f\left(\frac{t}{h}\right)$ in the class of polynomials orthogonal with weight $\beta(t)$, we write (3.1.37) as

$$hF(\gamma_0+ph)=\int_0^\infty e^{-pt}\beta(t)\,\varphi(t)\,dt \tag{3.1.38}$$

where

$$\beta(t)\,\varphi(t)=f\left(\frac{t}{h}\right) e^{-\gamma_0 t/h} \tag{3.1.39}$$

or

$$\varphi(t)=[\beta(t)]^{-1} f\left(\frac{t}{h}\right) e^{-\gamma_0 t/h}$$

For the function $\varphi(t)$ we obtain an expansion in terms of orthogonal polynomials with weight $\beta(t)$ via schemes described in the preceding subsections.

Then the expansion for the function $f\left(\frac{t}{h}\right)$ is obtained by multiplying the expansion for $\varphi(t)$ into $\beta(t)\,e^{\gamma_0 t/h}$, as is seen from formula (3.1.39).

Set $h = \gamma_0 > \gamma_a$ and consider the expansions of the function $f\left(\frac{t}{h}\right) = \beta(t)\,e^t\,\varphi(t)$ in terms of shifted Legendre polynomials and shifted Chebyshev polynomials of the first and second kinds.

For the Legendre polynomials $P_n^*(e^{-t})$, the expansion of the function $\varphi(t)$ is of the form (3.1.22), the weight function $\beta(t) = e^{-t}$, and so

$$f\left(\frac{t}{h}\right) = \beta(t)\,e^t\varphi(t) = \varphi(t) = \sum_{k=0}^{\infty}(2k+1)\,a_k P_k^*(e^{-t}) \quad (3.1.40)$$

For Chebyshev polynomials of the first kind $T_n^*(e^{-t})$, the weight function $\beta(t)$ is of the form

$$\beta(t) = e^{-t/2}(1-e^{-t})^{-1/2}$$

and the function $f\left(\frac{t}{h}\right)$ can be computed from the formula

$$f\left(\frac{t}{h}\right) = \beta(t)\,e^t\varphi(t) = e^{t/2}(1-e^{-t})^{-1/2}\,\varphi(t) \quad (3.1.41)$$

If we pass to trigonometric notation of the polynomials $T_n^*(e^{-t})$ and make the change of variable $t = -2\ln\cos\frac{\theta}{2}$, then (3.1.41) takes the form

$$f\left(-\frac{2}{h}\ln\cos\frac{\theta}{2}\right) = \frac{2}{\sin\theta}\,\varphi\left(-2\ln\cos\frac{\theta}{2}\right)$$

Since the expansion of $\varphi(\theta)$ is of the form (3.1.25), the expansion of $f\left(-\frac{2}{h}\ln\cos\frac{\theta}{2}\right)$ will be

$$f\left(-\frac{2}{h}\ln\cos\frac{\theta}{2}\right)=\frac{2}{\pi\sin\theta}\left[a_0+2\sum_{k=1}^{\infty}a_k\cos k\theta\right] \quad (3.1.42)$$

For Chebyshev polynomials of the second kind, $U_n^*(e^{-t})$, the weight function $\beta(t)$ is of the form

$$\beta(t) = e^{-3t/2}(1-e^{-t})^{1/2}$$

and the function $f\left(\frac{t}{h}\right)$ can be computed in the following manner:

$$f\left(\frac{t}{h}\right)=\beta(t)\,e^{t}\varphi(t)=e^{-t/2}(1-e^{-t})^{1/2}\varphi(t)$$

If we pass to trigonometric notation of the polynomials $U_n^*(e^{-t})$ and make the change of variable as in the preceding case, then we get

$$f\left(-\frac{2}{h}\ln\cos\frac{\theta}{2}\right)=\frac{1}{2}\sin\theta\varphi\left(-2\ln\cos\frac{\theta}{2}\right)$$

Since for φ we have expansion (3.1.28), it follows that

$$f\left(-\frac{2}{h}\ln\cos\frac{\theta}{2}\right)=\frac{4}{\pi}\sum_{k=0}^{\infty}a_k\sin(k+1)\theta \quad (3.1.43)$$

The coefficients a_k of the expansions (3.1.40), (3.1.42), (3.1.43) are computed from the corresponding triangular systems of equations, and for the moments $F(k)$ [see (3.1.4)] we take the numbers

$$\mu_k = hF(h(k+1)), \qquad k = 0, 1, 2, \ldots$$

In the expansion (3.1.42) the factor $1/\sin\theta$ can exert a strong influence on computational errors for values of θ

close to zero. And so this expansion can be used when it is required to compute the value of $f(t)$ not on the whole axis but only at the point $t = t_0$ and in its neighbourhood. In this case we can choose the parameter h so that the substitution $x = e^{-ht}$ translates the point $t = t_0$ to $x = 1/2$, that is, $\theta = \pi/2$. Then the effect of the factor $1/\sin\theta$ will be reduced to a minimum. The value of h will be equal to $(\ln 2)/t_0$. Here, if the abscissa of absolute convergence of the integral $\gamma_0 \leqslant 0$, then no restrictions are imposed on the quantity t_0; but if $\gamma_a > 0$, then this method can be used if $0 < t_0 < (\ln 2)/\gamma_a$ since $h = \gamma_0 > \gamma_a$.

All the methods described in this section have an essential drawback, which consists in the following. The coefficients of the Legendre polynomials, the Chebyshev polynomials of first and second kinds, and other Jacobi polynomials, and also the coefficients of triangular systems of linear algebraic equations in the sought-for coefficients a_k all grow very rapidly with increasing k. For this reason, in order for the coefficients of the series expansion to be computed at least with a moderate degree of accuracy, the starting values of μ_k must be specified with great accuracy.

Remark. The series (3.1.43) is the Fourier sine series of the function $f\left(-\frac{2}{h}\ln\cos\frac{\theta}{2}\right)$. It may happen that the coefficients a_k of this series will not decrease fast enough and the series will converge slowly. We can then improve the convergence of the series if the functions $F(p)$ and $f(t)$ satisfy certain conditions. From the theory of Fourier series it is known that if the function $f\left(-\frac{2}{h}\ln\cos\frac{\theta}{2}\right)$ is differentiable a sufficient number of times and vanishes at the endpoints of the interval $[0, \pi]$, then the coefficients of the expansion of this function in a Fourier sine series are of the order of $(1/k^3)$. We can therefore attempt to represent the image function $F(p)$ as a sum so that the original function for one term is computed exactly and for the other satisfies the conditions given above. If the equations

$$\lim_{p\to\infty} pF(p) = \lim_{t\to 0} f(t) = f_0$$

$$\lim_{p\to 0} pF(p) = \lim_{t\to\infty} f(t) = f_\infty$$

hold and the limits are finite, then $F(p)$ can be represented at

$$F(p) = F_1(p) + \left(\frac{f_\infty}{p} + \frac{f_0 - f_\infty}{p+1}\right)$$

For the term in parentheses the original function is computed exactly; the original function $f_1(t)$ for the function $F_1(p)$ is equal to $f(t) - f_\infty - (f_0 - f_\infty)\, e^{-t}$ and, like the function of θ, vanishes at the endpoints of the interval $[0, \pi]$. The Fourier sine series of the function $f_1(t)$ will converge faster than the series for the function $f(t)$.

3.2 *Computing inverse Laplace transforms with the aid of the Fourier sine series*

In this section we give another method for finding the original function from the values of the image function at equidistant points on the real axis. This method is based on two assumptions, which however do not restrict its generality. First, it is assumed that the image function $F(p)$ exists for $\operatorname{Re} p > 0$. This can always be done if, instead of $F(p)$, we consider the image function $F(p + a)$ for large enough a. The latter is equivalent to multiplying $f(t)$ by e^{-at}. Second, it is assumed that $f(0) = 0$. This can be attained if we put $f_1(t) = f(t) - f(0)\, e^{-t}$, which is tantamount to substituting $F(p) - \frac{f(0)}{p+1}$ for the image function $F(p)$.

We transform the Laplace integral

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{3.2.1}$$

via the substitution

$$e^{-\sigma t} = \cos\theta \tag{3.2.2}$$

where σ is an arbitrary number exceeding zero. Then

$$f(t) = f\left(-\frac{1}{\sigma}\ln\cos\theta\right) = \varphi(\theta) \quad \left(0 \leqslant \theta < \frac{\pi}{2}\right) \tag{3.2.3}$$

$$\sigma F(\sigma) = \int_0^{\pi/2} (\cos\theta)^{p/\sigma - 1}\sin\theta\,\varphi(\theta)\,d\theta \tag{3.2.4}$$

Expand the function $\varphi(\theta)$ in a Fourier sine series of odd multiple arcs

$$\varphi(\theta) = \sum_{k=0}^{\infty} c_k \sin(2k+1)\theta \tag{3.2.5}$$

The coefficients c_k here are determined in the usual way:

$$c_k = \frac{4}{\pi}\int_0^{\pi/2} \varphi(\theta)\sin(2k+1)\theta\,d\theta \tag{3.2.6}$$

In order to express the values of the coefficients c_k in terms of the values of the image function $F(p)$, we do as follows. In the integral (3.2.4) put $p = (2n+1)\sigma$ $(n = 0, 1, \ldots)$ to get

$$\sigma[F(2n+1)\sigma] = \int_0^{\pi/2} \cos^{2n}\theta\sin\theta\,\varphi(\theta)\,d\theta \tag{3.2.7}$$

The kernel of this integral can be represented as a linear combination of the functions $\sin(2k+1)\theta$:

$$\cos^{2n}\theta\sin\theta$$

$$= 2^{-2n}\sum_{k=0}^{n}\left[\binom{2n}{k} - \binom{2n}{k-1}\right]\sin[2(n-k)+1]\theta, \quad \binom{2n}{-1} = 0 \tag{3.2.8}$$

Substitute into (3.2.7) the expressions (3.2.5) and (3.2.8). Since

$$\int_0^{\pi/2} \sin(2\mu+1)\theta \sin(2\nu+1)\theta \, d\theta = \begin{cases} 0, & \mu \neq \nu \\ \pi/4, & \mu = \nu \end{cases}$$

then for fixed n only such terms remain for which $\nu = n-k$ $(k = 0, 1, \ldots, n)$, that is,

$$\sigma F[(2n+1)\sigma] = 2^{-2n}\frac{\pi}{4}\sum_{k=0}^{n}\left[\binom{2n}{k} - \binom{2n}{k-1}\right] c_{n-k}$$

or

$$\left[\binom{2n}{n} - \binom{2n}{n-1}\right] c_0 + \ldots + \left[\binom{2n}{k} - \binom{2n}{k-1}\right] c_{n-k} + \ldots + c_n = \frac{4^{n+1}}{\pi}\sigma F[(2n+1)\sigma]$$

or

$$\sum_{k=0}^{n} \frac{2k+1}{2n+1}\binom{2n+1}{n-k} c_k = \frac{4^{n+1}}{\pi}\sigma F[(2n+1)\sigma] \qquad (3.2.9)$$

Substituting into this equation in succession $n = 0, 1, \ldots$, we get a linear system of equations with a triangular matrix for determining the coefficients c_k:

$$c_0 = \frac{4}{\pi}\sigma F(\sigma)$$

$$c_0 + c_1 = \frac{4^2}{\pi}\sigma F(3\sigma)$$

$$2c_0 + 3c_1 + c_2 = \frac{4^3}{\pi}\sigma F(5\sigma)$$

$$\cdots\cdots\cdots\cdots\cdots$$

The choice of the value σ is determined by the magnitude of the interval for which it is necessary to compute the value of the original function $f(t)$; σ should be taken small for large t and large for small t.

Remark. The equation (3.2.3) defines the function $\varphi(\theta)$ on the interval $0 \leqslant \theta < \pi/2$. The trigonometric series on the right of (3.2.5) constitutes functions having two peculiarities: these functions are odd or, to put it differently, in a plane with Cartesian coordinate axes (θ, φ) they have a graph symmetric about the origin and such that their graph is symmetric about the straight line $\theta = \pi/2$. The trigonometric series (3.2.5) yields a continuation of the function $\varphi(\theta)$, which continuation possesses the indicated symmetry properties and is also 2π-periodic; what is more, it is a Fourier series of the continued function. All the familiar theorems on convergence of a Fourier series are applicable to any determination of its convergence.

3.3 *Computing inverse Laplace transforms with the aid of series in terms of generalized Chebyshev-Laguerre polynomials*

In the preceding sections of this chapter we considered methods of inverting the Laplace transforms in which the original function $f(t)$ was found from the values of the image function $F(p)$ at equidistant points of the real axis. In this section we consider a method of recovering the original function by using the value of the image function $F(p)$ and the values of its derivatives at a single point.

Given the Laplace transform

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \qquad (3.3.1)$$

Suppose the function $f(t)$ satisfies the condition

$$\int_0^\infty e^{-t} t^{-\lambda} |f(t)|^2\, dt < \infty \qquad (3.3.2)$$

where $\lambda > -1$.

We know from the general theory of series expansions of functions in terms of orthogonal polynomials that the Fourier series of the function $g(t) = t^{-\lambda} f(t)$ in terms of the generalized Chebyshev-Laguerre polynomials

$$\sum_{k=0}^{\infty} \frac{a_k}{r_k} L_k^{(\lambda)}(t) \tag{3.3.3}$$

converges to this function in the mean. This means that for the partial sums

$$S_N(t) = \sum_{k=0}^{N} \frac{a_k}{r_k} L_k^{(\lambda)}(t)$$

we have the equation

$$\lim_{N\to\infty} \int_0^{\infty} |g(t) - S_N(t)|^2 t^{\lambda} e^{-t}\, dt = 0$$

For certain supplementary restrictions on $g(t)$, which we do not give[1], the following equation is valid:

$$g(t) = \sum_{k=0}^{\infty} \frac{a_k}{r_k} L_k^{(\lambda)}(t) \tag{3.3.4}$$

The generalized Chebyshev-Laguerre polynomials are orthogonal on the half axis $0 \leqslant x < \infty$ with weight $p(x) = x^{\lambda} e^{-x}$ and can be represented by the formulas

$$L_k^{(\lambda)}(t) = \frac{1}{k!} t^{-\lambda} e^{t} \frac{d^k}{dt^k} (e^{-t} t^{k+\lambda})$$

$$L_k^{(\lambda)}(t) = \sum_{m=0}^{k} \frac{\Gamma(k+\lambda+1)}{\Gamma(m+\lambda+1)} \frac{(-t)^m}{m!\,(n-m)!}$$

---

[1] See [21], Theorem 9.1.5.

The orthogonality property of the polynomials $L_k^{(\lambda)}(t)$ is

$$\int_0^\infty e^{-t}t^\lambda L_k^{(\lambda)}(t)\,L_m^{(\lambda)}(t)\,dt=0 \quad (k\neq m) \tag{3.3.5}$$

$$r_k=\int_0^\infty e^{-t}t^\lambda\,[L_k^{(\lambda)}(t)]^2\,dt=\frac{\Gamma(k+\lambda+1)}{k!} \tag{3.3.6}$$

If the function $f(t)$ satisfies the condition (3.3.2), then the Laplace transform $F(p)$ is an analytic function in the half-plane $\operatorname{Re} p>1/2$. True enough, by virtue of the Schwarz-Bunyakovsky inequality we can write a chain of inequalities:

$$\int_0^\infty |e^{-pt}f(t)|\,dt\leqslant\int_0^\infty t^\lambda e^{-t}e^{-(\operatorname{Re} p-1)t}\,|g(t)|\,dt$$

$$\leqslant\left\{\int_0^\infty t^\lambda e^{-t}e^{-2(\operatorname{Re} p-1)t}\,dt\right\}^{1/2}\left\{\int_0^\infty t^\lambda e^{-t}\,|g(t)|^2\,dt\right\}^{1/2}$$

$$=\left\{\int_0^\infty t^\lambda e^{-t(2\operatorname{Re} p-1)}\,dt\right\}^{1/2}\left\{\int_0^\infty e^{-t}t^\lambda\,|f(t)|^2\,dt\right\}^{1/2}$$

From this it is clear that the integral (3.3.1), under condition (3.3.2), converges absolutely and uniformly in the half-plane $\operatorname{Re} p>1/2$. Hence, the function $F(p)$ is analytic in the half-plane $\operatorname{Re} p>1/2$.

To determine the coefficients a_k of the expansion (3.3.4) we expand the function $e^{-(p-1)t}$ in a series in terms of the Chebyshev-Laguerre polynomials:

$$e^{-(p-1)t}=\sum_{k=0}^\infty\frac{b_k}{r_k}L_k^{(\lambda)}(t) \tag{3.3.7}$$

where r_k is computed from the formula (3.3.6) and

$$b_k = \int_0^\infty e^{-t} t^\lambda e^{-(p-1)t} L_k^{(\lambda)}(t)\, dt$$

This formula is then transformed as follows:

$$b_k = \int_0^\infty e^{-pt} t^\lambda L_k^{(\lambda)}(t)\, dt$$

From this we see that b_k is the Laplace transform of the function $t^\lambda L_k^{(\lambda)}(t)$ and, as we know (see [3]), it may be computed from the formula

$$b_k = \frac{1}{p^{\lambda+1}} \left(1 - \frac{1}{p}\right)^k \frac{\Gamma(k+\lambda+1)}{k!}$$

Thus, the expansion (3.3.7) takes the form

$$e^{-(p-1)t} = \sum_{k=0}^\infty \frac{1}{p^{\lambda+1}} \left(1 - \frac{1}{p}\right)^k L_k^{(\lambda)}(t) \tag{3.3.8}$$

We write the Laplace integral (3.3.1) as

$$F(p) = \int_0^\infty t^\lambda e^{-t} e^{-(p-1)t} g(t)\, dt \tag{3.3.9}$$

Since the function $g(t)$ can be expanded in the series (3.3.4) and the function $e^{-(p-1)t}$ in the series (3.3.8), then by applying to the integral (3.3.9) the generalized Parseval equation we get

$$F(p) = \sum_{k=0}^\infty a_k \frac{1}{p^{\lambda+1}} \left(1 - \frac{1}{p}\right)^k$$

Making the change of variable $1/p = z$, we find

$$\frac{1}{z^{\lambda+1}} F\left(\frac{1}{z}\right) = \sum_{k=0}^\infty a_k (1-z)^k \tag{3.3.10}$$

Since we have shown that the function $F(p)$ is analytic in the half-plane $\operatorname{Re} p > 1/2$, the function $F\left(\frac{1}{z}\right)$ is analytic in the circle $|z-1| < 1$; hence, also is $\frac{1}{z^{\lambda+1}} F\left(\frac{1}{z}\right)$ analytic in this circle. Consequently, the coefficients a_k in the formula (3.3.10) are coefficients of the Taylor series of the function $\frac{1}{z^{\lambda+1}} F\left(\frac{1}{z}\right)$ at the point $z = 1$. And so the coefficients a_k can be computed from the formula

$$a_k = \frac{(-1)^k}{k!} \frac{d^k}{dz^k} \left\{ \frac{1}{z^{\lambda+1}} F\left(\frac{1}{z}\right) \right\}_{z=1} \qquad (3.3.11)$$

Thus the expansion of the original function $f(t)$ in a series in terms of generalized Chebyshev-Laguerre polynomials is of the form

$$f(t) = t^{\lambda} \sum_{k=0}^{\infty} a_k \frac{k!}{\Gamma(k+\lambda+1)} L_k^{(\lambda)}(t) \qquad (3.3.12)$$

and the coefficients a_k can be computed from the formula (3.3.11).

Note that if the function $F(p)$ has singularities far away from the origin, the transformation $z = 1/p$ will carry them into the neighbourhood of the point $z = 0$, which can reduce the radius of convergence of the series (3.3.10). And this, generally, can reduce the rate of decrease of the coefficients a_k. Thus, if the Laplace transform has singular points located at a distance from the origin, the series in terms of generalized Chebyshev-Laguerre polynomials of the function $f(t)$ may converge slowly and will therefore be of little use in practical applications; and, contrariwise, we can expect a rapid convergence of such a series if the singular points of the function $F(p)$ are located in a small neighbourhood of the origin.

Chapter 4

Methods of Computing the Mellin Integral with the Aid of Interpolation Quadrature Formulas

4.1 *The general theory of interpolation methods*

Let us consider methods of computing the Mellin integral

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}F(p)\,dp \qquad (4.1.1)$$

based on replacing the integrand $F(p)$ by another function that interpolates $F(p)$ via its values at certain points.

The error of computation of the integral (4.1.1) will depend mainly on the accuracy with which we can interpolate the function $F(p)$. To obtain good accuracy, it is important to bring into agreement the mode of interpolation and the properties of $F(p)$, which is not an arbitrary function but an image function.

To interpolate $F(p)$ we can work via a choice of points p_k at which the values of $F(p)$ are taken, and also through the choice of functions $\{\omega_\nu(p)\}$ that form the foundation of the interpolation. As we know, the image function $F(p)$ tends to zero if the point p goes off to infinity in a manner such that the real part of p increases without bound in the process. Of primary interest here is the case where $F(p)$ decreases by a power law. And so we assume that $F(p)$ can be represented in the form $F(p)=\frac{1}{(p-a)^s}\varphi(p)$ $(s>0)$, where the function $\varphi(p)$ is regular in the half-plane $\operatorname{Re} p > \alpha$ and is bounded in the half-plane $\operatorname{Re} p \geqslant c$ $(c > \alpha)$. If s is a fractional number, then by $(p - a)^s$ is meant the branch for which $\arg(p - a) = 0$ when real $p > a$. The parameter a must satisfy the condition $\operatorname{Re} a \leqslant \alpha$ and is chosen so that the function $\varphi(p)$ in the half-plane $\operatorname{Re} p > \alpha$ has "better properties" than $F(p)$. What we mean by improving the properties of a function is that the function $F(p)$, being an image function, is regular in the half-plane $\operatorname{Re} p > \alpha$. The peculiarities of behaviour of $F(p)$ are determined by its singular points lying in the half-plane $\operatorname{Re} p \leqslant \alpha$, in particular, by how they are situated. The variation of $F(p)$ for $\operatorname{Re} p > \alpha$ will, generally, be the smoother, the fewer singular points $F(p)$ has, the simpler they are, and the farther they are from the straight line $\operatorname{Re} p = \alpha$.

We can always make $a = 0 \leqslant \alpha < c$ by a parallel translation of the coordinate axes. For this reason we will assume that the function $F(p)$ is of the form

$$F(p) = \frac{1}{p^s} \varphi(p) \tag{4.1.2}$$

where $\varphi(p)$ is regular for $\operatorname{Re} p > \alpha$ and is continuous in the half-plane $\operatorname{Re} p \geqslant \alpha$, including the point at infinity. By replacing the function $F(p)$ in the integral (4.1.1) by the expression (4.1.2), we get

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} p^{-s} \varphi(p)\, dp \tag{4.1.3}$$

For the integral (4.1.3) we will construct an interpolation quadrature formula based on interpolation of the function $\varphi(p)$. This is performed with the aid of linear combinations of a certain system of functions $\omega_\nu(p)$ $(\nu = 0, 1, \ldots)$.

We subject the choice of the system to the following completeness condition: no matter what the function $\varphi(p)$ of the type indicated above, for any[1] $c > \alpha$ and $\varepsilon > 0$ there must be a linear combination $S_n(p) = \sum_{\nu=0}^{n} a_\nu \omega_\nu(p)$ such that in the region $\operatorname{Re} p \geqslant c$ the inequality

$$\left| \varphi(p) - \sum_{\nu=0}^{n} a_\nu \omega_\nu(p) \right| < \varepsilon$$

holds true. Under our assumptions concerning the function $\varphi(p)$, for $\omega_\nu(p)$ it is natural to take rational functions (instead of polynomials) of p that are bounded as $p \to \infty$ and whose poles lie in the half-plane $\operatorname{Re} p \leqslant \alpha$. Besides that, the functions $\omega_\nu(p)$ have to satisfy yet another requirement that is essential: computations involving them must be sufficiently simple. Now the simplest computations occur if for $\omega_\nu(p)$ we take negative powers, i.e. $p^{-\nu}$ ($\nu = 0, 1, \ldots$), and if we interpolate the function $\varphi(p)$ by polynomials in $1/p$.

As for the points p_k, we will assume them to be arbitrary and located to the right of the straight line $\operatorname{Re} p = \alpha$. Particular cases of the location of points on the real axis will be considered in the following sections of this chapter.

Take the points $p_0, p_1, \ldots, p_n$ lying in the half-plane $\operatorname{Re} p > \alpha$ and use them to construct a polynomial $P_n\left(\frac{1}{p}\right)$ that interpolates the functions $\varphi(p)$:

$$\varphi(p) = P_n\left(\frac{1}{p}\right) + r_n(p) = \sum_{k=0}^{n} l_k\left(\frac{1}{p}\right) \varphi(p_k) + r_n(p) \quad (4.1.4)$$

[1] As will be seen later on, the requirement that the completeness condition be fulfilled for arbitrary $c > \alpha$ may be relaxed and replaced by the assumption that the condition holds for sufficiently large c.

where

$$l_k\left(\frac{1}{p}\right)=\frac{\omega_k\left(\frac{1}{p}\right)}{\omega_k\left(\frac{1}{p_k}\right)} \tag{4.1.5}$$

$$\omega_k\left(\frac{1}{p}\right)=\frac{\omega\left(\frac{1}{p}\right)}{\left(\frac{1}{p}-\frac{1}{p_k}\right)}, \quad \omega\left(\frac{1}{p}\right)=\prod_{i=1}^{n}\left(\frac{1}{p}-\frac{1}{p_i}\right)$$

Substituting (4.1.4) into the integral (4.1.3), we get the following computation formula:

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}p^{-s}\left[\sum_{k=0}^{n} l_k\left(\frac{1}{p}\right)\varphi(p_k)+r_n(p)\right]dp$$

$$=\sum_{k=0}^{n} A_k(t)\,\varphi(p_k)+R_n \tag{4.1.6}$$

where

$$\left.\begin{aligned} A_k(t)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}p^{-s}l_k\left(\frac{1}{p}\right)dp\\ R_n&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}p^{-s}r_n(p)\,dp \end{aligned}\right\} \tag{4.1.7}$$

Discarding the remainder R_n in (4.1.6), we get an approximate formula for computing the original function from the image.

Now let us take up the computation of the coefficients $A_k(t)$. Expand the polynomial $l_k\left(\frac{1}{p}\right)$ in powers of $\frac{1}{p}$:

$$l_k\left(\frac{1}{p}\right)=a_{k0}+\frac{a_{k1}}{p}+\frac{a_{k2}}{p^2}+\ldots+\frac{a_{kn}}{p^n}=\sum_{j=0}^{n} a_{kj}p^{-j}$$

Then

$$A_k(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}p^{-s}\sum_{j=0}^{n} a_{kj}p^{-j}\,dp$$

$$=\sum_{j=0}^{n} a_{kj}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}p^{-s-j}\,dp=\sum_{j=0}^{n}\frac{a_{kj}t^{s+j-1}}{\Gamma(s+j)} \qquad (4.1.8)$$

Using (4.1.8), it is easy to compute the coefficients $A_k(t)$ for arbitrary values of t. All we have to know is the values of a_{kj}, which depend solely on the chosen points p_k. For the most frequently encountered methods of choosing p_k, the values of a_{kj} can be computed beforehand.

4.2 *The equal-interval interpolation method*

Let us consider the case of equally spaced points p_k located on the real half axis $[\alpha, \infty)$:

$$p_k=\alpha+(k+1)h \quad (h>0,\ k=0,\ 1,\ \ldots,\ n)$$

Without restricting the generality of the problem, we can always assume $h=1$, for which purpose it suffices to make the change of variable $p=\alpha+p'h$. Then the points p_k will become integers: $p'_k=k+1$ $(k=0,\ 1,\ \ldots,\ n)$.

In this case the formula (4.1.6) becomes [without the remainder R_n and with account taken of (4.1.8)]

$$f(t)\approx\sum_{k=0}^{n} A_k(t)\,\varphi(k+1)=\sum_{k=0}^{n}\left\{\sum_{j=0}^{n}\frac{a_{kj}t^{s+j-1}}{\Gamma(s+j)}\right\}\varphi(k+1) \qquad (4.2.1)$$

Here the polynomials $l_k\left(\frac{1}{p}\right)$ are appreciably simplified and are equal to

$$l_k\left(\frac{1}{p}\right)$$

$$=\frac{\left(\frac{1}{p}-\frac{1}{1}\right)\cdots\left(\frac{1}{p}-\frac{1}{k}\right)\left(\frac{1}{p}-\frac{1}{k+2}\right)\cdots\left(\frac{1}{p}-\frac{1}{n+1}\right)}{\left(\frac{1}{k+1}-\frac{1}{1}\right)\cdots\left(\frac{1}{k+1}-\frac{1}{k}\right)\left(\frac{1}{k+1}-\frac{1}{k+2}\right)\cdots\left(\frac{1}{k+1}-\frac{1}{n+1}\right)}$$

$$=\frac{(k+1)^n}{p^n}\,\frac{(p-1)\ldots(p-k)(p-k-2)\ldots(p-n-1)}{k(k-1)\ldots 2\cdot 1(-1)(-2)\ldots(k-n)}$$

$$=\frac{(-1)^{n-k}(k+1)^n}{k!\,(n-k)!}\,\frac{(p-1)(p-2)\ldots(p-n-1)}{p^n(p-k-1)} \qquad (4.2.2)$$

The coefficients a_{kj} in the expansion of the polynomial $l_k\left(\frac{1}{p}\right)$ in powers of $1/p$ can be computed with ease.

A table of values of a_{kj} (k, $j=0, 1, \ldots, n$) for $n = 1\,(1)\,15$ is given in [13].

For specific values of the parameter s we can tabulate values of $b_{kj}=\frac{a_{kj}}{\Gamma(s+j)}$, instead of the coefficients a_{kj}.

The formula (4.2.1) was constructed by H. E. Salzer for the special case of the parameter $s = 1$. He also computed the values of the coefficients $A_k(t)$ for certain values of t. For the same formula, Shirtliffe and Stephenson computed the values of the coefficients $c_{kj}=\frac{(k+1)\,a_{kj}}{j!}$ for $n = 1\,(1)\,9$.

4.3 *The unequal-interval interpolation method*

The equally spaced points that were chosen in Sec. 4.2 for interpolating the function $\varphi(p)$ when computing the Mellin integral will clearly be the simplest and most convenient but will not apparently yield the most exact result.

Since computing this integral is frequently rather difficult and complicated, it is desirable to try to choose the points on the real axis so that interpolation of the function $\varphi(p)$ is more exact than equal-interval interpolation and, hence, computing the Mellin integral is more exact as well.

We now consider a possible solution of this problem.

As in Sec. 4.1, we assume that the image function $F(p)$ can be represented as

$$F(p)=\frac{1}{(p-a)^s}\varphi(p)$$

Then the integral (4.1.1) becomes

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\frac{\varphi(p)}{(p-a)^s}\,dp \tag{4.3.1}$$

To transform into a finite interval the infinite half axis $[\alpha, \infty)$ on which the interpolation points are chosen, perform the linear-fractional transformation

$$p=\frac{A+(A-2\alpha)x}{1-x} \tag{4.3.2}$$

where A is a real number exceeding α. This transformation carries the half axis $[\alpha, \infty)$ into the interval $[-1, 1]$; the line $\operatorname{Re} p=\alpha$ goes into the unit circle $|x|=1$ and the half-plane $\operatorname{Re} p\geqslant\alpha$ goes into the unit circle $|x|\leqslant 1$. The point A transforms to the centre $x=0$ of the unit circle. The line of integration $\operatorname{Re} p=c$ in the integral (4.3.1) goes into a circle lying inside the unit circle and touching its boundary at the point $x=1$. The length of the radius of this circle will depend on the value of c. If c approaches α, then the radius will approach unity. Contrariwise, if c increases, then it will decrease and can become arbitrarily small. The function $\varphi(p)$ is transformed into the function

$$\varphi(p)=\varphi\left(\frac{A+(A-2\alpha)x}{1-x}\right)=\Phi(x) \tag{4.3.3}$$

Since $\varphi(p)$ was regular in the half-plane $\operatorname{Re} p > \alpha$, it follows that the function $\Phi(x)$ will be regular in the circle $|x| < 1$.

To compute the integral (4.3.1) we will now interpolate the function $\Phi(x)$ on the line of integration with respect to its values at $n + 1$ points x_k $(k = 0, 1, \ldots, n)$ of the diameter d of the unit circle $|x| \leqslant 1$ lying on the real axis. Here we attempt to choose the points x_k so as to minimize the interpolation error.

Note that the integral (4.3.1) is a contour integral of an analytic function of the complex variable p. It is but slightly dependent on the line of integration $\operatorname{Re} p = c$ $(c > \alpha)$. In particular, c can be chosen arbitrarily large. Then, as has already been pointed out, the line of integration under the transformation (4.3.2) passes into a circle of small radius that is symmetric with respect to the diameter d of the unit circle $|x| \leqslant 1$ and tangent to the circumference at the point $x = 1$. For this reason, when interpolating the function $\Phi(x)$ we will want to obtain a good accuracy, especially near the point $x = 1$.

We consider two ways of choosing the points x_k and state pictorial (not very rigorous) reasons for taking precisely these points.

When investigating the convergence of interpolation processes for analytic functions, of great importance is the **limit distribution function of the points.** This notion requires some explanation. Before going into it, we begin with the more general concept of the mass **distribution function.** Let a unit mass be arbitrarily distributed on the interval $[-1, 1]$. Take an arbitrary point x on $[-1, 1]$ to the left of 1 and denote by $\mu(x)$ the mass lying strictly to the left of the point x, that is to say, the mass belonging to the interval $[-1, x)$. Complete the definition of $\mu(x)$ by putting $\mu(1) = 1$.

Clearly the function $\mu(x)$ has the following properties:

(1) $\mu(-1) = 0$;

(2) $\mu(x)$ is a monotone nondecreasing function of x continuous on the left for $x \in [-1, 1)$;

(3) $\mu(1) = 1$.

Any function $\mu(x)$, irrespective of its physical meaning, that has the indicated three properties is termed a *distribution function* for the interval $[-1, 1]$.

Suppose we have a sequence of distribution functions $\mu_n(x)$ $(n = 0, 1, 2, \ldots)$. We say the distribution function $\mu(x)$ is a *limit function* for the given sequence if at every continuity point of $\mu(x)$ it is true that $\mu_n(x) \to \mu(x)$ $(n \to \infty)$.

Now suppose we take $n+1$ interpolation points x_k $(k = 0, 1, \ldots, n)$ on the interval $[-1, 1]$. To each point we assign a mass $1/(n+1)$. This defines a certain mass distribution function $\mu(x)$ which is called the distribution function of the given set of points.

Finally, let us consider an interpolation process defined by the following infinite triangular array of points:

$$X = \begin{pmatrix} x_0^{(0)} & & \\ x_0^{(1)} & x_1^{(1)} & \\ x_0^{(2)} & x_1^{(2)} & x_2^{(2)} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{pmatrix}$$

The rows of the array contain interpolation points at the separate steps of the process. Take the nth row with points $x_k^{(n)}$ $(k = 0, 1, \ldots, n)$ and state the corresponding distribution function $\mu_n(x)$. We assume the array to be such that the sequence $\mu_n(x)$ $(n = 0, 1, 2, \ldots)$ has a limit distribution function $\mu(x)$. The function $\mu(x)$ is called the *limit distribution function of the interpolation points.*

In the interpolation of analytic functions a special role is played by a limit distribution function called the *Chebyshev function*:

$$\mu(x) = \frac{1}{\pi} \int_{-1}^{x} \frac{dt}{\sqrt{1-t^2}}$$

If for the sake of pictorialness we assume that $\mu(x)$ is connected with the mass distribution on $[-1, 1]$, then the mass-distribution density will be $\rho(x) = \mu'(x) = \pi^{-1} \times \times (1 - x^2)^{-1/2}$ and the masses will, obviously, be arranged symmetrically about the point $x = 0$—rarefied near the midpoint of the interval $[-1, 1]$ and compressed at its endpoints.

It turns out that interpolation processes with points having the Chebyshev function for the limit distribution function are the best (in the following sense) for interpolating analytic functions of $[-1, 1]$: no matter what the function $g(x)$ that is analytic on $[-1, 1]$, the interpolation process for this function will converge everywhere on $[-1, 1]$ and the convergence will be uniform in x. Besides, the uniform convergence will also hold in a certain neighbourhood of the interval $-1 \leqslant x \leqslant 1$, but the form of the domain of convergence will depend on the properties of $g(x)$ (see [10]).

For this reason, when interpolating the function $\Phi(x)$, it is natural to take points x_k ($k = 0, 1, 2, \ldots, n$) on the interval $[-1, 1]$ for which the limit distribution function is the Chebyshev function.

It is also a fact that the roots of any system of orthogonal polynomials on $[-1, 1]$ with any summable and almost everywhere positive weight-function will have the Chebyshev function for the limit distribution function.

When interpolating $\Phi(x)$, we can take for the points x_k ($k = 0, 1, \ldots, n$) the roots of a polynomial of degree $n + 1$ from any known system of orthogonal polynomials, in particular the roots of Chebyshev polynomials of the first and second kinds, Legendre polynomials, and Jacobi polynomials.

We refer to the end of the section the problem of a definite choice of points x_k, for the present we assume them to be arbitrary and located on $[-1, 1]$.

Using the values of the function $\Phi(x)$ at the points x_k ($k = 0, 1, \ldots, n$), we construct the interpolating polynomial

$$\Phi(x) \approx \sum_{k=0}^{n} L_k(x)\Phi(x_k)$$

$$= \sum_{k=0}^{n} \frac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}\Phi(x_k) \quad (4.3.4)$$

Now we return from the variable x to the old variable p:

$$x = \frac{p-A}{p+A-2\alpha}$$

Then

$$\varphi(p) \approx \sum_{k=0}^{n} l_k(p)\varphi(p_k) \quad (4.3.5)$$

where

$$p_k = \frac{A+(A-2\alpha)x_k}{1-x_k}$$

$$l_k(p) = L_k(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{\dfrac{p-A}{p+A-2\alpha} - \dfrac{p_i-A}{p_i+A-2\alpha}}{\dfrac{p_k-A}{p_k+A-2\alpha} - \dfrac{p_i-A}{p_i+A-2\alpha}}$$

Let us simplify the expression for $l_k(p)$. Since

$$\frac{p-A}{p+A-2\alpha} - \frac{p_i-A}{p_i+A-2\alpha} = \frac{2(A-\alpha)(p-p_i)}{(p+A-2\alpha)(p_i+A-2\alpha)}$$

$$\frac{p_k-A}{p_k+A-2\alpha} - \frac{p_i-A}{p_i+A-2\alpha} = \frac{2(A-\alpha)(p_k-p_i)}{(p_k+A-2\alpha)(p_i+A-2\alpha)}$$

it follows that

$$l_k(p) = \frac{(p_k+A-2\alpha)^n \omega_k(p)}{(p+A-2\alpha)^n \omega_k(p_k)} \quad (4.3.6)$$

where

$$\omega_k(p) = \frac{\omega(p)}{p-p_k}, \quad \omega(p) = (p-p_0)(p-p_1)\dots(p-p_n)$$

For the function $\varphi(p)$ substitute the expression (4.3.5) into the integral (4.3.1) to obtain the following formula for its approximate value:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \frac{\varphi(p)}{(p-a)^s}\, dp$$

$$\approx \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{(p-a)^s} \sum_{k=0}^{n} l_k(p)\, \varphi(p_k)\, dp$$

$$= \sum_{k=0}^{n} A_k(t)\, \varphi(p_k) \quad (4.3.7)$$

where

$$A_k(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \frac{l_k(p)}{(p-a)^s}\, dp \quad (4.3.8)$$

We can reduce the integral in the last formula to an integral that can be computed with the aid of the usual tables for inversion of Laplace transforms. Indeed, expand the polynomial $\omega_k(p)$ in powers of $p + A - 2\alpha$:

$$\omega_k(p) = \sum_{j=0}^{n} b_{kj}\, (p + A - 2\alpha)^{n-j}$$

Then

$$l_k(p) = \sum_{j=0}^{n} a_{kj}\, (p + A - 2\alpha)^{-j} \quad (4.3.9)$$

where

$$a_{kj} = \frac{(p_k + A - 2\alpha)^n}{\omega_k(p_k)}\, b_{kj} \quad (4.3.10)$$

Substituting (4.3.9) into (4.3.8), we get

$$A_k(t) = \sum_{j=0}^{n} a_{kj} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{pt}}{(p+A-2\alpha)^j (p-a)^s}\, dp \quad (4.3.11)$$

This integral is a tabular integral expressed in terms of a confluent hypergeometric function (see [2], p. 231).

Finally, for $A_k(t)$ we get the following expression:

$$A_k(t) = \sum_{j=0}^{n} a_{kj} \frac{t^{s+j-1}}{\Gamma(s+j)} e^{(2\alpha-A)t}\, {}_1F_1(s, s+j, (a+A-2\alpha)t) \quad (4.3.12)$$

Here, ${}_1F_1$ is a confluent hypergeometric function:

$${}_1F_1(\alpha, \beta, z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{\nu=0}^{\infty} \frac{\Gamma(\alpha+\nu)}{\Gamma(\beta+\nu)\,\nu!} z^{\nu}$$

$$(|z| < \infty)$$

The equation (4.3.12) can be simplified for certain particular values of a and A. For instance, if the points a and A are located symmetrically about α, that is, if they are connected by the relation $\alpha = \frac{1}{2}(A+a)$, then

$$A_k(t) = \sum_{j=0}^{n} a_{kj} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \frac{dp}{(p-a)^{s+j}}$$

$$= \sum_{j=0}^{n} a_{kj} \frac{t^{s+j-1} e^{at}}{\Gamma(s+j)} \quad (4.3.13)$$

The formulas (4.3.12) and (4.3.13) permit determining the coefficients $A_k(t)$ of the quadrature rule (4.3.7) for any value of t. For convenience in using them, we can set up a table of values of a_{kj} that depend on the choice of the points x_k and also on the parameters α and A.

For α and A we can take the values 0 and 1, respectively; this is no restriction of generality since any other values lead to the given values by the change of variable $p = \alpha + (A - \alpha) p'$.

As for the points x_k, they can, as indicated at the beginning of this section, be put equal to the roots of a polynomial of degree $n + 1$ taken from any system of polynomials orthogonal on the interval $[-1, 1]$.

Reference [13] lists values of a_{kj} in the following two cases.

1. For the points x_k $(k = 0, 1, \ldots, n)$ one takes roots of the Chebyshev polynomial of the first kind

$$T_{n+1}(x) = \cos[(n+1) \arccos x]$$

The coefficients a_{kj} of (4.3.12) or (4.3.13), that is, the coefficients of the expansion $l_k(p)$ in inverse powers of $p + 1$, can be computed in the following manner. Pass from the variable p to the variable $x = \frac{p-1}{p+1}$ to find $L_k(x)$:

$$L_k(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)} = \frac{T_{n+1}(x)}{(x-x_k)T'_{n+1}(x_k)}$$

Expand the polynomial $\frac{T_{n+1}(x)}{x-x_k}$ in powers of $1-x$:

$$\frac{T_{n+1}(x)}{x-x_k} = \sum_{j=0}^{n} c_{kj}(1-x)^j \tag{4.3.14}$$

We get

$$L_k(x) = \frac{\sum_{j=0}^{n} c_{kj}(1-x)^j}{T'_{n+1}(x_k)} = \sum_{j=0}^{n} b_{kj}(1-x)^j$$

where

$$b_{kj}=\frac{c_{kj}}{T'_{n+1}(x_k)}$$

Returning to the old variable p, we find the expansion of $l_k(p)$ in powers of $\frac{1}{p+1}$:

$$\left.\begin{aligned} l_k(p) &= \sum_{j=0}^{n} b_{kj}\left(1-\frac{p-1}{p+1}\right)^j \\ &= \sum_{j=0}^{n} b_{kj}\frac{2^j}{(p+1)^j}=\sum_{j=0}^{n} a_{kj}\frac{1}{(p+1)^j}, \\ a_{kj} &= b_{kj}2^j=\frac{2^j c_{kj}}{T'_{n+1}(x_k)} \end{aligned}\right\} \quad (4.3.15)$$

To compute the coefficients a_{kj} we have to know the coefficients c_{kj} of the expansion of $\frac{T_{n+1}(x)}{x-x_k}$ in powers of $(1-x)$. One way of finding them is as follows. In (4.3.14) put $x=1$ and then

$$c_{k0}=\frac{T_{n+1}(1)}{1-x_k}$$

Now rewrite (4.3.14) as

$$\frac{T_{n+1}(x)}{x-x_k}-c_{k0}=\sum_{j=1}^{n} c_{kj}(1-x)^j$$

The value $x=1$ is a root of the last polynomial and we can lower the degree of the polynomial by unity to get

$$\left[\frac{T_{n+1}(x)}{x-x_k}-c_{k0}\right](1-x)^{-1}=\sum_{j=1}^{n} c_{kj}(1-x)^{j-1}$$

Again putting $x = 1$, we get

$$c_{k1} = \left[\frac{T_{n+1}(x)}{x - x_k} - c_{k0}\right] (1-x)^{-1} \big|_{x=1}$$

We continue this process until all the c_{kj} are found. Then all the a_{kj} are found from the formulas (4.3.15).

The numerical values of a_{kj} and p_k for $n = 1$ (1) 14 are listed in Table 6 of [13].

2. For the points x_k, one takes the roots of a Legendre polynomial of degree $n + 1$. The coefficients a_{kj} and the points $p_k = \frac{1+x_k}{1-x_k}$ can be computed in exactly the same way as in the preceding case.

Table 7 of [13] lists the respective values of a_{kj} and of the points $p_k = \frac{1+x_k}{1-x_k}$ for $n = 1$ (1)14.

4.4 Other interpolation methods. Using the truncated Taylor series

To compute the integral (4.3.1) we made use above of interpolation with respect to the values of the function at several points. But the same can be attained by making use of a different type of interpolation, for instance, by interpolation with multiple points, interpolation via the values of the function and its derivatives at different points, and so on.

We now take up the case when interpolation is carried out with a single multiple point. Then the interpolating polynomial will, as we know, coincide with the truncated Taylor series.

We now come back to interpolation of the function $\Phi(x)$ that is regular in the circle $|x| < 1$. On the interval $0 \leqslant x \leqslant 1$ choose a point ξ ($\xi < 1$). The function $\Phi(x)$ is regular in a circle with centre ξ and radius not less than $1 - \xi$. For the approximate value of $\Phi(x)$ in this circle let us take its truncated Taylor series at the point ξ:

$$\Phi(x) \approx \sum_{\nu=0}^{n} \frac{(x-\xi)^{\nu}}{\nu!} \Phi^{(\nu)}(\xi)$$

From the variable x pass to the old variable p and from the function $\Phi(x)$ to $\varphi(p)$; note that for what follows it will be more convenient to pass from $\varphi(p)$ to the function

$$\psi(p) = (p+A-2\alpha)^n \varphi(p) \approx \sum_{j=0}^{n} \frac{(p-\zeta)^j}{j!} \psi^{(j)}(\zeta)$$

$$\zeta = \frac{A+(A-2\alpha)\xi}{1-\xi}$$

Substitute the last expression into the integral (4.3.1) to get the following formula for its computation:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \frac{\varphi(p)}{(p-a)^s} dp \approx \sum_{j=0}^{n} B_j(t) \psi^{(j)}(\zeta) \quad (4.4.1)$$

where

$$B_j(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \frac{(p-\zeta)^j}{j!(p+A-2\alpha)^n (p-a)^s} dp \quad (4.4.2)$$

If $(p-\zeta)^j$ is expanded in powers of $(p+A-2\alpha)$, the resulting integrals will be tabular integrals and the integral in (4.4.2) can be computed.

In the special case where the function $\varphi(p)$ is regular at the point at infinity, it can be expanded in a series in negative powers of p of the form $\varphi(p) = \sum_{\nu=0}^{\infty} a_{\nu} p^{-\nu}$.

In this case, to compute the integral (4.3.1), we get the formula

$$f(t) \approx \sum_{\nu=0}^{\infty} a_{\nu} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \frac{dt}{p^{\nu}(p-a)^s}$$

This is a tabular integral and 1268 4

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\frac{dt}{p^{\nu}(p-a)^{s}}=\frac{t^{s+\nu-1}}{\Gamma(s+\nu)}\,{}_1F_1(s,\ s+\nu,\ at)$$

4.5 *Some theorems on convergence of interpolation*

4.5a. Introduction. In the preceding sections we considered the interpolation quadrature formulas (4.1.6) and (4.3.7) for approximation of the Mellin integral. The remainder terms of these formulas are

$$R_n(\varphi,\ t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}p^{-s}r_n(p)\,dp \qquad (4.5.1)$$

$$R_n(\varphi,\ t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}(p-a)^{-s}\,r_n(p)\,dp \qquad (4.5.2)$$

where $r_n(p)$ are the errors of interpolating the functions $\varphi(p)$.

The quadrature process (4.1.6) or (4.3.7) for the function $\varphi(p)$ is convergent if the remainder terms (4.5.1) or (4.5.2) tend to zero as $n \to \infty$. The convergence or divergence of this process depends both on the properties of the function $\varphi(p)$ and on the choice of the points p_k. The problem of investigating the convergence consists in determining the relations between the properties of $\varphi(p)$ and the points p_k, under which we can be assured that the remainder R_n tends to zero.

Below, in Sec. 4.6, we consider the solution of this problem for some specified points p_k and for certain particular classes of functions $\varphi(p)$.

From the formulas (4.5.1) and (4.5.2) it is clear that the quadrature process will converge only if the interpolation converges. And so we begin the investigation by studying the convergence of interpolation.

Starting with Sec. 4.3, when constructing rules for computation, we performed the interpolation of the auxiliary function $\Phi(x) = \varphi(p)\left(p = \frac{A+(A-2\alpha)x}{1-x}\right)$ with the aid of an integral polynomial in x. Under the given conditions $\alpha = 0$ and $A = 1$, which do not restrict the generality of the results, the relationship between Φ and φ is $\Phi(x) = \varphi\left(\frac{1+x}{1-x}\right)$. Interpolation of $\Phi(x)$ is equivalent to interpolation of $\varphi(p)$ with the aid of rational functions which are polynomials in $x = \frac{p-1}{p+1}$. An approximate expression for $\varphi(p)$ is given in the equations (4.3.5), (4.3.6).

In the integral representation of the original function (4.3.1), the function $\varphi(p)$ along the line of integration $p = c + i\sigma$ $(-\infty < \sigma < \infty,\ c > 0)$ and to the right of it is everywhere regular except possibly at the point at infinity. Under the transformation $p = \frac{1+x}{1-x}$, the line $p = c + i\sigma$ goes into a circle that is orthogonal to the real axis x of the complex plane and passes through the points $\left(\frac{c-1}{c+1}\right)$, 1. The centre of the circle lies at the point $1 - \frac{1}{c+1} = 1 - \varepsilon$ and the radius is equal to $\frac{1}{c+1} = \varepsilon$. For $c > 0$ this circle, which we denote by Γ_ε, lies inside the circle $|x| \leqslant 1$, with the exception of the point $x = 1$, and it contracts to $x = 1$ when c increases without bound.

On the circle Γ_ε and inside it the function $\Phi(x)$ is everywhere regular with the possible exception of the point $x = 1$, where even though $\Phi(x)$ is, by the assumption concerning $\varphi(p)$, continuous, it may not be holomorphic.

The convergence of algebraic interpolation of analytic functions in a closed region with singularities on the boundary has not been investigated sufficiently. In particular this refers to the case, to which the authors confine themselves in this chapter, where the interpolation points are taken on the real diameter of the circle Γ_ε, which is equivalent to the location of the points on the half axis Re $p \geqslant 0$ of the p-plane. The difficulty of this problem of convergence and the fact that it has not been studied in depth prompted the authors to confine themselves to considering a narrow class of image functions $F(p)$ and original functions $f(t)$ when $\varphi(p)$ is a regular function at the point at infinity of the p-plane. In this case the appropriate function $\Phi(x)$ is regular not only in the unit circle $|x| < 1$, but also in the neighbourhood of the point $x = 1$.

As will become clear later on, the assumption just made permits answering many questions concerning the convergence of the interpolation of $\Phi(x)$ and, hence, also questions on the convergence of the process of approximate computation of the original function $f(t)$.

Remark. Here we speak of a narrow set of original functions $f(t)$ for the following reasons. If $\varphi(p)$ is regular when $|p| > R$, then the image $F(p) = \varphi(p)p^{-s}$ can be represented in this region by a power series

$$F(p) = p^{-s} \sum_{n=0}^{\infty} c_n p^{-n}, \quad |p| > R \tag{*}$$

For the coefficients c_n of the series the estimate given below is true: for any $\varepsilon > 0$ there is a number $N = N(\varepsilon)$ such that the inequality

$$|c_n| \leqslant N(R + \varepsilon)^n \tag{**}$$

holds.

Throughout the foregoing, the exponent s in (*) was considered to be positive. But in the problem of finding the original function $f(t)$ for $F(p)$ that has the representation (*) we can assume that $s > 1$ since for $0 < s \leqslant 1$ it is possible to isolate the first term $c_0 p^{-s}$ in the series on the right. The original function of that term is known, it is tabular and is equal to $c_0 \frac{t^{s-1}}{\Gamma(s)}$. Then we have to find the original function for the imagefunction $F(p) - \frac{c_0}{p^s} = p^{-s-1} \times \times \sum_{n=0}^{\infty} c_{n+1} p^n$ that corresponds to the exponent $s+1$.

In the expression (4.1.1), for $f(t)$ choose $c > R$. Then the line of integration $(c - i\infty, c + i\infty)$ will lie inside the region of regularity of $\varphi(p)$ and the series $\sum_{n=0}^{\infty} c_n p^{-n}$ on it will converge uniformly. Besides, since the kernel $e^{pt} p^{-s}$ in

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \frac{1}{p^s} \sum_{n=0}^{\infty} c_n p^{-n} \, dp \qquad (t \geqslant 0) \qquad (***)$$

is bounded on the line of integration,

$$\left| \frac{1}{2\pi i} e^{pt} p^{-s} \right| \leqslant \frac{1}{2\pi} e^{ct} \frac{1}{|p^s|}$$

and is absolutely integrable because $s > 1$, termwise integration is possible in (***). Besides, since the originals of the functions $c_n \frac{1}{p^{s+n}}$ are known and have the values $c_n \frac{t^{n+s-1}}{\Gamma(n+s)}$, for $f(t)$ we have a representation in the form of a power series:

$$f(t) = t^{s-1} \sum_{n=0}^{\infty} c_n \frac{t^n}{\Gamma(n+s)}$$

Now the estimate (**) for the coefficient c_n indicates that the power series in the right-hand member converges for all finite values of t. The original function $f(t)$ differs from the entire function of a special type $\sum_{n=0}^{\infty} c_n \frac{t^n}{\Gamma(n+s)}$ by the power factor t^{s-1} alone. Functions of this type do not by far exhaust all practically important types of original functions.

4.5b. Convergence of an interpolation process of the form (4.3.4). In the construction of such a process we had to interpolate the function $\Phi(x)$ on the line of integration with respect to its values at $n+1$ points x_k $(k = 0, 1, \ldots, n)$ of the diameter d of the circle $|x| \leqslant 1$, into which the half-plane $\operatorname{Re} p \geqslant \alpha$ is carried under the linear transformation

$$p = \frac{A + (A - 2\alpha)x}{1 - x}$$

We assume that in the given transformation $A = 1$ and $\alpha = 0$. This does not restrict the generality, as was pointed out in Sec. 4.3.

Note that the line of integration on which we interpolate the function $\Phi(x)$ (if the number c is taken sufficiently large) is a circle of small radius lying inside the unit circle and tangent to the circumference at the point 1:

$$|x - (1 - \varepsilon)| = \varepsilon \qquad (0 < \varepsilon \leqslant 1)$$

Let us first pose the following problem: to determine that class of functions $\Phi(x)$ for which the interpolation process converges uniformly on the line of integration for an arbitrary choice of interpolation points on the diameter d of the unit circle. Uniform convergence of interpolation will mainly depend on the magnitude of the region of regularity of $\Phi(x)$. If this region is sufficiently broad near the diameter d, then we can predict that the interpolation process will most certainly converge uniformly, no matter how we choose the points x_k on this diameter. We now indicate the smallest region in which the function $\Phi(x)$ must be regular in order to be able to ensure uniform convergence of interpolation on the line of integration for arbitrary points on $[-1, 1]$.

The following theorem holds.

Theorem 1. *If a function* $\Phi(\mathrm{x})$ *is regular in a closed circle* $|\mathrm{x} + 1| \leqslant 2$ *and in the neighbourhood* $|\mathrm{x} - 1| \leqslant 2\varepsilon$ $(0 < \varepsilon \leqslant 1)$ *of the point* $\mathrm{x} = 1$, *then the interpolation process* (4.3.4), *which is constructed on the basis of any points lying on the diameter* d *of the unit circle, will converge uniformly to* $\Phi(x)$ *on the line* $|x-(1-\varepsilon)|=\varepsilon$. *The indicated region of regularity is the smallest one that ensures convergence of interpolation for any system of points on the diameter* d *of the circle* $|x|\leqslant 1$.

Proof. This theorem follows directly from a general theorem proved by V. I. Smirnov and N. A. Lebedev [20]. The following is the statement in which the problem is solved by Smirnov and Lebedev.

Let F, B, G *be three nonempty sets of points of the complex plane* z, $F \subset G$, $B \subset G$. *We say that the condition* {F, B, G} *is fulfilled if for any function* f (z), *regular on* G, *for any choice of interpolation points* $z_k^{(n)}$ $(k=1, 2, \ldots, n+1;\ n=1, 2, \ldots)$ *in any bounded subset* $F^* \subset F$, *a sequence of interpolation polynomials* $P_n(z)$ *of the function* f (z) *converges uniformly to* f (z) *as* $n \to \infty$ *on any bounded subset* $B^* \subset B$.

It is proved that if F *and* B *are two closed bounded sets of points of the z-plane, and* K_ξ *is the smallest closed circle containing the set* B *and having the centre at the point* $\xi \in F$, *then the set* $G=\bigcup_{\xi \in F} K_\xi$ *is the smallest closed set for which the condition* {F, B, G} *holds true.*

In our case, the set F is the interval $-1 \leqslant x \leqslant 1$, and the set B is the circle $|x-(1-\varepsilon)|=\varepsilon$. To find the set G, we construct the two smallest closed circles centred at the points $x=-1$, $x=1$ that contain the line $|x-(1-\varepsilon)|=\varepsilon$. The first circle is determined by the inequality $|x+1|\leqslant 2$, the second one by the inequality $|x-1|\leqslant 2\varepsilon$. Their sum is the sought-for set G.

Remark. Under the conditions of Theorem 1, uniform convergence will occur not only on the circle $|x-(1-\varepsilon)|=\varepsilon$ but inside it as well.

If we pass from the x-plane to the p-plane and from the function $\Phi(x)$ to the function $\varphi(p)$, then Theorem 1 can be stated as follows.

Theorem 1a. *If the function* $\varphi(p)$ *is regular in the half-plane* $\operatorname{Re} p \geqslant -1/2$ *and in the region* $|p+1| \geqslant 1/\varepsilon$ $(0 < \varepsilon \leqslant 1)$, *then the interpolation process* (4.3.5) *constructed on the basis of arbitrary points lying on the real half axis* $\operatorname{Re} p \geqslant 0$ *will converge uniformly to* $\varphi(p)$ *in the half-plane* $\operatorname{Re} p \geqslant 1/\varepsilon$. *The indicated region of regularity* $\varphi(p)$ *will be the smallest one ensuring convergence of interpolation for* $\operatorname{Re} p \geqslant 1/\varepsilon$ *relative to any system of points on the nonnegative real half axis.*

We now pose a different problem. We know that the function $\Phi(x)$ is regular in the unit circle $|x| \leqslant 1$. Suppose that it is also regular in the neighbourhood $|x-1| \leqslant 2\varepsilon$ of the point $x = 1$. Let us attempt to determine the largest interval including the diameter d of the circle $|x| \leqslant 1$ such that the interpolation process (4.3.4) based on any points of this interval would converge uniformly to the function $\Phi(x)$ on the line $|x-(1-\varepsilon)| = \varepsilon$ for any function $\Phi(x)$ that is regular in the earlier indicated region.

Theorem 2. *If the function* $\Phi(x)$ *is regular in the circle* $|x| \leqslant 1$ *and in the neighbourhood* $|x-1| \leqslant 2\varepsilon$ $(0 < \varepsilon \leqslant 1)$ *of the point* $x = 1$, *then the interpolation process* (4.3.4) *constructed on the basis of arbitrary points lying on the interval* $[0, 1]$ *will converge uniformly to* $\Phi(x)$ *on the circle* $|x-(1-\varepsilon)| = \varepsilon$.

The interval $[0, 1]$ *will be the largest set belonging to the diameter* d *of the circle and ensuring uniform convergence of interpolation with respect to arbitrary points taken in this set for functions that are regular in the indicated region.*

This theorem can be proved on the basis of the result obtained by Smirnov and Lebedev [20]. They have established that if B and G are two sets of points of a finite z-plane (a finite plane is a complex plane with the point ∞ deleted) — B is a closed bounded nonempty set, G is different from a finite plane, and $B \subset G$ — then the largest closed set F for which the condition $\{F, B, G\}$ holds is the set of centres of all closed circles K such that $K \subset G$ and $B \subset K$.

In our theorem, G is the sum of the circles $|x| \leqslant 1$ and $|x-1| \leqslant 2\varepsilon$, B is the circle $|x-(1-\varepsilon)| = \varepsilon$. The desired set F is the set of the centres of circles containing B and contained in G. Besides, we imposed the requirement that these centres belong to the diameter d of the unit circle. Then it is clear that the set F is the closed interval $[0, 1]$.

If we again pass from the x-plane to the p-plane, then Theorem 2 becomes

Theorem 2a. *If the function* $\varphi(\mathrm{p})$ *is regular in the half-plane* $\mathrm{Re}\, \mathrm{p} \geqslant 0$ *and also in the region* $|\mathrm{p}+1| \geqslant 1/\varepsilon$, *then the interpolation process* (4.3.5), *constructed on the basis of any points located on the real axis so that* $\mathrm{p}_k \geqslant 1$ $(\mathrm{k}=1, 2, \ldots, \mathrm{n})$, *converges uniformly to* $\varphi(\mathrm{p})$ *on the straight line* $\mathrm{Re}\, \mathrm{p} = \frac{1}{\varepsilon} - 1$. *The half axis* $[1, \infty)$ *will be the largest region on the real axis that ensures uniform convergence of interpolation relative to arbitrary points lying on it for functions* $\varphi(\mathrm{p})$ *that are regular in the above-indicated region.*

In Theorem 1 we indicated the smallest region in which the function $\Phi(x)$ must be regular for the interpolation process to converge uniformly on the line of integration for arbitrary points located on the diameter d of the unit circle. Now suppose that the points are not chosen arbitrarily on this diameter but have a very definite distribution. Then there arises the problem of determining the region of regularity of the function that ensures uniform convergence of interpolation on the line of integration $|x-(1-\varepsilon)| = \varepsilon$.

Very important for investigating the convergence of interpolation is the following logarithmic potential (see [10])

$$u(x) = \int_{-1}^{1} \ln \frac{1}{|x-t|}\, d\mu(t)$$

where $\mu(t)$ is the limit distribution function of the points.

We consider the level line $u(x) = c_1$. For an absolutely large negative value c_1, such a line will contain the interval

$[-1, 1]$ and a sufficiently large region near it, in particular the line of integration $|x-(1-\varepsilon)|=\varepsilon$ as well. Let us call this level line l_{c_1}; the portion of the plane bounded by it we denote by B_{c_1}. When c_1 increases, B_{c_1} decreases. We define the number λ as the least upper bound of values of c_1 for which the interval $[-1, 1]$ and the integration line $|x-(1-\varepsilon)|=\varepsilon$ lie inside B_{c_1}. For $c_1<\lambda$ the level line l_{c_1} will contain $[-1, 1]$ and the integration line. We denote by $\varkappa$ the open region of the x-plane in which $u(x)<\lambda$ and by β the complement.

Theorem 3. *If the function* $\Phi(x)$ *is regular in some domain* D *containing* β, *then the interpolation process* (4.3.4) *constructed on the basis of points having a limit distribution function* $\mu(x)$ *will, as* $n\to\infty$, *converge uniformly on the integration line* $|x-(1-\varepsilon)|=\varepsilon$; *and what is more, it will converge uniformly throughout the region* β.

The proof of this theorem is analogous to the proof of the theorem on the convergence of interpolation on the interval $[a, b]$ with points located on the same interval (see [10]).

Consider the special case where the limit distribution function of interpolation points is the Chebyshev function. For example, that will be the case when the points are roots of the polynomials of Chebyshev, Legendre, and Jacobi. The logarithmic potential will then be of the form (see [10])

$$u(x)=\frac{1}{\pi}\int_{-1}^{1}\ln\frac{1}{|x-t|}\frac{dt}{\sqrt{1-t^2}}=\ln\frac{2}{|x+\sqrt{x^2-1}|}$$

The level lines $u(x)=c_1$ for $c_1<\ln 2$ will be ellipses with foci $(-1, 1)$ and semiaxes $a=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)$, $b=\frac{1}{2}\left(\rho-\frac{1}{\rho}\right)$, where $c_1=\ln\frac{2}{\rho}$, $\rho>1$.

When $\rho=\frac{\sqrt{2\varepsilon}+\varepsilon}{\sqrt{2\varepsilon-\varepsilon^2}}$ the level line $u(x)=c_1=\ln\frac{2}{\rho}$ will contain the interval $[-1, 1]$ and the circle $|x-(1-\varepsilon)|=\varepsilon$, the latter being tangent to the level line at two points symmetric about the real axis. Hence the set β will consist of an ellipse with foci (—1, 1) and semiaxes

$$a=\sqrt{\frac{2}{2-\varepsilon}}, \quad b=\sqrt{\frac{\varepsilon}{2-\varepsilon}}$$

and a portion of the plane lying inside it. This enables us to state the following theorem.

Theorem 4. *If a function* $\Phi(x)$ *is regular in a closed region* β *consisting of an ellipse with foci at the points* —1, 1 *and semiaxes* $a=\sqrt{\frac{2}{2-\varepsilon}}$, $b=\sqrt{\frac{\varepsilon}{2-\varepsilon}}$, *and the region lying within it, then the interpolation process* (4.3.4), *constructed on the basis of points which for the limit distribution function have the Chebyshev function, will, as* $n\to\infty$, *converge uniformly in the above-indicated ellipse and, in particular, on the circle* $|x-(1-\varepsilon)|=\varepsilon$.

This theorem is a special case of Theorem 3.

Remark. If the number ε is taken sufficiently small—this can be done by choosing c sufficiently large—then the ellipse indicated in the theorem will go outside the circle $|x|=1$ only in the neighbourhood of the points —1 and 1. And since the function $\Phi(x)$ is regular in the circle $|x|<1$, then to satisfy Theorem 4 it suffices to demand the regularity of $\Phi(x)$ in the neighbourhood of the points $x=-1$, $x=1$, for instance, in the neighbourhood $|x-1|\leqslant 2\varepsilon$, $|x+1|\leqslant 2\varepsilon$.

We again pass from the variable x to the variable p and from the function $\Phi(x)$ to the function $\varphi(p)$. The following theorem holds.

Theorem 4a. *If the function* $\varphi(p)$ *is regular in the half-plane* $\operatorname{Re} p>0$, *in the neighbourhood* $|p|\leqslant 1/R$ *of the point* $p=0$, *and in the neighbourhood* $|p|\geqslant R$ *of the point at infinity, then the interpolation process* (4.3.5) *constructed on the basis of the points* $p_k=\frac{1+x_k}{1-x_k}$, *where the points* x_k *on the in-*

terval $[-1, 1]$ *have the Chebyshev function for the limit distribution function of the points, will converge uniformly, as* $n \to \infty$, *on the line* $\mathrm{Re}\, p = c$, *where* $c \geqslant R_1$ *and* R_1 *is a number not less than* R.

Remark. Under the conditions of the theorem, the uniform convergence of interpolation will hold not only on the line of integration $\mathrm{Re}\, p = c$, but also in a certain region D into which the domain β of Theorem 4 passes under the transformation $p = \frac{1+x}{1-x}$; in particular, uniform convergence holds in the half-plane $\mathrm{Re}\, p \geqslant c$, in the neighbourhood of the real half axis $0 < p < \infty$, and in the neighbourhoods of the points $p = 0$ and $p = \infty$; that is, $|p| \leqslant 1/R_1$ and $|p| \geqslant R_1$ for some R_1.

4.5c. Convergence of an interpolation process of the form (4.1.4). First of all perform the transformation $p = 1/x$. It carries the half-plane $\mathrm{Re}\, p \geqslant \alpha$ into a circle of radius $1/(2\alpha)$ with centre at the point $1/(2\alpha)$. The half-line $\alpha \leqslant p < \infty$ passes into the diameter d_1 of this circle, which diameter lies on the real axis; and the line of integration $\mathrm{Re}\, p = c$, where $c > \alpha$, on which we interpolate the function is carried into a circle lying inside the above-indicated circle and tangent to its circumference at the point $x = 0$. If c is chosen sufficiently large, then the radius of this circle may be made arbitrarily small. The function $\varphi(p)$ which is regular in the half-plane $\mathrm{Re}\, p > \alpha$ is transformed into the function $\Phi(x)$ which is regular in the circle $\left| x - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}$.

Interpolation of the function $\varphi(p)$ [see (4.1.4)] with respect to points lying on the real axis $\mathrm{Re}\, p > \alpha$ becomes algebraic interpolation of the function $\Phi(x)$ with respect to points lying on the diameter d_1 of the circle $\left| x - \frac{1}{2\alpha} \right| \leqslant \frac{1}{2\alpha}$.

$$\Phi(x) \approx \sum_{k=0}^{n} l_k(x)\, \Phi(x_k) \tag{4.5.3}$$

In this case we can formulate the following theorems.

Theorem 5. *If the function* $\Phi(x)$ *is regular in the circle* $\left|x - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}$ *and in the neighbourhood* $|x| \leqslant 2\varepsilon$ $\left(2\varepsilon < \frac{1}{2\alpha}\right)$ *of the point* $x = 0$, *then the interpolation process* (4.5.3) *constructed on the basis of any points lying on the interval* $[0, 1/(2\alpha)$ *will converge uniformly to* $\Phi(x)$ *on the line* $|x - \varepsilon| = \varepsilon$, *which may be taken for the line of integration. The interval* $\left[0, \frac{1}{2\alpha}\right]$ *will be the largest one belonging to the diameter* d_1 *and ensuring uniform convergence of interpolation with respect to arbitrary points lying on it for functions are regular in the indicated region.*

This theorem is proved on the basis of the theorem of Smirnov and Lebedev that we gave when proving Theorem 2. Here, the set G is the sum of the circle $\left|x - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}$ and $|x| \leqslant 2\varepsilon$, and the set B is the circle $|x - \varepsilon| = \varepsilon$. To construct F it is necessary to find the set of the centres of the circles containing B and belonging to G and, besides, such that the centres lie on the interval $[0, 1/\alpha]$. Clearly, this is the interval $[0, 1/(2\alpha)]$, and the theorem is proved.

Remark. Under the conditions of Theorem 5, uniform convergence of interpolation occurs not only on the contour $|x - \varepsilon| = \varepsilon$ but inside it as well.

If we pass to the old variable p and the function $\varphi(p)$, then Theorem 5 can be stated thus:

Theorem 5a. *If the function* $\varphi(p)$ *is regular in the half-plane* $\operatorname{Re} p > \alpha$ *and in the neighbourhood* $|p| \geqslant R$ *of the point at infinity, then the interpolation process* (4.1.4), *which is constructed on the basis of arbitrary points* p_k $(k = 0, 1, \ldots, n)$ *lying on the real axis so that* $p_k \geqslant 2\alpha$, *converges uniformly to* $\varphi(p)$ *in the half-plane* $\operatorname{Re} p \geqslant c$, *if* c *is chosen so that* $c \geqslant R$. *The half axis* $[2\alpha, \infty)$ *is the largest region on the real axis that ensures uniform convergence of interpolation relative to arbitrary points lying on it for the functions* $\varphi(p)$ *that are regular in the above-indicated region.*

Corollary. *If the function* $\varphi(p)$ *is regular in the half-plane* $\mathrm{Re}\, p \geqslant 1/2$ *and in the neighbourhood* $|p| \geqslant R$ *of the point at infinity, then by* Theorem 5a *the interpolation process will converge uniformly in the half-plane* $\mathrm{Re}\, p \geqslant c \geqslant R$ *with respect to arbitrary points lying on the real half axis* $[1, \infty)$, *in particular, it will converge uniformly also for the equidistant points* $p_k = k + 1$ $(k = 0, 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ *considered in* Sec. 4.2.

In proving the theorem on the convergence of the quadrature process for equidistant points, we will need, as in the case above, uniform convergence of interpolation not only in the half-plane $\mathrm{Re}\, p \geqslant c$, but also in some neighbourhood $|p| \geqslant R_1$ of the point at infinity. To achieve this, suppose, as we did in Sec. 4.2, that the function $\varphi(p)$ is regular in the half-plane $\mathrm{Re}\, p > 0$. Also assume that it is regular in the region $|p| \geqslant R$. Then uniform convergence of interpolation both for equidistant points and for any other points lying on the half axis $[1, \infty)$ will occur not only in the half-plane $\mathrm{Re}\, p \geqslant c$ but also in a broader region. To demonstrate this, pass to the variable $x = 1/p$; then the function $\Phi(x)$ will be regular in the half-plane $\mathrm{Re}\, x > 0$ and in the region $|x| \leqslant 1/R$. The interpolation points $x_k = \frac{1}{p_k} = \frac{1}{k+1}$ $(k = 0, 1, \ldots, n)$ will lie on the interval $[0, 1]$.

The following theorem is valid.

Theorem 6. *If the function* $\Phi(x)$ *is regular in the half-plane* $\mathrm{Re}\, x > 0$ *and in the neighbourhood* $|x| \leqslant 1/R$ *of the zero point, then the interpolation process* (4.5.3) *constructed on the basis of the points* $x_k = \frac{1}{k+1}$ $(k = 0, 1, \ldots, n)$ *or any other points lying on the interval* $[0, 1]$ *will converge uniformly in the region* B, *which is the intersection of two circles:* $|x| \leqslant 1/R$ *and* $|x - 1| \leqslant \sqrt{1 + 1/R^2}$. *The region* B *will be the largest region for which uniform convergence of interpolation occurs for any set of points in* $[0, 1]$.

Proof. This theorem follows immediately from the theorem of Smirnov and Lebedev [20], where it is established that if F and G are two closed sets of points of the z-plane and $F \subset G$ while K_ξ is the largest closed circle contained in G and having its centre at the point $\xi \in F$, then the set $B = \bigcap_{\xi \in F} K_\xi$ is the largest set for which the condition $\{F, B, G\}$ (see p. 97) holds.

In our case, the set G is the right half-plane and the region $|x| \leqslant 1/R$; the set F is the interval $[0, 1]$. To find the set B, construct the two largest closed circles contained in G and with centres at the points $x = 0$ and $x = 1$. These circles are $|x| \leqslant 1/R$ and $|x - 1| \leqslant \sqrt{1 + 1/R^2}$. The desired set B is the intersection of these circles. The proof of the theorem is complete.

Thus, uniform convergence of interpolation will take place not only on the contour of integration $\left| x - \frac{1}{2c} \right| = \frac{1}{2c}$ and inside it, if $c \geqslant R$, but also in a larger region, in particular in the circle $|x| \leqslant 1/R_1 < 1/R$, where

$$\frac{1}{R_1} = \sqrt{1 + \frac{1}{R^2}} - 1 = \frac{\sqrt{R^2+1} - R}{R}$$

If we now pass to the variable p, then we can say that uniform convergence occurs not only in the half-plane $\operatorname{Re} p \geqslant c$, but also in the region $|p| \geqslant R_1$, where

$$R_1 = \frac{R}{\sqrt{R^2+1} - R} > R$$

4.6 *Theorems on the convergence of interpolation methods of inversion*

The results obtained above on the convergence of interpolation permit stating certain theorems on the convergence, as $n \to \infty$, of the quadrature processes (4.3.7) and (4.1.6).

On the basis of Theorem 4a we can prove the following theorem.

Theorem 7. *Let the function* $\varphi(p)$ *be regular in the half-plane* $\operatorname{Re} p > 0$ *and also in the neighbourhood* $|p| \geqslant R$ *of the point at infinity and in the neighbourhood* $|p| \leqslant 1/R$ *of the zero point.*

Then the interpolation quadrature process (4.3.7) *constructed on the basis of the points* $p_k = \frac{1+x_k}{1-x_k}$, *where the points* x_k *have the Chebyshev distribution on the interval* $[-1, 1]$, *will converge to* $f(t)$ *as* $n \to \infty$ *for all values of* t, *the convergence being uniform with respect to* t *on any finite interval* $0 \leqslant t \leqslant T < \infty$, *that is,*

$$R_n(\varphi, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} (p-a)^{-s} r_n(p)\, dp \to 0 \qquad (4.6.1)$$

as $n \to \infty$ *uniformly with respect to* t *for* $0 \leqslant t \leqslant T < \infty$ *for all values of* T.

Proof. First consider the case $s > 1$. The representation (4.6.1) of the remainder $R_n(\varphi, t)$ has a peculiarity that simplifies investigating convergence: the integral on the right is largely independent of the choice of c due to the regularity of the function being integrated in the half-plane $\operatorname{Re} p > 0$ and due to the boundedness of $r_n(p)$ in the neighbourhood of the point at infinity. In particular the number c can be taken arbitrarily large. We choose $c \geqslant R$ and reserve the possibility of increasing c if necessary. On the basis of Theorem 4a we can say that the remainder $r_n(p)$ of interpolation willconverge uniformly to zero on the line of integration as $n \to \infty$, and for $\varepsilon > 0$ there is a number N independent of p such that for $n \geqslant N$ it will be true that $|r_n(p)| \leqslant \varepsilon$.

We transform the integral expressing the remainder term $R_n(\varphi, t)$ by putting $p = c + i\sigma$:

$$R_n(\varphi, t) = \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} e^{i\sigma t} (c - a + i\sigma)^{-s} r_n(c + i\sigma)\, d\sigma$$

and estimate it:

$$|R_n(\varphi, t)| = \frac{e^{ct}}{2\pi}\left|\int_{-\infty}^{\infty} e^{i\sigma t}(c-a+i\sigma)^{-s}\, r_n(c+i\sigma)\, d\sigma\right|$$

$$\leqslant \frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty} |e^{i\sigma t}(c-a+i\sigma)^{-s}\, r_n(c+i\sigma)\, d\sigma|$$

$$\leqslant \varepsilon\,\frac{e^{cT}}{2\pi}\int_{-\infty}^{\infty}\frac{d\sigma}{[(c-a)^2+\sigma^2]^{s/2}} \quad (0\leqslant t\leqslant T) \qquad (4.6.2)$$

The last improper integral is convergent since $s > 1$. Thus from (4.6.2) it follows that $R_n(\varphi, t)$ will tend to zero as $n\to\infty$.

It remains to consider the case $0 < s \leqslant 1$. We transform the remainder term $R_n(\varphi, t)$ thus:

$$R_n(\varphi, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}(p-a)^{-s}\, r_n(p)\, dp$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}(p-a)^{-s}[r_n(\infty)+r_n(p)-r_n(\infty)]\, dp$$

$$= \frac{r_n(\infty)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}(p-a)^{-s}\, dp$$

$$+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}(p-a)^{-s}[r_n(p)-r_n(\infty)]\, dp \qquad (4.6.3)$$

The integral in the first term is equal to $\frac{e^{at}t^{s-1}}{\Gamma(s)}$ and the interpolation error $r_n(\infty)$ tends to zero as $n\to\infty$; hence, the first term tends to zero as $n\to\infty$.

Rewrite the second term thus:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}(p-a)^{-s}p^{-1}\{p[r_n(p)-r_n(\infty)]\}\,dp \qquad (4.6.4)$$

We show that the function $p\,[r_n(p) - r_n(\infty)]$ will converge uniformly to zero as $n \to \infty$ on the line of integration $\operatorname{Re} p = c$.

By hypothesis, the function $\varphi(p)$ is regular in the neighbourhood $|p| \geqslant R$ of the point at infinity; hence both the interpolation error $r_n(p)$ and the function $r_n(p) - r_n(\infty)$ will be regular in this neighbourhood. Besides, in this neighbourhood the function $r_n(p) - r_n(\infty)$ tends to zero as $1/p$, and hence the function $p\,[r_n(p) - r_n(\infty)]$ is regular in the region $|p| \geqslant R$. From the remark concerning Theorem 4a it is known that $r_n(p)$ converges uniformly to zero as $n \to \infty$ in the region $|p| \geqslant R_1$ for a certain $R_1 > R$. Consider the value of the function $p\,[r_n(p) - r_n(\infty)]$ on the boundary of this region as $n \to \infty$. The error $r_n(p$, tends to zero uniformly with respect to p and, besides) $r_n(\infty) \to 0$, while the modulus of p remains equal to R_1. Hence the whole function converges uniformly to zero on the boundary of the region $|p| \geqslant R_1$. And since this function is regular in the closed region $|p| \geqslant R_1$, uniform convergence inside the region follows immediately from the maximum modulus principle.

If $c \geqslant R_1$, then it has been proved that the function $p\,[r_n(p) - r_n(\infty)]$ converges uniformly to zero on $\operatorname{Re} p = c$.

It has thus been proved that the function $p\,[r_n(p) - r_n(\infty)]$ converges uniformly to zero as $n \to \infty$ on the line of integration $\operatorname{Re} p = c$, that is, for any $\varepsilon > 0$ there is an N independent of p such that for $n \geqslant N$ the inequality $|p\,[r_n(p) - r_n(\infty)]| \leqslant \varepsilon$ holds true.

Let us estimate the integral (4.6.4):

$$\left| \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} (p-a)^{-s} p^{-1} p \, [r_n(p) - r_n(\infty)] \, dp \right|$$

$$\leqslant \varepsilon \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} \frac{d\sigma}{[(c-a)^2+\sigma^2]^{s/2} (c^2+\sigma^2)^{1/2}} \qquad (4.6.5)$$

The last improper integral converges for $s > 0$ and so the second term in (4.6.3) will tend to zero as $n \to \infty$ for any $s > 0$. The proof is complete.

Remark. In Theorem 7 the convergence of the quadrature was proved for points p_k for which $x_k = \frac{p_k - 1}{p_k + 1}$ have a limit distribution function $\mu(x)$ that coincides with the Chebyshev function. A similar theorem can also be demonstrated for the points p_k that have a general limit distribution function of interpolation points x_k. The sole difference here is that the region of regularity of the function $\varphi(p)$ must be different, namely, $\varphi(p)$ must be regular in the region D into which the domain β passes under the transformation $x = \frac{p-1}{p+1}$.

On the basis of Theorem 6 we can prove the following theorem for the quadrature process (4.1.6).

Theorem 8. *If the function* $\varphi(p)$ *is regular in the half-plane* $\operatorname{Re} p > 0$ *and also in the neighbourhood* $|p| \geqslant R$ *of the point at infinity, then the interpolation quadrature process* (4.1.6) *constructed on the basis of the points* $p_k = k + 1$ $(k = 0, 1, \ldots, n)$ *will converge if* c *is chosen so that* $c \geqslant R$, *that is,*

$$R_n(\varphi, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} p^{-s} r_n(p) \, dp \to 0$$

as $n \to \infty$.

The proof is strictly analogous to that of Theorem 7.

Note that under the conditions of Theorem 8 the quadrature process (4.1.6) converges not only for equidistant points but also for any other points located on the real half axis $[1, \infty)$.

Chapter 5

Methods of Numerical Inversion of Laplace Transforms via Quadrature Formulas of Highest Accuracy

5.1 *The theory of quadrature fomulas*

To compute the Mellin integral

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}F(p)\,dt; \qquad (5.1.1)$$

in Chap. 4 we constructed interpolation quadrature formulas that are exact for polynomials of degree $n-1$ in the arguments $\frac{1}{p}$ or $\frac{1}{p-A-2\alpha}$. This degree of accuracy for the given interpolation points in the half-plane Re $p>\alpha$ was attained via a choice of the quadrature coefficients A_k.

In constructing quadrature formulas it is natural to choose not only the coefficients but the points as well. We may hope in this way to increase the accuracy of the formula. In this chapter we will construct a quadrature formula of highest accuracy in the class of rational functions of a special type.

But before constructing such a formula, let us transform the integral (5.1.1) so that the parameters of the quadrature formula do not depend on α and t. To do this make the change of variable $p=p'/t+\alpha$. Then the integral (5.1.1) becomes

$$f(t)=\frac{1}{2\pi i}\,\frac{e^{\alpha t}}{t}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{p'}F^*(p')\,dp', \quad \varepsilon>0$$

$$F^*(p')=F\left(\frac{p'}{t}+\alpha\right)$$

Since the function $F(p)$ was regular in the half-plane $\operatorname{Re} p > \alpha$, the function $F^*(p')$ will be regular to the right of the imaginary axis $\operatorname{Re} p' > 0$, and ε can be any positive number. Thus, computing the Mellin integral reduces to computing the integral

$$J=\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{p}F^*(p)\,dp \tag{5.1.2}$$

where the variable of integration is again p.

The function $F^*(p)$, being the image function, has regularity in the right half-plane and tends to zero as p goes to infinity in a manner so that $\operatorname{Re} p \to \infty$. Suppose too that $F^*(p)$ tends to zero like a certain power of $1/p$, that is, suppose that $F^*(p)$ can be represented in the form

$$F^*(p)=\frac{1}{p^s}\varphi(p) \tag{5.1.3}$$

where $s>0$ and the function $\varphi(p)$ is regular in the half-plane $\operatorname{Re} p>0$ and has a finite limiting value as $p\to\infty$:

$$\lim_{p\to\infty}\varphi(p)=\varphi(\infty)$$

Substitute the expression (5.1.3) into the integral (5.1.2):

$$J(s)=\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{p}p^{-s}\varphi(p)\,dp \tag{5.1.4}$$

To compute this integral we construct a quadrature formula of the following kind:

$$J(s) \approx \sum_{k=1}^{n} A_k \varphi(p_k) \tag{5.1.5}$$

In (5.1.5) the coefficients A_k and the points p_k are arbitrary. We can deal with them as we like. Let us try to choose them so that formula (5.1.5) is exact for any polynomial of degree $2n - 1$ in the variable $1/p$. A necessary and sufficient condition for this is provided by the following theorem.

Theorem 1. *For the quadrature formula* (5.1.5) *to be exact for all polynomials of degree* $2n-1$ *in the variable* $x = 1/p$ *it is necessary and sufficient that the following two conditions hold:*

1. *The formula* (5.1.5) *must be interpolatory, that is, its coefficients* A_k *must have the values*

$$A_k = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^p p^{-s} l_k\left(\frac{1}{p}\right) dp \tag{5.1.6}$$

where

$$l_k\left(\frac{1}{p}\right) = \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(\frac{1}{p} - \frac{1}{p_j}\right) \Big/ \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(\frac{1}{p_k} - \frac{1}{p_j}\right)$$

2. *For every polynomial* $Q\left(\frac{1}{p}\right)$ *of degree not greater than* $n-1$, *the following equality must hold:*

$$\frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^p p^{-s} \omega_n\left(\frac{1}{p}\right) Q\left(\frac{1}{p}\right) dp = 0 \tag{5.1.7}$$

where

$$\omega_n\left(\frac{1}{p}\right) = \prod_{k=1}^{n} \left(\frac{1}{p} - \frac{1}{p_k}\right).$$

The *proof* is carried out in a manner strictly analogous to the proof of the corresponding theorem on quadratures of highest algebraic degree of accuracy (see [10]).

Necessity. If (5.1.5) holds for polynomials of degree $2n-1$ in the variable $x = 1/p$, then it also holds for polynomials of degree $n-1$ in $1/p$, and for this reason it must be an interpolation formula. This completes the necessity proof of the first condition.

Now let $Q\left(\frac{1}{p}\right)$ be any polynomial of degree not greater than $n-1$. The product $\psi\left(\frac{1}{p}\right) = \omega_n\left(\frac{1}{p}\right) Q\left(\frac{1}{p}\right)$ is a polynomial of degree not greater than $2n-1$, and for it the formula (5.1.5) must be exact:

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\psi\left(\frac{1}{p}\right) dp$$

$$=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\omega_n\left(\frac{1}{p}\right) Q\left(\frac{1}{p}\right) dp$$

$$=\sum_{k=1}^{n} A_k\omega_n\left(\frac{1}{p_k}\right) Q\left(\frac{1}{p_k}\right)$$

This sum is equal to zero since $\omega_n\left(\frac{1}{p_k}\right) = 0$, which proves the necessity of (5.1.7).

Sufficiency. Let $\psi\left(\frac{1}{p}\right)$ be an arbitrary polynomial of degree $2n-1$. Dividing it by $\omega_n\left(\frac{1}{p}\right)$, we get

$$\psi\left(\frac{1}{p}\right) = Q\left(\frac{1}{p}\right)\omega_n\left(\frac{1}{p}\right) + \rho\left(\frac{1}{p}\right)$$

where Q and ρ are polynomials in $1/p$ of degree not exceeding $n-1$. Since $\omega_n\left(\frac{1}{p_k}\right)=0$, it follows that

$$\psi\left(\frac{1}{p_k}\right)=\rho\left(\frac{1}{p_k}\right) \quad (k=1, 2, \ldots, n) \qquad (5.1.8)$$

We represent the integral of the function $\psi\left(\frac{1}{p}\right)$ as the sum of the following two integrals:

$$\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\psi\left(\frac{1}{p}\right)dp = \frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\omega_n\left(\frac{1}{p}\right)Q\left(\frac{1}{p}\right)dp + \frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\rho\left(\frac{1}{p}\right)dp \qquad (5.1.9)$$

The first integral in the right member is zero by the orthogonality condition. Since the degree of $\rho\left(\frac{1}{p}\right)$ is not greater than $n-1$ and the formula (5.1.5) is an interpolation formula, the following equation has to be exact:

$$\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\rho\left(\frac{1}{p}\right)dp = \sum_{k=1}^{n} A_k\rho\left(\frac{1}{p_k}\right)$$

Taking into account (5.1.8) and (5.1.9), we get

$$\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\psi\left(\frac{1}{p}\right)dp = \sum_{k=1}^{n} A_k\psi\left(\frac{1}{p_k}\right)$$

and (5.1.5) is indeed exact for arbitrary polynomials of degree $2n-1$ in $1/p$. The proof is complete.

Thus the question of the possibility of constructing a quadrature formula (5.1.5) that is exact for arbitrary polynomials of degree $2n-1$ is connected with the existence of a polynomial $\omega_n\left(\frac{1}{p}\right)$ of degree n having the property of orthogonality (5.1.7).

We now show that this polynomial exists and the condition (5.1.7) defines it uniquely. We seek the polynomial $\omega_n(x) = \omega_n\left(\frac{1}{p}\right)$ in the form of an expansion in powers of x:

$$\omega_n(x) = x^n + a_1 x^{n-1} + \ldots + a_n$$

The orthogonality condition (5.1.7) is equivalent to fulfillment of the system of equations

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p \omega_n(x)\, x^{s+m}\, dp = 0, \quad m = 0, 1, \ldots, n-1 \qquad (5.1.10)$$

Since $\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p x^u\, dp = \frac{1}{\Gamma(u)}$, it follows that the system (5.1.10) transforms to the system

$$\frac{1}{\Gamma(s+n+m)} + \frac{a_1}{\Gamma(s+n+m-1)} + \ldots + \frac{a_n}{\Gamma(s+m)} = 0 \qquad (5.1.11)$$
$$m = 0, 1, \ldots, n-1$$

Multiply the equations (5.1.11) by $\Gamma(s+n+m-1)$ and write down the system obtained as

$$\left.\begin{aligned}
&\frac{1}{s+n-1} + a_1 + (s+n-2)\,a_2 \\
&\qquad + \ldots + (s+n-2)(s+n-3)\ldots s a_n = 0,\\
&\frac{1}{s+n} + a_1 + (s+n-1)\,a_2 \\
&+ \ldots + (s+n-1)(s+n-2)\ldots(s+1)\,a_n = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{1}{s+2n-2} + a_1 + (s+2n-3)\,a_2 \\
&\qquad + \ldots + (s+2n-3)(s+2n-4)\\
&\qquad\qquad \ldots (s+n-1)\,a_n = 0
\end{aligned}\right\} \qquad (5.1.12)$$

The determinant of this system is

$$\Delta = \begin{vmatrix} 1 & s+n-2 & \dots & [(s+n-2)(s+n-3) \dots s] \\ 1 & s+n-1 & \dots & [(s+n-1)(s+n-2) \dots (s+1)] \\ \dots & \dots & \dots & \dots \\ 1 & s+2n-3 & \dots & [(s+2n-3)(s+2n-4) \dots (s+n-1)] \end{vmatrix}$$

It suffices to see that $\Delta \neq 0$, since in that case the system (5.1.12) will have the solution $a_1, a_2, \dots, a_n$ and only that solution. Consider a system of n functions $x^{s+n-2}, x^{s+n-1}, \dots$ $\dots, x^{s+2n-3}$. They are linearly independent on any interval that does not reduce to a point. Let us construct a linear differential equation of order n for which these functions form a complete system of independent solutions:

$$L_n(y) = \begin{vmatrix} y & y' & \dots & y^{(n)} \\ x^{s+n-2} & (x^{s+n-2})' & \dots & (x^{s+n-2})^{(n)} \\ x^{s+n-1} & (x^{s+n-1})' & \dots & (x^{s+n-1})^{(n)} \\ \dots & \dots & \dots & \dots \\ x^{s+2n-3} & (x^{s+2n-3})' & \dots & (x^{s+2n-3})^{(n)} \end{vmatrix} = 0$$

If we expand the determinant in terms of the elements of the first row and divide both sides of the equation by $x^{n(n+s-3)}$, then the last equation can be written as

$$c_0^n x^n y^{(n)} + c_1^n x^{n-1} y^{(n-1)} + \dots + c_n^n y = 0 \qquad (5.1.13)$$

where c_i^n = constant.

Indeed, consider the cofactor of the element y:

$$\begin{vmatrix} (x^{s+n-2})' & (x^{s+n-2})'' & \dots & (x^{s+n-2})^{(n)} \\ (x^{s+n-1})' & (x^{s+n-1})'' & \dots & (x^{s+n-1})^{(n)} \\ \dots & \dots & \dots & \dots \\ (x^{s+2n-3})' & (x^{s+2n-3})'' & \dots & (x^{s+2n-3})^{(n)} \end{vmatrix}$$

From the elements of the first, second, $\dots$, nth columns, take x^{s+n-3}, x^{s+n-4}, $\dots$, x^{s-2}, respectively, outside the determinant. Then from the elements of the rows of the remaining determinant take out the factors $1, x, \ldots, x^{n-1}$. This yields the determinant $x^{n(n+s-3)} D$, where D is

$$\begin{vmatrix} s+n-2 & (s+n-2)M & \ldots [(s+n-2) & \ldots (s-1)] \\ s+n-1 & (s+n-1)M & \ldots [(s+n-1) & \ldots s] \\ \cdots & \cdots & \cdots & \cdots \\ s+2n-3 & (s+2n-3)M & \ldots [(s+2n-3) & \ldots (s+n-2)] \end{vmatrix} = c_n^n$$

and the letter M denotes a factor less by unity than the preceding one.

In strictly analogous fashion we establish that the cofactors of the elements $y', y'', \ldots, y^{(n)}$ are equal, respectively, to

$$c_{n-1}^n x^{n(s+n-3)+1}, \quad c_{n-2}^n x^{n(s+n-3)+2}, \quad \ldots, \quad c_0^n x^{n(s+n-3)+n}$$

This proves the expansion (5.1.13).

The equation (5.1.13) is Euler's equation and it has two singular points: $x = 0$, $x = \infty$.

Write out the Wronskian determinant for the solutions $x^{s+n-2}, x^{s+n-1}, \ldots, x^{s+2n-3}$ of this equation:

$$W(x^{s+n-2}, \ldots, x^{s+2n-3})$$
$$= \begin{vmatrix} x^{s+n-2} & (s+n-2)x^{s+n-3} & \ldots [(s+n-2) & \ldots sx^{s-1}] \\ x^{s+n-1} & (s+n-1)x^{s+n-2} & \ldots [(s+n-1) & \ldots (s+1)x^{s}] \\ \cdots & \cdots & \cdots & \cdots \\ x^{s+2n-3} & (s+2n-3)x^{s+2n-4} & \ldots [(s+2n-3) & \ldots (s+n-1)x^{s+n-2}] \end{vmatrix}$$
$$(5.1.14)$$

Since the solutions $x^{s+n-2}, x^{s+n-1}, \ldots, x^{s+2n-3}$ of (5.1.13) are linearly independent, the determinant (5.1.14) can vanish only at the singularities of the equation, that is, at the points $x = 0$ and $x = \infty$. It is nonzero at all other points, say at $x = 1$. But for $x = 1$ the determinant $W(x^{s+n-2}, \ldots$

. . ., x^{s+2n-3}) coincides with the determinant Δ and therefore $\Delta \neq 0$; hence, the system (5.1.12) has a unique solution.

This proves the existence and uniqueness of the polynomial $\omega_n\left(\frac{1}{p}\right)$.

Since in (5.1.7) the weight function depends on the parameter s, the polynomial $\omega_n\left(\frac{1}{p}\right)$ too will depend on s. We denote it by $\omega_n^{(s)}\left(\frac{1}{p}\right)$. To finish investigating the possibility of constructing the formula (5.1.5), which is exact for polynomials of degree $2n-1$ in $1/p$, it is necessary to show that all roots of the polynomials $\omega_n^{(s)}\left(\frac{1}{p}\right)$ for arbitrary $s > 0$ lie in the right half-plane. This question will be dealt with in the next section.

The most exact quadrature formula (5.1.5) for the integral (5.1.4) was constructed for the special case $s = 1$ by H. E. Salzer (see [13]).

5.2 *Orthogonal polynomials connected with the quadrature formula of highest accuracy*

5.2a. An explicit expression of the polynomials $\omega_n^{(s)}\left(\frac{1}{p}\right)$. To obtain an explicit expression for $\omega_n^{(s)}\left(\frac{1}{p}\right)$, let us consider the following polynomial of degree n:

$$P_n^{(s)}\left(\frac{1}{p}\right) = (-1)^n e^{-p} p^{n+s-1} \frac{d^n}{dp^n} \left(e^p p^{-n-s+1}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} (n+s-1) \ldots (n+s+k-2)}{p^k} \qquad (5.2.1)$$

which we wrote down in a form similar to the Rodrigues formula for the Legendre polynomials.

We now show that for this polynomial the orthogonality condition

$$\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s} P_n^{(s)}\left(\frac{1}{p}\right) p^{-m}\, dp = 0 \quad (m = 0, 1, \ldots, n-1) \quad (5.2.2)$$

holds true and this is equivalent to the condition (5.1.7).

To prove this, in the integral (5.2.2) we replace $P_n^{(s)}\left(\frac{1}{p}\right)$ by its expression (5.2.1) and integrate by parts:

$$\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s} P_n^{(s)}\left(\frac{1}{p}\right) p^{-m}\, dp$$

$$= (-1)^n \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} p^{n-m-1} \frac{d^n}{dp^n}\left(e^p p^{-n-s+1}\right) dp$$

$$= (-1)^n p^{n-m-1} \frac{d^{n-1}}{dp^{n-1}}\left(e^p p^{-n-s+1}\right)\Big|_{\varepsilon-i\infty}^{\varepsilon+i\infty}$$

$$-(-1)^n (n-m-1) \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} p^{n-m-2} \frac{d^{n-1}}{dp^{n-1}}\left(e^p p^{-n-s+1}\right) dp$$

It is easy to show that the first term is zero, since each summand obtained after differentiating the product $e^p p^{-n-s+1}$ will be of the form $e^p p^{-k}$ $(k > s)$ and, since the function e^p remains bounded on the line of integration, each summand will tend to zero as p goes to infinity along the line of integration.

Integrating by parts $n - m - 1$ times, we obtain the following expression for the integral (5.2.2):

$$\pm (n-m-1)! \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{d^{m+1}}{dp^{m+1}}\left(e^p p^{-n-s+1}\right) dp$$

$$= \pm (n-m-1)! \frac{d^m}{dp^m}\left(e^p p^{-n-s+1}\right)\Big|_{\varepsilon-i\infty}^{\varepsilon+i\infty}$$

And since $s > 0$, $n \geqslant 1$, the expression obtained is also zero. Thus the polynomial $P_n^{(s)}\left(\frac{1}{p}\right)$ given by (5.2.1) satisfies the orthogonality condition (5.1.7), and, because of the uniqueness of a polynomial satisfying this condition we can conclude that the polynomial $P_n^{(s)}\left(\frac{1}{p}\right)$ will differ from $\omega_n^{(s)}\left(\frac{1}{p}\right)$ only by a constant factor equal to the leading coefficient of $P_n^{(s)}\left(\frac{1}{p}\right)$:

$$P_n^{(s)}\left(\frac{1}{p}\right) = (n+s-1)(n+s)\dots(2n+s-2)\,\omega_n^{(s)}\left(\frac{1}{p}\right)$$

5.2b. A recurrence relation for the polynomials $P_n^{(s)}\left(\frac{1}{p}\right)$. For the polynomials $P_n^{(s)}\left(\frac{1}{p}\right)$ we can obtain a recurrence formula connecting the three polynomials $P_{n-1}^{(s)}\left(\frac{1}{p}\right)$, $P_n^{(s)}\left(\frac{1}{p}\right)$, $P_{n+1}^{(s)}\left(\frac{1}{p}\right)$, as is done for ordinary orthogonal polynomials. Take the product $\frac{1}{p}P_n^{(s)}\left(\frac{1}{p}\right)$; it is a polynomial of degree $n+1$ in $1/p$ and can be represented as a linear combination of the polynomials $P_0^{(s)}\left(\frac{1}{p}\right)$, $P_1^{(s)}\left(\frac{1}{p}\right)$, ..., $P_{n+1}^{(s)}\left(\frac{1}{p}\right)$:

$$\frac{1}{p}P_n^{(s)}\left(\frac{1}{p}\right) = \sum_{k=0}^{n+1} c_{nk}' P_k^{(s)}\left(\frac{1}{p}\right) \tag{5.2.3}$$

The coefficients of this expansion can be determined from

$$c_{nk} = \frac{\dfrac{1}{2\pi i}\displaystyle\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\frac{1}{p}P_n^{(s)}\left(\frac{1}{p}\right)P_k^{(s)}\left(\frac{1}{p}\right)dp}{\dfrac{1}{2\pi i}\displaystyle\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\left[P_k^{(s)}\left(\frac{1}{p}\right)\right]^2 dp} \tag{5.2.4}$$

from which it is evident that if $k < n-1$, then $\frac{1}{p} P_k^{(s)}\left(\frac{1}{p}\right)$ is a polynomial of degree lower than n, and by the condition (5.1.7) the integral in the numerator of (5.2.4) is zero. Thus, in (5.2.3) only $c_{n,n-1}$, c_{nn}, $c_{n,n+1}$ can be different from zero:

$$\frac{1}{p} P_n^{(s)}\left(\frac{1}{p}\right) = c_{n,\,n-1} P_{n-1}^{(s)}\left(\frac{1}{p}\right) + c_{nn} P_n^{(s)}\left(\frac{1}{p}\right) + c_{n,\,n+1} P_{n+1}^{(s)}\left(\frac{1}{p}\right) \qquad (5.2.5)$$

Denote the variable $1/p$ by x and rewrite the relation (5.2.5) as

$$P_{n+1}^{(s)}(x) = (a_n x + b_n) P_n^{(s)}(x) + c_n P_{n-1}(x) \qquad (5.2.6)$$

Since we know the explicit expression of $P_n^{(s)}(x)$, the coefficients a_n, b_n, c_n can be determined with ease. Equating the coefficients of x^{n+1} in the right and left members of (5.2.6), we get $(n+s)(n+s+1)\ldots(2n+s) = a_n(n+s-1)(n+s)\ldots(2n+s-2)$, whence

$$a_n = \frac{(n+s)(n+s+1)\ldots(2n+s-2)(2n+s-1)(2n+s)}{(n+s-1)(n+s)\ldots(2n+s-2)} = \frac{(2n+s)(2n+s-1)}{n+s-1}$$

To find b_n equate the coefficients of x^n in both members of (5.2.6):

$$-(n+1)(n+s)(n+s-1)\ldots(2n+s-1) = -a_n n(n+s-1)(n+s)\ldots(2n+s-3) + b_n(n+s-1)(n+s)\ldots(2n+s-2)$$

From this follows

$$b_n = -\frac{(n+1)(n+s)\dots(2n+s-2)(2n+s-1)}{(n+s-1)(n+s)\dots(2n+s-2)}$$
$$+\frac{(2n+s-1)(2n+s)}{(n+s-1)}\,\frac{n(n+s-1)(n+s)\dots(2n+s-3)}{(n+s-1)(n+s)\dots(2n+s-3)(2n+s-2)}$$
$$= -\frac{(n+1)(2n+s-1)}{n+s-1}+\frac{(2n+s-1)(2n+s)\,n}{(2n+s-2)(n+s-1)}$$
$$= -\frac{(2n+s-1)(s-2)}{(2n+s-2)(n+s-1)}$$

Finally, we determine c_n by comparing, for example, the constant terms in the right and left members of (5.2.6):

$$(-1)^{n+1} = b_n(-1)^n + c_n(-1)^{n-1}$$

$$c_n = 1 + b_n = 1 - \frac{(2n+s-1)(s-2)}{(2n+s-2)(n+s-1)}$$
$$= \frac{(2n+s-2)(n+s-1)-(2n+s-1)(s-2)}{(2n+s-2)(n+s-1)} = \frac{n(2n+s)}{(2n+s-2)(n+s-1)}$$

We can thus state the following theorem.

Theorem 2. *Any three successive polynomials* $P_n^{(s)}(x)$ *are connected by the recurrence relation*

$$(2n+s-2)(n+s-1)\,P_{n+1}^{(s)}(x)$$
$$= [(2n+s)(2n+s-1)(2n+s-2)\,x$$
$$-(s-2)(2n+s-1)]\,P_n^{(s)}(x) + n(2n+s)\,P_{n-1}^{(s)}(x). \quad (5.2.7)$$

5.2c. A differential equation whose solution are the polynomials $P_n^{(s)}(x)$. We can indicate a linear differential equation of order two with variable coefficients satisfied by the polynomials $P_n^{(s)}(x)$

Indeed, as we have already established, $P_n^{(s)}\left(\frac{1}{p}\right)$ obey the orthogonality condition

$$\frac{1}{2\pi i}\int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s} P_n^{(s)}\left(\frac{1}{p}\right) Q\left(\frac{1}{p}\right) dp = 0$$

where $Q\left(\frac{1}{p}\right)$ is an arbitrary polynomial of degree not greater than $n-1$. If we pass to the variable $x = 1/p$, then this condition takes the form

$$\frac{1}{2\pi i}\int\limits_C e^{1/x} x^{s-2} P_n^{(s)}(x)\, Q(x)\, dx = 0 \tag{5.2.8}$$

where C is a circle of radius $1/(2\varepsilon)$ with centre at the point $x = 1/(2\varepsilon)$.

To derive the differential equation, consider the following integral:

$$J = \frac{1}{2\pi i}\int\limits_C [e^{1/x} x^s P_n^{(s)\prime}(x)]'\, x^k\, dx$$

Integrating by parts, we get

$$J = \frac{1}{2\pi i}[x^k e^{1/x} x^s P_n^{(s)\prime}(x)]_C - \frac{k}{2\pi i}\int\limits_C x^{k-1} e^{1/x} x^s P_n^{(s)\prime}(x)\, dx$$

The first term on the right vanishes because if we again go to the variable p, the expression $\frac{1}{p^{k+s}} e^p P_n^{(s)\prime}\left(\frac{1}{p}\right)$ will tend to zero as p tends to infinity along the straight line $\operatorname{Re} p = \varepsilon$. If $k = 0$, then the integral J is zero too.

But if $k > 0$, then we integrate by parts once again:

$$J = -\frac{k}{2\pi i}[x^{k-1} e^{1/x} x^s P_n^{(s)}(x)]_C$$
$$+ \frac{k}{2\pi i}\int\limits_C e^{1/x} x^{s-2} P_n^{(s)}(x)\, [(k+s-1)\, x^k - x^{k-1}]\, dx$$

The first summand on the right again vanishes. We consider the second summand. In the square brackets under the integral sign we have a polynomial of degree k; hence by virtue of the condition (5.2.8), the integral is zero when $k = 1, 2, \ldots, n-1$. We have thus proved that for $k = 0, 1, \ldots, n-1$ the integral J is equal to zero:

$$J = \frac{1}{2\pi i} \int_C [e^{1/x} x^s P_n^{(s)\prime}(x)]' x^k \, dx$$

$$= \frac{1}{2\pi i} \int_C [-e^{1/x} x^{s-2} P_n^{(s)\prime}(x) + s e^{1/x} x^{s-1} P_n^{(s)\prime}(x) + e^{1/x} x^s P_n^{(s)\prime\prime}(x)] x^k \, dx$$

$$= \frac{1}{2\pi i} \int_C e^{1/x} x^{s-2} [x^2 P_n^{(s)\prime\prime}(x) + (sx-1) P_n^{(s)\prime}(x)] x^k \, dx = 0$$

This last equation means that the polynomial of degree n in the square brackets is orthogonal to x^k with weight $e^{1/x}x^{s-2}$ for $k = 0, 1, \ldots, n-1$. From this we conclude that the polynomial differs from $P_n^{(s)}(x)$ only by a constant factor γ_n:

$$x^2 P_n^{(s)\prime\prime}(x) + (sx-1) P_n^{(s)\prime}(x) = \gamma_n P_n^{(s)}(x)$$

To determine the factor γ_n, one need only compare the coefficients of x^n in the last formula:

$$n(n-1)(n+s-1)\ldots(2n+s-2) + sn(n+s-1)\ldots(2n+s-2) = \gamma_n (n+s-1)\ldots(2n+s-2)$$

From this, $n(n-1) + sn = \gamma_n$ or $\gamma_n = n(n+s-1)$. This proves

Theorem 3. *The polynomial* $P_n^{(s)}(x)$ *defined by formula* (5.2.1) *is a solution of the linear differential equation with variable* *coefficients*

$$x^2P_n^{(s)''}(x)+(sx-1)P_n^{(s)'}(x)-n(n+s-1)P_n^{(s)}(x)=0 \quad (5.2.9)$$

5.2d. An integral representation of the polynomials $P_n^{(s)}(x)$. We now show that the polynomials $P_n^{(s)}(x)$ have the following integral representation:

$$P_n^{(s)}(x)=\frac{(-1)^n}{\Gamma(n+s-1)}\int_0^\infty t^{n+s-2}(1-xt)^n e^{-t}\,dt \quad (5.2.10)$$

We verify this by evaluating the integral in the right-hand member of (5.2.10):

$$\frac{(-1)^n}{\Gamma(n+s-1)}\int_0^\infty t^{n+s-2}(1-xt)^n e^{-t}\,dt$$

$$=\frac{(-1)^n}{\Gamma(n+s-1)}\int_0^\infty t^{n+s-2}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(xt)^{n-k}e^{-t}\,dt$$

$$=\frac{(-1)^n}{\Gamma(n+s-1)}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^{n-k}\int_0^\infty t^{2n+s-k-2}e^{-t}\,dt$$

$$=\frac{(-1)^n}{\Gamma(n+s-1)}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\Gamma(2n+s-k-1)\,x^{n-k}$$

The expression obtained coincides with (5.2.1) for the polynomials $P_n^{(s)}(x)$, which proves the assertion (5.2.10).

5.2e. The generating function for the polynomials $P_n^{(s)}(x)$. The polynomials $P_n^{(s)}(x)$ can be regarded as coefficients in the Taylor series expansion of some analytic function, which is called the *generating function* of these polynomials.

To find it consider the function $\frac{e^z}{z^{n+s-1}}$. It is analytic throughout the z-plane, except at the points $z=0$ and $z=\infty$; for this reason it can be represented by the Cauchy integral at any point p of the z-plane, except at the two indicated points:

$$\frac{e^p}{p^{n+s-1}}=\frac{1}{2\pi i}\int_l \frac{e^z}{z^{n+s-1}}\frac{dz}{z-p}$$

where l is a closed contour enclosing the point p and lying in the region of analyticity of the function $\frac{e^z}{z^{n+s-1}}$.

The nth derivative of this function is given by the formula

$$\frac{d^n}{dp^n}\left(\frac{e^p}{p^{n+s-1}}\right)=\frac{n!}{2\pi i}\int_l \frac{e^z}{z^{n+s-1}}\frac{dz}{(z-p)^{n+1}} \tag{5.2.11}$$

Perform the transformation

$$z=\frac{p}{2}(\sqrt{1-t}+1) \tag{5.2.12}$$

It carries the point $z = p$ to the point $t = 0$. Make a cut in the t-plane along the positive real axis from $t = 1$ to infinity and consider the branch $\sqrt{1-t}$ for which $\arg(1-t) = 0$ for real-valued $t < 1$. The contour of integration l will go into the contour λ that encloses the point $t = 0$. Following the transformation (5.2.12), the integral (5.2.11) becomes

$$\frac{d^n}{dp^n}\left(\frac{e^p}{p^{n+s-1}}\right)$$

$$=-\frac{n!}{2\pi i}\int_\lambda \frac{e^{\frac{p}{2}(\sqrt{1-t}+1)}}{\left(\frac{p}{2}\right)^{n+s-1}(\sqrt{1-t}+1)^{n+s-1}}\;\frac{\frac{p}{4}\frac{dt}{\sqrt{1-t}}}{\left(\frac{p}{2}\right)^{n+1}(\sqrt{1-t}-1)^{n+1}}$$

$$=-\frac{n!\,\frac{p}{4}}{2\pi i\left(\frac{p}{2}\right)^{2n+s}}\int_\lambda \frac{e^{\frac{p}{2}(\sqrt{1-t}+1)}}{(\sqrt{1-t}+1)^{s-2}\sqrt{1-t}}\,\frac{dt}{(-t)^{n+1}}$$

$$=\frac{(-1)^n\,n!}{4\pi i\left(\frac{p}{2}\right)^{2n+s-1}}\int_\lambda \frac{e^{\frac{p}{2}(\sqrt{1-t}+1)}}{(\sqrt{1-t}+1)^{s-2}\sqrt{1-t}}\,\frac{dt}{t^{n+1}} \tag{5.2.13}$$

Now, substituting into the explicit expression (5.2.1) for the polynomials $P_n^{(s)}\left(\frac{1}{p}\right)$ the expression (5.2.13) in place of the nth derivative, we get

$$P_n^{(s)}\left(\frac{1}{p}\right)=(-1)^n e^{-p} p^{n+s-1} \frac{(-1)^n n!}{4\pi i\left(\frac{p}{2}\right)^{2n+s-1}}$$

$$\times \int_\lambda \frac{e^{\frac{p}{2}(\sqrt{1-t}+1)}}{(\sqrt{1-t}+1)^{s-2}\sqrt{1-t}} \frac{dt}{t^{n+1}}$$

$$=\frac{2^{2n+s-2}n!}{p^n 2\pi i} \int_\lambda \frac{e^{\frac{p}{2}(\sqrt{1-t}-1)}}{(\sqrt{1-t}+1)^{s-2}\sqrt{1-t}} \frac{dt}{t^{n+1}}$$

or

$$\frac{P_n^{(s)}\left(\frac{1}{p}\right)p^n}{2^{2n}n!}=\frac{1}{2\pi i}\int_\lambda \frac{2^{s-2}e^{\frac{p}{2}(\sqrt{1-t}-1)}}{(\sqrt{1-t}+1)^{s-2}\sqrt{1-t}} \frac{dt}{t^{n+1}} \quad (5.2.14)$$

Consider the function

$$F(t)=\frac{2^{s-2}e^{\frac{p}{2}(\sqrt{1-t}-1)}}{(\sqrt{1-t}+1)^{s-2}\sqrt{1-t}}$$

It is analytic at the point $t=0$; hence in the neighbourhood of this point it can be represented by its Taylor series

$$F(t)=\sum_{n=0}^{\infty} c_n t^n$$

where

$$c_n=\frac{1}{2\pi i}\int_\lambda \frac{F(t)}{t^{n+1}}\,dt$$

$$=\frac{1}{2\pi i}\int_\lambda \frac{2^{s-2}e^{\frac{p}{2}(\sqrt{1-t}-1)}}{(\sqrt{1-t}+1)^{s-2}\sqrt{1-t}} \frac{dt}{t^{n+1}}=\frac{P_n^{(s)}\left(\frac{1}{p}\right)p^n}{2^{2n}n!}$$

Consequently

$$\frac{2^{s-2}e^{\frac{p}{2}(\sqrt{1-t}-1)}}{(\sqrt{1-t}+1)^{s-2}\sqrt{1-t}}=\sum_{n=0}^{\infty}\frac{p^nP_n^{(s)}\left(\frac{1}{p}\right)}{2^{2n}n!}t^n$$

Introducing the new variable $t_1 = pt/4$ and setting $1/p = x$, we can rewrite the last formula as

$$\frac{2^{s-2}e^{\frac{1}{2x}(\sqrt{1-4tx}-1)}}{(\sqrt{1-4tx}+1)^{s-2}\sqrt{1-4tx}}=\sum_{n=0}^{\infty}\frac{P_n^{(s)}(x)}{n!}t^n \qquad (5.2.15)$$

where the variable is again denoted by t.

The function on the left of (5.2.15) is the generating function for the polynomials $P_n^{(s)}(x)$.

5.2f. The distribution of roots of the polynomials $P_n^{(s)}(x)$. At the end of the preceding section we pointed out that in order to complete the investigation of possibilities for constructing the quadrature formula (5.1.5) that is exact for polynomials of degree $2n - 1$ in $1/p$, it is necessary to show that the roots of the polynomials $\omega_n^{(s)}\left(\frac{1}{p}\right)$ or the roots of the polynomials $P_n^{(s)}(x)$ that differ from $\omega_n^{(s)}(x)$ only by a constant factor lie in the right half-plane for all values of $s > 0$.

In this subsection we consider the question for certain particular values of s. We prove the following theorem.

Theorem 4. *All roots of the polynomials*

$$P_n^{(s)}(x)=P_n^{(s)}\left(\frac{1}{p}\right)=(-1)^n e^{-p}p^{n+s-1}\frac{d^n}{dp^n}\left(\frac{e^p}{p^{n+s-1}}\right)$$

lie in the right half-plane for all integral values of s $\geqslant 2$; *that is, the real parts of all roots are positive.*

Proof. First take $s = 2$. In this proof we appeal to some theorems of algebra. Let us recall them (see [8] pp. 112, 114):

(a) A necessary and sufficient condition for the real parts of all roots of the polynomial

$$Q_n(x) = b_n x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_0$$

with real coefficients to be of one sign is that the roots of the polynomials

$$f(x) = b_n x^n - b_{n-2} x^{n-2} + b_{n-4} x^{n-4} - \dots$$
$$\varphi(x) = b_{n-1} x^{n-1} - b_{n-3} x^{n-3} + b_{n-5} x^{n-5} - \dots$$

be all real and distinct.

(b) If all the roots of the polynomials $F(x) = \lambda f(x) + \mu \varphi(x)$ are real for arbitrary real values of λ and μ, then the roots of the polynomials $f(x)$ and $\varphi(x)$ are real and distinct.

Write the polynomial $P_n^{(2)}(x)$ as

$$P_n^{(2)}(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 \tag{5.2.16}$$

We have to prove that the real parts of all roots of the polynomial (5.2.16) are of the same sign. To do this, it suffices, on the basis of the theorems (a) and (b) above, to establish the truth of the following facts:

(1) all roots of the polynomials

$$Q_n(x) = a_n x^n - a_{n-2} x^{n-2} + a_{n-4} x^{n-4} - \dots$$
$$R_{n-1}(x) = a_{n-1} x^{n-1} - a_{n-3} x^{n-3} + a_{n-5} x^{n-5} - \dots$$

are real and distinct;

(2) all roots of the polynomials

$$P_n^{(\lambda, \mu)}(x) = \lambda a_n x^n + \mu a_{n-1} x^{n-1} - \lambda a_{n-2} x^{n-2} - \mu a_{n-3} x^{n-3} + \lambda a_{n-4} x^{n-4} + \mu a_{n-5} x^{n-5} - \dots$$

are real for arbitrary real values of λ and μ.

From (5.2.7) it is clear that the polynomial $P_n^{(2)}(x)$ satisfies the recurrence relation

$$P_n^{(2)}(x) = 2(2n-1)\,xP_{n-1}^{(2)}(x) + P_{n-2}^{(2)}(x) \qquad (5.2.17)$$

Here $P_0 = 1$, $P_1(x) = 2x - 1$.

Now let us find the recurrence relation for the polynomials $P_n^{(\lambda,\mu)}(x)$. Write the polynomial $P_n^{(2)}(x)$ in the form

$$P_n^{(2)}(x) = A_n(x) + B_n(x) + C_n(x) + D_n(x)$$

where

$$\left.\begin{aligned} A_n(x) &= a_n x^n + a_{n-4}x^{n-4} + a_{n-8}x^{n-8} + \ldots \\ B_n(x) &= a_{n-1}x^{n-1} + a_{n-5}x^{n-5} + a_{n-9}x^{n-9} + \ldots \\ C_n(x) &= a_{n-2}x^{n-2} + a_{n-6}x^{n-6} + a_{n-10}x^{n-10} + \ldots \\ D_n(x) &= a_{n-3}x^{n-3} + a_{n-7}x^{n-7} + a_{n-11}x^{n-11} + \ldots \end{aligned}\right\} \qquad (5.2.18)$$

Then $P_n^{(\lambda,\mu)}(x)$ can be written as follows:

$$P_n^{(\lambda,\mu)}(x) = \lambda[A_n(x) - C_n(x)] + \mu[B_n(x) - D_n(x)]$$

The recurrence relation (5.2.17) takes the form

$$\begin{aligned} &A_n(x) + B_n(x) + C_n(x) + D_n(x) \\ &\quad = 2(2n-1)\,x\,[A_{n-1}(x) + B_{n-1}(x) + C_{n-1}(x) + D_{n-1}(x)] \\ &\qquad + A_{n-2}(x) + B_{n-2}(x) + C_{n-2}(x) + D_{n-2}(x) \end{aligned}$$

From this and from the equations (5.2.18) and similar equations for $P_{n-1}^{(2)}(x)$ and $P_{n-2}^{(2)}(x)$ we get

$$\begin{aligned} A_n(x) &= 2(2n-1)\,x\,A_{n-1}(x) + C_{n-2}(x) \\ B_n(x) &= 2(2n-1)\,x\,B_{n-1}(x) + D_{n-2}(x) \\ C_n(x) &= 2(2n-1)\,x\,C_{n-1}(x) + A_{n-2}(x) \\ D_n(x) &= 2(2n-1)\,x\,D_{n-1}(x) + B_{n-2}(x) \end{aligned}$$

Multiply the first and third equations by λ and $-\lambda$ respectively and the second and fourth equations by μ and $-\mu$, and add them:

$$\lambda [A_n(x) - C_n(x)] + \mu [B_n(x) - D_n(x)]$$
$$= 2(2n-1)x\lambda [A_{n-1}(x) - C_{n-1}(x)] + \lambda [C_{n-2}(x) - A_{n-2}(x)]$$
$$+ 2(2n-1)x\mu [B_{n-1}(x) - D_{n-1}(x)] + \mu [D_{n-2}(x) - B_{n-2}(x)]$$

Rearranging we have

$$\lambda [A_n(x) - C_n(x)] + \mu [B_n(x) - D_n(x)]$$
$$= 2(2n-1)x\{\lambda [A_{n-1}(x) - C_{n-1}(x)] + \mu [B_{n-1}(x) - D_{n-1}(x)]\}$$
$$- \{\lambda [A_{n-2}(x) - C_{n-2}(x)] + \mu [B_{n-2}(x) - D_{n-2}(x)]\}$$

This last equation is nothing but the recurrence relation for $P_n^{(\lambda, \mu)}(x)$, namely,

$$P_n^{(\lambda, \mu)}(x) = 2(2n-1)x P_{n-1}^{(\lambda, \mu)}(x) - P_{n-2}^{(\lambda, \mu)}(x) \qquad (5.2.19)$$

where

$$P_0^{(\lambda, \mu)}(x) = \lambda, \quad P_1^{(\lambda, \mu)}(x) = 2\lambda x - \mu \quad \text{for } \lambda \neq 0$$
$$P_0^{(0, \mu)}(x) = 0, \quad P_1^{(0, \mu)}(x) = -\mu \quad \text{for } \lambda = 0$$

The case $\mu = 0$, $\lambda = 0$ is trivial and we disregard it. We thus have the following sequence of polynomials:

$$P_n^{(\lambda, \mu)}(x), \ P_{n-1}^{(\lambda, \mu)}(x), \ \ldots, \ P_0^{(\lambda, \mu)}(x) \ \text{for } \lambda \neq 0 \qquad (5.2.20)$$

or

$$P_n^{(0, \mu)}(x), \ P_{n-1}^{(0, \mu)}(x), \ \ldots, \ P_1^{(0, \mu)}(x) \ \text{for } \lambda = 0 \qquad (5.2.21)$$

In the latter sequence, $P_n^{(0, \mu)}(x)$ is a polynomial of degree $n-1$.

This sequence of polynomials has the following properties:

(1) The last polynomial of the sequence is a nonzero constant; namely, in the first sequence it is λ, in the second it is μ.

(2) For no value of x do two adjacent polynomials vanish.

Indeed, let x_1 be a root of $P_n^{(\lambda, \mu)}(x)$ and $P_{n-1}^{(\lambda, \mu)}(x)$. Then, due to (5.2.19), x_1 will be a root of $P_{n-2}^{(\lambda, \mu)}(x)$ as well. Continuing such reasoning, we would finally find that this common root would also be a root of $P_0^{(\lambda, \mu)}(x)$ in the sequence (5.2.20) or of $P_1^{(\lambda, \mu)}(x)$ in the sequence (5.2.21), which is impossible since these polynomials are nonzero constants.

(3) If some polynomial of a sequence vanishes for a real value of x, then the two adjacent polynomials have values of different sign for this x. True enough, from the formula (5.2.19) it is evident that if $P_{n-1}^{(\lambda, \mu)}(x) = 0$, then

$$P_n^{(\lambda, \mu)}(x) = -P_{n-2}^{(\lambda, \mu)}(x)$$

Thus this sequence of polynomials forms a generalized Sturm sequence. From algebra we know that if $u(x)$ denotes the number of sign changes in the polynomial sequence for a given value of x, and r is the number of real roots of the polynomial $P_n^{(\lambda, \mu)}(x)$, then we have $r \geqslant u(-\infty) - u(\infty)$. And since the sequence $P_n^{(\lambda, \mu)}(x)$ contains polynomials of all degrees with leading coefficients of the same sign, it follows that $u(\infty) = 0$ and $u(-\infty) = n$ in the sequence (5.2.20) and $u(-\infty) = n-1$ in the sequence (5.2.21). Hence all roots of the polynomial $P_n^{(\lambda, \mu)}(x)$ are not only real for arbitrary real values of λ and μ, but they are also simple and the roots of $P_{n-1}^{(\lambda, \mu)}(x)$ are mutually distinct from the roots of $P_n^{(\lambda, \mu)}(x)$ (see [9]).

From this, on the basis of theorems (a) and (b) of algebra (see p. 129), we can draw the conclusion that the real parts of all roots of the polynomial $P_n^{(2)}(x)$ are of the same sign. This sign can only be plus since the coefficient of x^{n-1} in $P_n^{(2)}(x)$ is negative. We have thus proved that all the roots of $P_n^{(s)}(x)$ for $s = 2$ lie in the right half-plane.

Now let us take the parameter $s = 3$. From the explicit expression (5.2.1) for the polynomials $P_n^{(s)}(x)$ it is clear that the polynomial $P_n^{(3)}(x)$ may be obtained from $P_n^{(2)}(x)$ if the coefficients a_k of the latter are multiplied by $\frac{n+k+1}{n+1}$.

Suppose for $s = 2$ the polynomial $P_n^{(2)}(x)$ has the form (5.2.16); then for $s = 3$ the polynomial $P_n^{(3)}(x)$ is of the form

$$P_n^{(3)}(x) = \frac{2n+1}{n+1} a_n x^n + \frac{2n}{n+1} a_{n-1} x^{n-1} + \frac{2n-1}{n+1} a_{n-2} x^{n-2} + \ldots + \frac{n+2}{n+1} a_1 x + \frac{n+1}{n+1} a_0 \qquad (5.2.22)$$

For what follows we will need the following familiar theorem of calculus [18].

(c) Let

$$f(z) = b_0 + \binom{n}{1} b_1 z + \binom{n}{2} b_2 z^2 + \binom{n}{3} b_3 z^3 + \ldots + \binom{n}{n-1} b_{n-1} z^{n-1} + b_n z^n$$

be a polynomial of degree n, all zeros of which lie in the "circle"[1] K; furthermore,

$$g(z) = c_0 + \binom{n}{1} c_1 z + \binom{n}{2} c_2 z^2 + \binom{n}{3} c_3 z^3 + \ldots + \binom{n}{n-1} c_{n-1} z^{n-1} + c_n z^n$$

is an nth degree polynomial with zeros $\beta_1, \beta_2, \ldots, \beta_n$. Then each zero γ of the polynomial $h(z)$, made up of $f(z)$ and $g(z)$,

$$h(z) = b_0 c_0 + \binom{n}{1} b_1 c_1 z + \binom{n}{2} b_2 c_2 z^2 + \binom{n}{3} b_3 c_3 z^3 + \ldots + \binom{n}{n-1} b_{n-1} c_{n-1} z^{n-1} + b_n c_n z^n \qquad (5.2.23)$$

is of the form $\gamma = -\beta_\nu k$, where ν is an index $(1 \leqslant \nu \leqslant n)$ and k is a point in K.

---

[1] The term "circle" is to be understood as meaning either the closed interior or the closed exterior region of a circle, or a closed half-plane.

Forming a combination like (5.2.23) made up of the polynomial $P_n^{(2)}(x)$ and the polynomial

$$Q(x) = \frac{2n+1}{n+1} x^n + \binom{n}{n-1} \frac{2n}{n+1} x^{n-1} + \binom{n}{n-2} \frac{2n-1}{n+1} x^{n-2} + \ldots + \binom{n}{1} \frac{n+2}{n+1} x + \frac{n+1}{n+1} \quad (5.2.24)$$

we get the polynomial $P_n^{(3)}(x)$.

The roots of the polynomial $P_n^{(2)}(x)$ lie in the right half-plane, as has already been demonstrated. Now let us find the roots of the polynomial $Q(x)$. We will prove that $Q(x)$ is of the form

$$Q(x) = \frac{1}{n+1} (x+1)^{n-1} [(2n+1)x + n + 1] \quad (5.2.25)$$

First compare the coefficients of x^k in the expressions (5.2.24) and (5.2.25) for the polynomial $Q(x)$. The coefficient of x^k in (5.2.24) is equal to $\frac{n+k+1}{n+1} C_n^{(k)}$. Now compute the coefficient of x^k in the expression (5.2.25):

$$\frac{1}{n+1} [(n+1) C_{n-1}^k + (2n+1) C_{n-1}^{k-1}]$$

$$= \frac{1}{n+1} \left[(n+1) \frac{n-k}{n} C_n^k + (2n+1) \frac{k}{n} C_n^k \right]$$

$$= \frac{C_n^k}{n(n+1)} [(n+1)(n-k) + k(2n+1)]$$

$$= \frac{C_n^k}{n(n+1)} (n^2 + nk + n) = \frac{n+k+1}{n+1} C_n^k$$

Consequently the polynomial $Q(x)$ does indeed have the form (5.2.25). From (5.2.25) it is immediately apparent that all roots of $Q(x)$ are negative: $x_1 = -\frac{n+1}{2n+1}$ and $x = -1$ is a root of multiplicity $n-1$.

Thus the roots of the polynomial $P_n^{(2)}(x)$ lie in the right half-plane and the roots of $Q(x)$ are negative. On the basis of theorem (c) of calculus, the roots of $P_n^{(3)}(x)$, which is a combination of $P_n^{(2)}(x)$ and $Q(x)$, will also lie in the right half-plane.

Knowing that the roots of the polynomial $P_n^{(3)}(x)$ have positive real parts, we can prove in exactly the same fashion that the roots of $P_n^{(4)}(x)$ also have positive real parts, and so on.

This can also be proved differently. For the polynomials $P_n^{(s)}(x)$ we have the integral representation (5.2.10). Differentiate the indicated equation with respect to x to get

$$P_n^{(s)\prime}(x) = \frac{(-1)^{n-1}}{\Gamma(n+s-1)} \int_0^\infty n t^{n+s-1} (1-xt)^{n-1} e^{-t}\, dt$$

$$= n(n+s-1) \frac{(-1)^{n-1}}{\Gamma(n+s)} \int_0^\infty t^{n+s-1} (1-xt)^{n-1} e^{-t}\, dt$$

$$= n(n+s-1) P_{n-1}^{(s+2)}(x) \qquad (5.2.26)$$

We know that if the roots of a polynomial are located in a certain half-plane, then all roots of its derivative are located in the same half-plane. And so from (5.2.26) we see that if the roots of the polynomials $P_n^{(s)}(x)$ for $s = 2$ and $s = 3$ are located in the right half-plane, then all the roots of all polynomials for integral s exceeding 2 are also located in the right half-plane. The proof of Theorem 4 is complete.

In Theorem 4 it was established that the roots of the polynomials $P_n^{(s)}(x)$ have positive real parts for all integers $s \geqslant 2$. This question remains open for other positive values of the parameter s, in particular for positive rational values. However, in the calculations carried out for $s = 0.01\ (0.01)3$ and $n = 1(1)10$ (see [13]) the roots always lay in the right half-plane, and their real parts increased with increasing s.

With this we conclude our investigation of the properties of orthogonal polynomials connected with the quadrature formula of highest accuracy for inverting Laplace transforms.

Note that these polynomials are a special case of the so-called Bessel polynomials y_n (x, a, b) for $a = s$, $b = -1$ that have been investigated in the works of H. L. Krall, O. Frink and W. A. Al-Salam.

5.3 *Methods for computing the coefficients and points of a quadrature formula*

In Sec. 5.1 it was pointed out that coefficients A_k of a quadrature formula of highest accuracy for inverting Laplace transforms have the values (5.1.6) or, in different notation,

$$A_k = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^p p^{-s} \frac{P_n^{(s)}\left(\frac{1}{p}\right)}{\left(\frac{1}{p} - \frac{1}{p_k}\right) P_n^{(s)\prime}\left(\frac{1}{p_k}\right)} dp \qquad (5.3.1)$$

where the derivative of the polynomial $P_n^{(s)}\left(\frac{1}{p}\right)$ is taken with respect to the variable $x = 1/p$.

To compute this integral we make use of an equation that is an analog of the familiar Christoffel-Darboux identity. Let us derive this equation.

Rewrite the recurrence relation (5.2.7) for the polynomials $P_n^{(s)}(x)$ in the form

$$xP_n(x) = B_n P_{n+1}(x) + C_n P_n(x) + D_n P_{n-1}(x) \qquad (5.3.2)$$

where

$$\left.\begin{aligned} B_n &= \frac{n+s-1}{(2n+s)(2n+s-1)} \\ C_n &= \frac{s-2}{(2n+s)(2n+s-2)} \\ D_n &= -\frac{n}{(2n+s-1)(2n+s-2)} \end{aligned}\right\} \qquad (5.3.3)$$

Here and henceforth the index s is dropped for the sake of simplicity.

Multiply (5.3.2) by $P_n(t)$:

$$xP_n(x)P_n(t) = B_nP_{n+1}(x)P_n(t) + C_nP_n(x)P_n(t) + D_nP_{n-1}(x)P_n(t) \quad (5.3.4)$$

Interchange the variables x and t in (5.3.4) and subtract the resulting equation from (5.3.4). Then the middle terms of the right members cancel out and we get

$$(x-t)P_n(x)P_n(t) = B_n[P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t)] + D_n[P_{n-1}(x)P_n(t) - P_n(x)P_{n-1}(t)]$$

Write down similar equations for $n-1, n-2, \ldots, 1, 0$:

$$(x-t)P_{n-1}(x)P_{n-1}(t) = B_{n-1}[P_n(x)P_{n-1}(t) - P_{n-1}(x)P_n(t)] + D_{n-1}[P_{n-2}(x)P_{n-1}(t) - P_{n-1}(x)P_{n-2}(t)],$$

$$(x-t)P_{n-2}(x)P_{n-2}(t) = B_{n-2}[P_{n-1}(x)P_{n-2}(t) - P_{n-2}(x)P_{n-1}(t)] + D_{n-2}[P_{n-3}(x)P_{n-2}(t) - P_{n-2}(x)P_{n-3}(t)],$$

. . . .

$$(x-t)P_1(x)P_1(t) = B_1[P_2(x)P_1(t) - P_1(x)P_2(t)] + D_1[P_0(x)P_1(t) - P_1(x)P_0(t)],$$

$$(x-t)P_0(x)P_0(t) = B_0[P_1(x)P_0(t) - P_0(x)P_1(t)]$$

We now transform the resulting system of equations into an equivalent system by changing its coefficients by the method indicated below:

$$(x-t) P_{n-1}(x) P_{n-1}(t)$$
$$= B_{n-1}[P_n(x) P_{n-1}(t) - P_{n-1}(x) P_n(t)]$$
$$+ D_{n-1}[P_{n-2}(x) P_{n-1}(t) - P_{n-1}(x) P_{n-2}(t)],$$

$$\frac{D_{n-1}}{B_{n-2}}(x-t) P_{n-2}(x) P_{n-2}(t)$$
$$= D_{n-1}[P_{n-1}(x) P_{n-2}(t) - P_{n-2}(x) P_{n-1}(t)]$$
$$+ \frac{D_{n-1}D_{n-2}}{B_{n-2}}[P_{n-3}(x) P_{n-2}(t) - P_{n-2}(x) P_{n-3}(t)],$$

. . . .

$$\frac{D_{n-1}D_{n-2}\ldots D_2}{B_{n-2}B_{n-3}\ldots B_1}(x-t) P_1(x) P_1(t)$$
$$= \frac{D_{n-1}D_{n-2}\ldots D_2}{B_{n-2}B_{n-3}\ldots B_2}[P_2(x) P_1(t) - P_1(x) P_2(t)]$$
$$+ \frac{D_{n-1}D_{n-2}\ldots D_1}{B_{n-2}B_{n-3}\ldots B_1}[P_0(x) P_1(t) - P_1(x) P_0(t)],$$

$$\frac{D_{n-1}D_{n-2}\ldots D_1}{B_{n-2}B_{n-3}\ldots B_0}(x-t) P_0(x) P_0(t)$$
$$= \frac{D_{n-1}D_{n-2}\ldots D_1}{B_{n-2}B_{n-3}\ldots B_1}[P_1(x) P_0(t) - P_0(x) P_1(t)]$$

Now, combining these equations termwise, we get the desired identity:

$$(x-t)\sum_{m=0}^{n-1}\alpha_m P_m(x) P_m(t)$$
$$= B_{n-1}[P_n(x) P_{n-1}(t) - P_{n-1}(x) P_n(t)] \qquad (5.3.5)$$

where

$$\alpha_m = \frac{D_{n-1}D_{n-2}\ldots D_{1+m}}{B_{n-2}B_{n-3}\ldots B_m}, \quad m = 0, 1, \ldots, n-2; \quad \alpha_{n-1} = 1$$

In (5.3.5) put $t = x_k = \frac{1}{p_k}$, where x_k is a root of the polynomial $P_n(x)$. Dividing by $x - x_k = \frac{1}{p} - \frac{1}{p_k}$ yields

$$\sum_{m=0}^{n-1} \alpha_m P_m(x) P_m(x_k) = B_{n-1} \frac{P_n(x) P_{n-1}(x_k)}{x - x_k}$$

Multiply by $e^p p^{-s}$ and integrate. The integral

$$P_m(x_k) \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^p p^{-s} P_m\left(\frac{1}{p}\right) dp$$

is equal to zero when $m \geqslant 1$ by the orthogonality condition (5.1.7) and is equal to $\frac{1}{\Gamma(s)}$ for $m = 0$. Therefore, after integrating we have

$$\alpha_0 \frac{1}{\Gamma(s)} = B_{n-1} P_{n-1}(x_k) \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^p p^{-s} \frac{P_n\left(\frac{1}{p}\right)}{\left(\frac{1}{p} - \frac{1}{p_k}\right)} dp$$

Divide by $P_n'(x_k)$ to get

$$\alpha_0 \frac{1}{\Gamma(s) P_n'(x_k)} \cdot$$

$$= B_{n-1} P_{n-1}(x_k) \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^p p^{-s} \frac{P_n\left(\frac{1}{p}\right)}{\left(\frac{1}{p} - \frac{1}{p_k}\right) P_n'\left(\frac{1}{p_k}\right)} dp$$

This integral is nothing more than the coefficient A_k of the quadrature formula, hence

$$A_k = \frac{\alpha_0}{B_{n-1} \Gamma(s) P_{n-1}(x_k) P_n'(x_k)} \tag{5.3.6}$$

It now remains to compute α_0 / B_{n-1}:

$$\frac{\alpha_0}{B_{n-1}} = \frac{D_{n-1} D_{n-2} \ldots D_1}{B_{n-1} B_{n-2} \ldots B_1 B_0}$$

To do this we first compute the ratio D_{n-1}/B_{n-1}:

$$\frac{D_{n-1}}{B_{n-1}} = -\frac{(n-1)(2n+s-2)(2n+s-3)}{(2n+s-3)(2n+s-4)(n+s-2)} = -\frac{(n-1)(2n+s-2)}{(2n+s-4)(n+s-2)}, \quad B_0 = \frac{1}{s}$$

whence

$$\frac{\alpha_0}{B_{n-1}} = \frac{(-1)^{n-1}(n-1)!\,(2n+s-2)(2n+s-4)\dots(2+s)s}{(2n+s-4)\dots(2+s)s(n+s-2)(n+s-3)\dots(1+s)s}$$
$$= \frac{(-1)^{n-1}(n-1)!\,(2n+s-2)}{(n+s-2)(n+s-3)\dots(1+s)s}$$
$$= \frac{(-1)^{n-1}(n-1)!\,(2n+s-2)\,\Gamma(s)}{\Gamma(n+s-1)} \qquad (5.3.7)$$

Putting (5.3.7) into (5.3.6) for A_k, we get the following formula:

$$A_k = \frac{(-1)^{n-1}(n-1)!\,(2n+s-2)}{\Gamma(n+s-1)\,P_{n-1}(x_k)\,P_n'(x_k)} \qquad (5.3.8)$$

which can be used to compute the coefficients of the quadrature formula.

Let us now go over to a method of computing the points p_k of the quadrature formula. The inverse powers of these points are the numbers $x_k = \frac{1}{p_k}$; as was found above, they are the roots of the polynomials $P_n^{(s)}(x)$. And so to compute p_k we can find the coefficients of the polynomials $P_n^{(s)}(x)$ and then, using some kind of method, find their roots. But this involves considerable computational difficulties: first, for each value of the parameter s we have to determine the coefficients of the polynomials $P_n^{(s)}(x)$ via a recurrence formula; second, computation of the roots of the polynomials $P_n^{(s)}(x)$ via the Newton method involves a great loss of accuracy, particularly for large values of n.

So as to avoid computing the coefficients of the polynomials $P_n^{(s)}(x)$ and the great loss of accuracy in the computations, we can take advantage of another method of finding the points of the quadrature formula. For the roots x_k of the polynomials $P_n^{(s)}(x)$ we can construct a simple system of algebraic equations containing only x_k and the parameter s. All we have to do is proceed from the differential equation that is satisfied by the polynomials $P_n^{(s)}(x)$:

$$x^2 P_n^{(s)''}(x) + (sx - 1)P_n^{(s)'}(x) - n(n + s - 1)P_n^{(s)}(x) = 0 \tag{5.3.9}$$

Put $x = x_k$ in (5.3.9) and then the third term on the left will vanish. Divide the resulting equation by $x_k^2 P_n^{(s)'}(x_k)$, which is possible since $P_n^{(s)}(x)$ does not have a zero root or any multiple roots. Indeed, from the explicit expression (5.2.1) it is evident that the constant term of $P_n^{(s)}(x)$ is not equal to zero, hence $x_k \neq 0$. Furthermore, from (5.3.9) it follows that if $P_n^{(s)'}(x_k) = 0$, then also $P_n^{(s)''}(x_k) = 0$. Differentiating the equation (5.3.9) n times, we then have

$$P_n^{(s)'''}(x_k) = 0, \quad P_n^{(s)\mathrm{IV}}(x_k) = 0, \quad \ldots, \quad P_n^{(s)(n)}(x_k) = 0$$

This last equation is invalid since $P_n^{(s)(n)}(x_k)$ is a nonzero constant. Hence $P_n^{(s)'}(x_k) \neq 0$.

After the indicated manipulations, (5.3.9) becomes

$$\frac{P_n^{(s)''}(x_k)}{P_n^{(s)'}(x_k)} + \frac{s}{x_k} - \frac{1}{x_k^2} = 0 \tag{5.3.10}$$

Let us transform the expression $\dfrac{P_n^{(s)''}(x_k)}{P_n^{(s)'}(x_k)}$. To do this write the polynomial $P_n^{(s)}(x)$ in the form

$$P_n^{(s)}(x) = a(x - x_1)(x - x_2)\ldots(x - x_n) = a\prod_{m=1}^{n}(x - x_m)$$

Then

$$P_n^{(s)\prime}(x) = a \sum_{i=1}^{n} \prod_{\substack{m=1 \\ m \neq i}}^{n} (x - x_m)$$

$$P_n^{(s)\prime}(x_k) = a \prod_{\substack{m=1 \\ m \neq k}}^{n} (x_k - x_m)$$

$$P_n^{(s)\prime\prime}(x) = a \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} \prod_{\substack{m=1 \\ m \neq i,\, j}}^{n} (x - x_m)$$

$$P_n^{(s)\prime\prime}(x_k) = 2a \sum_{\substack{j=1 \\ j \neq k}}^{n} \prod_{\substack{m=1 \\ m \neq j,\, k}}^{n} (x_k - x_m)$$

From this it follows that

$$\frac{P_n^{(s)\prime\prime}(x_k)}{P_n^{(s)\prime}(x_k)} = \frac{2a \sum\limits_{\substack{j=1 \\ j \neq k}}^{n} \prod\limits_{\substack{m=1 \\ m \neq j,\, k}}^{n} (x_k - x_m)}{a \prod\limits_{\substack{m=1 \\ m \neq k}}^{n} (x_k - x_m)} = \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{2}{(x_k - x_j)}$$

Now equation (5.3.10) can be written as

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{2}{x_k - x_j} + \frac{s}{x_k} - \frac{1}{x_k^2} = 0 \qquad (5.3.11)$$

Replace x_k by $\frac{1}{p_k}$ and perform a few simple manipulations in (5.3.11) to get

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2p_k p_j}{p_j - p_k} + sp_k - p_k^2 = 0,$$

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2p_j}{p_j - p_k} + s - p_k = 0,$$

$$\sum_{\substack{j=1\\ j\neq k}}^{n} 2\left(1 + \frac{p_k}{p_j - p_k}\right) + s - p_k = 0,$$

$$2(n-1) + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2p_k}{p_j - p_k} + s - p_k = 0,$$

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2}{p_k - p_j} + 1 - \frac{2n+s-2}{p_k} = 0 \qquad (5.3.12)$$

Writing down the equation (5.3.12) for all values of k ($k = 1, 2, \ldots, n$), we get a system of equations for determining the points p_k of the quadrature formula.

Equation (5.3.12) takes on a simple physical meaning if we take advantage of a certain electrostatical analogy. Suppose an electric charge of negative mass $-(2n + s - 2)$ is placed at point 0 of the complex plane. Also we consider n free charges with positive mass 2 whose complex coordinates we call p_k ($k = 1, 2, \ldots, n$). We will assume they act on one another with the ordinary force of a plane electric field, where the numerical value of the force is inversely proportional to the first power of the distance and the proportionality factor is equal to the product of masses of the charges. We also assume that to each free charge p_k is applied an external force of magnitude 2 that is parallel to the real axis and in the positive direction of the real axis. In a state of equilibrium of the system, the resultants of all forces applied to each of the free charges must equal zero:

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2\times 2}{\overline{p_k - p_j}} - \frac{2(2n+s-2)}{\overline{p_k - 0}} + 2 = 0$$

If we cancel the factor 2 and go over to complex conjugate quantities, we get the system (5.3.12).

The equation (5.3.12) is a system of equations that permits obtaining the points of the quadrature formula. Since the points p_k are complex numbers, it is possible, by representing p_k as $p_k = s_k + i\sigma_k$, to transform the system (5.3.12) by separating the real and imaginary parts. We then get the following system of equations for finding s_k and σ_k:

$$\left.\begin{aligned} &\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2(s_k - s_j)}{(s_k - s_j)^2 + (\sigma_k - \sigma_j)^2} + 1 - \frac{(2n+s-2)\,s_k}{s_k^2 + \sigma_k^2} = 0 \\ &\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{2(\sigma_k - \sigma_j)}{(s_k - s_j)^2 + (\sigma_k - \sigma_j)^2} - \frac{(2n+s-2)\,\sigma_k}{s_k^2 + \sigma_k^2} = 0 \end{aligned}\right\} \quad (5.3.13)$$

The system (5.3.13) consists of $2n$ equations. Actually there will be only n equations since p_k are complex conjugate numbers:

$$p_2 = \bar{p}_1, \quad p_4 = \bar{p}_3, \quad \ldots$$

Hence

$$s_2 = s_1, \quad s_4 = s_3, \quad \ldots; \quad \sigma_2 = -\sigma_1, \quad \sigma_4 = -\sigma_3, \quad \ldots$$

Tables 1 and 2 of [13] give the values of the points and coefficients of the quadrature formula (5.1.5) that has the highest degree of accuracy: for $s = 1, 2, 3, 4, 5$, $n = 1(1)15$ accurate to 20 digits and for $s = 0.01(0.01)3$, $n = 1(1)10$ to 7-8 significant digits.

Chapter 6

Methods of Inverting Laplace Transforms via Quadrature Formulas with Equal Coefficients

6.1 *Constructing a computation formula*

We assume that the problem of inverting Laplace transforms has again been reduced to computing the integral

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\varphi(p)\,dp \tag{6.1.1}$$

To evaluate it we construct a quadrature formula with equal coefficients:

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\varphi(p)\,dp \approx C_n \sum_{k=1}^{n} \varphi(p_k) \tag{6.1.2}$$

The unknown quantities that we can deal with in (6.1.2) are the numbers C_n and p_k ($k = 1, 2, \ldots, n$). We choose them so that (6.1.2) is exact for any polynomial of degree n in the variable $1/p$. This requirement is equivalent to formula (6.1.2) being exact for the functions $\varphi(p) = 1/p^k$ ($k = 0, 1, \ldots, n$). The factor C_n is determined from the condition that (6.1.2) be exact for the function $\varphi(p) \equiv 1$:

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\,dp = nC_n$$

Hence

$$C_n = \frac{1}{n}\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\,dp = \frac{1}{n\Gamma(s)} \tag{6.1.3}$$

For the value of C_n thus found, the rule (6.1.2) becomes

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}\varphi(p)\,dp \approx \frac{1}{n\Gamma(s)}\sum_{k=1}^{n}\varphi(p_k) \qquad (6.1.4)$$

If we pass from the variable p to the variable $x = 1/p$ and set $x_k = 1/p_k$, then we get the following system of equations for the unknowns x_k:

$$\left.\begin{aligned}
x_1+x_2+\ldots+x_n &= \frac{n\Gamma(s)}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}p^{-1}\,dp = \frac{n\Gamma(s)}{\Gamma(s+1)},\\
x_1^2+x_2^2+\ldots+x_n^2 &= \frac{n\Gamma(s)}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}p^{-2}\,dp = \frac{n\Gamma(s)}{\Gamma(s+2)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x_1^n+x_2^n+\ldots+x_n^n &= \frac{n\Gamma(s)}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^p p^{-s}p^{-n}\,dp = \frac{n\Gamma(s)}{\Gamma(s+n)}
\end{aligned}\right\} \qquad (6.1.5)$$

From this system we could obtain the values of x_k and hence also p_k. But since this system is nonlinear, its solution can give rise to certain computational difficulties. And so we try to find a different method of computing x_k similar to what was done for the Chebyshev quadratures in the field of the real variable (see [10]). We introduce the nth degree polynomial $\omega_n(x)$ whose roots are the numbers x_k:

$$\omega_n(x) = (x-x_1)(x-x_2)\ldots(x-x_n)$$

Expand this polynomial in powers of x:

$$\omega_n(x) = x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + A_n$$

As we know, the coefficients A_k will be elementary symmetric functions of the roots. Now the equations (6.1.5) yield sums of the powers of the roots x_k:

$$S_i = \sum_{k=1}^{n} x_k^i = \frac{n\Gamma(s)}{\Gamma(i+s)} \quad (i = 1, 2, \ldots, n) \qquad (6.1.6)$$

Now, appealing to polynomial theory, we write down the familiar relationships between elementary symmetric functions of the roots A_k $(k = 1, 2, \ldots, n)$ and the functions S_i $(i = 1, 2, \ldots, n)$. They are defined by the so-called *Newtonian relations*

$$\left.\begin{aligned} &S_1 + A_1 = 0, \\ &S_2 + A_1 S_1 + 2A_2 = 0, \\ &S_3 + A_1 S_2 + A_2 S_1 + 3A_3 = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &S_n + A_1 S_{n-1} + A_2 S_{n-2} + \ldots + nA_n = 0 \end{aligned}\right\} \qquad (6.1.7)$$

Substituting the values of S_i from (6.1.6) into the system (6.1.7), we get the following system of equations:

$$\left.\begin{aligned} &\frac{1}{\Gamma(s+1)} + \frac{1}{n\Gamma(s)} A_1 = 0, \\ &\frac{1}{\Gamma(s+2)} + \frac{1}{\Gamma(s+1)} A_1 + \frac{2}{n\Gamma(s)} A_2 = 0, \\ &\frac{1}{\Gamma(s+3)} + \frac{1}{\Gamma(s+2)} A_1 + \frac{1}{\Gamma(s+1)} A_2 + \frac{3}{n\Gamma(s)} A_3 = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{1}{\Gamma(s+n)} + \frac{1}{\Gamma(s+n-1)} A_1 + \frac{1}{\Gamma(s+n-2)} A_2 \\ &\qquad + \ldots + \frac{1}{\Gamma(s+1)} A_{n-1} + \frac{n}{n\Gamma(s)} A_n = 0 \end{aligned}\right\} \qquad (6.1.8)$$

All the values of A_k can be found in succession from this system. The value of A_k can be written out in explicit form as a determinant of order k. Since the determinant of the system (6.1.8) is triangular and equal to $\frac{k!}{n^k [\Gamma(s)]^k}$, then

$$A_k = -\frac{n^k [\Gamma(s)]^k}{k!} \Delta, \text{ where}$$

$$\Delta = \begin{vmatrix} \frac{1}{n\Gamma(s)} & 0 & 0 & \dots & 0 & \frac{1}{\Gamma(s+1)} \\ \frac{1}{\Gamma(s+1)} & \frac{2}{n\Gamma(s)} & 0 & \dots & 0 & \frac{1}{\Gamma(s+2)} \\ \frac{1}{\Gamma(s+2)} & \frac{1}{\Gamma(s+1)} & \frac{3}{n\Gamma(s)} & \dots & 0 & \frac{1}{\Gamma(s+3)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{\Gamma(s+k-2)} & \frac{1}{\Gamma(s+k-3)} & \frac{1}{\Gamma(s+k-4)} & \dots & \frac{k-1}{n\Gamma(s)} & \frac{1}{\Gamma(s+k-1)} \\ \frac{1}{\Gamma(s+k-1)} & \frac{1}{\Gamma(s+k-2)} & \frac{1}{\Gamma(s+k-3)} & \dots & \frac{1}{\Gamma(s+1)} & \frac{1}{\Gamma(s+k)} \end{vmatrix}$$

for $2 \leqslant k \leqslant n$, and $A_1 = -n/s$.

Having found all the A_k, we can construct a polynomial $\omega_n(x)$. After we find its roots x_k, we can determine all the points $p_k = 1/x_k$ of the quadrature formula (6.1.4).

6.2 *Remark on the spacing of points*

The formula (6.1.4) was constructed by H. E. Salzer for the particular value of the parameter $s = 1$. He also computed for this parameter s the polynomials $\omega_n(x)$ for $n = 1(1)10$ and he determined their roots x_k and, hence, the points $p_k = 1/x_k$ of the quadrature formula. The points p_k are tabulated to 8 significant digits in [13].

From the meaning of the problem, the points p_k of formula (6.1.4) must lie in the domain of definition and regularity of the function $\varphi(p)$, that is, in the right half-plane.

As may be seen from the table in [13], beginning with a certain n, two of the points p_k pass into the left half-plane. Apparently, as n increases, more and more points p_k will pass into the left half-plane since the real parts of p_k decrease with increasing n. This is a drawback when using formula (6.1.4) for computations.

Part Two

FOURIER TRANSFORMS AND THEIR APPLICATION TO INVERSION OF LAPLACE TRANSFORMS

Chapter 7

Introduction

As is well known and as will be demonstrated below, the problem of computing the Mellin integral can be reduced to the Fourier transformation, which is the classical machinery for investigating a broad range of applied problems. Many methods have been devised for handling this transformation numerically. In principle, each of these methods can be applied to solving the inversion problem. A survey of the most important of them will be given in Chapters 9 and 10. Here we wish to note once again that all these methods have one fundamental drawback; namely, they fail to account for the fact that the integrand in the Mellin integral is not an arbitrary function $F(p)$ with complex values but an image function that definitely has certain properties known beforehand. These properties include, for instance, regularity in the half-plane Re $p > \alpha$ ($\alpha \leqslant c$), a tending to zero as the point goes to infinity, and so on. In this respect, the computational methods of inversion that were considered in the first part of this book are preferable since to a certain extent they take into account the image properties.

Despite the foregoing drawback, we have retained in this book the question of computing the Mellin integral by reducing it to the Fourier integral; this is done for the following reasons. First, it is one of several possible methods of computation when the points of the quadrature formula are taken on the line of integration $p = c + i\tau$ ($-\infty < \tau < \infty$). Second, this method may be useful in practice at least as a supplementary one for checking computations involving other methods.

7.1 *Fourier transforms*

Below we consider the Fourier double integral

$$\frac{1}{\pi} \int_0^{\infty} du \int_{-\infty}^{\infty} f(t) \cos u(x-t)\, dt \qquad (7.1.1)$$

and assume the function f to be absolutely integrable on the number axis $-\infty < t < \infty$. The inner integral with respect to the argument t will be absolutely convergent for all real values of x and u, and the convergence will be uniform as well.

As for the convergence of the double integral (7.1.1) and its numerical value, Theorem 1 given below is sufficient (see, for example, [22]). The following remark will help the reader to get a better grasp of the situation.

Remark. Suppose a function $f(x)$ with finite values is given on an interval $[a, b]$. Partition $[a, b]$ into a finite number of parts using the points $x_0 = a < x_1 < \ldots < x_n = b$.

Form the sum

$$V(x_0, x_1, \ldots, x_n) = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$$

It has the meaning of the sum of absolute values of variation of f on the subintervals $[x_{k+1}, x_k]$ and depends on the points x_k $(k = 1, 2, \ldots, n-1)$. The upper bound of possible sums $V(x_0, x_1, \ldots, x_n)$,

$$\operatorname*{Var}_{a \leqslant x \leqslant b}(f) = \sup_{x_1, \ldots, x_{n-1}} V(x_0, x_1, \ldots, x_n)$$

is called the *total absolute variation of* f *on the interval* [a, b]. If $\operatorname*{Var}_{a \leqslant x \leqslant b}(f)$ has a finite value, then we say that f is a *function of bounded (finite) variation on* [a, b].

Theorem 1. *Let* f (t) *be absolutely integrable on the number axis* $-\infty < t < \infty$. *If on a certain interval* [a, b] *containing the point* x *the function* f *has bounded variation, then the following equation is valid:*

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{\pi} \int_0^\infty du \int_{-\infty}^\infty f(t) \cos u(x-t)\, dt \qquad (7.1.2)$$

But if f *is of bounded variation on* [a, b] *and is continuous there, then*

$$f(x) = \frac{1}{\pi} \int_0^\infty du \int_{-\infty}^\infty f(t) \cos u(x-t)\, dt \qquad (7.1.3)$$

Here, the double integral converges to f (x) *uniformly with respect to* x *in any closed interior portion of* [a, b].

The equations (7.1.2) and (7.1.3) are called the *Fourier formulas.*

In what follows we will assume everywhere that $f(t)$ is of bounded variation in any finite interval of the t axis.

Then (7.1.2) will hold for all finite values of x. Besides, to simplify notation we will assume that all discontinuities of f are "regular" and the relation $f(x) = \frac{1}{2}[f(x+0) + f(x-0)]$ holds at all points x. Then under this condition the equations (7.1.2) and (7.1.3) will have the same aspect and from now on we will make use of (7.1.3).

We can make the Fourier integral more symmetric if we take advantage of complex quantities and replace the trigonometric function by its expression in terms of exponential functions:

$$\frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t)\cos u(x-t)\,dt$$

$$= \frac{1}{2\pi}\int_0^\infty du \int_{-\infty}^\infty f(t)\left[e^{iu(x-t)} + e^{-iu(x-t)}\right] dt$$

If one performs termwise integration and if in the first of the integrals one replaces the variable u by $-u$, it is possible to reduce the Fourier formula (7.1.3) to the form[1]

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ixu}\,du \int_{-\infty}^\infty f(t)\,e^{iut}\,dt \qquad (7.1.4)$$

This equation is termed, for short, the *complex Fourier formula*. With it are connected the following reciprocal (dual)

[1] In the Fourier formula (7.1.3), by the integral with respect to the variable u taken along the half axis $0 \leqslant u < \infty$ is meant the limit to which the integral over the interval $0 \leqslant u < A$ tends when A increases without bound. Accordingly, in (7.1.4) the integral over the axis $-\infty < u < \infty$ is taken in its principal-value meaning as the limit to which the integral taken along the symmetric interval $-A \leqslant u \leqslant A$ tends when $A \to \infty$.

relations between pairs of functions:

$$\varphi(u) = \int_{-\infty}^{\infty} f(t)\, e^{iut}\, dt \tag{7.1.5}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(u)\, e^{-ixu}\, du \tag{7.1.6}$$

The former is called the *complex Fourier transformation* [1] and carries the original function f into the image function φ. The latter equation gives a rule of transition from the image φ to the original f.

We now give two special-type formulas of Fourier which together are equivalent to the general formula (7.1.3). If we take advantage of the familiar expression for the cosine of a difference of two arguments, then (7.1.3) can be cast in a form similar to the Fourier-series expansion of the

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[1] In order to make the transform and its inversion formula more symmetric, one often modifies the equations (7.1.5) and (7.1.6):

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{iut}\, dt$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(u)\, e^{-ixu}\, du$$

Since the numerical factor in the first equation is of no importance in our discussion and only complicates the notation, we have chosen the form (7.1.5) and (7.1.6). Similar remarks may be applied to the Fourier cosine and sine formulas.

function:

$$f(x)=\int_{0}^{\infty}[a(u)\cos ux+b(u)\sin ux]\,du, \qquad (7.1.7)$$

$$a(u)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\cos ut\,dt,$$

$$b(u)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\sin ut\,dt$$

When f is an even function, then

$$a(u)=\frac{2}{\pi}\int_{0}^{\infty}f(t)\cos ut\,dt, \quad b(u)=0$$

and (7.1.7) becomes the *Fourier cosine formula*:

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\cos xu\,du\int_{0}^{\infty}f(t)\cos ut\,dt \quad (0\leqslant x<\infty) \qquad (7.1.8)$$

Similarly, if f is an odd function, then (7.1.7) becomes the *Fourier sine formula*:

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\sin ux\,du\int_{0}^{\infty}f(t)\sin ut\,dt \quad (0\leqslant x<\infty) \qquad (7.1.9)$$

The general Fourier formula (7.1.3) may be regarded as a combination of the particular formulas (7.1.8) and (7.1.9). Indeed, every function f may be represented as the sum of its even and odd parts:

$$f(x)=g(x)+h(x), \quad g(x)=\frac{1}{2}[f(x)+f(-x)],$$

$$h(x)=\frac{1}{2}[f(x)-f(-x)]$$

The inner integral of (7.1.3) will have the following expression in terms of g and h:

$$\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt = 2\cos xu \int_{0}^{\infty} g(t)\cos ut\,dt + 2\sin xu \int_{0}^{\infty} h(t)\sin ut\,dt$$

And so from (7.1.3) we get

$$f(x) = g(x) + h(x) = \frac{2}{\pi}\int_{0}^{\infty} \cos xu\,du \int_{0}^{\infty} g(t)\cos ut\,dt + \frac{2}{\pi}\int_{0}^{\infty} \sin xu\,du \int_{0}^{\infty} h(t)\sin ut\,dt$$

and the general Fourier formula is indeed the sum of the cosine formula for $g(x)$ and the sine formula for $h(x)$.

Connected with the Fourier cosine formula (7.1.8) is a relationship between the pair of functions f and φ_c:

$$\varphi_c(u) = \int_{0}^{\infty} f(t)\cos ut\,dt \quad (0 \leqslant x < \infty) \tag{7.1.10}$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \varphi_c(u)\cos xu\,du \tag{7.1.11}$$

The first is the cosine transform of the original function f into the image function φ_c, the second is the inversion.

Also connected with the Fourier sine formula (7.1.9) is the reciprocal relation between the functions f and φ_s:

$$\varphi_s(u) = \int_0^\infty f(t) \sin ut \, dt \quad (0 \leqslant x < \infty) \qquad (7.1.12)$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \varphi_s(u) \sin xu \, du \qquad (7.1.13)$$

The equation (7.1.12) is the Fourier sine transform, and (7.1.13) is its inversion.

It will be seen that the complex Fourier transform (7.1.5) can easily be reduced to the transforms (7.1.10) and (7.1.12). In (7.1.5), replace $f(t)$ by its decomposition into even and odd parts: $f(x) = g(x) + h(x)$, where $g(x)$ and $h(x)$ are indicated above:

$$\varphi(u) = \int_{-\infty}^{\infty} f(t) e^{iut} \, dt$$

$$= \int_{-\infty}^{\infty} [g(t) + h(t)] [\cos ut + i \sin ut] \, dt$$

$$= 2 \int_0^\infty g(t) \cos ut \, dt + 2i \int_0^\infty h(t) \sin ut \, dt$$

$$= 2g_c(u) + 2ih_s(u)$$

Hence the complex transformation (7.1.5) is a simple linear combination of the Fourier cosine and Fourier sine transforms.

7.2 *Reducing integrals of the Mellin type to the Fourier transformation*

We now investigate a connection between inversion of Laplace transforms and the Fourier transformation. Consider the Mellin integral

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(p)\,e^{xp}\,dp \quad (-\infty<x<\infty) \qquad (7.2.1)$$

under the sole assumption concerning the function $F(p)$ that it is specified and absolutely integrable on the straight line $p=c+i\tau$ $(-\infty<\tau<\infty)$. The integral converges uniformly with respect to x on the axis $-\infty<x<\infty$ and is a continuous function of x on that axis.

If in (7.2.1) we put in place of p its value $p=c+i\tau$ on the line of integration and note that in the quantity $e^{xp}=e^{cx}e^{ix\tau}$ the factor e^{cx} does not depend on the variable of integration, then we can transform (7.2.1) to the form

$$e^{-cx}f(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} F(c+i\tau)\,e^{ix\tau}\,d\tau \qquad (7.2.2)$$

which shows that computing the Mellin integral $f(x)$ leads to the complex Fourier transformation of the function $F(c+i\tau)$.

This simple fact permits us to apply to the Mellin integral all known methods of the numerical Fourier transformation.

In Sec. 7.1. we noted that the Fourier transform of the function $F(c+i\tau)$ can readily be reduced to the simpler cosine and sine transforms of the even and odd parts of the function $F(c+i\tau)$:

$$F(c+i\tau)=g(\tau)+h(\tau),$$

$$g(\tau)=\frac{1}{2}[F(c+i\tau)+F(c-i\tau)],$$

$$h(\tau)=\frac{1}{2}[F(c+i\tau)-F(c-i\tau)],$$

$$e^{-cx}f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[g(\tau)+h(\tau)]e^{ix\tau}\,d\tau$$

$$=\frac{1}{\pi}\int_{0}^{\infty}g(\tau)\cos x\tau\,d\tau+\frac{i}{\pi}\int_{0}^{\infty}h(\tau)\sin x\tau\,d\tau$$

$$=\frac{1}{\pi}[g_c(x)+ih_s(x)] \quad (7.2.3)$$

The foregoing permits reducing the computation of Mellin integrals to that of the Fourier cosine and sine transforms. It is principally these transforms that we will be dealing with from now on.

Chapter 8

Inversion of Laplace Transforms by Means of the Fourier Series

We start our investigation of the problem of applying harmonic analysis to inversion of Laplace transforms with a consideration of cases that may be regarded as degenerate, when the computational problem is simplified and the Fourier integral transformation can be replaced by a Fourier series. Most likely this can be done in two cases: either when the function $f(x)$ rapidly diminishes in absolute value as x goes to infinity or when its image function $F(c + i\tau)$ under the integral sign of (7.2.2) rapidly tends to zero as $|\tau|$ increases. Here we confine ourselves to a description of computation schemes since the problems of convergence and error estimates are involved and require further study.

8.1 *The case of a rapidly decreasing original function $f(x)$*

To abbreviate notation, we introduce

$$f(x)\, e^{-cx} = g(x), \qquad F(c + i\tau) = G(\tau)$$

Then the Laplace transformation

$$F(c+i\tau) = \int_0^\infty f(t)\, e^{-(c+i\tau)t}\, dt$$

and its inverse (7.2.2) can be written as

$$G(\tau) = \int_0^{\infty} g(t)\, e^{-i\tau t}\, dt$$

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\tau)\, e^{ix\tau}\, d\tau \qquad (8.1.1)$$

Now suppose that the original function $f(t)$ and, hence, the function $g(t)$ vanish or have negligibly small values everywhere outside the finite interval $[0, T]$. Expand $g(t)$ in a Fourier series on $[0, T]$ and write the expansion in complex form:

$$g(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t} \qquad (8.1.2)$$

where $\omega = 2\pi/T$ and

$$c_k = \frac{1}{T} \int_0^{T} g(t)\, e^{-ik\omega t}\, dt \qquad (8.1.3)$$

Since $g(t)$ has a vanishingly small value outside $[0, T]$, we can assume approximately that

$$c_k \approx \frac{1}{T} \int_0^{\infty} g(t)\, e^{-ik\omega t}\, dt = \frac{1}{T} G(k\omega) \qquad (8.1.4)$$

The error of this equation has the value

$$\frac{1}{T} G(k\omega) - c_k = \frac{1}{T} \int_T^{\infty} g(t)\, e^{-ik\omega t}\, dt$$

and can be estimated by the following inequality:

$$\left| \frac{1}{T} G(k\omega) - c_k \right| \leqslant \frac{1}{T} \int_T^{\infty} |g(t)|\, dt \qquad (8.1.5)$$

Take (8.1.2); if instead of c_k we introduce their approximate values (8.1.4), we get the following expression of $g(t)$ in terms of the values of the image F at equidistant points $c + ik\omega$ ($k = 0, \pm 1, \pm 2, \ldots$):

$$f(t)\, e^{-ct} = g(t) \approx \frac{\omega}{2\pi} \sum_{k=-\infty}^{\infty} F(c + ik\omega)\, e^{ik\omega t} \qquad (8.1.6)$$

When using this equation, it is important that the function $F(c + ik\omega)$ fall off sufficiently rapidly as the absolute value of k increases without bound. In certain cases this can be attained via a preliminary preparation of the image $F(p)$ and an acceleration of approach of $F(p)$ to zero as $p \to \infty$. This preparation is discussed in Part Three of this book.

8.2 *The case of rapid decrease of the modulus of the image function $F(p)$*

We consider the function $G(\tau) = F(c + i\tau)$ to be absolutely integrable on the axis $-\infty < \tau < \infty$ and negligibly small outside the finite interval $[-T \leqslant \tau \leqslant T]$. As to its behaviour on the interval $[-T, T]$, we find that $G(\tau)$ is a regular analytic function with values equal to zero to acceptable accuracy at the endpoints of the interval. It can be expanded in a Fourier series, which we write down in complex form:

$$G(\tau) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\Omega t}, \quad \Omega = \frac{\pi}{T} \qquad (8.2.1)$$

$$c_m = \frac{1}{2T} \int_{-T}^{T} G(t)\, e^{im\Omega t}\, dt \qquad (8.2.2)$$

Since outside $[-T, T]$ the function $G(\tau)$ is regarded as negligibly small, the following equation holds true within the accepted accuracy:

$$g(x) \approx \frac{1}{2\pi} \int_{-T}^{T} G(\tau) e^{ix\tau} d\tau \quad (-T \leqslant x \leqslant T) \qquad (8.2.3)$$

The error can be roughly estimated with the aid of the inequality

$$\left| g(x) - \frac{1}{2\pi} \int_{-T}^{T} G(\tau) e^{ix\tau} d\tau \right|$$

$$= \frac{1}{2\pi} \left| \int_{-\infty}^{-T} G(\tau) e^{ix\tau} d\tau + \int_{T}^{\infty} G(\tau) e^{ix\tau} d\tau \right|$$

$$\leqslant \frac{1}{2\pi} \int_{T}^{\infty} [\, |G(\tau)| + |G(-\tau)| \,] \, d\tau$$

In the integral (8.2.3) we introduce the expansion (8.2.1) instead of $G(\tau)$ and perform term-by-term integration of the series [1]. If we take into account the equation

$$\int_{-T}^{T} e^{i(x-k\Omega)\tau} d\tau = \frac{1}{i(x-k\Omega)} [e^{i(xT-k\pi)} - e^{-i(xT-k\pi)}]$$

$$= (-1)^k \, 2T \frac{\sin xT}{xT - k\pi}, \quad x \neq k\Omega \qquad (8.2.4)$$

---

[1] In the inversion problem, the function $G(\tau) = F(c + i\tau)$ will be analytic on the interval $-T \leqslant \tau \leqslant T$, but can assume distinct values at the endpoints. The Fourier series will converge uniformly to $G(\tau)$ with respect to τ on an interval of the type $[-T + \delta, T - \delta]$ for any positive δ and will converge without bound to $G(\tau)$ on the semi-intervals $(-T, -T + \delta]$, $[T - \delta, T)$. Termwise integration of the series is possible.

then for the original function $f(x)$ we get the following approximate expression:

$$f(x)\,e^{-cx}=g(x)\approx\frac{T}{\pi}\sum_{k=-\infty}^{\infty}c_k\,(-1)^k\,\frac{\sin xT}{xT-k\pi} \tag{8.2.5}$$

$$x\neq k\frac{\pi}{T},\quad -T<x<T$$

When $x=m\frac{\pi}{T}=m\Omega$, then, as follows immediately from (8.2.2) and (8.2.3), we get the following value for the function $f(x)$:

$$f(m\Omega)\,e^{-cm\Omega}\approx\frac{T}{\pi}c_m \tag{8.2.6}$$

Applying (8.2.5) to computing the original function requires computation of the Fourier coefficients (8.2.2) of the function $F(c+i\tau)$. Methods of such computations are discussed in the literature (see [11], [19]).

Chapter 9

Interpolation Formulas for Computing Fourier Integrals

9.1 *Some preliminary remarks*

In order to carry out the Fourier transformation numerically, that is, to compute the integrals

$$\varphi_c(u) = \int_0^\infty f(t)\cos ut\,dt \qquad (9.1.1)$$

$$\varphi_s(u) = \int_0^\infty f(t)\sin ut\,dt \qquad (9.1.2)$$

$$\varphi(u) = \int_{-\infty}^\infty f(t)\,e^{iut}\,dt \qquad (9.1.3)$$

use can be made of many familiar classical rules of integration, such as, for instance, the trapezoidal, parabolic, and other rules based on the formulas of Cotes, Gauss, and so on.

We will not dwell on these rules for two reasons: first, these formulas and their errors are well known and have been thoroughly studied; second, they all have an essential drawback, to which it is worth calling attention in order to explain why one should not confine himself to these rules but should construct other formulas that are more convenient for computing Fourier integrals.

The formulas that we have just been speaking about are obtained by replacing the integrand over the entire interval of integration or any of its parts by an algebraic polynomial of low degree. For this reason, they will most likely produce good accuracy if the function being integrated is sufficiently smooth and does not vary rapidly.

The function to be integrated in (9.1.1) is the product $f(t) \cos ut$. If the parameter u is a large number, the function $\cos ut$ will oscillate rapidly and so as to more or less precisely follow the variations of the product $f(t) \cos ut$ and obtain the value of the integral with admissible error, in the quadrature formula one has to take a large number of points even for a slowly varying function $f(t)$. This can make the computations difficult or even impossible of solution. Something quite similar may be said of the integrals (9.1.2) and (9.1.3).

The use of general quadrature formulas may be only of restricted value: they may, practically speaking, be useful for computing the integrals (9.1.1), (9.1.2), (9.1.3) only for small values of u.

To set up rules of computation that suit variations of u over wide ranges, it is necessary beforehand to take into account the factors $\cos ut$, $\sin ut$, e^{iut}. One way this can be done is by taking such factors for the weight functions.

Besides, since u can assume many values, some of which cannot be foreseen, it is advisable to set up the computation rules so that they contain the parameter u literally and are convenient in computations for arbitrary (even large) values of u.

We will assume that the function $f(t)$ tends to zero so fast, as $|t| \to \infty$, that the convergence of the integrals $\int_0^\infty |f(x)|\,dx$ or $\int_{-\infty}^\infty |f(x)|\,dx$ is ensured. For the sake of definiteness we will assume that for large values of $|t|$ the inequality

$$|f(t)| \leqslant A|t|^{-1-\varepsilon}, \quad \varepsilon > 0 \qquad (9.1.4)$$

holds.

9.2 *Algebraic interpolation of the function $f(x)$*

Algebraic interpolation, that is, interpolation with the aid of integral algebraic polynomials, is satisfactorily used for approximating continuous and sufficiently smooth functions on finite intervals. If the interval, over which a function f has to be approximated, is a half axis of the whole axis and the function is absolutely integrable there or even satisfies the contidion (9.1.4), then algebraic interpolation cannot yield a good approximation. Then the interpolation must be based on a different system of functions, for instance a system of rational functions that are continuous on the axis of the half axis and tend to zero with unbounded growth of $|t|$. We will discuss that type of interpolation a bit later. As for using algebraic interpolation, we are forced to partition the half axis or the axis into an infinite number of finite intervals and resort to piecewise interpolation, and not even necessarily "spliced" interpolation (now called *spline interpolation* [1]).

9.2a. Auxiliary formulas. We begin by obtaining simple auxiliary formulas that will serve to compute integrals of functions containing trigonometric factors.

Let $[a, b]$ be an arbitrary finite interval and $l(x)$ an algebraic polynomial of degree n. Using n-fold integration by

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[1] That is the term used for algebraic interpolation when a specific interpolating polynomial is built on each interval, and the polynomials are chosen so that at the points of contact of two adjacent intervals the corresponding polynomials and their derivatives up to a certain order have the same values.

parts it is easy to obtain the following equation:

$$\int_a^b l(x)\, e^{ipx}\, dx$$

$$= e^{ipb}\left[-\frac{il(b)}{p}+\frac{l'(b)}{p^2}+\frac{il''(b)}{p^3}-\frac{l'''(b)}{p^4}-\dots\right]$$

$$-e^{ipa}\left[-\frac{il(a)}{p}+\frac{l'(a)}{p^2}+\frac{il''(a)}{p^3}-\frac{l'''(a)}{p^4}-\dots\right]$$

$$= e^{ip\frac{b+a}{2}}\left\{\left[-\frac{il(b)}{p}+\frac{l'(b)}{p^2}+\frac{il''(b)}{p^3}-\frac{l'''(b)}{p^4}-\dots\right]e^{ip\frac{b-a}{2}}\right.$$

$$\left.-\left[-\frac{il(a)}{p}+\frac{l'(a)}{p^2}+\frac{il''(a)}{p^3}-\frac{l'''(b)}{p^4}-\dots\right]e^{-ip\frac{b-a}{2}}\right\} \tag{9.2.1}$$

If we consider $l(x)$ to be a real polynomial, and if we replace the exponential functions by their Euler expressions in terms of trigonometric functions and then compare the real and imaginary parts, we get useful formulas for computing the integrals containing trigonometric factors:

$$\int_a^b l(x) \cos px\, dx$$

$$= \cos p\,\frac{a+b}{2}\left\{\left[\frac{l(b)+l(a)}{p}-\frac{l''(b)+l''(a)}{p^3}+\dots\right]\sin p\,\frac{b-a}{2}\right.$$

$$\left.+\left[\frac{l'(b)-l'(a)}{p^2}-\frac{l'''(b)-l'''(a)}{p^4}+\dots\right]\cos p\,\frac{b-a}{2}\right\}$$

$$+\sin p\,\frac{b+a}{2}\left\{\left[\frac{l(b)-l(a)}{p}-\frac{l''(b)-l''(a)}{p^3}+\dots\right]\cos p\,\frac{b-a}{2}\right.$$

$$\left.-\left[\frac{l'(b)+l'(a)}{p^2}-\frac{l'''(b)+l'''(a)}{p^4}+\dots\right]\sin p\,\frac{b-a}{2}\right\} \tag{9.2.2}$$

$$\int_a^b l(x)\sin px\,dx$$

$$=\sin p\frac{b+a}{2}\left\{\left[\frac{l(b)+l(a)}{p}-\frac{l''(b)+l''(a)}{p^3}+\dots\right]\sin p\frac{b-a}{2}\right.$$

$$\left.+\left[\frac{l'(b)-l'(a)}{p^2}-\frac{l'''(b)-l'''(a)}{p^4}+\dots\right]\cos p\frac{b-a}{2}\right\}$$

$$-\cos p\frac{b+a}{2}\left\{\left[\frac{l(b)-l(a)}{p}-\frac{l''(b)-l''(a)}{p^3}+\dots\right]\cos p\frac{b-a}{2}\right.$$

$$\left.-\left[\frac{l'(b)+l'(a)}{p^2}-\frac{l'''(b)+l'''(a)}{p^4}+\dots\right]\sin p\frac{b-a}{2}\right\} \qquad (9.2.3)$$

The equations (9.2.2) and (9.2.3) were obtained on the assumption that $l(x)$ is an arbitrary polynomial with real coefficients. But since the equations hold for arbitrary real coefficients of the polynomial $l(x)$, they will also hold for complex coefficients and, hence, for any complex $l(x)$.

9.2b. Constructing formulas for computations. The idea behind the construction of interpolation rules is simple in the extreme. For the sake of definiteness, let us consider the cosine transformation (9.1.1). We divide the half axis of integration $[0, \infty)$ into finite intervals by the points $0 = a_0 < a_1 < \dots < a_k < \dots$. Take one of the intervals $[a_k, a_{k+1}]$ and on it interpolate the function f, assuming it to be sufficiently smooth, with the aid of an algebraic polynomial. Let us for instance choose interpolation with respect to the values of the function. On $[a_k, a_{k+1}]$ choose n_k+1 arbitrarily spaced points x_j^k ($j = 0, 1, \dots, n_k$; $a_k \leqslant x_0^k < \dots < x_{n_k}^k \leqslant a_{k+1}$) and perform the interpolation with respect to the values $f(x_j^k) = f_j^{(k)}$ by means of a polynomial

$P_k(x)$ of degree n_k:

$$P_k(x) = \sum_{j=0}^{n_k} \frac{\omega_k(x)}{(x - x_j^k)\,\omega_k'(x_j^k)}\, f_j^{(k)} = \sum_{j=0}^{n_k} l_j^{(k)}(x)\, f_j^{(k)},$$

$$\omega_k(x) = (x - x_0^k) \ldots (x - x_{n_k}^k),$$

$$f(x) = P_k(x) + r_k(x) \qquad (9.2.4)$$

Now consider an integral of the form (9.1.1), taking it over the interval $[a_k, a_{k+1}]$ instead of over the half axis $[0, \infty)$, and replace the function $f(x)$ on the interval by the interpolating polynomial $P_k(x)$. This gives us the approximate equation

$$\int_{a_k}^{a_{k+1}} f(t) \cos ut\, dt \approx \int_{a_k}^{a_{k+1}} P_k(t) \cos ut\, dt$$

$$= \sum_{j=0}^{n_k} f_j^{(k)} \int_{a_k}^{a_{k+1}} l_j^{(k)}(t) \cos ut\, dt$$

Summing such equations over all subintervals, we construct an approximate expression for the Fourier cosine transform:

$$\varphi_c(u) = \int_0^{\infty} f(t) \cos ut\, dt$$

$$\approx \sum_{k=0}^{\infty} \sum_{j=0}^{n_k} f_j^{(k)} \int_{a_k}^{a_{k+1}} l_j^{(k)}(t) \cos ut\, dt \qquad (9.2.5)$$

Each of the integrals under the sign of double summation can be computed, for example, from the formula (9.2.2).

If we denote by $r_k(x) = f(x) - P_k(x)$ the error of interpolation on the interval $[a_k, a_{k+1}]$, then the error of the equation (9.2.5) will be

$$R_c(u) = \sum_{k=0}^{\infty} \int_{a_k}^{a_{k+1}} r_k(t) \cos ut \, dt \tag{9.2.6}$$

At least in certain cases this representation of $R_c(u)$ can be used to obtain an error estimate of the approximate representation (9.2.5) for $\varphi_c(u)$. For example, from (9.2.6) there quite obviously follows the following estimate $R_c(u)$ uniform with respect to u:

$$|R_c(u)| \leqslant \sum_{k=0}^{\infty} \int_{a_k}^{a_{k+1}} |r_k(t)| \, dt \tag{9.2.7}$$

In this notation, we assume that the right side of the inequality has a finite value. The value of the right side depends on the choice of the points a_k, the numbers n_k, the interpolation points $x_j^{(k)}$, and the properties of the function f, in particular, on whether it has derivatives of sufficiently high order, and on the rate of their approach to zero as t increases without bound.

The investigation of the approximate representation (9.2.5) for $\varphi_c(u)$ in the general form is complicated and is mainly of theoretical value. We confine ourselves to considering this representation in a few simple and special cases.

Suppose the half axis $[0, \infty)$ is partitioned by the points $x_k = kh$ $(h > 0,\ k = 0, 1, 2, \ldots)$ into equal intervals of length h. We also assume known the values of the function f at the division points: $f(x_k) = f(kh) = f_k$.

I. ***Computation rules based on linear interpolation.*** We first consider an analog of the trapezoidal rule. Take an interval $[kh\ (k+1)h]$ and perform linear interpolation of the function f with respect to the two values at the endpoints of the interval:

$$f(x)=\frac{x-(k+1)h}{-h}f_k+\frac{x-kh}{h}f_{k+1}+r_k(x, f) \quad (9.2.8)$$

If f has a continuous second derivative, the error of interpolation $r_k(x, f)$ can, as we know, be represented as [1]

$$r_k(x, f)=h^2\int_0^1 f''(kh+h\tau)\,[(\xi-\tau)E(\xi-\tau)-\xi(1-\tau)]\,d\tau$$

$$x=x_k+h\xi=h(k+\xi) \quad (9.2.9)$$

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[1] This representation can be obtained if we take advantage of Taylor's formula with remainder in the form of the integral

$$f(x)=f_k+(x-x_k)f'_k+\int_{x_k}^{x} f''(t)(x-t)\,dt=f_k+(x-x_k)f'_k+\varphi(x)$$

Since the linear function $f_k+(x-x_k)f'_k$ is interpolated exactly, the errors of interpolation of f and φ coincide. But

$$r_k(x, \varphi)=\varphi(x)+\frac{x-x_{k+1}}{h}\varphi(x_k)-\frac{x-x_k}{h}\varphi(x_{k+1})$$

and since $\varphi(x_k)=0$, it follows that

$$r_k(x, f)=r_k(x, \varphi)=\int_{x_k}^{x} f''(t)(x-t)\,dt-\int_{x_k}^{x_{k+1}} f''(t)\,\frac{x-x_k}{h}\,(x_{k+1}-t)\,dt$$

$$=\int_{x_k}^{x_{k+1}} f''(t)\left[(x-t)E(x-t)-\frac{1}{h}(x-x_k)(x_k+h-t)\right]dt$$

Now it remains to replace the variables x and t by putting $x=x_k+h\xi$, $t=x_k+h\tau$ $(0\leqslant\xi, \tau\leqslant 1)$ to obtain the equation (9.2.9).

where $E(x)$ is the "quenching function". This function serves to eliminate extra intervals of integration and is defined by

$$E(x) = \begin{cases} 1, & x > 0 \\ 1/2, & x = 0 \\ 0, & x < 0 \end{cases}$$

Multiply both sides of (9.2.8) by $\cos ux$ and integrate over the interval $[kh, (k+1)h]$. By means of integration by parts or on the basis of (9.2.2), it is easy to compute the integral with the interpolating polynomial:

$$\int_{kh}^{(k+1)h} \left[-\frac{x-(k+1)h}{h} f_k + \frac{x-kh}{h} f_{k+1} \right] \cos ux\, dx$$

$$= \frac{1}{u} f_{k+1} \sin u(k+1)h - \frac{1}{u} f_k \sin ukh$$

$$+ \frac{1}{u^2 h} (f_{k+1} - f_k) [\cos u(k+1)h - \cos ukh]$$

It can be seen that when summing over all intervals $[kh, (k+1)h]$ (if we take into account that $f_k \to 0$ as $k \to \infty$) the terms involving sines vanish. And the terms involving cosines yield summands outside the integral for the cosine transformation.

Finally, if by putting $x = kh + h\xi$ we introduce the variable ξ into the integral with error $r_k(x, f)$ instead of the integration variable x, then we get an expression for the error in the form of a double integral, and after summing over the intervals $[kh, (k+1)h]$ $(k = 0, 1, \ldots)$ we get the following exact representation [1] for $\varphi_c(u)$:

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[1] In this notation we considered the function $f(x)$ and its second derivative $f''(x)$ as decreasing so rapidly, as x goes to infinity, that the series participating in the representation should converge absolutely and uniformly. The same is assumed also for other representations in Sec. 9.2.

$$\varphi_c(u) = \int_0^\infty f(t) \cos ut \, dt$$

$$= \frac{1-\cos uh}{u^2 h} f_0 + 2 \frac{1-\cos uh}{u^2 h} \sum_{k=1}^{\infty} f_k \cos ukh + R_c(u) \qquad (9.2.10)$$

where

$$R_c(u) = h^3 \int_0^1 d\xi \int_0^1 d\tau \, [(\xi-\tau) E(\xi-\tau) - \xi(1-\tau)]$$

$$\times \sum_{k=0}^{\infty} f''(kh+h\tau) \cos u(kh+h\xi)$$

If we discard the remainder $R_c(u)$, we get a formula for the approximate cosine transformation with respect to values of the original function f at equidistant points.

The foregoing expression for the remainder R_c of the formula permits obtaining its estimates and indicating the order of smallness relative to h for certain assumptions concerning the function f. We now obtain the simplest of these estimates. First note that the kernel of the double integral in square brackets has the values

$$(\xi-\tau) E(\xi-\tau) - \xi(1-\tau) = \begin{cases} -\tau(1-\xi) & \text{for} \quad \tau < \xi \\ -\xi(1-\tau) & \text{for} \quad \tau > \xi \end{cases}$$

It is negative in the region of integration $0 < \xi, \tau < 1$. The double integral of the kernel is readily computed:

$$\int_0^1 d\xi \int_0^1 [(\xi-\tau) E(\xi-\tau) - \xi(1-\tau)] \, d\tau = -\frac{1}{12}$$

This permits estimating R_c in the following manner:

$$|R_c| \leqslant h^3 \int_0^1 d\xi \int_0^1 [\xi(1-\tau)-(\xi-\tau)E(\xi-\tau)] \sum_{k=0}^{\infty} |f''(kh+h\tau)|\, d\tau$$

$$= \frac{h^3}{12} \sum_{k=0}^{\infty} |f''(kh+h\theta)|, \quad 0<\theta<1 \quad (9.2.11)$$

The estimate of the sum depends on the properties of the second derivative f''. Suppose that for f'' the inequality

$$|f''(x)| \leqslant \frac{B}{(a+x)^{\alpha}}, \quad \alpha>1, \; a>0$$

holds. Then

$$\sum_{k=0}^{\infty} |f''(kh+h\tau)| \leqslant B \sum_{k=0}^{\infty} \frac{1}{(a+kh+h\tau)^{\alpha}} < B \sum_{k=0}^{\infty} \frac{1}{(a+kh)^{\alpha}}$$

If h and a have been chosen, the value of the last sum can be found by computation, and in certain cases with the aid of tabulated functions. This estimate can be replaced by a simpler but less exact one.

The inequalities given below are obvious and may be obtained if t is replaced in the left-hand integral by its least value and in the right-hand one by its largest value:

$$\int_k^{k+1} \frac{dt}{(a+ht)^{\alpha}} < \frac{1}{(a+kh)^{\alpha}} < \int_{k-1}^{k} \frac{dt}{(a+ht)^{\alpha}}$$

Then from this, if we sum over the values of k from 1 to ∞ and then add

$$\int_0^1 \frac{dt}{(a+ht)^{\alpha}} < \frac{1}{a^{\alpha}}$$

to the left sides of the resulting inequalities and $\frac{1}{a^\alpha}$ to the middle and last parts, we find

$$\int_0^\infty \frac{dt}{(a+ht)^\alpha}=\frac{1}{(\alpha-1)\,ha^{\alpha-1}}<\sum_{k=0}^{\infty}\frac{1}{(a+kh)^\alpha}<\frac{1}{a^\alpha}+\frac{1}{(\alpha-1)\,ha^{\alpha-1}}$$

The thus constructed inequalities show that the sum at hand is of the order of $1/h$ for small h. For R_c, from the inequalities thus found, there follows the estimate

$$|R_c|\leqslant\frac{h^2}{12}\left[\frac{1}{(\alpha-1)\,a^{\alpha-1}}+\frac{h}{a^\alpha}\right] \tag{9.2.12}$$

The representation (9.2.11) for R_c can serve as a source of estimates of a different type, which can also turn out to be useful in certain cases. From the foregoing expression for the kernel of a double integral it is evident that it assumes values not exceeding 1/2 in absolute value. And so for R_c the following inequality holds:

$$|R_c(u)|\leqslant\frac{1}{2}h^3\int_0^1 d\xi\int_0^1\sum_{k=0}^{\infty}|f''(kh+h\tau)|\,d\tau$$

$$=\frac{1}{2}h^3\int_0^1\sum_{k=0}^{\infty}|f''(kh+h\tau)|\,d\tau$$

But

$$\int_0^1|f''(kh+h\tau)|\,d\tau=\frac{1}{h}\int_{kh}^{(k+1)h}|f''(t)|\,dt=\frac{1}{h}\operatorname*{Var}_{kh\leqslant t\leqslant(k+1)h}f'(t)$$

and since

$$\sum_{k=0}^{\infty}\operatorname*{Var}_{kh\leqslant t\leqslant(k+1)h}f'(t)=\operatorname*{Var}_{0\leqslant t<\infty}f'(t)$$

for $R_c(u)$ we get an estimate of the form

$$|R_c(u)| \leqslant \frac{1}{2} h^2 \underset{0 \leqslant t < \infty}{\mathrm{Var}} f'(t) \qquad (9.2.13)$$

Its use is particularly simple when $f'(t)$ is a monotonic or piecewise monotonic function with simply determinable intervals of monotonicity.

Using a similar approach, we can obtain rules based on the linear interpolation of f for approximate Fourier sine and complex transformations. We will simply state the rules and refrain from all reasoning involved in their derivation.

For the sine transform (9.1.2), the following representation in terms of values of the function f at the points $x_k = kh$ $(k = 0, 1, \ldots)$ holds true:

$$\varphi_s(u) = \frac{1}{u}\left(1 - \frac{1}{uh}\sin uh\right) f_0 + 2\,\frac{1-\cos uh}{u^2 h}\sum_{k=1}^{\infty} f_k \sin ukh + R_s(u) \qquad (9.2.14)$$

$$R_s(u) = h^3 \int_0^1 d\xi \int_0^1 d\tau\,[(\xi-\tau)\,E(\xi-\tau) - \xi(1-\tau)] \sum_{k=0}^{\infty} f''(kh+h\tau)\sin u(kh+h\xi)$$

The rule of approximation is obtained if in (9.2.14) we drop the remainder term $R_s(u)$. It is the error of the method.

The remainder R_s is, in aspect, similar to R_c in (9.2.10) and differs from it in that in its representation the function $\cos u(kh + h\xi)$ is replaced by $\sin u(kh + h\xi)$. When obtaining the estimates of R_c, the absolute value of $\cos u(kh + h\xi)$ was replaced by unity. The same may be done with the function $\sin u(kh + h\xi)$ when estimating R_s. And so for

$R_s(u)$ the estimate inequalities (9.2.11), (9.2.12), (9.2.13) hold true. What we have is this.

If the function f has a continuous second derivative on $[0, \infty)$ that decreases sufficiently rapidly as $x \to \infty$, then

$$|R_s(u)| \leqslant \frac{h^3}{12} \sum_{k=0}^{\infty} |f''(kh+h\theta)|, \quad 0<\theta<1 \quad (9.2.15)$$

$$|R_s(u)| \leqslant \frac{1}{2} h^2 \operatorname*{Var}_{0\leqslant t<\infty} f'(t) \quad (9.2.16)$$

If the inequality

$$|f''(x)| \leqslant B(a+x)^{-\alpha} \quad (\alpha>1,\ a>0)$$

holds for the second derivative, then

$$|R_s(u)| \leqslant \frac{1}{12} h^2 \left[\frac{1}{(\alpha-1)\,a^{\alpha-1}} + \frac{h}{a^{\alpha}}\right] \quad (9.2.17)$$

Finally, for the complex transformation (9.2.3) we have a representation in terms of the values of f at the equally spaced points $x_k = kh$ $(k = 0, \pm 1, \pm 2, \ldots)$:

$$\varphi(u) = \int_{-\infty}^{\infty} f(t)\, e^{ikt}\, dt$$

$$= 2\,\frac{1-\cos uh}{u^2 h} \sum_{k=-\infty}^{\infty} f_k e^{-iukh} + R(u) \quad (9.2.18)$$

$$R(u) = h^3 \int_0^1 d\xi \int_0^1 d\tau\, [(\xi-\tau)\, E(\xi-\tau) - \xi(1-\tau)] \times$$

$$\times \sum_{k=-\infty}^{\infty} f''(kh+h\tau)\, e^{-iu(kh+h\xi)}$$

The rule of computation is obtained if in (9.2.18) we drop the remainder term $R(u)$. The latter has the meaning of the error of the rule and for it, from the indicated representation, there follow the estimates

$$|R(u)| \leqslant \frac{1}{12} h^3 \sum_{k=-\infty}^{\infty} |f''(kh + h\theta)|, \quad 0 < \theta < 1 \quad (9.2.19)$$

$$|R(u)| \leqslant \frac{1}{2} h^2 \operatorname*{Var}_{-\infty < t < \infty} f'(t) \quad (9.2.20)$$

If the inequality

$$|f''(x)| \leqslant B(a + |x|)^{-\alpha} \quad (\alpha > 1, \; a > 0, \; -\infty < x < \infty)$$

holds for the second derivative of $f(x)$, then

$$|R(u)| \leqslant \frac{1}{6} h^2 \left[\frac{1}{(\alpha - 1) a^{\alpha - 1}} + \frac{h}{a^{\alpha}} \right] \quad (9.2.21)$$

II. ***Second-degree (quadratic) interpolation.*** Take the interval $[kh, (k+2)h]$ of length $2h$ and interpolate the function f with respect to its values f_k, f_{k+1}, f_{k+2} at the points kh, $(k+1)h$, $(k+2)h$ with a polynomial of degree two:

$$f(x) = \frac{(x - x_{k+1})(x - x_{k+2})}{(-h)(-2h)} f_k + \frac{(x - x_k)(x - x_{k+2})}{(-h)h} f_{k+1} + \frac{(x - x_k)(x - x_{k+1})}{h \cdot 2h} f_{k+2} + r_k(x) \quad (9.2.22)$$

To obtain the error $r_k(x)$ in the necessary form, take advantage of Taylor's formula with remainder in the form of an integral, using the quenching function E:

$$f(x) = f_k + (x - x_k) f'_k + \frac{1}{2}(x - x_k)^2 f''_k + \frac{1}{2} \int_{x_k}^{x} f'''(t)(x - t)^2 \, dt$$

$$= P_2(x) + \frac{1}{2} \int_{x_k}^{x_{k+2}} f'''(t)(x - t)^2 E(x - t) \, dt = P_2(x) + \psi(x)$$

Since the polynomial $P_2(x)$ is interpolated exactly, the interpolation errors of $f(x)$ and $\psi(x)$ coincide. Now the integral expression of $\psi(x)$ permits reducing this error to the interpolation error of the elementary function $(x-t)^2 E(x-t)$:

$$r_h(x)=\frac{1}{2}\int_{x_k}^{x_{k+2}} f'''(t)\Big[(x-t)^2 E(x-t)$$
$$+\frac{(x-x_k)(x-x_{k+2})}{h^2}(x_{k+1}-t)^2 E(x_{k+1}-t)$$
$$-\frac{1}{2}\frac{(x-x_k)(x-x_{k+1})}{h^2}(x_{k+2}-t)^2\Big]dt$$

The first term in the interpolation polynomial that corresponds to the point x_k,

$$\frac{(x-x_{k+1})(x-x_{k+2})}{(-h)(-2h)}(x_k-t)^2 E(x_k-t),$$

is dropped under the integral sign since $E(x_k-t)=0$ for $x_k<t\leqslant x_{k+2}$. The factor $E(x_{k+2}-t)$ is not indicated in the last term in square brackets since it is equal to unity for all values of $t<x_{k+2}$.

Let us simplify the representation of the error $r_k(x)$ by introducing the variables ξ, τ and putting $x=x_k+h\xi$, $t=x_k+h\tau$ $(0\leqslant\xi,\ \tau\leqslant 2)$:

$$r_k(x)=\frac{h^3}{2}\int_0^2 f'''(x_k+h\tau)\Big[(\xi-\tau)^2 E(\xi-\tau)$$
$$+\xi(\xi-2)(1-\tau)^2 E(1-\tau)-\frac{1}{2}\xi(\xi-1)(2-\tau)^2\Big]dt \quad (9.2.23)$$

Multiply both sides of (9.2.22) by $\cos ux$, integrate over the interval $[x_k, x_{k+2}]$, and sum the result term by term over the even values of k $(k=0, 2, 4, \ldots)$. We then get the following exact representation of the Fourier cosine transform in terms of the values of the functions f_k ($k = 0, 1, \ldots$):

$$\left.\begin{aligned} \varphi_c(u) &= \int_0^\infty f(x)\cos ux\,dx \\ &= \alpha_2 f_0 + \gamma_2 \sum_{k=0}^{\infty} f_{2k+1}\cos(2k+1)\theta \\ &\qquad + 2\alpha_2 \sum_{k=1}^{\infty} f_{2k}\cos 2k\theta + R_c, \\ \theta &= uh,\ h^{-1}\alpha_2 = \frac{3}{2}\theta^{-2} + \frac{1}{2}\theta^{-2}\cos 2\theta - \theta^{-3}\sin 2\theta, \\ h^{-1}\gamma_2 &= 4\theta^{-2}[\theta^{-1}\sin\theta - \cos\theta], \end{aligned}\right\} \quad (9.2.24)$$

$$R_c = \int_{x_k}^{x_{k+2}} r_k(x)\,dx = \frac{h^4}{2}\int_0^2 d\xi \int_0^2 d\tau \Big[(\xi-\tau)^2 E(\xi-\tau) + \xi(\xi-2)(1-\tau)^2 E(1-\tau) - \frac{1}{2}\xi(\xi-1)(2-\tau)^2\Big] \sum_{k=0}^{\infty} f'''(x_{2k}+h\tau)\cos u(x_{2k}+h\xi) \quad (9.2.25)$$

To carry out the estimate R_c, it is first necessary to learn about certain properties of the kernel (of the double integral) in square brackets. The region of integration is the square $0 \leqslant \xi, \tau \leqslant 2$ and for our purposes it is convenient to divide it into 6 portions by the straight lines $\xi = \tau$, $\tau = 1$, $\xi = 1$. These portions are appropriately numbered in Fig. 1. The signs of the kernel on the boundaries of the portions are readily determined from the indicated expression of the kernel (9.2.25) and they too are labelled in the figure. The signs and estimates of the kernel inside the portions are investigated below.

Portion I. It is defined by the inequalities $0 \leqslant \tau \leqslant \xi \leqslant 1$. The kernel, which we denote by $K(\xi, \tau) = K$, here has the value of

$$K = (\xi-\tau)^2+\xi(\xi-2)(1-\tau)^2-\frac{1}{2}\xi(\xi-1)(2-\tau)^2$$

$$=\frac{1}{2}(\xi-1)(\xi-2)\tau^2 \quad (9.2.26)$$

It is clearly nonnegative since the factors $\xi - 1$ and $\xi - 2$

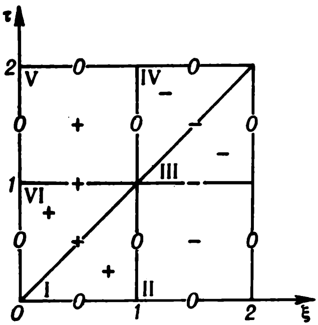

Fig. 1

are both nonpositive. Besides, since $(\xi - 1)(\xi - 2) < 2$ and $\tau^2 \leqslant 1$, the inequalities

$$0 \leqslant K(\xi, \tau) \leqslant 1$$

hold true for the kernel.

Portion II. Here $0 \leqslant \tau \leqslant 1$, $1 \leqslant \xi \leqslant 2$. The kernel K also has the expression (9.2.26); the difference is that the factor $\xi - 1$ assumes nonnegative values and the kernel K is hence nonpositive. Since $\tau^2 \leqslant 1$, $2 - \xi \leqslant 1$ and $\xi - 1 \leqslant 1$, we have for the kernel the estimate

$$0 \geqslant K(\xi, \tau) \geqslant -1$$

Portion III. Here $1 \leqslant \tau \leqslant \xi \leqslant 2$. The kernel has the value

$$K(\xi, \tau) = (\xi - \tau)^2 - \frac{1}{2}\xi(\xi - 1)(2 - \tau)^2$$
$$= \frac{1}{2}(2 - \xi)(1 + \xi)\tau^2 + 2\xi(\xi - 2)\tau - \xi(\xi - 2)$$

For each fixed value of ξ in the half-interval $1 \leqslant \xi < 2$, the graph of the function $K(\xi, \tau)$ in the plane with axes K, τ is the parabola[1] concave towards positive values of K. On the boundary of the portion, the kernel K assumes non-positive values since for $\tau = \xi$,

$$K(\xi, \xi) = -\frac{1}{2}\xi(\xi - 1)(2 - \xi)^2 \leqslant 0$$

For $\tau = 1$,

$$K(\xi, 1) = (\xi - 1)^2 - \frac{1}{2}\xi(\xi - 1) = (\xi - 1)\left(\frac{1}{2}\xi - 1\right) \leqslant 0$$

For $\xi = 2$,

$$K(2, \tau) = (2 - \tau)^2 - (2 - \tau)^2 = 0$$

And so at all points of Portion III

$$K(\xi, \tau) \leqslant 0$$

On the other hand,

$$K(\xi, \tau) \geqslant -\frac{1}{2}\xi(\xi - 1)(2 - \tau)^2 \geqslant -\frac{1}{2} \times 2 \times 1 \times 1^2 = -1$$

and, hence, at all points of the portion

$$0 \geqslant K(\xi, \tau) \geqslant -1$$

Portion IV. Here, $1 \leqslant \xi \leqslant \tau \leqslant 2$, $K(\xi, \tau) = -\frac{1}{2}\xi(\xi - 1) \times \times (2 - \tau)^2$.

---

[1] Here and henceforth is meant a parabola with axis of symmetry parallel to the K axis.

Clearly

$$0 \geqslant K(\xi, \tau) \geqslant -1$$

Portion V. Here, $0 \leqslant \xi \leqslant 1 \leqslant \tau \leqslant 2$, $K(\xi, \tau) = -\frac{1}{2}\xi(\xi-1)(2-\tau)^2$.

The kernel has the same expression as for Portion IV with the difference that here the factor $\xi - 1$ is negative. The kernel $K(\xi, \tau)$ will be nonnegative and will satisfy the inequalities

$$0 \leqslant K(\xi, \tau) \leqslant 1$$

Portion VI. Here, $0 \leqslant \xi \leqslant \tau \leqslant 1$,

$$K(\xi, \tau) = \xi(\xi-2)(1-\tau)^2 - \frac{1}{2}\xi(\xi-1)(2-\tau)^2$$

$$= \xi\left[-\frac{1}{2}(3-\xi)\tau^2 + 2\tau - \xi\right]$$

For each fixed value of ξ the graph of the kernel in the $K\tau$-plane is a parabola convex towards positive values of K.

On the boundary of the portion, K is everywhere nonnegative since

$$K(\xi, \xi) = \frac{1}{2}(1-\xi)(2-\xi)\xi^2 \geqslant 0 \quad \text{for } \xi = \tau$$

$$K(0, \tau) = 0 \quad \text{for } \xi = 0$$

$$K(\xi, 1) = \frac{1}{2}\xi(1-\xi) \geqslant 0 \quad \text{for } \tau = 1$$

And so at all points of the portion the kernel is nonnegative. Finally, in the above-indicated expression of the kernel, the term $\xi(\xi - 2)(1 - \tau)^2$ is nonpositive and hence

$$0 \leqslant K(\xi, \tau) \leqslant \frac{1}{2}\xi(1-\xi)(2-\tau)^2 \leqslant 1$$

From this consideration of the kernel on all portions follows the required inequality

$$|K(\xi, \tau)| \leqslant 1$$

It permits obtaining the estimate, given below, of the remainder term R_c:

$$|R_c(u)| \leqslant \frac{h^4}{2} \int_0^2 d\xi \int_0^2 d\tau \sum_{k=0}^{\infty} |f'''(x_{2k}+h\tau)|$$

$$= h^4 \int_0^2 d\tau \sum_{k=0}^{\infty} |f'''(x_{2k}+h\tau)| \quad (9.2.27)$$

If we note that

$$\int_0^2 d\tau \sum_{k=0}^{\infty} |f'''(x_{2k}+h\tau)| = \frac{1}{h} \int_0^{\infty} |f'''(x)|\, dx = \frac{1}{h} \operatorname*{Var}_{0\leqslant x<\infty} f''(x)$$

we then get the estimate

$$|R_c(u)| \leqslant h^3 \operatorname*{Var}_{0\leqslant x<\infty} f''(x) = h^3 \int_0^{\infty} |f'''(x)|\, dx \quad (9.2.28)$$

In this estimate, the integral may be replaced by a greater quantity obtained in the following manner:

$$\int_0^{\infty} |f'''(x)|\, dx = h \int_0^1 \sum_{k=0}^{\infty} |f'''(x_k+ht)|\, dt$$

$$\leqslant h \max_{0\leqslant t\leqslant 1} \sum_{k=0}^{\infty} |f'''(x_k+ht)|$$

and this permits replacing (9.2.28) by a rougher estimate, but one which is more convenient computationally, at least in certain cases:

$$|R_c(u)| \leqslant h^4 \max_{0\leqslant t\leqslant 1} \sum_{k=0}^{\infty} |f'''(x_k+ht)| \quad (9.2.29)$$

For the Fourier sine transform, similar formulas for computations and for error estimates will be of the form

$$\varphi_s(u)=\int_0^\infty f(x)\sin ux\,dx$$

$$=\beta_2 f_0+\gamma_2\sum_{k=0}^\infty f_{2k+1}\sin(2k+1)\theta$$

$$+2\alpha_2\sum_{k=1}^\infty f_{2k}\sin 2k\theta+R_s(u), \quad (9.2.30)$$

$$h^{-1}\beta_2=\theta^{-1}-\theta^{-3}+\frac{1}{2}\theta^{-2}\sin 2\theta+\theta^{-3}\cos 2\theta$$

[the values of the parameters θ, α_2, γ_2 are indicated in (9.2.24)],

$$R_s(u)=\frac{h^4}{2}\int_0^2 d\xi\int_0^2 d\tau\,[(\xi-\tau)^2 E(\xi-\tau)$$

$$+\xi(\xi-2)(1-\tau)^2 E(1-\tau)-\frac{1}{2}\xi(\xi-1)(2-\tau)^2]$$

$$\times\sum_{k=0}^\infty f'''(x_{2k}+h\tau)\sin u(x_{2k}+h\xi),$$

$$|R_s(u)|\leqslant h^3\int_0^\infty |f'''(x)|\,dx=h^3\underset{0\leqslant x<\infty}{\operatorname{Var}} f''(x),$$

$$|R_s(u)|\leqslant h^4\max_{0\leqslant t\leqslant 1}\sum_{k=0}^\infty |f'''(x_k+ht)|$$

Finally, for the Fourier complex transform the analogous rules and estimates are:

$$\varphi(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$$

$$= 2\alpha_2 \sum_{k=-\infty}^{\infty} f_{2k} e^{-i2k\theta} + \gamma_2 \sum_{k=-\infty}^{\infty} f_{2k+1} e^{-i(2k+1)\theta} + R(u) \quad (9.2.31)$$

$$R(u) = \frac{h^4}{2} \int_0^2 d\xi \int_0^2 d\tau \Big[(\xi-\tau)^2 E(\xi-\tau)$$

$$+ \xi(\xi-2)(1-\tau)^2 E(1-\tau)$$

$$- \frac{1}{2} \xi(\xi-1)(2-\tau)^2 \Big] \sum_{k=-\infty}^{\infty} f'''(x_{2k} + h\tau)\, e^{-i(x_{2k}+h\xi)u} \quad (9.2.32)$$

$$|R(u)| \leqslant h^3 \int_{-\infty}^{\infty} |f'''(x)|\, dx = h^3 \operatorname*{Var}_{-\infty<x<\infty} f''(x)$$

$$|R(u)| \leqslant h^4 \max_{0\leqslant\tau\leqslant 1} \sum_{k=-\infty}^{\infty} |f'''(x_k + h\tau)|$$

III. ***Interpolation of degree three.*** Above we considered rules underlying linear and quadratic interpolation. They are analogs of the Cotes trapezoidal and parabolic rules. Quite obviously, it is possible for the Fourier transformations to construct analogs of the Cotes rules of any order. The higher the degree of accuracy of such rules, the more complicated they will be, and their complexity increases rapidly with increasing degree. We give rules here corresponding to interpolation of degree three, which are analogs of the "three-eighths rule" of Newton and Cotes.

Take 4 points x_k, x_{k+1}, x_{k+2}, x_{k+3} and perform interpolation of f with respect to the values at these points:

$$f(x) = \frac{(x-x_{k+1})(x-x_{k+2})(x-x_{k+3})}{-h(-2h)(-3h)} f_k + \frac{(x-x_k)(x-x_{k+2})(x-x_{k+3})}{h(-h)(-2h)} f_{k+1} + \frac{(x-x_k)(x-x_{k+1})(x-x_{k+3})}{2h\cdot h(-h)} f_{k+2} + \frac{(x-x_k)(x-x_{k+1})(x-x_{k+2})}{3h\cdot 2h\cdot h} f_{k+3} + r_3(x)$$

Multiplication of this equation by $\cos ux$ and integration over the interval $[x_k, x_k + 3h]$ with subsequent summation over the values $k = 0, 3, 6, \ldots$ leads to a representation of the cosine transform in terms of the values $f_0, f_1 \ldots,$

$$\varphi_c(u) = \int_0^\infty f(x)\cos ux\, dx$$

$$= \alpha_3 f_0 + \sum_{k=0}^{\infty} (\gamma_3 \cos 3k\theta - \delta_3 \sin 3k\theta) f_{3k+1} + \sum_{k=1}^{\infty} (\gamma_3 \cos 3k\theta + \delta_3 \sin 3k\theta) f_{3k-1} + 2\alpha_3 \sum_{k=1}^{\infty} f_{3k} \cos 3k\theta + R_c(u) \quad (9.2.33)$$

$$R_c(u) = \frac{h^5}{6} \int_0^3 d\xi \int_0^3 d\tau \, [(\xi-\tau)^3 E(\xi-\tau) - \frac{1}{2}\xi(\xi-2)(\xi-3)(1-\tau)^3 E(1-\tau)$$

$$+\frac{1}{2}\xi(\xi-1)(\xi-3)(2-\tau)^3 E(2-\tau)$$

$$-\frac{1}{6}\xi(\xi-1)(\xi-2)(3-\tau)^3]$$

$$\times\sum_{k=0}^{\infty} f^{IV}(x_{3k}+h\tau)\cos u(x_{3k}+h\xi)$$

For the Fourier sine and complex transforms, we have the following representations:

$$\varphi_s(u)=\int_0^{\infty} f(x)\sin ux\,dx$$

$$=\beta_3 f_0+\sum_{k=0}^{\infty}(\gamma_3\sin 3k\theta+\delta_3\cos 3k\theta)f_{3k+1}$$

$$+\sum_{k=1}^{\infty}(\gamma_3\sin 3k\theta-\delta_3\cos 3k\theta)f_{3k-1}$$

$$+2\alpha_3\sum_{k=1}^{\infty} f_{3k}\sin 3k\theta+R_s(u) \quad (9.2.34)$$

$$R_s(u)=\frac{h^5}{6}\int_0^3 d\xi\int_0^3 d\tau\Big[(\xi-\tau)^3 E(\xi-\tau)$$

$$-\frac{1}{2}\xi(\xi-2)(\xi-3)(1-\tau)^3 E(1-\tau)$$

$$+\frac{1}{2}\xi(\xi-1)(\xi-3)(2-\tau)^3 E(2-\tau)$$

$$-\frac{1}{6}\xi(\xi-1)(\xi-2)(3-\tau)^3\Big]\sum_{k=0}^{\infty} f^{IV}(x_{3k}+h\tau)\sin u(x_{3k}+h\xi);$$

$$\psi(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx = (\gamma_3 - i\delta_3) \sum_{k=-\infty}^{\infty} f_{3k+1} e^{-i3k\theta}$$

$$+ (\gamma_3 + i\delta_3) \sum_{k=-\infty}^{\infty} f_{3k-1} e^{-i3k\theta}$$

$$+ 2\alpha_3 \sum_{k=-\infty}^{\infty} f_{3k} e^{-i3k\theta} + R(u) \quad (9.2.35)$$

$$R(u) = \frac{h^5}{6} \int_0^3 d\xi \int_0^3 d\tau \Big[(\xi - \tau)^3 E(\xi - \tau)$$

$$- \frac{1}{2} \xi (\xi - 2)(\xi - 3)(1 - \tau)^3 E(1 - \tau)$$

$$+ \frac{1}{2} \xi (\xi - 1)(\xi - 3)(2 - \tau)^3 E(2 - \tau)$$

$$- \frac{1}{6} \xi (\xi - 1)(\xi - 2)(3 - \tau)^3 \Big] \sum_{k=-\infty}^{\infty} f^{\mathrm{IV}}(x_k + h\tau)\, e^{-i(x_{3k} + h\xi)u}$$

In (9.2.33), (9.2.34), (9.2.35) the coefficients α_3, β_3, γ_3, δ_3 have the following values:

$$\left.\begin{aligned}
&\theta = uh \\
&h^{-1}\alpha_3 = \frac{11}{6}\theta^{-2} - \theta^{-4} + \left(\theta^{-4} - \frac{1}{3}\theta^{-2}\right)\cos 3\theta + \theta^{-3}\sin 3\theta \\
&h^{-1}\beta_3 = \theta^{-1} - 2\theta^{-3} + \left(\theta^{-4} - \frac{1}{3}\theta^{-2}\right)\sin 3\theta - \theta^{-3}\cos 3\theta \\
&h^{-1}\gamma_3 = 3\theta^{-4} - 3\theta^{-2} + 3\left(\frac{1}{2}\theta^{-2} - \theta^{-4}\right)\cos 3\theta - 4\theta^{-3}\sin 3\theta \\
&h^{-1}\delta_3 = 5\theta^{-3} + 3\left(\frac{1}{2}\theta^{-2} - \theta^{-4}\right)\sin 3\theta + 4\theta^{-3}\cos 3\theta
\end{aligned}\right\} \quad (9.2.36)$$

Above we gave computation formulas obtained with the aid of interpolation of the function f with respect to the values it assumes at the points x_k. We can attempt to increase the accuracy of the computations by appealing to interpolation not only with respect to the values of the function f but also to the values of its derivatives up to a certain order. Then the interpolation will be with "multiple points" or of the Hermitian type. In general form it is involved and to simplify notation we will examine the case of double points only, when the interpolation is carried out with respect to the values of f and of the first derivative f'. As before, we consider the points to be equally spaced: $x_k = kh$ ($k = 0, 1, \ldots; h > 0$). Take $n + 1$ points $x_k, x_{k+1}, \ldots, x_{k+n}$ and construct a polynomial $P_{2n+1}(x)$ of degree $2n + 1$ that satisfies the conditions

$$P_{2n+1}(x_{k+j}) = f_{k+j}, \quad P'_{2n+1}(x_{k+j}) = f'_{k+j}$$

$$j = 0, 1, \ldots, n$$

The explicit expression of such a polynomial is well known in the theory of interpolation and we will not derive it here (see [1], Vol. 1, Chap. 2, Sec. 11, Subsec. 1). The appropriate representation of the function f is

$$f(x) = \sum_{j=0}^{n} \frac{\omega^2(x)}{(x - x_{k+j})^2 [\omega'(x_{k+j})]^2} \times \left\{ \left[1 - \frac{\omega''(x_{k+j})}{\omega'(x_{k+j})} (x - x_{k+j}) \right] f_{k+j} + (x - x_{k+j}) f'_{k+j} \right\} + r_n^{(k)}(x)$$

$$= \sum_{j=0}^{n} H_j^{(k)}(x) + r_n^{(k)}(x) \quad (9.2.37)$$

$$\omega(x) = (x - x_k)(x - x_{k+1}) \ldots (x - x_{k+n})$$

If we multiply both sides of (9.2.37) by $\cos ux$, and then integrate over the interval $[kh, (k+n)h]$, and sum the results over the values k, which are multiples of n ($k = 0, n, 2n, \ldots$), we get a representation of the Fourier cosine transform in terms of the values of f_k and f'_k

$$\varphi_c(u) = \sum_{k=1}^{\infty} [\alpha_k^0(u) f_k + \alpha_k^1(u) f'_k] + R_c(u) \quad (9.2.38)$$

The coefficients $\alpha_k^0(u)$ and $\alpha_k^1(u)$ may be expressed in terms of the kernel of the transformation $\cos ux$ and the coefficients $H_j^{(k)}(x)$ of interpolation, the remainder $R_c(u)$ of the representation is expressed in terms of $\cos ux$ and the error $r_n^{(k)}(x)$ on all intervals of interpolation $[jn, (j+1)n]$ ($j = 0, 1, \ldots$).

The foregoing general arguments are given so as to illuminate the ideas underlying computation formulas of the type (9.2.38). As for computations, of greater importance are the particular cases of such formulas that correspond to the first few values of n.

IV. ***Third-degree interpolation with two double points.*** Take the interval $[kh, (k+1)h]$ and interpolate f over it with respect to the values $f_k, f_{k+1}, f'_k, f'_{k+1}$ with the aid of a third-degree polynomial:

$$f(x) = \frac{(x-x_{k+1})^2}{h^2}\left[\left(1+2\frac{x-x_k}{h}\right) f_k + (x-x_k) f'_k\right]$$

$$+\frac{(x-x_k)^2}{h^2}\left[\left(1-2\frac{x-x_{k+1}}{h}\right) f_{k+1}\right.$$

$$\left.+(x-x_{k+1}) f'_{k+1}\right] + r_k(x) \quad (9.2.39)$$

To obtain the required expression for the error $r_k(x)$ we can, as before, take advantage of Taylor's formula:

$$f(x) = f_k + (x - x_k) f'_k + \frac{1}{2}(x - x_k)^2 f''_k$$

$$+ \frac{1}{6}(x - x_k)^3 f'''_k + \frac{1}{6}\int_{x_k}^{x} f^{\mathrm{IV}}(t)(x-t)^3\,dt$$

$$= P_3(x) + \frac{1}{6}\int_{x_k}^{x_{k+1}} f^{\mathrm{IV}}(t)(x-t)^3 E(x-t)\,dt$$

Since the polynomial $P_3(x)$ is interpolated exactly, the errors of the interpolation of f and of the integral term of the right-hand member of the equation coincide. And this reduces finding the remainder for f to finding the remainder for the function $(x - t)^3\ E(x - t)$:

$$r_k(x) = \frac{1}{6}\int_{x_k}^{x_{k+1}} f^{\mathrm{IV}}(t)\Big\{(x-t)^3 E(x-t)$$

$$- \frac{(x-x_k)^2}{h^2}\Big[\Big(1 - 2\frac{x - x_{k+1}}{h}\Big)(x_{k+1} - t)^3$$

$$+ 3(x - x_{k+1})(x_{k+1} - t)^2\Big]\Big\}\,dt$$

To simplify the notation, set $x = x_k + h\xi$, $t = x_k + h\tau$ $(0 \leqslant \xi,\ \tau \leqslant 1)$ to get

$$r_k(x) = \frac{h^4}{6}\int_0^1 f^{\mathrm{IV}}(x_k + h\tau)\{(\xi-\tau)^3 E(\xi-\tau)$$

$$- \xi^2[(3-2\xi)(1-\tau)^3 + 3(\xi - 1)(1-\tau)^2]\}\,d\tau$$

$$= \frac{h^4}{6}\int_0^1 f^{\mathrm{IV}}(x_k + h\tau)\{(\xi-\tau)^3 E(\xi-\tau)$$

$$+ \xi^2(1-\tau)^2[(3-2\xi)\tau - \xi]\}$$

By multiplying (9.2.39) into $\cos ux$, integrating over the interval $[kh, (k+1)h]$ and summing the results over $k = 0, 1, 2, \ldots$, we can construct a representation of $\varphi_c(u)$, from which rejecting the remainder term yields a computation rule for finding $\varphi_c(u)$ from the values of f_k and f'_k:

$$\varphi_c(u) = \int_0^\infty f(x)\cos ux\,dx = \alpha'_1 f_0 - \gamma'_1 f'_0 + 2\alpha'_1 \sum_{k=1}^\infty f_k \cos k\theta - 2\delta'_1 \sum_{k=1}^\infty f'_k \sin k\theta + R_c(u) \quad (9.2.40)$$

where

$$\left.\begin{aligned} \theta &= uh \\ h^{-1}\alpha'_1 &= 12\theta^{-4}(1-\cos\theta) - 6\theta^{-3}\sin\theta \\ h^{-2}\gamma'_1 &= \theta^{-2} - 6\theta^{-4} + 2\theta^{-3}\sin\theta + 6\theta^{-4}\cos\theta \\ h^{-2}\delta'_1 &= 4\theta^{-3} - 6\theta^{-4}\sin\theta + 2\theta^{-3}\cos\theta \end{aligned}\right\} \quad (9.2.41)$$

In the representation of $R_c(u)$, the variable x is replaced by the canonical variable $\xi = h^{-1}(x - x_k)$, $x = x_k + h\xi$:

$$R_c(u) = \frac{h^5}{6}\int_0^1\int_0^1 \{(\xi-\tau)^3 E(\xi-\tau) + \xi^2(1-\tau)^2[(3-2\xi)\tau - \xi]\} \times \sum_{k=0}^\infty f^{IV}(x_k + h\tau)\cos u(x_k + h\xi)\,d\tau\,d\xi \quad (9.2.42)$$

To estimate R_c, let us first determine certain properties of the kernel of the double integral in braces. We denote it by $K(\xi, \tau)$:

$$K(\xi, \tau) = \begin{cases} (\xi-\tau)^3 + \xi^2(1-\tau)^2[(3-2\xi)\tau - \xi], & 0<\tau<\xi<1 \\ \xi^2(1-\tau)^2[(3-2\xi)\tau - \xi], & 0<\xi<\tau<1 \end{cases}$$

First note that the limiting values of the kernel on the boundary of the square of integration $[0 \leqslant \xi, \tau \leqslant 1]$ are equal to zero (Fig. 2). This follows immediately from the

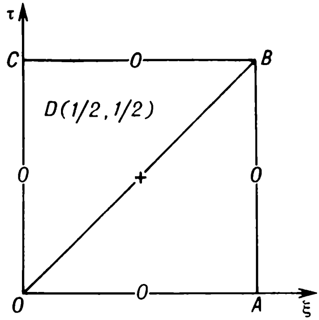

Fig. 2

indicated expressions of the kernel. Also note that on the diagonal OB of the square, the kernel has positive values:

$$K(\xi, \xi) = 2[\xi(1-\xi)]^3 > 0 \quad (0 < \xi < 1)$$

The sign of the kernel in the triangle OBC above the diagonal OB is determined by the sign of the bilinear polynomial $(3-2\xi)\tau - \xi$. For any fixed ξ $(0 < \xi < 1)$, as τ grows from ξ to 1, the polynomial increases from the value $(3-2\xi)\xi - \xi = 2\xi(1-\xi) > 0$ to $3(1-\xi) > 0$. For this reason, the kernel $K(\xi, \tau) \geqslant 0$ when $\xi \leqslant \tau \leqslant 1$, $0 \leqslant \xi \leqslant 1$. Besides, ordinary tools of analysis can be used to prove that the kernel in the $\triangle OBC$ attains its maximum value at the midpoint of the diagonal, $D(1/2, 1/2)$, and

$$\max_{0 \leqslant \xi \leqslant \tau \leqslant 1} K(\xi, \tau) = K\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{32}$$

In the triangle OAB below the diagonal OB, the kernel is a third-degree polynomial in τ:

$$(\xi-\tau)^3+\xi^2(1-\tau)^2[(3-2\xi)\tau-\xi] = (1-\xi)^2\tau^2[3\xi-(1+2\xi)\tau]$$

Its sign is determined by the expression in square brackets. Since $\xi \leqslant 1$ and $\tau \leqslant \xi$, it follows that

$$3\xi-(1+2\xi)\tau \geqslant 2\xi-2\xi^2 \geqslant 0$$

For this reason the kernel is nonnegative in the region $0 \leqslant \tau \leqslant \xi \leqslant 1$.

It can be demonstrated, by the usual methods for finding extrema of functions in closed regions, that the kernel attains its greatest value at the point D (1/2, 1/2) and

$$\max_{0\leqslant\tau\leqslant\xi\leqslant 1} K(\xi,\tau) = K\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{32}$$

Finally, it is easy to compute the double integral of the kernel:

$$\int_0^1 d\xi \int_0^1 d\tau\,\{(\xi-\tau)^3 E(\xi-\tau)+\xi^2(1-\tau)^2[(3-2\xi)\tau-\xi]\}$$
$$= \int_0^1 d\xi\left\{\int_0^\xi (\xi-\tau)^3+\xi^2\int_0^1 (1-\tau)^2[(3-2\xi)\tau-\xi]\,d\tau\right\} = \frac{1}{120}$$

(9.2.43)

The foregoing properties of the kernel permit obtaining estimates $R_c(u)$ similar to those that were found in the case of interpolation via the function values. If in the representation (9.2.42) for $R_c(u)$ we replace the kernel and $|\cos u(x_k+h\xi)|$ by the greater quantities 1/32 and 1, respectively, then what we get is the estimate

$$| R_c(u) | \leqslant \frac{h^5}{6} \cdot \frac{1}{32} \int_0^1 d\xi \int_0^1 d\tau \sum_{k=0}^{\infty} | f^{IV}(x_k + h\tau) |$$

$$= \frac{h^4}{192} \int_0^{\infty} | f^{IV}(x) | \, dx = \frac{h^4}{192} \operatorname*{Var}_{0 \leqslant x < \infty} f'''(x) \quad (9.2.44)$$

Now if in the integral we replace all summands of the infinite sum by their absolute values and $| \cos u (x_k + h\xi) |$ by unity, and if, by using the positivity of the kernel, we apply to the integral the mean-value theorem for the weight, then we get the inequality

$$| R_c(u) | \leqslant \frac{h^5}{6} \sum_{k=0}^{\infty} | f^{IV}(x_k + h\vartheta) |$$

$$\times \int_0^1 d\xi \int_0^1 d\tau \{ (\xi - \tau)^3 E(\xi - \tau) + \xi^2 (1 - \tau)^2 [(3 - 2\xi)\tau - \xi] \}$$

$$= \frac{h^5}{720} \sum_{k=0}^{\infty} | f^{IV}(x_k + h\vartheta) |, \quad 0 < \vartheta < 1 \quad (9.2.45)$$

Similar reasoning concerning the Fourier sine and complex transforms yields the following expressions in terms of the values of f_k and f'_k:

$$\varphi_s(u) = \int_0^{\infty} f(x) \sin ux \, dx = \beta'_1 f_0 + \delta'_1 f'_0 + 2\alpha'_1 \sum_{k=1}^{\infty} f_k \sin k\theta$$

$$+ 2\delta'_1 \sum_{k=1}^{\infty} f'_k \cos k\theta + R_s(u) \quad (9.2.46)$$

$$R_s(u) = \frac{h^5}{6}\int_0^1 d\xi \int_0^1 d\tau \{(\xi-\tau)^3 E(\xi-\tau)$$

$$+ \xi^2(1-\tau)^2[(3-2\xi)\tau-\xi]\}$$

$$\times \sum_{k=0}^{\infty} f(x_k+h\tau)\sin u(x_k+h\xi) \quad (9.2.47)$$

$$|R_s(u)| \leqslant \frac{h^4}{192}\int_0^{\infty} |f^{IV}(x)|\,dx = \frac{h^4}{192}\operatorname*{Var}_{0\leqslant x<\infty} f'''(x),$$

$$|R_s(u)| \leqslant \frac{h^5}{720}\sum_{k=0}^{\infty} |f^{IV}(x_k+h\vartheta)|, \quad 0<\vartheta<1,$$

$$\varphi(u) = \int_{-\infty}^{\infty} f(x)\,e^{-iux}\,dx = 2\alpha_1' \sum_{k=-\infty}^{\infty} f_k e^{-ik\theta}$$

$$-2i\delta_1' \sum_{k=-\infty}^{\infty} f_k' e^{-ik\theta} + R(u) \quad (9.2.48)$$

The values of α_1', γ_1', δ_1', θ are indicated in the equations (9.2.41), and, besides,

$$h^{-1}\beta_1' = 6\theta^{-3}\cos\theta - 12\theta^{-4}\sin\theta + \theta^{-3}(6+\theta^2)$$

$$R(u) = \frac{h^5}{6}\int_0^1 d\xi \int_0^1 d\tau \{(\xi-\tau)^3 E(\xi-\tau)$$

$$+ \xi^2(1-\tau)^2[(3-2\xi)\tau-\xi] \sum_{k=-\infty}^{\infty} f(x_k+h\tau)\,e^{-iu(x_k+h\xi)}\}$$

((9.2.49)

$$|R(u)| \leqslant \frac{h^4}{192} \int_{-\infty}^{\infty} |f^{\mathrm{IV}}(x)|\, dx = \frac{h^4}{192} \operatorname*{Var}_{-\infty<x<\infty} f'''(x)$$

$$|R(u)| \leqslant \frac{h^5}{720} \sum_{k=-\infty}^{\infty} |f^{\mathrm{IV}}(x_k + h\vartheta)|, \quad 0 < \vartheta < 1$$

V. ***Interpolation of the function with three double points.*** Take the points x_k, x_{k+1}, x_{k+2} and interpolate f with respect to the values of f_k, f'_k, f_{k+1}, f'_{k+1}, f_{k+2}, f'_{k+2}:

$$f(x) = \sum_{j=0}^{2} \frac{\omega_k^2(x)}{(x - x_{k+j})^2 [\omega'_k(x_{k+j})]^2} \times \left\{ \left[1 - \frac{\omega''_k(x_{k+j})}{\omega'(x_{k+j})} (x - x_{k+j}) \right] f_{k+j} + (x - x_{k+j}) f'_{k+j} \right\} + r_k(x), \qquad (9.2.50)$$

$$\omega_k(x) = (x - x_k)(x - x_{k+1})(x - x_{k+2})$$

In order to study the interpolation error, let us drop its integral representation (due to its relative complexity) that was used in the foregoing cases and take advantage of the familiar expression of $r_k(x)$ in the Lagrange remainder form:

$$r_k(x) = \frac{\omega^2(x)}{6!} f^{\mathrm{VI}}(x_k + \vartheta_k h), \quad 0 \leqslant \vartheta_k \leqslant 2 \qquad (9.2.51)$$

It will bring us there faster but will yield a somewhat rougher estimate, since the exact value of ϑ_k is not known.

Multiplying (9.2.50) termwise by $\cos ux$, integrating over the interval $[x_k, x_{k+2}]$, and then summing the results over even values of k ($k = 0, 2, 4, \ldots$) yield the following expression for $\varphi_c(u)$:

$$\varphi_c(u) = \int_0^\infty f(x) \cos ux\, dx = \alpha_2' f_0 + \gamma_2' \sum_{k=0}^{\infty} f_{2k+1} \cos(2k+1)\theta$$

$$+ 2\alpha_2' \sum_{k=1}^{\infty} f_{2k} \cos 2k\theta - \delta_2' f_0' - \eta_2' \sum_{k=0}^{\infty} f_{2k+1}' \sin(2k+1)\theta$$

$$- 2\zeta_2' \sum_{k=1}^{\infty} f_{2k}' \sin 2k\theta + R_c(u) \qquad (9.2.52)$$

The values of the coefficients α_2', γ_2', ... are given at the end of the section, and

$$R_c(u) = \frac{h^7}{6!} \int_0^2 \xi^2 (\xi - 1)^2 (\xi - 2)^2 \sum_{k=0}^{\infty} f^{VI}(x_{2k} + \vartheta_{2k} h)$$

$$\times \cos u(x_{2k} + h\xi)\, d\xi \qquad (9.2.53)$$

From this there follows the uniform estimate $R_c(u)$ with respect to u:

$$|R_c(u)|$$

$$\leqslant \frac{h^7}{6!} \int_0^2 \xi^2 (\xi - 1)^2 (\xi - 2)^2 d\xi \sum_{k=0}^{\infty} \max_{0 \leqslant \vartheta \leqslant 2} |f^{VI}(x_k + \vartheta h)|$$

$$= \frac{1}{9450} h^7 \sum_{k=0}^{\infty} \max_{0 \leqslant \vartheta \leqslant 2} |f^{VI}(x_k + \vartheta h)| \qquad (9.2.54)$$

For the Fourier sine transform, a similar rule of computation and its error are:

$$\varphi_s(u) = \int_0^\infty f(x)\sin ux\,dx = \beta_2' f_0 + \gamma_2' \sum_{k=0}^{\infty} f_{2k+1}\sin(2k+1)\theta$$

$$+ 2\alpha_2' \sum_{k=1}^{\infty} f_{2k}\sin 2k\theta + \zeta_2' f_0' + \eta_2' \sum_{k=0}^{\infty} f_{2k+1}'\cos(2k+1)\theta$$

$$+ 2\zeta_2' \sum_{k=1}^{\infty} f_{2k}'\cos 2k\theta + R_s(u) \quad (9.2.55)$$

$$R_s(u) = \frac{h^7}{6!}\int_0^2 \xi^2(\xi-1)^2(\xi-2)^2$$

$$\times \sum_{k=0}^{\infty} f^{\mathrm{VI}}(x_{2k}+h\vartheta_{2k})\sin u(x_{2k}+h\xi)\,d\xi \quad (9.2.56)$$

$$|R_s(u)| \leqslant \frac{1}{9450} h^7 \sum_{k=0}^{\infty} \max_{0\leqslant\vartheta\leqslant 2} |f^{\mathrm{VI}}(x_{2k}+h\vartheta)|$$

For the Fourier complex transform the analogous rule and estimate are of the form

$$\varphi(u) = \int_{-\infty}^{\infty} f(x)e^{-iux}\,dx = 2\alpha_2' \sum_{k=-\infty}^{\infty} f_{2k}e^{-i2k\theta}$$

$$+ \gamma_2' \sum_{k=-\infty}^{\infty} f_{2k+1}e^{-i(2k+1)\theta} - 2i\zeta_2' \sum_{k=-\infty}^{\infty} f_{2k}'e^{-i2k\theta}$$

$$- i\eta_2' \sum_{k=-\infty}^{\infty} f_{2k+1}'e^{-i(2k+1)\theta} + R(u) \quad (9.2.57)$$

$$R(u)=\frac{h^7}{6!}\int_0^2 \xi^2(\xi-1)^2(\xi-2)^2$$

$$\times \sum_{k=-\infty}^{\infty} f^{VI}(x_{2k}+h\vartheta_{2k})\, e^{-iu(x_{2k}+h\xi)}\, d\xi \quad (9.2.58)$$

$$|R(u)|\leqslant\frac{1}{9450}h^7\sum_{k=-\infty}^{\infty}\max_{0\leqslant\vartheta\leqslant 1}|f^{VI}(x_{2k}+h\vartheta)|$$

The parameters α_2', β_2', . . ., θ in (9.2.52), (9.2.55) and (9.2.57) have the following values:

$$\theta=uh,$$

$$h^5u^6\alpha_2'=\theta(156-7\theta^2)\sin\theta\cos\theta + 3(60+17\theta^2)\cos^2\theta-15(12-5\theta^2),$$

$$h^5u^6\beta_2'=\theta(\theta^4+8\theta^2-24)+\theta(7\theta^2-156)\cos^2\theta + 3(60-17\theta^2)\sin\theta\cos\theta,$$

$$h^5u^6\gamma_2'=16\theta(3-\theta^2)\sin\theta-48\theta^2\cos\theta,$$

$$h^4u^6\delta_2'=2\theta(\theta^2-24)\sin\theta\cos\theta + 15(\theta^2-4)\cos^2\theta+\theta^4+27\theta^2+6\theta,$$

$$h^4u^6\zeta_2'=\theta(5\theta^2-12)+15(4-\theta^2)\sin\theta\cos\theta + 2\theta(\theta^2-24)\cos^2\theta$$

$$h^4u^6\eta_2'=16\theta(\theta^2-15)\cos\theta+48(5-2\theta^2)\sin\theta$$

9.3 *Interpolation by rational functions*

9.3a. Introduction. Choice of interpolation and its error. At the beginning of this chapter we noticed that although interpolation by means of algebraic polynomials leads to practical rules of computation, it is not always a convenient computational device. It requires a partition of the half axis or the axis of integration into an infinite number of finite subintervals, the number of which depends, for one thing, on the rate of decrease of the function f as $|x| \to \infty$. If the rate of decrease of f is not fast enough, then many such subintervals will be required and this will complicate the computations or, in some cases, make them impossible.

In order to avoid having to partition the region of integration into finite parts, we must change the system of functions on which the interpolation is based. The choice of such a system depends, firstly, on the region of integration, which in our problem amounts to either the whole x axis or the half axis $x \geqslant 0$. We will only consider the Fourier cosine and sine transforms and, accordingly, assume the region of integration to be the half axis $x \geqslant 0$. Such an assumption is not a restriction of the problem, since the Fourier complex transform is readily reduced to the cosine and sine transforms. Secondly, the choice depends on the properties of the set of functions to be interpolated. Above, we agreed to consider functions f that satisfy, for large values of x, the condition $|f(x)| \leqslant Ax^{-s}$, $s > 1$. From among them we select those functions that are frequently encountered in applications and can be represented in the form

$$f(x) = \frac{F(x)}{(1+x)^s}, \quad s > 1 \tag{9.3.1}$$

where $F(x)$ is continuous on the half axis $[0, \infty)$ and has a finite limit, $\lim\limits_{x\to\infty} F(x) = F(\infty)$. The function $F(x)$ with such properties will be said to be continuous on the closed half axis $[0, \infty]$ and the limiting value of $F(\infty)$ will be considered its value at the point at infinity.

To approximate such functions F we can take many systems of elementary functions bounded on the half axis $[0, \infty)$. To simplify the computations, take for the basic functions the system of simple fractions $\frac{1}{(1+x)^m}$ $(m = 0, 1, 2, \ldots)$ and interpolate with the aid of polynomials in the argument $\frac{1}{1+x}$:

$$P_n(x) = \sum_{m=0}^{n} \frac{a_m}{(1+x)^m} \tag{9.3.2}$$

In the set of functions $F(x)$ continuous on the closed half axis $[0, \infty]$ the polynomials $P_n(x)$ form a complete system in the metric C. True enough, the transformation of the argument $z = \frac{1}{1+x}$ carries the half axis $[0, \infty]$ into the closed interval $[0, 1]$ of the z axis. The function $F(x)$ that is continuous on $[0, \infty]$ goes into the function $\psi(z)$ continuous on $[0, 1]$, and the rational functions $P_n(x)$ go into the polynomials $p_n(z)$ in z. Then it only remains to refer to the Weierstrass theorem on the completeness of a set of algebraic polynomials in the class of functions continuous on a finite closed interval.

Now on the half axis $[0, \infty)$ take $n+1$ points x_k $(0 \leqslant x_0 < x_1 < \ldots < x_n < \infty)$ and choose the coefficients a_k of the function P_n so that its values at the points x_k coincide with the values of F:

$$P_n(x_k) = \sum_{i=0}^{n} a_i (1+x_k)^{-i} = F(x_k) \quad (k=0, 1, \ldots, n) \tag{9.3.3}$$

These equations yield a linear system of equations for the coefficients a_i. The determinant is the Vandermonde determinant in the arguments $\frac{1}{1+x_k}$ $(k = 0, 1, \ldots, n)$, which is different from zero since all the x_k are distinct. The system has a unique solution and hence there exists a unique rational function $P_n(x)$ of the form (9.3.2) that satisfies the conditions (9.3.3).

When solving the system (9.3.3), the coefficients a_i will be found as linear functions of $F(x_k)$ $(k = 0, 1, \ldots, n)$. Their substitution into (9.3.2) will show that P_n is also a linear function of $F(x_k)$:

$$P_n(x) = l_0(x) F(x_0) + l_1(x) F(x_1) + \ldots + l_n(x) F(x_n) \tag{9.3.4}$$

Here $l_k(x)$ are polynomials of degree n in $\frac{1}{1+x}$. They are influence functions of the interpolation points x_k and obviously satisfy the conditions

$$l_i(x_k) = \begin{cases} 0 & \text{when } i \neq k \\ 1 & \text{when } i = k \end{cases}$$

that define them uniquely.

It is immediately apparent that

$$l_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \left(\frac{1}{1+x} - \frac{1}{1+x_j}\right) \left[\prod_{\substack{j=0 \\ j\neq k}}^{n} \left(\frac{1}{1+x_k} - \frac{1}{1+x_j}\right)\right]^{-1} \tag{9.3.5}$$

After simple manipulations for $l_k(x)$ we get other expressions that show that the coefficients $l_k(x)$ differ by extremely simple factors from the familiar interpolation multipliers of Lagrange:

$$\left.\begin{aligned} l_k(x) &= \frac{(1+x_k)^n}{(1+x)^n} \frac{\omega_{n+1}(x)}{(x-x_k)\,\omega'_{n+1}(x_k)} \\ \omega_{n+1}(x) &= \prod_{j=0}^{n} (x - x_j) \end{aligned}\right\} \tag{9.3.6}$$

In the future, it will be useful, for computing with the rational functions $P_n(x)$, to find an expansion of $P_n(x)$ in powers of $\frac{1}{1+x}$. It can conveniently be constructed in the following simple manner. Expand the polynomial $\frac{\omega_{n+1}(x)}{x-x_k}$ in powers of $1+x$:

$$\frac{\omega_{n+1}(x)}{x-x_k}=\sum_{l=0}^{n} c_l^{(k)}(1+x)^l$$

and put the expansion into (9.3.6) and (9.3.4):

$$P_n(x)=\sum_{k=0}^{n} F(x_k)\sum_{l=0}^{n} c_l^{(k)}\frac{(1+x_k)^n}{\omega'_{n+1}(x_k)}\frac{1}{(1+x)^{n-l}} \qquad (9.3.7)$$

In the inner sum, the factors in front of $(1+x)^{-n+l}$ depend solely on the points x_i ($i=0, 1, \ldots, n$), and for frequently used systems of points they can be computed beforehand and tabulated.

Let us now investigate the interpolation error $r_n(x) = F(x) - P_n(x)$. Its exact integral representations for classes of functions of sufficiently high smoothness in terms of the derivatives of the function F have been constructed for interpolation with respect to any system of coordinate functions (see [1], Vol. 1, Chap. 2, Sec. 4). The representation we need can be obtained from these general results as a special case. But this would require of the reader a knowledge of the general results or their exposition by the authors, which would take up a good deal of space, and so we prefer to obtain the necessary representations in a shorter way by taking advantage of the relationship between our problem and that of interpolation via algebraic polynomials. Recall that if we substitute $z=\frac{1}{1+x}$, $x=\frac{1}{z}-1$, then the half axis $0\leqslant x\leqslant\infty$ goes into the unit interval $1\geqslant z\geqslant 0$ of the z axis, the polynomial $P_n(x)$ [see (9.3.2)] goes into an entire algebraic polynomial of degree n in z:

$$P_n(x) = P_n\left(\frac{1}{z} - 1\right) = \sum_{k=0}^{n} a_k z^k = p_n(z) \qquad (9.3.8)$$

The function $F(x)$ transforms to a certain function of the argument z:

$$F(x) = F\left(\frac{1}{z} - 1\right) = \psi(z)$$

The interpolation points x_k go into the points $z_k = (1+x_k)^{-1}$ $(1 \geqslant z_0 > z_1 > \ldots > z_n > 0)$ and on the z axis we obtain the problem of interpolating $\psi(z)$ by the polynomial (9.3.8). In this new problem the interpolation error coincides with the error $r_n(x)$:

$$\rho_n(z) = \psi(z) - p_n(z) = F(x) - P_n(x) = r_n(x)$$

But the expression for the remainder $\rho_n(z)$ is familiar. We take advantage of the Taylor formula for $\psi(z)$ assuming that on $[0, 1]$ the function $\psi(z)$ has a continuous derivative of order $n+1$:

$$\psi(z) = \psi(1) + (z-1)\psi'(1) + \ldots + \frac{1}{n!}(z-1)^n \psi^{(n)}(1)$$

$$+ \frac{1}{n!}\int_1^z \psi^{(n+1)}(\tau)(z-\tau)^n \, d\tau$$

$$= \Pi_n(z) + \frac{(-1)^{n+1}}{n!}\int_z^1 \psi^{(n+1)}(\tau)(\tau - z)^n \, d\tau$$

$$= \Pi_n(z) + \frac{(-1)^{n+1}}{n!}\int_0^1 \psi^{(n+1)}(\tau)(\tau - z)^n E(\tau - z)\, d\tau$$

The polynomial Π_n interpolates exactly, and $\rho_n(z)$ coincides with the interpolation error of the integral term.

Since

$$\rho_n(z) = \psi(z) - \sum_{k=0}^{n} \Lambda_k(z)\,\psi(z_k)$$

$$\Lambda_k(z) = \frac{\Omega(z)}{(z-z_k)\,\Omega'(z_k)}, \quad \Omega(z) = \prod_{j=0}^{n}(z-z_j)$$

it follows that

$$\rho_n(z) = \frac{(-1)^{n+1}}{n!} \int_0^1 \psi^{(n+1)}(\tau) \Big\{ (\tau-z)^n E(\tau-z)$$

$$- \sum_{k=0}^{n} \Lambda_k(z)\,(\tau-z_k)^n E(\tau-z_k) \Big\}\, d\tau \quad (9.3.9)$$

To obtain $r_n(x)$ let us return to the earlier chosen x axis. Set $\tau = \frac{1}{1+t}$, $t = \frac{1}{\tau} - 1$. Since $d\tau = -\frac{1}{(1+t)^2}\,dt$, the equation

$$\frac{d}{d\tau} = -(1+t)^2 \frac{d}{dt} \quad (9.3.10)$$

between the operators of differentiation with respect to τ and with respect to h holds true.

Applying it $n+1$ times to the function $\psi(\tau) = F(t)$, we get the following rule for computing the derivative with respect to the variable τ:

$$\psi^{(n+1)}(\tau) = (-1)^{n+1}(1+t)^2 \frac{d}{dt} (1+t)^2 \frac{d}{dt} \ldots (1+t)^2 \frac{d}{dt} F(t)$$

It is easy to see that after performing the differentiation we get an equation of the form given below in which now we will not compute the coefficients $a_1, \ldots, a_n$:

$$\psi^{(n+1)}(\tau) = (-1)^{n+1}(1+t)^{n+1}[(1+t)^{n+1}F^{(n+1)}(t) + a_1(1+t)^n F^{(n)}(t) + a_2(1+t)^{n-1}F^{(n-1)}(t) + \ldots + a_{n-1}(1+t)^2F''(t) + a_n(1+t)F'(t)] = (-1)^{n+1}(1+t)^{n+1}L_{n+1}(F) \quad (9.3.11)$$

Consider the differential equation $\psi^{(n+1)}(\tau) = 0$. The general solution is a polynomial in τ of degree n with arbitrary coefficients, for the complete system of independent solutions we can take $1, \tau, \tau^2, \ldots, \tau^n$.

An equivalent equation is

$$L_{n+1}(F) = (1+t)^{n+1}F^{(n+1)}(t) + a_1(1+t)^n F^{(n)}(t) + a_2(1+t)^{n-1}F^{n-1}(t) + \ldots + a_n(1+t)F'(t) = 0 \quad (9.3.12)$$

It is Euler's equation[1] with the singular points $t = -1$ and $t = \infty$. The complete system of $n+1$ linearly independent solutions of it, into which solutions the powers τ^i $(i = 0, 1, \ldots, n)$ go under the transformation $\tau = (1+t)^{-1}$, is $1, (1+t)^{-1}, (1+t)^{-2}, \ldots, (1+t)^{-n}$. This points to a simple way of computing a_j $(j = 1, 2, \ldots, n)$.

If we write down the equation (9.3.12) and then adjoin the results of the substitutions into it of the solutions $(1+t)^{-1}, (1+t)^{-2}, \ldots, (1+t)^{-n}$, then after a few simplifications we get the following system of $n+1$ equations:

$$(1+t)^{n+1}F^{(n+1)} + a_1(1+t)^n F^{(n)} + a_2(1+t)^{n-1}F^{(n-1)} + \ldots + a_n(1+t)F' = 0,$$

$$(n+1)! - a_1 n! + a_2(n-1)! - \ldots + (-1)^n a_1 1! = 0,$$

[1] Euler's equation of order n with singular points $x = 0$ and $x = \infty$ is

$$A_0x^ny^{(n)} + A_1x^{n-1}y^{(n-1)} + \ldots + A_ny = 0$$

The equation (9.3.12) differs from it by the variable $x = 1 + t$.

$$(n+2)!-a_1(n+1)!+a_2n!-\ldots+(-1)^n a_1 2!=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(2n)!-a_1(2n-1)!+a_2(2n-2)!-\ldots+(-1)^n a_1 n!=0$$

It may be regarded as a linear homogeneous system of equations in the quantities 1, a_1, a_2, . . ., a_n that constitute a nonzero solution of the system. The determinant of the system must vanish, which permits writing (9.3.12) differently:

$$\begin{vmatrix} (1+t)^{n+1}F^{(n+1)} & (1+t)^nF^{(n)} & (1+t)^{n-1}F^{(n-1)} & \ldots(1+t)\,F' \\ (n+1)! & -n! & (n-1)! & (-1)^n\,1! \\ (n+2)! & -(n+1)! & n! & \ldots(-1)^n\,2! \\ \ldots & \ldots & \ldots & \ldots \\ (2n)! & -(2n-1)! & (2n-2)! & \ldots(-1)^n\,n! \end{vmatrix}=0 \tag{9.3.13}$$

If we expand the determinant in terms of the elements of the first row, an equation should emerge that differs from (9.3.12) by a constant factor. Therefore the coefficients a_1, a_2, . . . must be equal, respectively, to the ratio of the cofactors of the elements of the first row of the determinant (9.3.13), from the second onwards, to the cofactor of the first element of the row. This must be done since in (9.3.12) the coefficient of the derivative of highest order is reduced to $(1+t)^{n+1}$.

Let us now return to transforming the integral (9.3.9) to the old variables x, t; set $\tau=\frac{1}{1+t}$, $z=\frac{1}{1+x}$. Then $\psi^{(n+1)}(\tau)$ goes to $(-1)^{n+1}(1+t)^{n+1}L_{n+1}(F)$.

It will be useful to clarify the significance of the kernel of the integral (9.3.9) in the braces. We adjoin to the kernel the multiplier $(n!)^{-1}$.

In the function $\frac{1}{n!}(\tau-z)^n E(\tau-z)$, we consider z as the independent variable and τ as a parameter. For

$z < \tau$, when $E(\tau - z) = 1$, this function is a solution of the equation $\frac{d^{n+1}y}{dz^{n+1}} = 0$, which solution satisfies at the point $z = \tau$ the conditions $y(\tau) = y'(\tau) = \ldots = y^{(n-1)}(\tau) = 0$ and $y^{(n)}(\tau) = 1$. For $z > \tau$, when $E(\tau - z) = 0$, this solution continues to be identically zero. And the kernel itself is the error of interpolating such a function by an algebraic polynomial of degree n in the values at the points z_k $(k = 0, 1, \ldots, n)$.

The separate parts of the kernel in the variables x, t will have the following expressions:

$$(\tau - z)^n = \left(\frac{1}{1+t} - \frac{1}{1+x}\right)^n = \frac{(x-t)^n}{(1+t)^n (1+x)^n},$$

$$E(\tau - z) = E\left[\frac{x-t}{(1+t)(1+x)}\right] = E(x-t),$$

$$\Lambda_k(z) = \frac{(z-z_0)\ldots(z-z_{k-1})(z-z_{k+1})\ldots(z-z_n)}{(z_k-z_0)\ldots(z_k-z_{k-1})(z_k-z_{k+1})\ldots(z_k-z_n)}$$

$$= \prod_{j \neq k}\left(\frac{1}{1+x} - \frac{1}{1+x_j}\right)\left[\prod_{j \neq k}\left(\frac{1}{1+x_k} - \frac{1}{1+x_j}\right)\right]^{-1}$$

$$= \frac{(1+x_k)^n}{(1+x)^n} \frac{\omega_{n+1}(x)}{(x-x_k)\,\omega'_{n+1}(x_k)},$$

$$\omega_{n+1} = \prod_{j=0}^{n}(x - x_j), \quad d\tau = -\frac{dt}{(1+t)^2}$$

Their substitution into the integral (9.3.9) yields the following value for the interpolation error $r_n(x)$:

$$r_n(x) = \int_0^\infty L_{n+1}(F)\frac{1}{n!\,(1+x)^n}\left\{(x-t)^n E(x-t)\right.$$

$$\left. - \sum_{k=0}^{n}\frac{\omega_{n+1}(x)}{(x-x_k)\,\omega'_{n+1}(x_k)}(x_k-t)^n E(x_k-t)\right\}\frac{dt}{1+t} \qquad (9.3.14)$$

The quantity within the braces under the integral sign is nothing but the error of algebraic interpolation in the variable x of the function $(x-t)^n E(x-t)$ with respect to its values at the points x_k. When t is less than $x, x_0, \ldots, x_n$, then the interpolation is exact and the quantity in the braces in (9.3.14) vanishes. But when t is greater than $x, x_0, \ldots, x_n$, then all "quenching" functions $E(x-t)$, $E(x_k-t)$ are equal to zero, and the error too is zero. The quantity in braces can assume values different from zero only on the interval where $x, x_0, \ldots, x_n$ are located. For this reason, given continuity of $L_{n+1}(f)$, the integral over the half axis $0 \leqslant t < \infty$ is actually a proper integral.

9.3b. The general interpolation quadrature rule. For the sake of convenience, combine the Fourier cosine and sine transforms into a single integral with exponential function:

$$\varphi_e(u) = \int_0^\infty e^{iux} f(x)\, dx \tag{9.3.15}$$

If f is a real-valued function, the cosine and sine transforms of f will be the real and imaginary parts of φ_e, respectively.

It was assumed above that $f(x) = F(x)(1+x)^{-s}$ $(s > 1)$ and $F(x)$ is a continuous and sufficiently smooth function on the half axis $0 \leqslant x \leqslant \infty$.

Let us interpolate the function F with the aid of a polynomial $P_n(x)$ of degree n in $(1+x)^{-1}$ and write $P_n(x)$ in the form (9.3.7).

If in the integral (9.3.15) we replace the original function f by its expression

$$f(x) = (1+x)^{-s} F(x) = (1+x)^{-s} [P_n(x) + r_n(x)]$$

we get for $\varphi_e(u)$ the following representation which, after dropping the remainder term R_n, can serve as a rule of computation for φ_e:

$$\left.\begin{aligned}\varphi_e(u) &= \int_0^\infty e^{iux}(1+x)^{-s}F(x)\,dx\\ &= \int_0^\infty e^{iux}(1+x)^{-s}[P_n(x)+r_n(x)]\,dx\\ &= \sum_{k=0}^{n} F(x_k)\sum_{l=0}^{n}\frac{c_l^{(k)}(1+x_k)^n}{\omega'(x_k)}\\ &\quad\times\int_0^\infty e^{iux}(1+x)^{-n+l-s}\,dx+R_n(u),\\ R_n(u) &= \int_0^\infty e^{iux}(1+x)^{-s}r_n(x)\,dx\end{aligned}\right\}\qquad(9.3.16)$$

Here, the coefficients $A_{kl}=\dfrac{c_l^{(k)}(1+x_k)^n}{\omega'(x_k)}$ depend only on the points x_k and do not depend either on s or on the function F. They can be tabulated for the commonest systems of points.

The integrals $\int_0^\infty e^{iux}(1+x)^{-n+l-s}\,dx$ depend solely on the frequency u and on s, that is, on how rapidly $f(x)$ decreases as x increases without bound. In Subsection 9.3e we give rules for computing these integrals.

From the representation for the error $R_n(u)$ in (9.3.16) it is easy to obtain a uniform estimate with respect to u in terms of the interpolation error $r_n(x)$:

$$|R_n(u)|\leqslant\int_0^\infty\frac{|r_n(x)|}{(1+x)^s}\,dx\qquad(9.3.17)$$

From it follows the theorem on the convergence of the computation process corresponding to (9.3.16).

Let the process be defined by the infinite triangular array of interpolation points

$$X = \begin{pmatrix} x_0^{(0)} & & & \\ x_0^{(1)} & x_1^{(1)} & & \\ \cdot & \cdot & \cdot & \cdot \\ x_0^{(n)} & x_1^{(n)} & x_n^{(n)} & \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{9.3.18}$$

Suppose that the interpolation (9.3.7) of the function F is carried out with respect to the points $x_k^{(n)}$ $(k = 0, 1, \ldots, n)$ in row n of the array X. Assuming that $n \to \infty$, we have the following theorem.

Theorem 1. *Suppose the following conditions hold:*

(1) *the interpolation process* (9.3.7) *defined by the array of points* (9.3.18) *converges for the function* $F(x)$ *for almost all values of* x *on the half axis* $0 \leqslant x < \infty$;

(2) *for all sufficiently large values of* n, *the interpolation error* $r_n(x)$ *satisfies the condition*

$$|r_n(x)| \leqslant M < \infty \quad (0 \leqslant x < \infty)$$

Then the remainder $R_n(u)$ *of the appropriate computation process* (9.3.16) *for transformation of* $\varphi_e(u)$ *tends to zero uniformly with respect to* u *on the axis* $-\infty < u < \infty$ *as* $n \to \infty$.

This theorem is a direct consequence of the familiar theorem on passage to the limit in the Lebesgue integral (see for example [17]): if a sequence of functions $g_n(x)$ on a set E converges almost everywhere on E to the function $g(x)$ summable on E and there exists a function $h(x)$ summable on E such that for all n and $x \in E$ we have the inequality $|g_n(x)| \leqslant h(x)$, then

$$\int_E g_n(x)\,dx \to \int_E g(x)\,dx$$

A different representation of $R_n(u)$ obtained from (9.3.16) by replacing $r_n(x)$ by its representation in the form (9.3.14) may be useful, at least in certain cases, for finding the error estimates $R_n(x)$ depending on the properties of the function $F(x)$:

$$R_n(u) = \int_0^\infty dx\, e^{iux} \frac{1}{n!\,(1+x)^{n+s}} \int_0^\infty L_{n+1}(F) \times \left\{ (x-t)^n E(x-t) - \sum_{h=0}^{n} \frac{\omega_{n+1}(x)}{(x-x_h)\,\omega'_{n+1}(x_h)} \times (x_h-t)^n E(x_h-t) \right\} \frac{dt}{1+t} \quad (9.3.19)$$

A remark is in order concerning the sign of the kernel of the double integral:

$$K^*(x, t) = \frac{1}{n!\,(1+x)^{n+1}(1+t)} \left\{ (x-t)^n E(x-t) - \sum_{k=0}^{n} \frac{\omega_{n+1}(x)}{(x-x_k)\,\omega'_{n+1}} (x_k-t)^n E(x_h-t) \right\} = \frac{1}{n!\,(1+x)^{n+1}(1+t)} K(x, t)$$

Its sign coincides with the sign of the expression $K(x, t)$ in the braces. This expression was encountered in (9.3.9). Recall that it is the kernel in the integral representation of the error in the following problem of algebraic interpolation.

Let the points x_k ($k = 0, 1, \ldots, n$) and the interpolation point x lie on the interval $[a, b]$ and let the function

$g(x)$ be interpolated with respect to its values $g(x_k)$ by the polynomial $p_n(x)$ of degree n. If $g(x)$ has a continuous derivative of order $n+1$ on $[a, b]$, then the interpolation error $\rho_n(x) = g(x) - p_n(x)$ can be represented in terms of the derivative of order $n+1$ of g in the form

$$\rho_n(x) = \frac{1}{n!} \int_a^b g^{(n+1)}(t) K(x, t)\, dt \tag{9.3.20}$$

On the other hand, for $\rho_n(x)$ we have the familiar Lagrange representation

$$\rho_n(x) = \frac{\omega_{n+1}(x)}{(n+1)!} g^{(n+1)}(\xi), \quad a \leqslant \xi \leqslant b \tag{9.3.21}$$

whence it follows that if the derivative $g^{(n+1)}(x)$ is different from zero on $[a, b]$, then $\rho_n(x)$ does not vanish at any point x, except at x_k $(k = 0, 1, \ldots, n)$. Therefore, for every fixed value of x differing from the points x_k, the kernel $K(x, t)$, as a function of t, does not change sign for $a \leqslant t \leqslant b$, because if the kernel $K(x, t)$ changed sign, then there would be a function $g^{(n+1)}(\tau)$ that preserved sign and such that the integral (9.3.20) would vanish, which is impossible because $x \neq x_k$ $(k = 0, 1, \ldots, n)$.

Besides, if we assume $g(x)$ to be a polynomial of degree $n+1$ for which $g^{(n+1)}(x) = 1$, then from both representations of the error $\rho_n(x)$ it follows that

$$\frac{1}{n!} \int_a^b K(x, t)\, dt = \frac{\omega_{n+1}(x)}{(n+1)!}$$

Thus, for each fixed x the sign of the kernel K, (x,t) coincides with the sign of $\omega_{n+1}(x)$.

9.3c. Interpolation with equally spaced points. For the interpolation points we take the equally spaced points $x_k = kh$ $(k = 0, 1, 2, \ldots; h > 0)$. In this case $\omega_{n+1}(x) = x(x-h) \ldots (x-nh)$ and the coefficients $c_l^{(k)}$ are determined from

$$\frac{\omega_{n+1}(x)}{x-x_k} = x(x-h)\ldots[x-(k-1)h]$$

$$\times[x-(k+1)h]\ldots(x-nh) = \sum_{l=0}^{n} c_l^{(k)}(1+x)^l \quad (k=0, 1, \ldots, n)$$

In the case of uniform convergence

$$\omega'(x_k) = kh(k-1)h\ldots h(-h)(-2h)\ldots(-1)(n-k)h$$
$$= h^n(-1)^{n-k}k!\,(n-k)!,$$

$$A_{kl} = \frac{c_l^{(k)}(1+kh)^n}{(-1)^{n-k}h^n k!\,(n-k)!} \tag{9.3.22}$$

When compiling number tables for A_{kl}, one can always assume $h = 1$ since any other value of h leads to unity via a linear transformation of the independent variable[1] $x = hx'$.

The rule for computing (9.3.16) in the case of equally spaced points takes the form

$$\varphi_e(u) = \int_0^\infty e^{iux}\frac{F(x)}{(1+x)^s}\,dx$$

$$= \sum_{k=0}^{n} F(kh)\sum_{l=0}^{n} A_{kl}\int_0^\infty e^{iux}(1+x)^{-n-s+l}\,dx + R_n(u) \tag{9.3.23}$$

The convergence of the computational process when $R_n(u) \to 0$ as $n \to \infty$, in other words, the possibility of an arbitrarily exact computation of $\varphi_e(u)$ via the rule (9.3.23) is very peculiar here and requires some explanation. In the preceding subsection it was pointed out that

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[1] Tables of the values A_{kl} $(k,\ l = 0, 1, \ldots, n)$ for $n = 1\,(1)\,15$ and $h = 1$ to 10 places are given in [11].

underlying the convergence of $R_n(u)$ to zero is the convergence of interpolation of the function $F(x)$ by the rational function $P_n(x)$ [see (9.3.4)] and that it is sufficient, for a uniform approach of $R_n(u)$ to zero, that $P_n(x)$ boundedly converge to $F(x)$ almost everywhere on $[0, \infty]$.

To make the exposition more pictorial, let us return to the variable z and put $z = \frac{1}{1+x}$. The closed half axis $0 \leqslant x \leqslant \infty$ goes into the closed interval $1 \geqslant z \geqslant 0$. The function $F(x)$, which we assumed to be continuous on the half axis $0 \leqslant x \leqslant \infty$, transforms to a certain function $\psi(z) = F\left(\frac{1}{z} - 1\right)$ that is continuous on the unit interval $0 \leqslant z \leqslant 1$ and $P_n(x)$ goes into a certain algebraic polynomial $p_n(z)$ of degree n that interpolates $\psi(z)$ with respect to the values $\psi_k = \psi(z_k)$ at the points $z_k = \frac{1}{1+kh}$ $(k = 0, 1, \ldots, n)$.

As n increases, the old points remain and new ones are adjoined. If we examine the set of interpolation points z_k $(k = 0, 1, 2, \ldots)$, we find that they form a monotonic decreasing sequence converging to zero.

The convergence of the interpolation process has to be regarded only in those sets of functions where each function is fully determined by the values that it assumes on a countable set of all the points. If this condition is not fulfilled and if there exist several functions that assume the same values at all interpolation points and, hence, have the same interpolating polynomials, the question of the convergence of interpolation does not have the ordinary sense.

When considering the convergence of interpolation for the set of all functions continuous on the interval $[a, b]$ or for the set of functions having continuous derivatives up to a certain fixed order m, one takes into account that each such function is determined by its values on a countable set of points that is everywhere dense on $[a, b]$. Accordingly, in studies of the convergence of interpolation of such functions it is always assumed that the interpolation points $z_i^{(k)}$ $(i = 0, 1, \ldots, k;\ k = 0, 1, 2, \ldots)$ are everywhere dense on $[a, b]$.

In the problem at hand, the array of interpolation points is of the form

$$Z = \begin{pmatrix} z_0 & & \\ z_0 & z_1 & \\ z_0 & z_1 & z_2 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad (9.3.24)$$

Its rows are subsequences of the sequence of points z_0, z_1, z_2, $\ldots$, $[z_k = (1 + kh)^{-1}]$ convergent to the unique accumulation point $z = 0$. The simplest and most natural set of functions that are determined by the values on such a sequence is the set of analytic functions for which all interpolation points and their limiting point lie inside the domain of regularity. Let us now examine such a set of functions; we will assume that the function $\psi(z)$ is regular in the domain of the complex plane z that contains the interval $[0, 1]$. The broader the domain and hence the farther away from $[0, 1]$ the singular points of $\psi(z)$, the smoother the behaviour of $\psi(z)$ on the interval $[0, 1]$ and the more probable the convergence of interpolation to $\psi(z)$ on $[0, 1]$. For this reason it is natural to pose the question of finding the smallest region in which the regularity of $\psi(z)$ ensures such a convergence. In the theory of interpolation, proof is given that the smallest such region is a closed circle of unit radius centred at the origin (see [10], Ch. 12, Sec. 2): $|z| \leqslant 1$, and convergence in the circle and, in particular, on its radius $0 \leqslant r \leqslant 1$ will be uniform. We omit the proof of this result and merely confine ourselves to an explanation of the pictorial aspect of the matter, which fortunately is comparatively simple.

The question of convergence is determined by the behaviour of the interpolating polynomial $p_n(z)$ for large values of n. But if n is great, then most of the points will be close to the limiting point $z = 0$ and the interpolation will be close to interpolation with the unique point $z = 0$ that has multiplicity n. Now this is given by the truncated Taylor series

$$S_n(z) = \psi(0) + \frac{z}{1!}\psi'(0) + \ldots + \frac{z^n}{n!}\psi^{(n)}(0)$$

If the set of functions is characterized solely by the region of regularity and if no special assumptions are made about the behaviour of the function in this region, then the convergence of $S_n(z)$ to $\psi(z)$ on the closed interval $[0, 1]$ can, most likely, be guaranteed only when $\psi(z)$ is regular in the circle $|z| \leqslant 1$.

The foregoing reasoning cannot of course serve as proof of the assertion we need but we believe that it gives a rather simple and pictorial view at least of the probability of the assertion.

Let us now return to the old variable $x = \frac{1}{z} - 1$. The function $\psi(z)$ goes into the function $\psi(z) = \psi\left(\frac{1}{1+x}\right) = F(x)$ that is regular in a certain region containing the closed half axis $0 \leqslant x \leqslant \infty$, in particular the point at infinity $x = \infty$. The unit circle $|z| \leqslant 1$ transforms into the region $\frac{1}{|1+x|} \leqslant 1$, or $|1 + x| \geqslant 1$, which is the closed exterior of a circle of radius 1 with centre at the point -1.

This permits us to state the following.

When a function F(x) *is analytic and regular in a region* $|1 + x| \geqslant 1$, *then the interpolation process for it relative to equally spaced points* $x_k = kh$ $(k = 0, 1, \ldots)$ *by means of a polynomial* $P_n(x)$ *of degree* n *in* $\frac{1}{1+x}$ [see (9.3.4)] *converges uniformly in that region.*

This enables us to say that the following theorem holds true.

Theorem 2. *If a function* F(x) *is regular in the region* $|1+x| \geqslant 1$ *of the complex plane* x, *then the computation process obtained from the equation* (9.3.23) *when the remainder* $R_n(u)$ *is dropped converges to* $\varphi_e(u)$ *uniformly relative to* u *on the axis* $-\infty < u < \infty$ *as* $n \to \infty$.

9.3d. Computation rules associated with the roots of orthogonal polynomials. The convergence of the interpolation quadrature process (9.3.23) with equally spaced points for a very narrow class of functions suggested the construction of other computation rules that are more favourable as regards the region of convergence and sufficiently simple computationally. We can also strive to make them compatible with the use of available numerical tables. Such a construction can be carried out in a variety of ways, two of which will be given in this subsection and a third in Sec. 10.2.

When choosing a computation rule—if one wishes to preserve the interpolation-quadrature type—one has to take into account certain results established in the theory of interpolation and the theory of quadratures. The choice may be made on the basis of the following two arguments, the first of which was developed in Chap. 4.

1. When interpolating on a finite interval, say [—1, 1], the following arrays of points are particularly favourable with respect to convergence:

$$Z = \begin{pmatrix} z_0^{(0)} & & \\ z_0^{(1)} & z_1^{(1)} & \\ z_0^{(2)} & z_1^{(2)} & z_2^{(2)} \\ \dots & \dots & \dots \end{pmatrix} \qquad (9.3.25)$$

They have the limiting Chebyshev density distribution

$$\rho(z) = \frac{1}{\pi}(1-z^2)^{-1/2} \qquad (9.3.26)$$

For example, an interpolation process with such an array will converge uniformly on the interval [—1, 1] for any function that is analytic on $[-1, 1]$ including the endpoints -1 and 1.

The most thoroughly studied tables of this kind are tables of the roots of orthogonal polynomials.[1] Of particular importance in applications are the Jacobi polynomials corresponding to the weight function

$$p(x) = (1-x)^{\alpha}(1+x)^{\beta} \quad (\alpha, \beta > -1)$$

that permits taking into account the power singularities at the endpoints of the interval $[-1, 1]$. A particularly important role in the interpolation problem is played by the Chebyshev polynomials of the first kind. The interpolation process in which we take the roots of such a polynomial of degree n for the interpolation points converges uniformly on $[-1, 1]$ for any function $\varphi(z)$ whose modulus of continuity $\omega(\delta)$ satisfies the condition $\omega(\delta)\ln\delta \to 0$ $(\delta \to 0)$ (see [16], p. 542).

Proof is also given in the theory of interpolation that such an interpolation process will (as has already been pointed out) converge uniformly to $\varphi(z)$ if $\varphi(z)$ is a function absolutely continuous on $[-1, 1]$.

2. From the theory of approximate quadratures we know that when computing an integral with a weight function, it is possible, in the computation formula[2]

$$\int_a^b p(z)\varphi(z)\,dz \approx \sum_{k=1}^{n} A_k\varphi(z_k), \tag{9.3.27}$$

[1] We know that if a weight function $p(z)$ is almost everywhere positive on the interval $[-1, 1]$, then the table of the roots of the appropriate system of orthogonal polynomials always has the limiting density distribution (9.3.26).

[2] The weight $p(z)$ is assumed to be such that the integrals

$$\int_a^b p(z)z^m\,dz \quad (m = 0, 1, \ldots)$$

are absolutely convergent.

to substantially increase the algebraic degree of accuracy if one computes, in a specific way, the points z_k and the coefficients A_k; namely, proof is given that if the weight function $p(z)$ is of constant sign, then through a choice of z_k and A_k the equation (9.3.27) can be made to be exact when $\varphi(z)$ is an arbitrary polynomial of degree $2n-1$; here, z_k and A_k are determined uniquely: z_k must be the roots of a polynomial $P_n(z)$ of degree n taken from the orthogonal system of polynomials corresponding to the weight $p(z)$, and

$$A_k = \int_a^b p(z) \frac{P_n(z)}{(z-z_k) P'_n(z_k)} dz$$

This means that the quadrature rule (9.3.27) must be interpolatory.

Also note that the formula (9.3.27) converges on a very broad set of functions: if the interval $[a, b]$ is finite, and the weight function on it is of constant sign and is not equivalent to zero, then for the convergence of

$$\sum_{k=1}^{n} A_k \varphi(z_k) \to \int_a^b p(z) \varphi(z) dz \quad (n \to \infty)$$

it suffices that $\varphi(z)$ be bounded and the set of its discontinuity points be of measure zero. Now take the integral $\varphi_e(u)$ (9.3.15) in the form

$$\varphi_e(u) = \int_0^\infty e^{iux} F(x) \frac{dx}{(1+x)^s} \quad (s > 1) \qquad (9.3.28)$$

As before, we consider the function $F(x)$ to be continuous and sufficiently smooth on the closed half axis $[0, \infty]$. The factor $(1+x)^{-s}$ has a single singularity: zero to the power s at the point at infinity. We adjoin it below to the weight function. The quantity $e^{iux} = \cos ux + i \sin ux$ determines the oscillations of the integrand; e^{iux} is a complex quantity and its real and imaginary parts are of variable sign. All this makes it difficult to refer e^{iux} to the weight in the ordinary way, without a preliminary transformation, and then, when setting up the quadrature rule, to strive to attain the highest possible degree of algebraic accuracy. In the next section we will see what can be done about this; for the present we adjoin this factor to the function to be integrated.

In order to transform (9.3.28) to an integral with a classical weight function, make the substitution $x = \frac{1-t}{1+t}$ or $t = \frac{1-x}{1+x}$. The half axis $0 \leqslant x \leqslant \infty$ will go into the interval $[-1, 1]$ and the integral (9.3.28) will take the form

$$\varphi_e(u) = 2^{1-s} \int_{-1}^{1} e^{iu\frac{1-t}{1+t}} F\left(\frac{1-t}{1+t}\right)(1+t)^{s-2}\,dt \quad (9.3.29)$$

The power factor $(1+t)^{s-2}$ can be taken for the weight function $p(t) = (1+t)^{s-2}$. It is a special case of the Jacobi weight $(1-t)^{\alpha}(1+t)^{\beta}$ for $\alpha = 0$, $\beta = s-2$. We then take the remaining portion of the integrand for the function to be integrated:

$$\psi(t) = e^{iu\frac{1-t}{1+t}}\left(\frac{1-t}{1+t}\right)$$

Then the integral (9.3.29) can be computed by the rule for integration with the Jacobi weight:

$$\varphi_e(u) = 2^{1-s} \int_{-1}^{1} \psi(t)(1+t)^{s-2}\,dt \approx 2^{1-s} \sum_{k=1}^{n} A_k \psi(t_k) \quad (9.3.30)$$

Here, t_k are the roots of the Jacobi polynomial $J_n^{(0,\,s-2)}(t)$ of degree n and A_k are the quadrature coefficients corresponding to these roots. The numerical values of t_k and A_k may be taken from published tables (see [12], [14]).

Since it is assumed that $F(x) = F\left(\frac{1-t}{1+t}\right)$ is continuous on the half axis $0 \leqslant x \leqslant \infty$ and the exponential factor $\exp\left\{iu\,\frac{1-t}{1+t}\right\}$ is bounded in absolute value by unity and is continuous for $-1 < t \leqslant 1$, the function $\psi(t)$ will be bounded and continuous for $-1 < t \leqslant 1$. Hence, when n grows without bound, we have, for any value of u, convergence of the computation process (9.3.30):

$$\lim_{n\to\infty} 2^{1-s}\sum_{k=1}^{n} A_k\psi(t_k) = 2^{1-s}\int_{-1}^{1}\psi(t)(1+t)^{s-2}\,dt = \varphi_e(u)$$

The foregoing method of computation is clearly nothing but a transfer to the integral (9.3.29) of the idea of integration of highest degree of accuracy with Jacobi weight; here the weight merely takes into account the rate of decrease of $f(x) = \frac{F(x)}{(1+x)^s}$ as x goes to infinity. We speak of this method in order to point out its essential drawback for computing $\varphi_e(x)$ and to indicate a possible way of diminishing this drawback.

Recall that the rule of highest-accuracy integration (9.3.30) presumes the function $\psi(t)$ that is to be integrated to have, on the closed interval $[-1, 1]$, a continuous derivative of order not lower than $2n$ that takes on small values. When this condition holds, we can hope to obtain a good accuracy. But if such a derivative is absent, the rule may not yield a high accuracy and will be inferior in this respect to other rules having a lower degree of accuracy. We assumed the factor $F\left(\frac{1-t}{1+t}\right)$ to be sufficiently smooth; as for the other factor $\exp\left\{iu\frac{1-t}{1+t}\right\}$, we see that its oscillations, as $t \to -1$, accelerate without bound and for it the point

$t = -1$ is a point of discontinuity. For this reason, near $t = -1$ the function $\psi(t)$ will not be exactly approximated by algebraic polynomials, and although the rule (9.3.30) enables one, in principle, to compute $\varphi_e(u)$ with arbitrary precision, to attain the required accuracy might require a larger value of n.

Let us retain the points t_k in (9.3.30) considering them to be roots of the polynomial $J_n^{(0,\, s-2)}(t)$. In this way we will to some extent leave them coordinated with the rate of approach of $f(x)$ to zero as $x \to \infty$. As for the coefficients A_k, we choose them without striving to attain the highest degree of accuracy but taking into account the oscillations of the functions $\psi(t)$. Let us refer the oscillating factor $\exp\left\{iu\frac{1-t}{1+t}\right\}$ to the weight and set

$$p^*(t) = e^{iu\frac{1-t}{1+t}}(1+t)^{s-2}, \quad \psi^*(t) = F\left(\frac{1-t}{1+t}\right)$$

Now interpolate $\psi^*(t)$ with respect to the values at the interpolation points t_k:

$$\psi^*(t) = \sum_{k=1}^{n} l_k^*(t)\,\psi^*(t_k) + r_n^*(t) \tag{9.3.31}$$

$$l_k^*(t) = \frac{\Omega^*(t)}{(t-t_k)\,\Omega^{*\prime}(t_k)}$$

$$\Omega^*(t) = (t-t_1)\ldots(t-t_n) = \frac{1}{q_n} J_n^{(0,\, s-2)}(t)$$

Here q_n is the coefficient of the highest degree of t in the polynomial $J_n^{(0,\, s-2)}(t)$.

If in place of $\psi^*(t)$ and $p^*(t)$ we put their values into (9.3.29), then we obtain the following expression of $\varphi_e(u)$

in terms of the values of $\psi^* (t_k)$:

$$\varphi_e (u) = 2^{1-s} \int_{-1}^{1} p^* (t) \psi^* (t) \, dt = \sum_{k=1}^{n} A_k^* \psi^* (t_k) + R_n^* \qquad (9.3.32)$$

$$A_k^* = 2^{1-s} \int_{-1}^{1} p^* (t) \, l_k^* (t) \, dt$$

$$R_n^* = 2^{1-s} \int_{-1}^{1} p^* (t) \, r_n^* (t) \, dt$$

After R_n^* has been dropped, this expression gives an approximate calculation rule for $\varphi_e (u)$.

When constructing interpolation quadrature rules, their interpolation points can, in principle, be chosen arbitrarily or one can use this arbitrary character for certain specific purposes. In the rule (9.3.32) an attempt has been made to bring this arbitrary choice into agreement with the nature of decrease of $f (x)$ as x increases.

Here is another example of choosing the points t_k. But first we offer some general arguments. In the integral (9.3.29) we take $p^* (t)$ for the weight function and $\psi^* (t) = F \left(\frac{1-t}{1+t} \right)$ for the function to be integrated.

Let us take an arbitrary Jacobi polynomial, $J_n^{(\alpha, \beta)} (x)$, of degree n with indices α, $\beta > -1$. We take its roots, which, as before, we denote by t_k ($k = 1, 2, \ldots, n$), for the interpolation points of the function $\psi^* (t)$, without bothering for the time being about seeking the best points or, to put the matter more precisely, the most suitable values of α, β.

The interpolation formula will have the same form (9.3.31) as above but with a different meaning of the t_k:

$$\psi^*(t) = \sum_{k=1}^{n} L_k^*(t)\,\psi^*(t_k) + r_n^*(t)$$

$$L_k^*(t) = \frac{\Omega^*(t)}{(t-t_k)\,\Omega^{*\prime}(t_k)}$$

$$\Omega^*(t) = \prod_{k=1}^{n}(t-t_k) = \frac{1}{q_n^{(\alpha,\beta)}} J_n^{(\alpha,\beta)}(t)$$

$$L_k^*(t) = \frac{1}{t-t_k} J_n^{(\alpha,\beta)}(t)\,[J_n^{(\alpha,\beta)\prime}(t_k)]^{-1} \tag{9.3.33}$$

$q_n^{(\alpha,\beta)}$ is the coefficient of t^n in the polynomial $J_n^{(\alpha,\beta)}(t)$.

To obtain the necessary computation formulas,[1] let us return to the earlier variable x and set $x = \frac{1-t}{1+t}$. We denote the values of x that correspond to $t = t_k$ by $x_k = \frac{1-t_k}{1+t_k}$.

The computations that follow do not require any explanation:

$$\Omega^*(t) = \prod_{j=1}^{n}(t-t_j) = \prod_{j=1}^{n}\left(\frac{1-x}{1+x} - \frac{1-x_j}{1+x_j}\right)$$

$$= \frac{(-1)^n 2^n}{(1+x)^n}\prod_{j=1}^{n}\left(\frac{x-x_j}{1+x_j}\right) = \frac{2^n}{(1+x)^n}\,\frac{\omega_n(x)}{\omega_n(-1)},$$

$$\omega_n(x) = \prod_{j=1}^{n}(x-x_j),$$

[1] Certain results that will be obtained are contained as special cases in the equations (9.3.4) to (9.3.7). But since it is useful for computational purposes to retain the possibility of using the classical Jacobi polynomials, all the necessary formulas have been obtained irrespective of the indicated equations.

$$\Omega^{*\prime}(t_k) = \prod_{j\neq k}(t_k - t_j) = \prod_{j\neq k}\left(\frac{1-x_k}{1+x_k} - \frac{1-x_j}{1+x_j}\right)$$

$$= -\frac{2^{n-1}}{(1+x_k)^{n-2}}\,\frac{\omega_n'(x_k)}{\omega_n(-1)},$$

$$t - t_k = \frac{1-x}{1+x} - \frac{1-x_k}{1+x_k} = -\frac{2(x-x_k)}{(1+x)(1+x_k)},$$

$$L_k^*(t) = \frac{1}{t-t_k} J_n^{(\alpha,\beta)}(t)\,[J_n^{(\alpha,\beta)\prime}(t_k)]^{-1}$$

$$= \left(\frac{1+x_k}{1+x}\right)^{n-1} \frac{\omega_n(x)}{(x-x_k)\,\omega_n'(x_k)} = l_k(x), \qquad (9.3.34)$$

$$\psi^*(t_k) = F(x_k), \quad dx = -\frac{2}{(1+t)^2}\,dt$$

If in place of the function $\psi^*(t) = F\left(\frac{1-t}{1+t}\right) = F(x)$ we substitute into the integral (9.3.29) its interpolational representation (9.3.33), this will yield the following equation for $\varphi_e(u)$:

$$\varphi_e(u) = \sum_{k=1}^{n} 2^{1-s} F(x_k) \int_{-1}^{1} e^{iu\frac{1-t}{1+t}} L_k^*(t)(1+t)^{s-2}\,dt + R_n^*(u)$$

$$(9.3.35)$$

$$R_n^*(u) = 2^{1-s} \int_{-1}^{1} e^{iu\frac{1-t}{1+t}} r_n^*(t)(1+t)^{s-2}\,dt$$

The coefficient $L_k^*(t)$ is a polynomial of degree $n-1$ in t. Let us expand it in terms of powers of the binomial $t+1$:

$$L_k^*(t) = \sum_{l=0}^{n-1} a_l^{(k)} (t+1)^l$$

This expansion can be constructed by ordinary algebraic rules. If we introduce this expansion into (9.3.35), the computation of $\varphi_e(u)$ will be reduced to finding several simple integrals dependent on u and s:

$$\varphi_e(u)=\sum_{k=1}^{n} F(x_k)\sum_{l=0}^{n-1} a_l^{(k)}2^{1-s}\int_{-1}^{1} e^{iu\frac{1-t}{1+t}}(1+t)^{s+l-2}\,dt+R_n^*(u) \tag{9.3.36}$$

After returning to the variable x, the equations (9.3.35) and (9.3.36) become

$$\varphi_e(u)=\sum_{k=1}^{n} F(x_k)\int_{0}^{\infty} e^{iux}l_k(x)\frac{dx}{(1+x)^s}+R_n(u) \tag{9.3.37}$$

where $R_n(u) = R_n^*(u)$.

If $l_k(x)$ is expanded in powers of $\frac{1}{1+x}$,

$$l_k(x)=\left(\frac{1+x_k}{1+x}\right)^{n-1}\frac{\omega_n(x)}{(x-x_k)\,\omega_n'(x_k)}=\sum_{l=0}^{n-1} b_l^{(k)}(1+x)^{-l}$$

then after a termwise integration the representation of $\varphi_e(u)$ takes the form

$$\varphi_e(u)=\sum_{k=1}^{n} F(x_k)\sum_{l=0}^{n-1} b_l^{(k)}\int_{0}^{\infty} e^{iux}(1+x)^{-s-l}\,dx+R_n(u) \tag{9.3.38}$$

The evaluation of the integrals on the right will be discussed in the next subsection; for the present we will say a few words about the choice of the parameters α, β of the Jacobi weight. In contrast to the formula (9.3.30), where the choice of the points was coordinated with the nature of the decrease of $f(x)$ $(x\to\infty)$, we now pay special attention to the interpolation of $\psi^*(t) = F\left(\frac{1-t}{1+t}\right)$.

Under the accepted assumptions, $\psi^*(t)$ can be an arbitrary continuous and sufficiently smooth function on $[-1, 1]$. Therefore we must first of all consider the Chebyshev polynomials of the first kind $T_n(t)$ $(n = 1, 2, \ldots)$ and the roots of the polynomial of degree n that are taken for the interpolation points. The polynomials $T_n(x)$ are a special case of the Jacobi polynomials for $\alpha = \beta = -1/2$. A table of the values of the coefficients $b_l^{(k)}$ in (9.3.38) for the Chebyshev points is given in [11] (Table IV).

Of slightly less importance (though still very substantial) is the case where for the interpolation points we take the roots of a Legendre polynomial of degree n. The importance of the choice of such points is due to the fact that interpolation quadrature formulas with these points have the highest degree of accuracy for a constant weight function and are extremely useful when integrating functions without singularities. The same reference, [11] (Table III), gives a table of the coefficients $b_l^{(k)}$ for such points.

9.3e. On computing the integrals $J_{m+\alpha} = \int_0^\infty e^{iux} \times (1+x)^{-m-\alpha}\,dx$.

Here, m is a nonnegative integer and $0 \leqslant \alpha < 1$. We obtain a rule for computing only integrals with the exponential function e^{iux}. Similar rules for integrals with the trigonometric functions $\cos ux$ and $\sin ux$ can be obtained from this by separating the real and imaginary parts in it.

We can construct a recurrence rule for reducing the value of m by taking advantage of integration by parts:

$$J_{m+\alpha} = \int_0^\infty e^{iux} \frac{dx}{(1+x)^{m+\alpha}}$$

$$= -\frac{e^{iux}}{(m-1+\alpha)(1+x)^{m-1+\alpha}}\Bigg|_0^\infty$$

$$+\frac{iu}{m-1+\alpha}\int_0^\infty e^{iux}\frac{dx}{(1+x)^{m-1+\alpha}}$$

$$= \frac{1}{m-1+\alpha} + \frac{iu}{m-1+\alpha} \int_0^\infty e^{iux} \frac{dx}{(1+x)^{m-1+\alpha}},$$

$$J_{m+\alpha} = \frac{1}{m-1+\alpha} + \frac{iu}{m-1+\alpha} J_{m-1+\alpha} \qquad (9.3.39)$$

This relation enables us to confine ourselves to obtaining rules for computing J_α $(0 < \alpha \leqslant 1)$.

Let us transform J_α by the change of variable $1 + x = t$:

$$J_\alpha = e^{-iu} \int_1^\infty e^{iut} \frac{dt}{t^\alpha}$$

When $\alpha = 1$, we get the equation

$$J_1 = e^{-iu} \left[\int_1^\infty \frac{\cos ut}{t} dt + i \int_1^\infty \frac{\sin ut}{t} dt \right] = -e^{-iu} [\operatorname{Ci}(u) + i\operatorname{Si}(u)] \qquad (9.3.40)$$

Here, Ci (u) and Si (u) are the integral cosine and the integral sine for which detailed numerical tables have been compiled. When $0 < \alpha < 1$, from the equations given below we obtain the needed rule for computing the integral under study:[1]

$$\int_1^\infty e^{iux} \frac{dx}{x^\alpha} = \int_0^\infty e^{iux} \frac{dx}{x^\alpha} - \int_0^1 e^{iux} \frac{dx}{x^\alpha},$$

$$\int_0^\infty e^{iux} \frac{dx}{x^\alpha} = u^{\alpha-1} \Gamma(1-\alpha) e^{i\frac{\pi}{2}(1-\alpha)}, \quad u > 0, \qquad (9.3.41)$$

[1] See [15] concerning the relation (9.3.41).

$$\int_0^1 e^{iux} \frac{dx}{x^\alpha} = \sum_{k=0}^{\infty} \frac{(iu)^k}{k!} \int_0^1 x^{k-\alpha}\, dx = \sum_{k=0}^{\infty} \frac{(iu)^k}{k!} \frac{1}{(k+1-\alpha)}$$

$$J_\alpha = \int_0^\infty e^{iux} \frac{dx}{(1+x)^\alpha} = u^{\alpha-1}\Gamma(1-\alpha)\, e^{i\left[\frac{\pi}{2}(1-\alpha)-u\right]} + e^{-iu} \sum_{k=0}^{\infty} \frac{(iu)^k}{k!\,(k+1-\alpha)} \qquad (9.3.42)$$

Chapter 10

Highest-Accuracy Formulas for Computation

10.1 *Introduction*

The problems dealing with constructing formulas of highest accuracy for the Fourier cosine and sine transformations are studied in a similar manner but have different solutions.

We will assume the original function $f(x)$ to be representable in the form $f(x) = (1+x)^{-s}F(x)$, where $s > 1$ and the function $F(x)$ is continuous on the half axis $0 \leqslant x \leqslant \infty$.

We consider the Fourier cosine transform

$$\varphi_c(u) = \int_0^\infty f(x) \cos ux \, dx = \int_0^\infty \frac{\cos ux}{(1+x)^s} F(x) \, dx$$

For the weight function we take the factor $(1+x)^{-s} \cos ux$ and for computing the integral we construct a quadrature formula of the form

$$\int_0^\infty \frac{\cos ux}{(1+x)^s} F(x) \, dx \approx \sum_{k=1}^{n} A_k F(x_k) \qquad (10.1.1)$$

It has $2n$ parameters A_k and x_k, and we can attempt to choose them so that (10.1.1) is valid exactly when $F(x)$ is an arbitrary polynomial in $(1+x)^{-1}$ of degree $2n-1$ or, what is the same, that for the $2n$ simple fractions

$(1+x)^{-i}$ $(i=0, 1, \ldots, 2n-1)$ the following relations hold:

$$\int_0^\infty (1+x)^{-s-i}\cos ux\,dx = \sum_{k=1}^{n} A_k (1+x_k)^{-i} \quad (10.1.2)$$

$$(i=0, 1, \ldots, 2n-1)$$

These equations form a system of $2n$ equations for finding the parameters, the system being linear in A_k and nonlinear in x_k. If it is solved, then it will turn out that even for small values of n the points x_k will lie outside the half axis of integration $[0, \infty)$.

The reason for this drawback is the varying sign property[1] of the weight function $(1+x)^{-s}\cos ux$, and in order to circumvent this it is sufficient to make the weight function of constant sign.

In the problem at hand this can be done with the aid of an elementary transformation:

$$\int_0^\infty \cos ux\, f(x)\,dx = \int_0^\infty (1+\cos ux) f(x)\,dx - \int_0^\infty f(x)\,dx = I_c^1 - I_c^2$$

The integral I_c^2 has a simple form and via the transformation $x=\frac{1-t}{1+t}$ can be reduced to an integral with the

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[1] In the theory of quadrature rules of highest algebraic degree of accuracy, to which our problem reduces by the change of variable $x=\frac{1-t}{1+t}$, we know that if the weight function is of constant sign, then the points of the quadrature formula always lie inside the interval of integration.

Jacobi weight function with parameters 0, $s-2$:

$$I_{\mathrm{c}}^{2} = \int_{0}^{\infty} f(x)\,dx = \int_{0}^{\infty} \frac{F(x)}{(1+x)^{s}}\,dx$$

$$= 2^{1-s} \int_{-1}^{1} F\left(\frac{1-t}{1+t}\right)(1+t)^{s-2}\,dt$$

It can be computed with the aid of formulas of the highest algebraic degree of accuracy, the coefficients and points of which are tabulated over sufficiently broad ranges.

Let us concentrate on the integral I_{c}^{1}. It depends on the parameter u. In order to make the points and coefficients of the computation formula independent of u, perform the transformation $ux = x'$ ($u \geqslant 0$) that reduces the parameter under the cosine sign to unity and carries it from the weight to the function being integrated:

$${}^{1}_{\mathrm{c}} = \int_{0}^{\infty} (1+\cos x)\frac{1}{u} f\left(\frac{x}{u}\right) dx = \int_{0}^{\infty} (1+\cos x)\, X(x)\,dx$$

$$X(x) = \frac{1}{u} f\left(\frac{x}{u}\right) \tag{10.1.3}$$

Suppose $X(x)$ can be represented as

$$X(x) = \frac{F(x)}{(1+x)^{s}} \quad (s > 1) \tag{10.1.4}$$

where $F(x)$ is continuous on the half axis $0 \leqslant x \leqslant \infty$. Then the problem will be to establish a rule for computing the integral

$$I_{\mathrm{c}}^{1} = \int_{0}^{\infty} (1+\cos x)\, X(x) = \int_{0}^{\infty} \frac{1+\cos x}{(1+x)^{s}} F(x)\,dx \tag{10.1.5}$$

with the positive weight function $1 + \cos x$ or $(1 + \cos x) \times$ $\times\ (1 + x)^{-s}$.

10.2 *Constructing a formula of highest degree of accuracy*

Such formulas can have a variety of aspects depending primarily on the properties possessed by the integrand and on how, in connection with such properties, the weight function is chosen. We will speak of a rule suited to the set of functions $X(x)$ representable in the form (10.1.4).

For the weight we take the factor $1 + \cos x$. It takes into account the oscillations of the integrand in (10.1.3) but is not connected with the nature of decrease of $X(x)$ as $x \to \infty$, which will be taken into account when choosing the system of functions relative to which the highest degree of accuracy will be attained.

In accord with the chosen weight, let us consider the integral I_c^1 [see (10.1.5)] and let us construct for it a computation rule of the form

$$I_c^1 = \int_0^\infty (1+\cos x)\, X(x)\, dx \approx \sum_{k=1}^{n} A_k X(x_k) \quad (10.2.1)$$

We choose the parameters A_k, x_k so that the equation is exact for the functions $(1+x)^{-s-i}$ $(s > 0;\ i = 0, 1, 2, \ldots, 2n-1)$ or, what is the same thing, for all functions of the form

$$(1+x)^{-s} \sum_{j=0}^{2n-1} c_j (1+x)^{-j} = (1+x)^{-s-2n+1} P_{2n-1}(x)$$

where $P_{2n-1}(x)$ is an arbitrary polynomial of degree $2n-1$ in x:

$$\int_0^\infty \frac{1+\cos x}{(1+x)^{2n-1+s}} P_{2n-1}(x)\, dx$$

$$= \sum_{k=1}^{n} \frac{A_k}{(1+x_k)^{2n-1+s}} P_{2n-1}(x_k) = \sum_{k=1}^{n} B_k P_{2n-1}(x_k) \quad (10.2.2)$$

The problem of seeking B_k, x_k and, hence, the parameters A_k, x_k has reduced to the classical problem of constructing, on the half-open interval $[0, \infty)$, a quadrature formula of highest algebraic degree of accuracy with the positive weight function $(1 + \cos x)(1 + x)^{-2n+1-s}$.

All integrals participating in what follows are assumed to be absolutely convergent. Suppose we are considering an integral over a finite or infinite interval $[a, b]$ and suppose the weight function $p(x)$ is of constant sign and is not zero. Suppose we use the following formula to evaluate the integral:

$$\int_a^b p(x) f(x)\, dx \approx \sum_{k=1}^{n} B_k f(x_k) \qquad (10.2.3)$$

For the formula (10.2.3) to be exact for all polynomials of degree $2n - 1$, it is necessary and sufficient that the following conditions hold.

(1) The coefficients B_k have the values

$$B_k = \int_a^b p(x) \frac{\omega(x)}{(x - x_k)\, \omega'(x_k)}\, dx \qquad (10.2.4)$$

$$\omega(x) = \prod_{k=1}^{n} (x - x_k)$$

That is, the rule of integration (10.2.3) is interpolatory.

(2) The polynomial $\omega(x)$ is orthogonal on $[a, b]$ with weight $p(x)$ to any polynomial $Q(x)$ of maximum degree $n - 1$:

$$\int_a^b p(x)\, \omega(x)\, Q(x)\, dx = 0 \qquad (10.2.5)$$

It is very easy to see the necessity of these conditions. Let the equation (10.2.3) hold for all polynomials of maximum degree $2n-1$. Consider the polynomial $\omega_j(x) = \frac{\omega(x)}{(x-x_j)\,\omega'(x_j)}$. It is of degree $n-1$, and (10.2.3) should be exact for it. But $\omega_j(x)$ has the following properties: $\omega_j(x_k) = 0$ for $k \neq j$, and $\omega_j(x_j) = 1$. And so for $\omega_j(x)$ we should obtain from (10.2.3) the equation

$$\int_a^b p(x)\,\omega_j(x)\,dx = \sum_{k=1}^{n} B_k\ \omega_j(x_k) = B_j$$

which proves (10.2.4).

To see the necessity of condition (2), take a polynomial $Q(x)$ of maximum degree $n-1$. The polynomial $f(x) = \omega(x)\,Q(x)$ will be of maximum degree $2n-1$, and for it (10.2.3) must be exact; since $f(x_k) = \omega(x_k)\,Q(x_k) = 0 \cdot Q(x_k) = 0$, it follows that (10.2.3) coincides with (10.2.5).

Sufficiency is verified in just as simple a manner. Suppose conditions (1) and (2) hold. Take an arbitrary polynomial $f(x)$ of degree $2n-1$. Dividing it by $\omega(x)$ by the usual rules of algebra, we can represent f in the form $f(x) = \omega(x)\,Q(x) + r(x)$, where $Q(x)$ and $r(x)$ are polynomials of maximum degree $n-1$. Since $\omega(x_k) = 0$, we have $f(x_k) = r(x_k)$ and so

$$\int_a^b p(x)\,f(x)\,dx = \int_a^b p(x)\,\omega(x)\,Q(x)\,dx + \int_a^b p(x)\,r(x)\,dx$$

The first of the integrals on the right is equal to zero by orthogonality. Since by the first condition the rule (10.2.3) is interpolatory, it is exact for any polynomial of degree $n-1$, in particular for $r(x)$, so that

$$\int_a^b p(x)\,r(x)\,dx = \sum_{k=1}^{n} B_k r(x_k) = \sum_{k=1}^{n} B_k f(x_k)$$

And so

$$\int_a^b p(x) f(x)\,dx = \sum_{k=1}^{n} B_k f(x_k)$$

and the equation (10.2.3) holds exactly for $f(x)$.

It need only be shown that the rule (10.2.3) cannot be true for all polynomials of degree $2n$. It suffices to find at least one polynomial of that degree for which (10.2.3) cannot be exact. Let us consider $f(x) = \omega^2(x)$. This is a polynomial of degree $2n$. For it, the left-hand side of (10.2.3) is nonzero because of the constant sign of the weight $p(x)$, whereas the right-hand side is zero since $f(x_k) = 0$ when $k = 1, 2, \ldots, n$. Hence (10.2.3) cannot be exact for $f = \omega^2(x)$.

From this it follows that if the degree of accuracy $2n - 1$ is attainable in (10.2.3), then it is the highest degree. Now to prove the attainability, it suffices to establish the existence of a polynomial $\omega(x)$ that satisfies the requirement of orthogonality (10.2.5). Let us seek such a polynomial in the form of an expansion in powers of x:

$$\omega(x) = x^n + a_1 x^{n-1} + \ldots + a_n$$

The condition (10.2.5) is tantamount to the following n equations holding true:

$$\int_a^b p(x)\,\omega(x)\,x^i\,dx = 0, \quad i = 0, 1, \ldots, n-1$$

If in place of $\omega(x)$ we substitute its expansion in terms of x and to save space set $\int_a^b p(x)\,x^k\,dx = \alpha_k$, we get a system of equations for finding the coefficients $a_1, \ldots, a_n$:

$$\alpha_n + \alpha_{n-1}a_1 + \alpha_{n-2}a_2 + \ldots + \alpha_0 a_n = 0$$
$$\alpha_{n+1} + \alpha_n a_1 + \alpha_{n-1}a_2 + \ldots + \alpha_1 a_n = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\alpha_{2n-1} + \alpha_{2n-2}a_1 + \alpha_{2n-3}a_2 + \ldots + \alpha_{n-1}a_n = 0$$

To check the solvability of the system and the uniqueness of its solution, it is sufficient to establish that the homogeneous system

$$\begin{gathered}\alpha_{n-1}a_1+\alpha_{n-2}a_2+\ldots+\alpha_0 a_n=0\\ \alpha_n a_1+\alpha_{n-1}a_2+\ldots+\alpha_1 a_n=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \alpha_{2n-2}a_1+\alpha_{2n-3}a_2+\ldots+\alpha_{n-1}a_n=0\end{gathered}$$

has only a zero solution. Let us assume that $a_1, \ldots, a_n$ satisfy the equations of the system. Multiplying the equations by $a_n, \ldots, a_1$ respectively and adding, we get

$$\int_a^b p(x)(a_n+a_{n-1}x+\ldots+a_1x^{n-1})^2\,dx=0$$

Since the weight function $p(x)$ is not equivalent to zero and preserves sign, this equation is only possible when the polynomial in brackets is identically zero, which is only possible for $a_1 = a_2 = \ldots = a_n = 0$ and the homogeneous system thus only has a trivial solution.

Using the coefficients $a_1, \ldots, a_n$, we can construct a polynomial $\omega(x)$; and after finding its roots x_k ($k = 1, \ldots, n$) and computing the coefficients B_k with the aid of equations (10.2.4), we can set up the rule (10.2.3) which is exact for polynomials of degree $2n-1$. From what has been said, it is evident that such a rule will be unique since the polynomial $\omega(x)$ and the coefficients A_k are determined in unique fashion.

This result can be supplemented by proof of the fact that the roots x_k of the polynomials are distinct and all lie within the interval of integration $[a, b]$.

In the integral (10.2.2) the interval of integration $[a, b]$ is the half axis $[0, \infty)$ and the weight function is $p(x) = (1+x)^{-2n-s+1}(1+\cos x)$; it is positive everywhere except at the points $x = (2j+1)\pi$ ($j = 0, 1, \ldots$). The polynomial $\omega(x) = x^n + a_1 x^{n-1} + \ldots + a_n$ is determined by the orthogonality condition

$$\int_0^\infty (1+x)^{-2n-s+1} (1+\cos x)\, \omega(x)\, x^j\, dx = 0 \qquad (j = 0, 1, \ldots, n-1) \tag{10.2.6}$$

Its coefficients a_k can be found from the system

$$\sum_{i=0}^{n} a_i \int_0^\infty (1+x)^{-2n-s+1} (1+\cos x)\, x^{n-i+j}\, dx = 0 \qquad (j = 0, 1, \ldots, n-1;\ a_0 = 1) \tag{10.2.7}$$

The coefficients B_k that figure in (10.2.2) and in the quadrature rule of the form (10.2.3), if it is written for the integral

$$\int_0^\infty (1+x)^{-2n-s+1} (1+\cos x)\, f(x)\, dx,$$

must be found with the aid of a formula of the form (10.2.4). Now the coefficients A_k of (10.2.1), when it is exact for all functions of the form

$$X(x) = (1+x)^{-s} \sum_{j=0}^{2n-1} c_j (1+x)^{-j} = (1+x)^{-2n-s+1} P_{2n-1}(x),$$

differ from B_k, as can be seen from (10.2.2), by the factor $(1 + x_k)^{2n+s-1}$; for them we have the following values:

$$A_k = (1+x_k)^{2n+s-1} \int_0^\infty (1+x)^{-2n-s+1} (1+\cos x) \frac{\omega(x)}{(x-x_k)\,\omega'(x_k)}\, dx \tag{10.2.8}$$

A table of values of x_k and A_k for (10.2.1) when $s = 1.05\,(0.05)\,4$, $n = 1\,(1)\,10$ can be found in [11] (Table VI).

A similar table of x_k and A_k for the formula of highest degree of accuracy

$$\int_0^\infty (1+\sin x)\, f(x)\, dx \approx \sum_{k=1}^{n} A_k f(x_k) \qquad (10.2.9)$$

$$f(x) = \frac{F(x)}{(1+x)^s}, \quad |F(x)| \leqslant M$$

for the same parameters s and n can also be found in [11] (Table V).

In conclusion, one final remark is in order concerning the convergence of computational processes (10.2.1) and (10.2.9) of highest degree of accuracy as $n \to \infty$. For the sake of definiteness we will have in view (10.2.1). For our purposes it will suffice to reduce (10.2.1) to the familiar Gaussian-type rule in which the highest algebraic degree of accuracy is attained and then take advantage of the familiar theorems on the convergence of a quadrature process of this kind.

Let us take the integral I_c^1 [see (10.1.5)] in the form obtained when replacing the function $X(x)$ by its representation $X(x) = (1+x)^{-s} F(x)$.

Accordingly we now consider a rule that is equivalent to (10.2.1):

$$I_c^1 = \int_0^\infty \frac{1+\cos x}{(1+x)^s} F(x)\, dx$$

$$\approx \sum_{j=1}^{n} A_j (1+x_j)^{-s} F(x_j) = \sum_{j=1}^{n} A_j^* F(x_j) \qquad (10.2.10)$$

When constructing (10.2.1), the parameters x_k and A_k were chosen so that the equation held exactly when $X(x)$ was any function of the form $X(x) = \sum_{j=1}^{2n-1} c_j (1 + x)^{-s-j}$. For the rule (10.2.10), this is equivalent to its yielding an exact result for a function F of the form

$$F(x) = \sum_{j=1}^{2n-1} c_j (1+x)^{-j} = P_{2n-1}[(1+x)^{-1}]$$

Make a change of variable putting $\frac{1}{1+x} = t$, $x = \frac{1}{t} - 1$, $1 \geqslant t > 0$. Then (10.2.10) goes into the new rule for a finite interval [0, 1]

$$I_c^1 = \int_0^1 (1 + \cos x)\, t^{s-2} F^*(t)\, dt$$

$$\approx \sum_{j=1}^{n} A_j^* F^*(t_k) = Q_n^*(F^*) \qquad (10.2.11)$$

$$F^*(t) = F\left(\frac{1}{t} - 1\right) = F(x), \quad t_k = \frac{1}{1+x_k}$$

The equation holds exactly every time $F^*(t)$ is a polynomial of degree $2n - 1$ in t:

$$F^*(t) = P_{2n+1}(t)$$

From this it follows that (10.2.11) is a quadrature rule of the highest algebraic degree of accuracy for the interval [0, 1] and for the weight function $p(t) = (1 + \cos x)\, t^{s-2}$.

Relative to such rules we know that as $n \to \infty$ the sequence of approximate values $Q_n^*(F^*)$ converges to the exact value of the integral I_c^1 for any function $F^*(t)$ bounded on [0, 1] and such that the set of its points of discontinuity is of measure zero,[1] in particular for any function F^* bounded on [0, 1] and continuous inside that interval.

[1] For all Riemann-integrable (in the proper sense) functions on [0, 1].

This enables us to state a theorem on the convergence of the quadrature process (10.2.1) as $n \to \infty$.

Theorem 1. *If a function* X (x) *can be represented as* (10.1.4), *where* F (x) *is continuous on the half axis* $0 \leqslant x \leqslant \infty$, *then the quadrature process defined by the rule of highest degree of accuracy* (10.2.1) *converges to the exact value of the integral as* $n \to \infty$.

A similar theorem is valid for the quadrature rule (10.2.9) of the highest degree of accuracy for the Fourier sine transform.

Part Three

ISOLATING SINGULARITIES OF A FUNCTION IN COMPUTATIONS

In Part One and Part Two of this book we considered two interrelated problems: the inversion of Laplace transforms and computing Fourier integrals. Each of the problems is solved within the set of its own conditions and by specific methods, and for this reason the methods for isolating singularities of a function in computations involving these problems do not coincide and so will be considered here separately.

Chapter 11

Isolating Singularities of the Image Function $F(p)$

11.1 *Introduction*

For the sake of definiteness we will speak of methods of inversion based on computing the Mellin integral, but some of the arguments used in regard to this problem can be carried over to other inversion methods as well.

Computation of the original function $f(x)$ was based on interpolation of the image function $F(p)$ or the function

$\varphi(p)$ connected with the image by the equation $F(p) = \varphi(p)(p-a)^{-s}$. The interpolation was carried out with the aid of a polynomial in $(p-a)^{-1}$ or a more general rational function.

Recall that $F(p)$ is an analytic function of p that is regular in the half-plane $\operatorname{Re} p > \gamma$ and tends to zero as p goes to infinity in that half-plane. When computing the Mellin integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(p) e^{px} dp \qquad (11.1.1)$$

the line of integration $\operatorname{Re} p = c$ was chosen so that the inequality $c > \gamma$ held. The interpolation points were taken on the line of integration $\operatorname{Re} p = c$, either on the real axis or in the half-plane $\operatorname{Re} p > c$, and the interpolating rational function was chosen so that its poles were to the left of the straight line $\operatorname{Re} p > \gamma$ and besides so that the function tended to zero as $p \to \infty$.

We can foresee that such interpolation will, generally, be the more exact, the more smooth the variation of $F(p)$ and $\varphi(p)$ is in the half-plane $\operatorname{Re} p \geqslant c$. Now smoothness depends, firstly, on the position of the singular points of F: the farther they are removed from the half-plane $\operatorname{Re} p \geqslant c$, the more smooth will the variation of F and φ be there. The smoothness of behaviour of F and φ depends, secondly, on the nature of the singular points or, if to use a term that is not quite exact but one that describes the essence of the matter pictorially, on their influence upon the behaviour of F and φ.

Finally, the exactness of interpolation will be influenced by the behaviour of F and φ near the point at infinity[1]

[1] If we speak of the fundamental aspect of this question, all the facts mentioned above that affect the behaviour of F might be taken into account beforehand when constructing the computational method. This can be done, for example, via a choice of a system of functions underlying the interpolation. The authors rejected that approach

of the p plane, in particular the rate of decrease of $|F(p)|$ as $p \to \infty$.

We can attempt to improve the behaviour of the function $F(p)$, and with it $\varphi(p)$ as well, in the half-plane $\operatorname{Re} p \geqslant c$ if we eliminate from $F(p)$ the singular points closest to the straight line $\operatorname{Re} p = c$ or at least if we weaken their influence on the variation of F. This is usually done by decomposing the function F into two summands, $F(p) = F_1(p) + F_2(p)$, which are chosen so that the function $F_1(p)$ has those singularities that we wish to eliminate from $F(p)$, or the principal parts of those singularities when we wish to weaken them in $F(p)$. What is more, the function $F_1(p)$ must be such that it is the image function and that the original function for it, $f_1(x)$, could be found exactly. Now the second summand, $F_2(p)$, is defined by $F_2(p) = F(p) - F_1(p)$. We can expect that $F_2(p)$ will vary in the half-plane $\operatorname{Re} p = \sigma \geqslant c$ more smoothly than $F(p)$, and the original function $f_2(x)$ for it can be found approximately with greater accuracy than for $F(p)$.

11.2 *Removing and weakening the singularities of the image function $F(p)$*

11.2a. Removing poles in the image function. Suppose that at the points p_k $(k = 1, \ldots, n)$ the function $F(p)$ has poles of orders m_k $(k = 1, \ldots, n)$ respectively. Also assume that we know the polar parts of the power expansions in the neighbourhoods of the poles:

$$G_k\left(\frac{1}{p-p_k}\right) = \sum_{\nu=1}^{m_k} \frac{a_{k\nu}}{(p-p_k)^{\nu}} \qquad (11.2.1)$$

---

because it could lead to setting up a large number of narrowly specialized methods of computation and they took into account beforehand, in rough form, only the rate of decrease of F as the point p goes to infinity when they represented F in the form $F(p) = (p - a)^{-s} \times \varphi(p)$.

Set

$$F_1(p) = \sum_{k=1}^{n} G_k\left(\frac{1}{p-p_k}\right) \qquad (11.2.2)$$

and

$$F_2(p) = F(p) - F_1(p)$$

For $F_2(p)$ the points p_k $(k = 1, \ldots, n)$ will be points of regularity.

The original function for $F_1(p)$ can be found exactly (see, for example, [7]) and has the value

$$f_1(x) = \sum_{k=1}^{n} \sum_{\nu=1}^{m_k} a_{k\nu} \frac{x^{\nu-1}}{\Gamma(\nu)} e^{p_k x}$$

11.2b. Weakening the influence of branching singularities. Let us consider the case of a power branching singularity and assume that $F(p)$ is of the following form:

$$F(p) = (p-\alpha)^{\mu} G(p), \quad G(\alpha) \neq 0$$

where μ is a real number and α is a value that does not belong to the half-plane of regularity of $F(p)$ so that $\operatorname{Re} \alpha \leqslant \gamma < c$, and $G(p)$ is a function regular in the open simply connected region containing the half-plane $\operatorname{Re} p > \gamma$ and the point α. Choose an arbitrary point b lying to the left of the point α, that is, so that $\operatorname{Re}(\alpha - b) > 0$.

We will seek the function $F_1(p)$, which is close to $F(p)$ near the point α, in the form

$$F_1(p) = z^{\mu} \frac{c_0 + c_1 z + \ldots + c_m z^m}{(p-b)^r} = z^{\mu} R(z) \qquad (11.2.3)$$

where $z = p - \alpha$.

Since $F_1(p)$ must be an image function and hence must tend to zero as the point p goes to infinity, the exponent r of the divisor must satisfy the condition $r > m + \mu$.

Now let us choose the coefficients c_k ($k = 0, 1, \ldots, m$) so that in the expansions

$$G(p) = G(z+\alpha) = g_0 + g_1 z + \ldots + g_m z^m + \ldots$$

where $g_k = \frac{1}{k!} G^{(k)}(\alpha)$ and

$$R(z) = (p-b)^{-r}(c_0 + c_1 z + \ldots + c_m z^m)$$

the coefficients of powers of z from zero to m coincide.

If we note that

$$(p-b)^{-r} = (\alpha-b)^{-r}\left(1+\frac{z}{\alpha-b}\right)^{-r}$$
$$= (\alpha-b)^{-r}\sum_{n=0}^{\infty}(-1)^n \frac{r(r+1)\ldots(r+n-1)}{n!}\left(\frac{z}{\alpha-b}\right)^n$$

then it is easy to obtain the expansion of $R(z)$ in powers of z. And a comparison of the coefficients of z^k ($k = 0, 1, \ldots, m$) in the expansions of $G(z+\alpha)$ and $R(z)$ yields the following system of equations for computing c_j ($j = 0, 1, \ldots, m$):

$$\left.\begin{aligned}
&c_0 = g_0(\alpha-b)^r,\\
&c_1 - r(\alpha-b)^{-1}c_0 = g_1(\alpha-b)^r,\\
&c_2 - r(\alpha-b)^{-1}c_1 + \frac{r(r+1)}{2!}(\alpha-b)^{-2}c_0 = g_2(\alpha-b)^r,\\
&c_3 - r(\alpha-b)^{-1}c_2 + \frac{1}{2!}r(r+1)(\alpha-b)^{-2}c_1\\
&\qquad - \frac{1}{3}r(r+1)(r+2)(\alpha-b)^{-3}c_0 = g_3(\alpha-b)^r,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&c_m - r(\alpha-b)^{-1}c_{m-1} + \frac{1}{2!}r(r+1)(\alpha-b)^{-2}c_{m-2}\\
&\quad - \ldots + (-1)^m\frac{1}{m!}r(r+1)\ldots(r+m-1)\\
&\qquad\qquad \times(\alpha-b)^{-m}c_0 = g_m(\alpha-b)^r
\end{aligned}\right\}\qquad (11.2.4)$$

From this system we can successively obtain $c_0, c_1, \ldots$ $\ldots, c_m$. After finding c_k for $F_1(p)$ we get the following expression:

$$F_1(p) = (p-\alpha)^{\mu} R(p-\alpha) = \sum_{j=0}^{m} c_j \frac{(p-\alpha)^{\mu+j}}{(p-b)^r}$$

The original function of $F_1(p)$ can be computed exactly (see [7]):

$$f_1(x) = \sum_{j=0}^{m} c_j \frac{x^{r-\mu-j-1} e^{\alpha x}}{\Gamma(r-\mu-j)} {}_1F_1[r;\ r-\mu-j;\ (b-\alpha)x] \tag{11.2.5}$$

Here, ${}_1F_1$ is the confluent hypergeometric function

$${}_1F_1(\alpha;\ \beta;\ z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k) z^k}{\Gamma(\beta+k) k!}$$

At the point $p = \alpha$ the function $F(p) - F_1(p) = F_2(p)$ will have a derivative of order $m+1$ units higher than $F(p)$, and the original function $f_2(x)$ for it may be found with the aid of approximate methods with, as a rule, less effort and to a higher accuracy than for $F(p)$.

In the particular case when the exponent μ is negative, the weakening of the influence of the branching point on the variation of the function may be obtained in a more simple fashion without introducing the auxiliary point b. Choose m such that $\mu + m < 0$ and set

$$F_1(p) = \sum_{j=0}^{m} \frac{1}{j!} G^{(j)}(\alpha) (p-\alpha)^{j+\mu}$$

The original function f_1 for F_1 has a simpler aspect than that given above:

$$f_1(x) = \sum_{j=0}^{m} g_j \frac{x^{-j-\mu-1}}{\Gamma(-j-\mu)} e^{\alpha x}$$

In this case the remainder function $F_2(p) = F(p) - F_1(p)$ will be

$$F_2(p) = (p-\alpha)^{\mu}\left[G(p) - \sum_{j=0}^{m} \frac{1}{j!} G^{(j)}(\alpha)(p-\alpha)^j\right]$$

and if m is chosen so that $-1 < m + \mu < 0$, then it will be zero at the point $p = \alpha$.

Now consider the problem of weakening a logarithmic singularity. We confine ourselves to the simplest case where the image $F(p)$ is of the form

$$F(p) = (p-\alpha)^{-\nu} \ln(p-\alpha)\, G(p)$$
$$G(\alpha) \neq 0, \quad \operatorname{Re}\alpha \leqslant \gamma$$

Here, ν is a real number and $G(p)$ is a function regular in some simply connected open domain containing the half-plane $\operatorname{Re} p > \gamma$ and the point α.

The greater the value of ν, the faster the variation of F for p close to α, and in order to make the variation of F smoother (at least in the neighbourhood of the point α) we can resort to the following transformation of F.

Take an integer m such that $-\nu + m < 0$ and then construct the expansion of $G(p)$ about the point $p = \alpha$ in powers of $p - \alpha = z$ and consider the following portion of the expansion:

$$g_0 + g_1 z + \ldots + g_m z^m$$

Put

$$F_1(p) = \sum_{j=0}^{m} g_j (p-\alpha)^{-\nu+j} \ln(p-\alpha)$$

The original f_1 for F_1 is tabular (see [7]):

$$f_1(x) = \sum_{j=0}^{m} g_j \frac{1}{\Gamma(\nu-j)} x^{\nu-j-1} e^{\alpha x} [\Psi(\nu-j) - \ln x]$$

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

As for the function

$$F_2(p) = F(p) - F_1(p)$$
$$= (p-\alpha)^{-\nu} \ln (p-\alpha)\left(G(p) - \sum_{j=0}^{m} g_j (p-\alpha)^j\right)$$

we see that at the point $p = \alpha$ it has a power singularity weaker than in the case of $F(p)$ since the quantity in brackets (for values of p close to α) is of order at least $O[(p-\alpha)^{m+1}]$.

Up to now we have considered only certain typical ways of removing and weakening singularities of an image function by means of isolating the "singular part" $F_1(p)$ from that function. Now the aspect of $F_1(p)$ depends on the type of singularities of $F(p)$ and need not have the form given in our description. In Sec. 11.4 there is a small table of image functions F and their corresponding original functions f that may come in handy in certain cases when constructing $F_1(p)$.

11.3 *A remark on the increase in the rate of approach to zero of the image function F (p)*

For approximate inversion of Laplace transforms an important item is the rate at which the image function $F(p)$ tends to zero as $p \to \infty$. This can be illustrated pictorially in the problem of reducing the Mellin integral to the Fourier integral. The line of integration in the Mellin integral is the straight line $\operatorname{Re} p = c$ and we can put $p = c + i\tau$ $(-\infty < \tau < \infty)$. If we take τ for the new variable, we get the following complex Fourier integral:

$$f(x) = e^{cx} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(c+i\tau) e^{ix\tau} \, d\tau$$

Most likely, the faster $F(c + i\tau)$ decreases as $|\tau|$ grows without bound, the more convenient the integral will be computationally and the more accurate the result that can be obtained.

As was done in Sec. 11.2, we can attempt to accelerate the rate of approach to zero of $F(p)$, as $p \to \infty$, by decomposing $F(p)$ into two summands: $F(p) = F_1(p) + F_2(p)$ and choosing them so that the first, $F_1(p)$, tends to zero just as fast as $F(p)$, and the original of it is computed exactly, while the second summand, $F_2(p)$, tends to zero faster than $F(p)$. For example, when the function $F(p)$ can be represented in the form

$$F(p) = \frac{A}{p+a} + \frac{B}{p+b} + \ldots + \frac{L}{p+l} + \frac{\gamma(p)}{p^{1+\varepsilon}}$$

$$\gamma(p) \to 0 \quad (p \to \infty), \quad \varepsilon > 0$$

we can put

$$F_1(p) = \frac{A}{p+a} + \frac{B}{p+b} + \ldots + \frac{L}{p+l} \text{ and } F_2(p) = F(p) - F_1(p)$$

The original function for $F_1(p)$ is found exactly:

$$f_1(x) = Ae^{-ax} + Be^{-bx} + \ldots + Le^{-lx}$$

while the function $F_2(p)$ tends to zero faster than $F(p)$.

Similarly, when the image function $F(p)$ is of the form

$$F(p) = \sum_{i=1}^{m} \frac{A_i}{(p+a)^{k_i}} + \frac{\gamma(p)}{(p+a)^{k_m}}$$

$$\gamma(p) \to 0 \quad (p \to \infty), \quad 0 < k_1 < k_2 < \ldots < k_m$$

we can take

$$F_1(p) = \sum_{i=1}^{m} \frac{A_i}{(p+a)^{k_i}}, \quad F_2(p) = F(p) - F_1(p)$$

The original function for F_1 can also be computed exactly:

$$f_1(x) = \sum_{i=1}^{m} \frac{A_i}{\Gamma(k_i)} x^{k_i-1} e^{-ax}$$

11.4. *A table of image functions $F(p)$ and the corresponding original functions $f(x)$ for constructing the singular part of the image function $F_1(p)$*

$F(p)$	$f(x)$
$\dfrac{1}{p+a}$	e^{-ax}
$\dfrac{1}{\sqrt{p+a}}$	$\dfrac{e^{-ax}}{\sqrt{\pi x}}$
$(p+a)^{-n-1/2}$	$\dfrac{2^n x^n e^{-ax}}{1\times 3\ \dots\ (2n-1)\sqrt{\pi x}}$
$(p+a)^{-\nu}$ $\operatorname{Re}\nu > 0$	$\dfrac{x^{\nu-1}e^{-ax}}{\Gamma(\nu)}$
$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(ax)$
$(p^2+a^2)^{-n-1/2}$, $n=1, 2, \dots$	$\dfrac{1}{1\times 3\times 5\ \dots\ (2n-1)}\times$ $\times\left(\dfrac{x}{a}\right)^n J_n(ax)$
$(p^2+a^2)^{-\nu-1/2}$, $\operatorname{Re}\nu > -\dfrac{1}{2}$	$\dfrac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\dfrac{x}{2a}\right)^\nu J_\nu(ax)$
$\dfrac{1}{\sqrt{p^2-a^2}}$	$I_0(ax)$
$\dfrac{1}{(p^2-a^2)^{n+1/2}}$, $n=1, 2, \dots$	$\dfrac{1}{1\times 3\times 5\ \dots\ (2n-1)}\times$ $\times\left(\dfrac{x}{a}\right)^n I_n(ax)$

(Continued)

$F(p)$	$f(x)$
$\dfrac{1}{(p^2-a^2)^{\nu+1/2}}$, $\operatorname{Re}\nu > -\dfrac{1}{2}$	$\dfrac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\dfrac{x}{2a}\right)^{\nu} I_{\nu}(ax)$
$\dfrac{\sqrt{p+a}}{p+b}$	$\dfrac{e^{-ax}}{\sqrt{\pi x}}+\sqrt{a-b}\; e^{-bx}\times$ $\times \operatorname{erf}\sqrt{(a-b)x}$
$\dfrac{1}{(p+a)\sqrt{p+b}}$	$\dfrac{1}{\sqrt{b-a}}\, e^{-ax} \operatorname{erf}\sqrt{(b-a)x}$
$\dfrac{1}{(p+a)^{3/2}(p+b)}$	$(a-b)^{-3/2}e^{-bx} \operatorname{erf}\sqrt{(a-b)x}-$ $-\dfrac{2\sqrt{x}}{\sqrt{\pi}(a-b)}\, e^{-ax}$
$\dfrac{1}{\sqrt{(p+a)(p+b)}}$	$e^{-\frac{a+b}{2}x} I_0\left(\dfrac{a-b}{2}x\right)$
$\dfrac{1}{(p+a)^{1/2}(p+b)^{3/2}}$	$xe^{-\frac{a+b}{2}x}\left[I_0\left(\dfrac{a-b}{2}x\right)+\right.$ $\left.+I_1\left(\dfrac{a-b}{2}x\right)\right]$
$(p+a)^{1/2}(p+b)^{-3/2}$	$e^{-\frac{a+b}{2}x}\left\{(a-b)xI_1\times\right.$ $\times\left(\dfrac{a-b}{2}x\right)+$ $\left.+[1+(a-b)x]\, I_0\left(\dfrac{a-b}{2}x\right)\right\}$

(*Continued*)

$F(p)$	$f(x)$
$(p-a)^{-\nu}(p-b)^{-\mu}$, $\operatorname{Re}(\nu+\mu)<0$	$\frac{x^{\nu+\mu-1}}{\Gamma(\nu+\mu)}e^{bx}\times$ $\times\,{}_1F_1[\nu;\ \nu+\mu;\ (a-b)x]$
$\frac{\ln(p+a)}{p+a}$	$[\Psi(1)-\ln x]\,e^{-ax}$, $\Psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$
$\frac{\ln(p+a)}{(p+a)^2}$	$x\,[1+\Gamma'(1)-\ln x]\,e^{-ax}$
$\frac{\ln(p+a)}{\sqrt{p+a}}$	$-\frac{\ln x+C+\ln 4}{\sqrt{\pi x}}e^{-ax}$ $C=-\Gamma'(1)$ is Euler's constant
$\frac{\ln(p+a)}{(p+a)^{\nu}}$, $\operatorname{Re}\nu>0$	$\frac{\Psi(\nu)-\ln(x)}{\Gamma(\nu)}x^{\nu-1}e^{-ax}$
$\frac{1}{p}e^{-ap},\quad a>0$	$\begin{cases}0, & 0<x<a\\ 1, & a<x\end{cases}$
$\frac{1}{p}(1-e^{-ap}),\quad a>0$	$\begin{cases}1, & 0<x<a\\ 0, & a<x\end{cases}$
$\frac{1}{p}(e^{-ap}-e^{-bp})$, $0\leqslant a<b$	$\begin{cases}0, & 0<x<a\\ 1, & a<x<b\\ 0, & b<x\end{cases}$
$\frac{1}{p^2}(e^{-ap}-e^{-bp})$, $0\leqslant a<b$	$\begin{cases}0 & 0<x<a\\ x-a, & a<x<b\\ b-a, & b<x\end{cases}$

(Continued)

$F(p)$	$f(x)$
$\frac{1}{p^2}(e^{-ap}-e^{-bp})^2,$ $0 \leqslant a < b$	$\begin{cases} 0, & 0 < x < 2a \\ x-2a, & 2a < x < a+b \\ 2b-x, & a+b < x < 2b \\ 0, & 2b < x \end{cases}$
$\frac{1}{p+\beta} e^{-ap}, \quad a > 0$	$\begin{cases} 0, & 0 < x < a \\ e^{-\beta(x-a)}, & a < x \end{cases}$
$\frac{\lambda p+\mu}{p^2+\beta^2} e^{-ap}, \quad a > 0$	$\begin{cases} 0, \quad 0 < x < a \\ \lambda \cos[\beta(x-a)] + \\ + \mu\beta^{-1} \sin[\beta(x-a)], \quad a < x \end{cases}$
$\frac{\lambda p+\mu}{p^2-\beta^2} e^{-ap}, \quad a > 0$	$\begin{cases} 0, \quad 0 < x < a \\ \lambda \cosh[\beta(x-a)] + \\ + \mu\beta^{-1} \sinh[\beta(x-a)], \quad a < x \end{cases}$

Chapter 12

Isolating Singularities of a Function in the Fourier Transformation

Computing Fourier integrals from the values of the function f being transformed is based, just as computation of the Mellin integral is, mainly on interpolation of the function f by means of polynomials or rational functions, the interpolation being carried out either throughout the domain of integration or in its parts. The accuracy with which the integrals are computed depends both on the rule that is chosen and on the properties of the function f; ordinarily, however, the smoother the function f and the faster it tends to zero as its argument goes to infinity, the more exact the result that can be computationally obtained.

There are three directions in which we can carry out a preliminary preparation of the function f to computation of Fourier integrals:

(1) Improving the differential properties of the function f, in particular, raising the order of its differentiability.

(2) Improving the smoothness of variation of the function f. A few examples will suffice to illustrate what this means. The possibility of an arbitrarily exact and uniform approximation of a continuous function by a polynomial on a closed interval is well known. In certain cases, to attain a given accuracy of approximation it is necessary to construct a polynomial of high degree with large and strongly varying values of its derivatives. A difficulty of this nature

is encountered even in the case of an analytic function f when its singularities lie near the interval over which the approximating polynomial is being constructed.

In cases of this kind, it is natural to attempt to simplify the problem of approximation by a preliminary isolation from f of the "singular, unsmooth part" so that the computations can be performed with the more smoothly varying "remainder" . We have already encountered (in Sec. 11.2) a problem of this kind concerned with increasing the smoothness of variation.

(3) Accelerating the approach of the function f to zero when $|x|$ increases without bound.

We dwell here on two questions only: the removal of discontinuities of the first kind and the acceleration of approach to zero of the function undergoing transformation.

12.1 *Removing discontinuities of the first kind*

Let a function f be specified on an interval $[a, b]$ and let x be an interior point of this interval. Suppose that at the point x the function f has a limiting value on the right,

$$f(x+0) = \lim_{x' \to x,\ x' > x} f(x')$$

and a limiting value on the left,

$$f(x-0) = \lim_{x' \to x,\ x' < x} f(x')$$

If the values $f(x+0)$ and $f(x-0)$ exist but at least one of them is different from $f(x)$, then we say that f has a *discontinuity of the first kind at* x.

If x is one of the endpoints of the interval $[a, b]$, then when determining the discontinuity we consider only one of the limiting values: $f(x+0)$ or $f(x-0)$.

In almost the same way we determine discontinuities of the first kind of the derivatives of the function f. For example, consider the first derivative f' and assume that it exists at all points of some neighbourhood of the value x, with the possible exception of the point x itself. Also suppose that there exist the limiting values $f'(x+0)$ and $f'(x-0)$. If it turns out that $f'(x-0) \neq f'(x+0)$, then we say that the first derivative f' has at the point x a *discontinuity of the first kind with the jump* $f'(x+0) - f'(x-0)$.

For the sake of definiteness, we take the cosine transform. Suppose that f and its derivatives up to order m are everywhere continuous on the half axis $[0, \infty)$ except at q points $x_1, x_2, \ldots, x_q$ $(x_0 = 0 < x_1 < x_2 < \ldots < x_q)$ where they have discontinuities of the first kind. We denote by $k_j^{(i)}$ the magnitude of the jump of $f^{(i)}(x)$ at the point x_j:

$$k_j^{(i)} = f^{(i)}(x_j + 0) - f^{(i)}(x_j - 0)$$
$$(i = 0, 1, \ldots, m; \quad j = 1, 2, \ldots, q)$$

When it is necessary to remove the discontinuities only of the function f, it suffices to introduce a piecewise constant function $F_0(x)$ that is absolutely integrable on $[0, \infty)$ and has the same jumps as f. For such a function we can clearly take

$$F_0(x) = \sum_{j=1}^{q} [E(x - x_j) - 1]\, k_j^{(0)} \qquad (12.1.1)$$

where $E(x)$ is the "quenching function" (see p. 172).

On intervals of the half axis $[0, \infty)$ between points of discontinuity, F_0 has the values given below:

$$F_0(x) = \begin{cases} -(k_1^{(0)} + k_2^{(0)} + \ldots + k_q^{(0)}) = a_1^{(0)}, & 0 < x < x_1, \\ -(k_2^{(0)} + \ldots + k_q^{(0)}) = a_2^{(0)}, & x_1 < x < x_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ -k_q = a_q^{(0)}, & x_{q-1} < x < x_q, \\ 0, & x_q < x \end{cases} \qquad (12.1.2)$$

The cosine transform of F_0 is found very easily:

$$\int_0^\infty F_0(x)\cos px\,dx = \sum_{j=1}^{q} a_j^{(0)} \int_{x_{j-1}}^{x_j} \cos px\,dx$$

$$= \frac{1}{p}\sum_{j=1}^{q} a_j^{(0)}(\sin px_j - \sin px_{j-1}) = -\frac{1}{p}\sum_{j=1}^{q} k_j^{(0)} \sin px_j$$

The difference $\varphi_0(x) = f(x) - F_0(x)$ is everywhere continuous except at the points x_j, and at these points for it we have $\varphi_0(x_j+0) = \varphi_0(x_j-0)$. If we redefine it at the points x_j by putting $\varphi_0(x_j) = \varphi_0(x_j+0)$ there, it will be continuous everywhere on the half axis $[0, \infty)$.

Since outside the points x_j the function $F_0(x)$ is piecewise continuous, it follows that everywhere, except at the points x_j, the derivatives of all orders of $\varphi_0(x)$ and $f(x)$ coincide if the derivatives of f exist.

If we wish to remove from f only the discontinuities of the first derivative f' while leaving unchanged the discontinuities of the function f itself and of its derivatives above order one, we can take advantage of the function[1]

$$F_1(x) = \sum_{j=1}^{q} [E(x-x_j)-1]\,k_j^{(1)}(x-x_j) \qquad (12.1.3)$$

It is piecewise linear and continuous on $[0, \infty)$ and becomes identically zero when $x > x_q$. The first derivative of it,

$$F_1'(x) = \sum_{j=1}^{q} [E(x-x_j)-1]\,k_j^{(1)}$$

is piecewise constant outside the points x_j and at these points it has jumps equal to the quantities $k_j^{(1)}$.

---

[1] The reasoning behind the construction of $F_1(x)$ is so simple that we omit it.

The difference $\varphi_1(x) = f(x) - F_1(x)$ will have a derivative $\varphi_1'(x)$ everywhere for $x \neq x_j$ $(j = 1, 2, \ldots, q)$. Besides, at the points x_j we have $\varphi_1'(x_j + 0) = \varphi_1'(x_j - 0)$ and if we redefine $\varphi_1'(x)$ at the points x_j by putting $\varphi_1'(x_j) = \varphi_1'(x_j + 0)$ there, the function $\varphi_1'(x)$ will be continuous on the half axis $[0, \infty)$.

The cosine transform of the function $F_1(x)$ is found by means of simple manipulations. Since

$$\int_0^\infty [E(x - x_j) - 1](x - x_j)\cos px\,dx = \int_0^{x_j} (x_j - x)\cos px\,dx = \frac{1}{p^2}(1 - \cos px_j)$$

it follows that

$$\int_0^\infty F_1(x)\cos px\,dx = \sum_{j=1}^{q} k_j^{(1)} \int_0^\infty [E(x - x_j) - 1](x - x_j)\cos px\,dx = \frac{1}{p^2}\sum_{j=1}^{q} k_j^{(1)}(1 - \cos px_j) \qquad (12.1.4)$$

Also note that the difference

$$\varphi_{01}(x) = f(x) - F_0(x) - F_1(x)$$

will have the following properties: $\varphi_{01}(x_j + 0) = \varphi_{01}(x_j - 0)$, $\varphi_{01}'(x_j + 0) = \varphi_{01}'(x_j - 0)$, and if φ_{01} is redefined at the points x_j by the equations $\varphi_{01}(x_j) = \varphi_{01}(x_j + 0)$, then φ_{01} will be continuously differentiable for $x \geqslant 0$.

These arguments can be continued. For any $i = 0, 1, \ldots$ $\ldots, m$ put

$$F_i(x) = \sum_{j=1}^{q} [E(x-x_j)-1] \frac{(x-x_j)^i}{i!} k_j^{(i)} \qquad (12.1.5)$$

Such a function F_i is continuously differentiable with respect to x up to order $i-1$ inclusive, and the derivative of order i for $x \neq x_j$ ($j = 1, 2, \ldots, q$) has the value

$$F_i^{(i)}(x) = \sum_{j=1}^{q} [E(x-x_j)-1] k_j^{(i)}$$

and is a piecewise constant function with jumps at the points x_j, the magnitudes of the jumps being respectively $k_j^{(i)}$. The equation $F_i^{(i)}(x) \equiv 0$ holds for $x > x_q$.

With the aid of $F_i(x)$ we can remove from the function f the jumps of the ith derivative.

Now if we put

$$\varphi_{01\ldots m}(x) = f(x) - [F_0(x) + F_1(x) + \ldots + F_m(x)]$$

and then redefine $\varphi_{01\ldots m}$ at the points x_j by setting $\varphi_{01\ldots m}(x_j) = \varphi_{01\ldots m}(x_j+0)$, we get a function that is m times continuously differentiable on $[0, \infty)$. In this way we remove the discontinuities in f and its derivatives up to order m.

12.2 *Increasing the rate of approach to zero of the function undergoing transformation*

Suppose $f(x)$ tends to zero exponentially so that for a certain s we have $(x+a)^s f(x) \to C \neq 0$ $(a > 0)$. In many cases the parameter a, which depends on the behaviour of f near the origin, can be taken equal to unity. The function f can be represented as

$$f(x) = \frac{C}{(x+1)^s} + f_1(x)$$

where $f_1(x)$ tends to zero faster than $f(x)$ because $(x+1)^s \times$ $\times f_1(x) \to 0 \quad (x \to \infty)$.

The exact Fourier transform of the principal part $C(x+1)^{-s}$ of the function f may be found by the method indicated in Chap. 10.

If it turns out that the rate of approach to zero of f_1 is not sufficient for the Fourier transformation to be carried out, an attempt can in turn be made to isolate the principal part from f_1. Its aspect depends on the properties of the function f_1 but it may turn out that f_1, like f, tends to zero by a power law and from f_1 it is possible to isolate the principal part of the same type but with different values of the parameters C and s. By performing the operation of isolating the principal part several times, it is occasionally possible to construct a representation of f of the form

$$f(x) = \frac{C_1}{(x+a_1)^{s_1}} + \frac{C_2}{(x+a_2)^{s_2}} + \ldots + \frac{C_m + \gamma_m(x)}{(x+a_m)^{s_m}}$$

where

$$0 < s_1 < s_2 < \ldots < s_m \quad \text{and} \quad \gamma_m(x) \to 0 \quad (x \to \infty)$$

Now suppose that f tends to zero exponentially as $x \to \infty$ and there exists a positive number α such that $e^{\alpha x} f(x) \to$ $\to C \neq 0$. Then for f it is true that

$$f(x) = Ce^{-\alpha x} + f_1(x) \quad (\alpha > 0)$$

Here, $f_1(x)$ tends to zero faster than $f(x)$ so that the first term of the right side, $Ce^{-\alpha x}$, is the principal term.

The Fourier transformation of the principal term is fulfilled exactly:

$$\int_0^\infty e^{-\alpha x} \cos px \, dx = \frac{\alpha}{\alpha^2 + p^2}$$

$$\int_0^\infty e^{-\alpha x} \sin px \, dx = \frac{p}{\alpha^2 + p^2} \quad (\alpha > 0)$$

In Part Three we have discussed only the very simplest problems of preparing the function $F(p)$ to inversion of the Laplace transform and of the function $f(x)$ to the Fourier transformation. We believe this to be sufficient for the reader to get a general idea of preparation. Actually, what the preparation amounts to is this: one isolates from the functions $F(p)$ and $f(x)$ the singular or principal parts such that we can perform the computations for them with sufficient simplicity and arbitrary exactness.

This problem of isolation is not a standard one and if the reader encounters a case that is more complicated than those given in this book, we refer him to other texts that may be of help in preparing computations for a large number of cases (see [2] and [7]).

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